Additivity of states uniquely determined by marginals

Yi Shen [1,*] and Lin Chen [2,3,†]
[1] School of Science, Jiangnan University, Wuxi Jiangsu 214122, China
[2] School of Mathematical Sciences, Beihang University, Beijing 100191, China
[3] International Research Institute for Multidisciplinary Science, Beihang University, Beijing 100191, China

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The pure states that can be uniquely determined among all (UDA) states by their marginals are essential to efficient quantum state tomography. We generalize the UDA states from the context of pure states to that of arbitrary (whether pure or mixed) states, motivated by the efficient state tomography of low-rank states. The concept of additivity of k-UDA states for three different composite types of tensor product applies if the composite state of two k-UDA states is still uniquely determined by the k-partite marginals for the corresponding type of tensor product. We show that the additivity holds if one of the two initial states is pure and present the conditions under which the additivity holds for two mixed UDA states. One of the three composite types of tensor product is also adopted to construct genuinely multipartite entangled (GME) states. Therefore, it is effective to construct multipartite k-UDA states with genuine entanglement by uniting the additivity of k-UDA states and the construction of GME states.

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I. INTRODUCTION

In quantum mechanics the correlation between the whole and its parts reflects a significant difference from its classical counterpart [1]. A remarkable example is the Bell state, each of whose single-party reduced states is maximally mixed. This means that no useful information of each party can be obtained by local measurements. For a prepared global state, one can observe its parts by implementing measurements. More importantly, can we infer or even reconstruct the global state with given measurement results? This task is known as quantum state tomography (QST) [2]. Performing QST based on the reduced states is a common and efficient method to characterize the global state [3–5]. It is closely connected to another essential topic in quantum information, namely, the marginal problem, which stems from quantum chemistry [6]. The marginal problem asks whether there is a global state compatible with a given set of multipartite marginal reductions [7]. If the answer is positive, it is interesting to further study whether such a global state is uniquely determined. This uniqueness issue plays a core role in efficient QST [3,8], because the general case of QST requires a large number of observables as the dimension of the quantum system increases [9].

If the set of marginal reductions is generated from a given pure state, the uniqueness issue is specified as whether there is another global state sharing all the same reductions as the given state [10]. According to the scope of the discussion, this problem is divided into two cases, namely, uniquely determined among pure (UDP) states [11] and uniquely determined among all (UDA) states [12]. The former means there is no other pure state satisfying the desired requirement and the latter means there is no other state satisfying the desired requirement. Such uniquely determined states are of practical interest for the following reasons. First, they make tomography meaningful and efficient as we mentioned above. Second, they are closely related to the unique ground (UG) state of a Hamiltonian that may be obtained by engineering this Hamiltonian and then cooling down the system [13,14]. Third, the relation of the three classes of UDP, UDA, and UG states implies a hierarchy of topological order (see Fig. 4 in [15]). Thus, clarifying the relation between three such classes of states is helpful in identifying topological states with or without topological order. Topological states can be encoded as topological stabilizer codes, for example, the toric codes and the surface codes, which are widely used in quantum error correction to protect quantum information from unwanted environmental interactions (decoherence) and other forms of noise [16]. It is common to construct stabilizer codes by using graph states whose stabilizer generators are related to simple graphs [17,18]. These codes are typically defined in systems with a large number of parties. Hence, it is necessary to study the UDP and UDA states of a large number of parties. This is one of our motivations for proposing the concept of additivity, which allows us to generate UDA states of a larger number of parties.

Bipartite states generally cannot be fixed by their single-party reductions. Hence, the first nontrivial case of UDA states should be tripartite states. It was first shown that almost every pure three-qubit state is completely determined by its bipartite reduced states [10]. Subsequently, it was shown that almost every pure multipartite state (whose local dimensions are all equal) is a UDA state by its reductions of a fraction of the parties, and the fraction was specified as less

*yishen@jiangnan.edu.cn
†Corresponding author: linchen@buaa.edu.cn

than about two-thirds of the parties [12]. This fraction was further improved to be just over half the parties [19]. With the prior knowledge that the given state is pure, a stronger result ensues that almost every tripartite pure state is uniquely determined by only two of three bipartite marginals [11]. Similar conclusions were extended to the UDA states in some tripartite systems [4,20]. These results imply that only a partial reduction is sufficient to ensure the uniqueness. Thus, it is interesting to find which reductions are enough to ensure the uniqueness [21]. Note here that the term "almost every state" is adopted to characterize the set of UDP or UDA states, which means excluding a measure zero set from the overall set being considered. In some cases such measure zero set is explicit [21–23]. For example, only the n-qubit generalized Greenberger-Horne-Zeilinger (GHZ) states and their local unitary equivalents cannot be fixed by the $(n-1)$-qubit reductions [23].

Both UDP and UDA states are typically specified as pure states. Nevertheless, pure states are extremely unstable, which leads to mixed states being more common in the laboratory. This motivates us to consider the uniqueness issue in terms of arbitrary (pure or mixed) states. We define the mixed UDA states in Definition 2 by generalizing the typical definition. One can verify whether an arbitrary state is a UDA by the the flow chart depicted in Fig. 2. The QST of low-rank states has been studied recently [24–26]. Analogous to pure UDA states, mixed UDA states can also make the QST of low-rank states more efficient. In this paper we study the generation of UDA states in systems with a large number of parties and high local dimensions. Specifically, we consider three different composite types of tensor product defined in Definition 3. Each type of product corresponds to a type of system composition. We illustrate such system compositions in Fig. 3. The first two products are known as the tensor product and the Kronecker product, respectively, which can expand the number of parties and enlarge local dimensions, as shown in Figs. 3(a) and 3(b), respectively. The third one combines the first two types of products by applying the Kronecker product on a part of the subsystems, which can expand the number of parties and enlarge local dimensions simultaneously, as shown in Fig. 3(c). To characterize mixed UDA states, we extend two essential properties of pure UDA states to the case of mixed ones in Lemmas 1 and 2. The concept of additivity of UDA states for each type of product applies if the corresponding product of two UDA states is still a UDA state by the marginals of the same number of parties. By virtue of the specific expressions formulated in Lemma 3, we show that the UDA states admit additivity if one of the initial two states is pure, as stated in Theorem 1. Specifically, the additivity of UDA states holds in the context of pure states. If the initial two states are both mixed UDA states, we propose the conditions for the additivity in Lemma 4 and conjecture there is generally no additivity in this setting by Lemma 5. Furthermore, uniting the construction of GME states proposed in Ref. [27] and the additivity of UDA states, we derive an operational approach to construct multipartite UDA states with genuine entanglement in Proposition 1. We illustrate the construction process in Fig. 4. By repeating this process, we construct a class of GME states which are uniquely determined by two-particle correlations only, in Example 1.

The remainder of this paper is organized as follows. In Sec. II we clarify some notation and present necessary definitions and useful lemmas. In Sec. III we propose the concept of additivity of k-UDA states and show that k-UDA states admit the additivity in the case when one of the two initial states is pure. We further discuss the additivity of two mixed k-UDA states in this section. In Sec. IV we provide an effective method to construct genuinely entangled UDA states. A summary and prospects for future work are given in Sec. V.

II. PRELIMINARIES

In this section we clarify some notation for convenience, formulate necessary definitions of mixed UDA states and different types of tensor products, and directly extend two essential properties of pure k-UDA states to mixed k-UDA states.

First, we introduce some notation for clear expression. For any positive integer m, we denote by $[m]$ the set $\{1, 2, \dots, m\}$. Let $\mathcal{S}$ be a subset of $[m]$. Then we denote by $\mathcal{S}^c$ the complement of $\mathcal{S}$ in $[m]$, i.e., $[m]\backslash\mathcal{S}$. Suppose that $A_1, \dots, A_m$ are m systems associated with the Hilbert spaces $\mathcal{H}_{A_1}, \dots, \mathcal{H}_{A_m}$, respectively. For any subset $\mathcal{S} \subseteq [m]$, we denote the composite system $\bigotimes_{i\in\mathcal{S}} A_i$ by $A_\mathcal{S}$ associated with the Hilbert space $\bigotimes_{i\in\mathcal{S}} \mathcal{H}_{A_i}$. Next suppose $B_1, \dots, B_n$ are n other systems associated with the Hilbert spaces $\mathcal{H}_{B_1}, \dots, \mathcal{H}_{B_n}$, respectively. Let $\ell = \min\{m, n\}$ and $\mathcal{S}$ be a subset of $[\ell]$. Analogously, we denote the composite system $\bigotimes_{i\in\mathcal{S}}(A_i \otimes B_i)$ by $(AB)_\mathcal{S}$ associated with the Hilbert space $\bigotimes_{i\in\mathcal{S}}(\mathcal{H}_{A_i} \otimes \mathcal{H}_{B_i})$. For more simplicity, we denote the composite system $\bigotimes_{i\in[\ell]}(A_i \otimes B_i)$ by $(C_1, \dots, C_\ell)$, where $\ell = \max\{m, n\}$, $C_j = (A_j B_j)$ for each $1 \leqslant j \leqslant \min\{m, n\}$, and C_j is A_j or B_j for each $\min\{m, n\} < j \leqslant \ell$.

Second, we define the states that can be uniquely determined by their k-partite marginal reductions, in terms of arbitrary states rather than pure states only. Since they are generalized from the definitions for pure UDP states and pure UDA states, we also present the typical definitions of pure states as follows.

Definition 1. (i) For a pure state $|\psi\rangle$, if there is no pure state $|\phi\rangle$ $(\neq|\psi\rangle)$ having all the same k-partite marginals as $|\psi\rangle$, then $|\psi\rangle$ is called k uniquely determined among pure (k-UDP) states. (ii) For a pure state $\rho \equiv |\psi\rangle\langle\psi|$, if there is no (pure or mixed) state σ $(\neq\rho)$ having all the same k-partite marginals as ρ, then ρ is called k uniquely determined among all (k-UDA) states.

By definition, it is direct to conclude that a pure state $|\psi\rangle$ must be a k-UDP state if it is a k-UDA state. Nevertheless, the converse is not obvious. In Ref. [3] the authors constructed a four-qubit pure state that is a 2-UDP state but not a 2-UDA state. This example reveals that the set of k-UDA states is strictly included in the set of k-UDP states (see Fig. 1). Due to the existence of k-UDP states but not k-UDA states, the UDA property shows more essential uniqueness which could play a more valuable role in quantum information processing tasks, for example, the QST without prior knowledge.

According to Definition 1, the k-UDP states are required to be pure states, while the k-UDA states can be extended to the case of mixed states by supposing the initial state ρ is a mixed

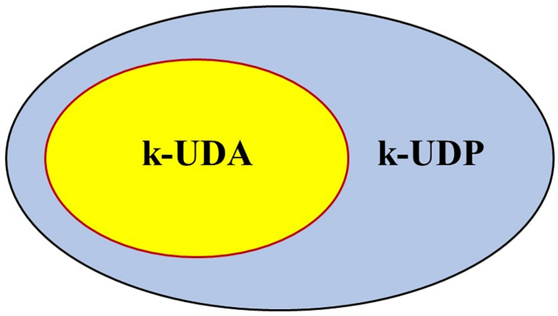

FIG. 1. The set of pure k-UDA states is strictly included in that of pure k-UDP states. An example that is a k-UDP state but not a k-UDA state has been proposed in Ref. [3].

one. We present the general definition for k-UDA states as follows.

Definition 2. For an arbitrary state ρ (whether pure or mixed), if there is no other state σ having all the same k-partite marginals as ρ, then ρ is called k-uniquely determined among all (k-UDA) states.

Note that, unless stated otherwise, the k-UDA states in this paper are arbitrary (whether pure or mixed). By Definition 2 one can verify whether a state is a k-UDA state by the flow chart depicted in Fig. 2. Here we mention that the mixed k-UDA states (rank greater than one) do exist and thus the generalization to the case of mixed states is appropriate. We give some examples to support this fact. First, it is known from Theorem 4 in [20] that almost every tripartite state ρ supported on $\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}\otimes\mathbb{C}^{d_3}$ ($d_1\geqslant d_2\geqslant d_3$) with rank no more than $\lfloor\frac{d_1}{d_3}\rfloor$ is a 2-UDA state by only two of three bipartite marginals. When focusing on the four-qubit system, this result also implies that almost all four-qubit states of rank 2 are 2-UDA states by regarding the first two qubits as a new single subsystem. Second, a tripartite 2-UDA state supported on $\mathbb{C}^d\otimes\mathbb{C}^d\otimes\mathbb{C}^d$ ($d\geqslant 3$) with rank d has been constructed in Ref. [28]. This example also contains genuine entanglement and inspires us to consider the existence of a subset of UDA states, namely, both UDA and genuinely entangled, in Sec. IV. Both the above two known examples are tripartite states, while there are few results on multipartite mixed UDA states of more than three parties. Third, in this paper we derive a method to construct mixed UDA states supported on multipartite Hilbert spaces with unequal local dimensions. We also provide a family of such states in Example 1, which directly shows the existence of multipartite mixed UDA states.

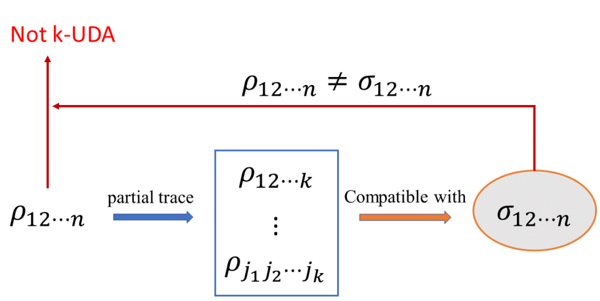

FIG. 2. The oval represents the compatible set of those k-partite marginals from $\rho_{1,\dots,n}$. The $\rho_{1,\dots,n}$ is not a k-UDA state if there exists a state $\sigma_{1,\dots,n}$ ($\neq\rho_{1,\dots,n}$) in the oval.

It follows from Ref. [19] that the marginals of fewer than half of the parties are not sufficient for the uniqueness among all states. This implies that some generic states of fewer parties may transition from a k-UDA state to not a k-UDA one as the number of parties increases, for a fixed integer k. Thus, it is necessary to construct k-UDA states in a system of more parties and higher local dimensions. In view of this, we introduce several constructions of multipartite states based on different composite types of tensor product. They were originally proposed to construct GME states and have been shown to be effective in constructing GME states [27]. These constructions are operational because the tensor product of two states can be physically realized. We formulate the definitions of three composite types of tensor product as follows.

Definition 3. Suppose that $\mathcal{H}_{A_1}\otimes\cdots\otimes\mathcal{H}_{A_m}$ and $\mathcal{H}_{B_1}\otimes\cdots\otimes\mathcal{H}_{B_n}$ are two multipartite Hilbert spaces and assume $m\leqslant n$ without loss of generality. Let ρ be an m-partite state supported on the former Hilbert space and σ be an n-partite state supported on the latter one. According to different composite types, there are generally three types of tensor products of ρ and σ.

(i) The first composite system is defined as

$$\mathcal{H}_{A_1}\otimes\cdots\otimes\mathcal{H}_{A_m}\otimes\mathcal{H}_{B_1}\otimes\cdots\otimes\mathcal{H}_{B_n}, \tag{1}$$

which is physically regarded as an $(m+n)$-partite system. The composite state denoted by $\rho\otimes\sigma$ supported on the Hilbert space in Eq. (1) is typically referred to as the tensor product of ρ and σ.

(ii) The second composite system is defined as

$$\begin{aligned}&\bigotimes_{j=1}^{m}(\mathcal{H}_{A_j}\otimes\mathcal{H}_{B_j})\bigotimes_{j=m+1}^{n}\mathcal{H}_{B_j}\\ &\quad:=\mathcal{H}_{(AB)_1}\otimes\cdots\otimes\mathcal{H}_{(AB)_m}\otimes\mathcal{H}_{B_{m+1}}\otimes\cdots\otimes\mathcal{H}_{B_n},\end{aligned} \tag{2}$$

which is physically regarded as an n-partite Hilbert space. The composite state denoted by $\rho\otimes_K\sigma$ supported on the Hilbert space in Eq. (2) is referred to as the Kronecker product of ρ and σ.

(iii) The third composite system is defined as

$$\begin{aligned}&\bigotimes_{j=1}^{\ell}(\mathcal{H}_{A_j}\otimes\mathcal{H}_{B_j})\bigotimes_{j=\ell+1}^{m}\mathcal{H}_{A_j}\bigotimes_{j=\ell+1}^{n}\mathcal{H}_{B_j}\\ &\quad:=\mathcal{H}_{(AB)_1}\otimes\cdots\otimes\mathcal{H}_{(AB)_\ell}\otimes\mathcal{H}_{A_{\ell+1}}\\ &\qquad\times\otimes\cdots\otimes\mathcal{H}_{A_m}\otimes\mathcal{H}_{B_{\ell+1}}\otimes\cdots\otimes\mathcal{H}_{B_n}\;\forall\,\ell<m,\end{aligned} \tag{3}$$

which is physically regarded as an $(m+n-\ell)$-partite Hilbert space. Assume the systems $(AB)_1,\dots,(AB)_\ell$ as $C_1,\dots,C_\ell$ for simplicity. Then we call the composite state denoted by $\rho\otimes_{K_c}\sigma$ supported on the Hilbert space in Eq. (3) as the K_c product of ρ and σ.

To better understand the three different composite types of tensor product, we illustrate Definition 3 in Fig. 3. Each panel in Fig. 3 corresponds to one composite type of tensor product defined by Definition 3. In particular, in Fig. 3(c) we observe that the proposed K_c product can expand the number of parties and enlarge local dimensions simultaneously.

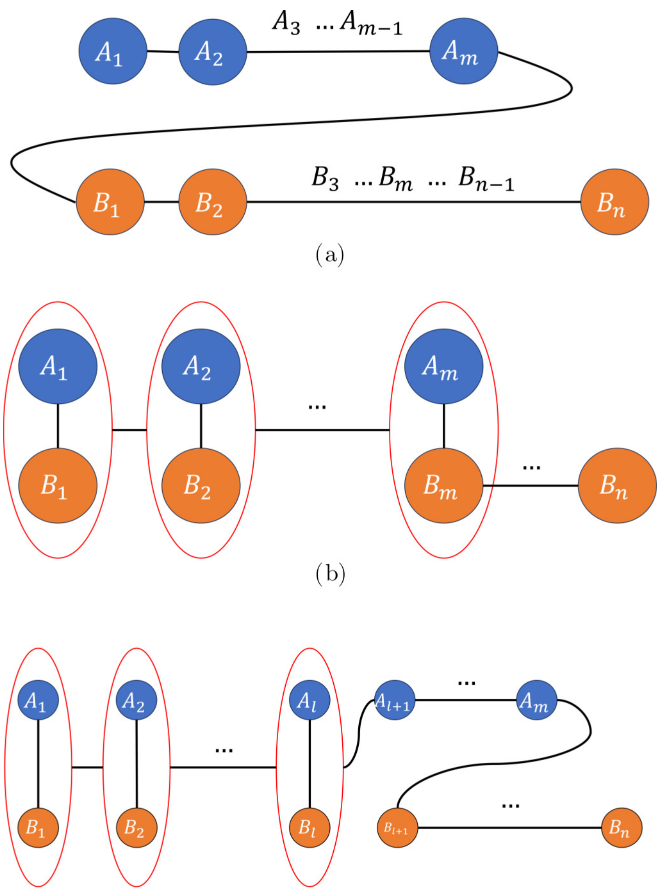

FIG. 3. Different composite types of tensor product by Definition 3. (a) Typical tensor product corresponding to Definition 3(i). It generates an $(m+n)$-partite system. (b) Kronecker product corresponding to Definition 3(ii). It increases the local dimensions of the first m subsystems. (c) Proposed K_c product corresponding to Definition 3(iii). It generates an $(m+n-\ell)$-partite system and increases part of the local dimensions.

In the final part of this section we show that two essential properties hold for arbitrary k-UDA states. First, it is known that the uniqueness among all states is maintained for a local unitary (LU) equivalence for pure k-UDA states. In Lemma 1 we show that this uniqueness is also maintained for a LU equivalence for mixed k-UDA states. Second, we derive the inclusion relation between the set of k-UDA states and that of $(k+1)$-UDA states in Lemma 2, in terms of arbitrary states. Such generalizations are direct. One may refer to the Appendix for detailed proofs of the two lemmas.

Lemma 1. If an n-partite (pure or mixed) state ρ is a k-UDA state, then any state LU equivalent to ρ is also a k-UDA state.

Lemma 2. If ρ is a k-UDA state, then ρ is also a $(k+1)$-UDA state. However, the converse is generally incorrect.

Due to Lemma 1, we conclude that all the LU equivalents of arbitrary k-UDA states remain k-UDA states. Thus, it is helpful to apply local unitary operators on the density operator before determining whether it is a k-UDA state. Lemma 2 implies that the set of k-UDA states is strictly included in the set of $(k+1)$-UDA states. We will use Lemma 2 to derive Corollary 1 in the next section. Moreover, the marginals of fewer parties are easier to measure in experiments. Thus, from an experimental perspective, it is more valuable to know the k-UDA states with the number of parties k as small as possible.

III. ADDITIVITY OF UDA STATES FOR DIFFERENT COMPOSITE TYPES OF TENSOR PRODUCT

In this section we propose the concept of additivity on k-UDA states and study whether the additivity of k-UDA states holds for different composite types of tensor product. For each composite type defined in Definition 3, the additivity of two k-UDA states holds if the corresponding product of them is still a k-UDA state.

We consider the additivity of k-UDA states for the following reasons. First, recall that the uniqueness by marginals depends on a fraction of parties sharing the marginal reductions [12,19]. As the number of all parties of the global state increases, only when the number of parties of the marginal is large enough can the uniqueness be ensured. Therefore, it is necessary to identify which states of a large number of parties can be UDA states by their k-partite marginals for k much smaller than the fraction of all parties. One direction is to enlarge the number of parties of the global system while fixing the number k. In this view, we adopt the method of constructing GME states proposed in Ref. [27] to generate k-UDA states in systems with a large number of parties. Furthermore, from an experimental perspective, measuring k-partite marginals for smaller k is more practical. It also leads us to consider fixing the number of parties of the marginals. Second, by Definition 3(i), we may expand the number of parties and enlarge local dimensions simultaneously via the proposed K_c product. Hence, this product provides a tool to study the k-UDA states in the systems of distinct local dimensions. In some problems related to the correlation between the whole and parts, whether the local dimensions of a system are the same will result in significant differences. For instance, the existence of absolutely maximally entangled states and k-uniform states in the systems of distinct local dimensions is quite different from that in the systems of equal local dimensions [29]. Third, since the K_c product is effective to construct GME states from Ref. [27], it is possible to construct multipartite k-UDA states with genuine entanglement by uniting such a construction of GME states and the additivity of k-UDA states. Due to the extensive use of GME states, it is valuable to perform QST based on the genuinely entangled k-UDA states. We will discuss the construction of genuinely entangled k-UDA states in detail in Sec. IV.

Based on Definition 3, we specifically consider whether three such composite states $\rho\otimes\sigma$, $\rho\otimes_K\sigma$, and $\rho\otimes_{K_c}\sigma$ are still k-UDA states for two k-UDA states ρ and σ. We start by assuming that one of ρ and σ is pure. Under this assumption, we may explicitly formulate the expressions of the composite states as follows, for each type of tensor product.

Lemma 3. (i) If the reduction of the system $(A_1,\dots,A_m)$ from the global state $\rho_{A_1\cdots A_m E}$ is a pure state, then the global state is in the form

$$\rho_{A_1\cdots A_m E}=|\psi\rangle\langle\psi|_{A_1\cdots A_m}\otimes\sigma_E.$$

(ii) If the reduction of system $(A_1,\dots,A_m)$ from an n-partite state $\rho_{(A_1B_1)\cdots(A_mB_m)B_{m+1}\cdots B_n}$ is a pure state, then the n-partite global state is in the form

$$\rho_{(A_1B_1)\cdots(A_mB_m)B_{m+1}\cdots B_n} = |\psi\rangle\langle\psi|_{A_1\cdots A_m} \otimes_K \gamma_{B_1\cdots B_n}.$$

(iii) If the reduction of system $(A_1,\dots,A_m)$ from an $(m+n-\ell)$-partite state $\rho_{C_1\cdots C_\ell A_{\ell+1}\cdots A_m B_{\ell+1}\cdots B_n}$ is a pure state, where $C_j = A_jB_j$ for $1 \leqslant j \leqslant \ell$ and $\ell < \min\{m,n\}$, then the $(m+n-\ell)$-partite global state is in the form

$$\rho_{C_1\cdots C_\ell A_{\ell+1}\cdots A_m B_{\ell+1}\cdots B_n} = |\psi\rangle\langle\psi|_{A_1\cdots A_m} \otimes_{K_c} \delta_{B_1\cdots B_n}.$$

Proof. (i) Suppose that the global state has the following decomposition:

$$\rho_{A_1\cdots A_m E} = \sum_j |x_j\rangle\langle x_j|_{A_1\cdots A_m E}. \tag{4}$$

Since the reduced state of the system $(A_1,\dots,A_m)$ is pure, we conclude that each $|x_j\rangle_{A_1\cdots A_m E}$ in Eq. (4) is a product vector in the bipartition $(A_1\cdots A_m)|E$. Otherwise, the reduced state of the system $(A_1,\dots,A_m)$ is of rank greater than one. Thus, we may assume

$$\rho_{A_1\cdots A_m E} = \sum_j |\phi_j\rangle\langle\phi_j|_{A_1\cdots A_m} \otimes |\alpha_j\rangle\langle\alpha_j|_E. \tag{5}$$

By calculation, $\mathrm{tr}_E(\rho_{A_1\cdots A_m E}) = \sum_j a_j|\phi_j\rangle\langle\phi_j|_{A_1\cdots A_m}$, where $a_j = \langle\alpha_j|\alpha_j\rangle$. This reduction is a pure state, and we may assume $\mathrm{tr}_E\rho_{A_1\cdots A_m E} = |\psi\rangle\langle\psi|_{A_1\cdots A_m}$. This implies that each $|\phi_j\rangle$ is proportional to $|\psi\rangle$. Then we conclude that $\rho_{A_1\cdots A_m E} = |\psi\rangle\langle\psi|_{A_1\cdots A_m} \otimes \sigma_E$.

(ii) The proof is similar to assertion (i). Suppose that the global state has the decomposition

$$\rho_{C_1\cdots C_m B_{m+1}\cdots B_n} = \sum_j |y_j\rangle\langle y_j|_{C_1\cdots C_m B_{m+1}\cdots B_n}, \tag{6}$$

where $C_j := A_jB_j$ for $1 \leqslant j \leqslant m$. Since the reduced state of the system $(A_1,\dots,A_m)$ is a pure state whose rank is only one, we conclude that each $|y_j\rangle$ in Eq. (6) is a Kronecker product of two vectors in systems $(A_1,\dots,A_m)$ and $(B_1,\dots,B_n)$, respectively. According to the similar discussion in (i), we derive that

$$\rho_{(A_1B_1)\cdots(A_mB_m)B_{m+1}\cdots B_n} = |\psi\rangle\langle\psi|_{A_1\cdots A_m} \otimes_K \gamma_{B_1\cdots B_n}.$$

(iii) Since the K_c product proposed in Definition 3(iii) is a joint use of the tensor product and the Kronecker product, we similarly derive assertion (iii) according to the discussion of the first two assertions.

This completes the proof. ■

By virtue of the essential expressions in Lemma 3, we can show that two k-UDA states admit the additivity under each type of tensor product in the case when one of the two initial states is pure.

Theorem 1. Suppose that α and β are two k-UDA states of systems $(A_1,\dots,A_m)$ and $(B_1,\dots,B_n)$, respectively. If one of α and β is pure, then (i) $\alpha\otimes\beta$ is an $(m+n)$-partite k-UDA state of the system $(A_1,\dots,A_m,B_1,\dots,B_n)$; (ii) $\alpha\otimes_K\beta$ is an ℓ-partite k-UDA state of the system $(C_1,\dots,C_\ell)$, where $\ell = \max\{m,n\}$ and $C_i := (A_iB_i)$ for $i = 1,\dots,\ell$; and (iii) $\alpha\otimes_{K_c}\beta$ is an $(m+n-\ell)$-partite k-UDA state of the system $(C_1,\dots,C_\ell,A_{\ell+1},\dots,A_m,B_{\ell+1},\dots,n)$, where $\ell \leqslant \min\{m,n\}$ and $C_i := (A_iB_i)$ for $i = 1,\dots,\ell$.

Proof. First of all, we may assume that $\alpha_{A_1\cdots A_m} = |\psi\rangle\langle\psi|$ without loss of generality. Denote by $A_{[m]}$ the m-partite system $(A_1,\dots,A_m)$ and similarly by $B_{[n]}$ the n-partite system $(B_1,\dots,B_n)$.

(i) Let $\rho_{A_{[m]}B_{[n]}} = \alpha_{A_1\cdots A_m}\otimes\beta_{B_1\cdots B_n}$. Suppose that $\sigma_{A_{[m]}B_{[n]}}$ is an $(m+n)$-partite state of the system $(A_{[m]},B_{[n]})$, which shares all the same k-partite marginals as $\rho_{A_{[m]}B_{[n]}}$. Let $\sigma_{A_{[m]}} = \mathrm{tr}_{B_{[n]}}(\sigma_{A_{[m]}B_{[n]}})$. We claim that $\sigma_{A_{[m]}}$ shares all the same k-partite marginals as $\alpha_{A_{[m]}}$ for the following reason. According to the assumption, we obtain that for any subset $\mathcal{S}\subset[m]$ with $|\mathcal{S}| = k$,

$$\begin{aligned}\sigma_{A_{\mathcal{S}}} &:= \mathrm{tr}_{A_{\mathcal{S}^c}}(\sigma_{A_{[m]}}) = \mathrm{tr}_{A_{\mathcal{S}^c}}[\mathrm{tr}_{B_{[n]}}(\sigma_{A_{[m]}B_{[n]}})] \equiv \mathrm{tr}_{A_{\mathcal{S}^c}B_{[n]}}(\sigma_{A_{[m]}B_{[n]}})\\ &= \mathrm{tr}_{A_{\mathcal{S}^c}B_{[n]}}(\rho_{A_{[m]}B_{[n]}}) = \mathrm{tr}_{A_{\mathcal{S}^c}}(\alpha_{A_{[m]}}) \equiv \alpha_{A_{\mathcal{S}^c}}.\end{aligned} \tag{7}$$

Recall that $\alpha_{A_{[m]}}$ is a k-UDA state. It follows from Eq. (7) that $\sigma_{A_{[m]}}$ has to be equal to $\alpha_{A_{[m]}}$.

Since $\alpha_{A_{[m]}}$ is pure, it follows from Lemma 3(i) that

$$\sigma_{A_{[m]}B_{[n]}} = |\psi\rangle\langle\psi|_{A_{[m]}}\otimes\gamma_{B_{[n]}}. \tag{8}$$

Similar to the discussion above, we also claim that $\gamma_{B_{[n]}}$ shares all the same k-partite marginals as $\beta_{B_{[n]}}$. Since $\beta_{B_{[n]}}$ is also a k-UDA state, it follows that $\gamma_{B_{[n]}}$ has to be equal to $\beta_{B_{[n]}}$. It follows from Eq. (8) that $\sigma_{A_{[m]}B_{[n]}}$ is the tensor product of $\alpha_{A_{[m]}}$ and $\beta_{B_{[n]}}$, which means $\sigma_{A_{[m]}B_{[n]}}$ must be identical to $\rho_{A_{[m]}B_{[n]}}$. Thus, by definition $\rho_{A_{[m]}B_{[n]}}$ is uniquely determined by its k-partite marginals.

(ii) The proof is similar to that of assertion (i). Without loss of generality, we may assume $m\leqslant n$, i.e., $n=\ell=\max\{m,n\}$. Let $\rho_{C_{[n]}} = \alpha_{A_{[m]}}\otimes_K\beta_{B_{[n]}}$, where the composite system $C_j = (A_jB_j)$ for $1\leqslant j\leqslant m$ and $C_j = B_j$ for $m+1\leqslant j\leqslant n$ for simplicity. Denote by $\rho_{\mathcal{S}}$ the reduced state of the system $C_{\mathcal{S}}\,\forall\,\mathcal{S}\subset[n]$. Suppose that $\sigma_{C_{[n]}}$ is an n-partite state compatible with the marginal set $\{\rho_{\mathcal{S}}\mid\forall\mathcal{S},\ |\mathcal{S}|=k\}$. Let $\sigma_{A_{[m]}} = \mathrm{tr}_{B_{[n]}}(\sigma_{C_{[n]}})$. We claim that $\sigma_{A_{[m]}}$ shares all the same k-partite marginals as $\alpha_{A_{[m]}}$ for the following reason. According to the assumption, for any subset $\tilde{\mathcal{S}}\subset[m]$ with $|\tilde{\mathcal{S}}| = k$ and its complementary set $\tilde{\mathcal{S}}^c = [m]-\tilde{\mathcal{S}}$, we obtain that

$$\begin{aligned}\sigma_{A_{\tilde{\mathcal{S}}}} &:= \mathrm{tr}_{A_{\tilde{\mathcal{S}}^c}}(\sigma_{A_{[m]}}) = \mathrm{tr}_{A_{\tilde{\mathcal{S}}^c}}[\mathrm{tr}_{B_{[n]}}(\sigma_{C_{[n]}})]\\ &= \mathrm{tr}_{B_{\tilde{\mathcal{S}}}}[\mathrm{tr}_{C_{\tilde{\mathcal{S}}^c}C_{[n\backslash m]}}(\sigma_{C_{[n]}})] = \mathrm{tr}_{B_{\tilde{\mathcal{S}}}}[\mathrm{tr}_{C_{\tilde{\mathcal{S}}^c}C_{[n\backslash m]}}(\rho_{C_{[n]}})]\\ &= \mathrm{tr}_{A_{\tilde{\mathcal{S}}^c}}[\mathrm{tr}_{B_{[n]}}(\rho_{C_{[n]}})] = \mathrm{tr}_{A_{\tilde{\mathcal{S}}^c}}(\alpha_{A_{[m]}}) \equiv \alpha_{A_{\tilde{\mathcal{S}}}},\end{aligned} \tag{9}$$

where $[n\backslash m] := [n]-[m] = \{m+1,\dots,n\}$. Recall that $\alpha_{A_{[m]}}$ is a k-UDA state. It follows from Eq. (9) that $\sigma_{A_{[m]}}$ has to be equal to $\alpha_{A_{[m]}}$.

Since $\sigma_{A_{[m]}}\equiv\alpha_{A_{[m]}} = |\psi\rangle\langle\psi|$ is pure, it follows from Lemma 3(ii) that

$$\sigma_{C_{[n]}} = |\psi\rangle\langle\psi|_{A_{[m]}}\otimes_K\gamma_{B_{[n]}}. \tag{10}$$

Similar to the discussion above, we claim that $\gamma_{B_{[n]}}$ shares all the same k-partite marginals as $\beta_{B_{[n]}}$. Since $\beta_{B_{[n]}}$ is also a k-UDA state, we obtain that $\gamma_{B_{[n]}} = \beta_{B_{[n]}}$. It follows from Eq. (10) that $\sigma_{C_{[n]}}$ has to be the Kronecker product of $\alpha_{A_{[m]}}$ and $\beta_{B_{[n]}}$, which means $\sigma_{C_{[n]}}$ must be identical to $\rho_{C_{[n]}}$. Thus, by definition $\rho_{C_{[n]}}$ is uniquely determined by its k-partite marginals.

(iii) For any $\ell\leqslant\min\{m,n\}$ let

$$\rho_{C_{[\ell]}A_{[m\backslash\ell]}B_{[n\backslash\ell]}} = \alpha_{[m]}\otimes_{K_c}\beta_{[n]},$$

where the composite system $C_j = A_jB_j$ for $1 \leqslant j \leqslant \ell$, $[m\backslash\ell] := [m] - [\ell]$, and $[n\backslash\ell] := [n] - [\ell]$. Suppose that $\sigma_{C_{[\ell]}A_{[m\backslash\ell]}B_{[n\backslash\ell]}}$ is an $(m+n-\ell)$-partite state compatible with all the k-partite marginals from $\rho_{C_{[\ell]}A_{[m\backslash\ell]}B_{[n\backslash\ell]}}$. Let $\sigma_{A_{[m]}} = \mathrm{tr}_{B_{[n]}}(\sigma_{C_{[\ell]}A_{[m\backslash\ell]}B_{[n\backslash\ell]}})$. We claim that $\sigma_{A_{[m]}}$ shares all the same k-partite marginals as $\alpha_{A_{[m]}}$ for the following reason. One can verify that for any subset $S \subset [m]$ with $|S| = k$,

$$\sigma_{A_{\mathcal{S}}} := \mathrm{tr}_{A_{\mathcal{S}^c}}(\sigma_{A_{[m]}}) = \mathrm{tr}_{A_{\mathcal{S}^c}}[\mathrm{tr}_{B_{[n]}}(\sigma_{C_{[\ell]}A_{[m\backslash\ell]}B_{[n\backslash\ell]}})]$$
$$= \mathrm{tr}_{C_{\mathcal{T}}A_{\mathcal{S}^c-\mathcal{T}}B_{[n]-\mathcal{T}}}(\sigma_{C_{[\ell]}A_{[m\backslash\ell]}B_{[n\backslash\ell]}}) = \mathrm{tr}_{B_{\mathcal{T}^c}}(\sigma_{C_{\mathcal{T}^c}A_{\mathcal{S}-\mathcal{T}^c}}), \tag{11}$$

where $\mathcal{T} = \mathcal{S}^c \cap [\ell]$ and $\mathcal{T}^c$ is the complementary set of $\mathcal{T}$ in $[\ell]$. Since $|\mathcal{T}^c| + |\mathcal{S} - \mathcal{T}^c| = |\mathcal{S}| = k$, it implies that $\sigma_{C_{\mathcal{T}^c}A_{\mathcal{S}-\mathcal{T}^c}}$ is a k-partite marginal of the $(m+n-\ell)$-partite state $\sigma_{C_{[\ell]}A_{[m\backslash\ell]}B_{[n\backslash\ell]}}$. According to the assumption that $\sigma_{C_{[\ell]}A_{[m\backslash\ell]}B_{[n\backslash\ell]}}$ shares all the same k-partite marginals as ρ, it follows from Eq. (11) that

$$\sigma_{A_{\mathcal{S}}} = \mathrm{tr}_{B_{\mathcal{T}^c}}(\rho_{C_{\mathcal{T}^c}A_{\mathcal{S}-\mathcal{T}^c}}) = \alpha_{A_{\mathcal{S}}} \;\forall\, |\mathcal{S}| = k. \tag{12}$$

Recall that $\alpha_{A_{[m]}}$ is a k-UDA state. It follows from Eq. (12) that $\sigma_{A_{[m]}}$ has to be equal to $\alpha_{A_{[m]}}$.

Since $\alpha_{A_{[m]}} = |\psi\rangle\langle\psi|$ is pure, it follows from Lemma 3(iii) that

$$\sigma_{C_{[\ell]}A_{[m\backslash\ell]}B_{[n\backslash\ell]}} = |\psi\rangle\langle\psi|_{A_{[m]}} \otimes_{K_c} \delta_{B_{[n]}}. \tag{13}$$

Similar to the above discussion, we claim that $\delta_{B_{[n]}}$ shares all the same k-partite marginals as $\beta_{B_{[n]}}$. Since $\beta_{B_{[n]}}$ is also a k-UDA state, we obtain that $\delta_{B_{[n]}} = \beta_{B_{[n]}}$, and $\sigma_{C_{[\ell]}A_{[m\backslash\ell]}B_{[n\backslash\ell]}}$ must be identical to $\rho_{C_{[\ell]}A_{[m\backslash\ell]}B_{[n\backslash\ell]}}$ from Eq. (13). Thus, by definition $\rho_{C_{[\ell]}A_{[m\backslash\ell]}B_{[n\backslash\ell]}}$ is uniquely determined by its k-partite marginals. This completes the proof. ■

Based on Theorem 1 and its proof, we obtain two direct corollaries as follows. First, recall that the k-UDA states are typically defined on pure states. Therefore, Theorem 1 reveals that the additivity of k-UDA states corresponding to each type of tensor product holds for the typical definition, i.e., Definition 1. Second, according to the proof, we observe that the composite states $\alpha \otimes \beta$, $\alpha \otimes_K \beta$, and $\alpha \otimes_{K_c} \beta$ can be completely determined by only a part of its k-partite marginals. Take $\alpha \otimes \beta$ in Theorem 1(i) as an example. It follows from Eqs. (7) and (8) that the composite state $\alpha \otimes \beta$ can be fixed, as long as the k-partite marginals of systems $A_{\mathcal{S}}$ and $B_{\mathcal{T}}$ can uniquely determine the two parts α and β, respectively. This implies that the k-partite marginals of systems $(A_{\mathcal{S}}B_{\mathcal{T}})$ are redundant, for $\mathcal{S} \subset [m]$, $\mathcal{T} \subset [n]$, and $|\mathcal{S}| + |\mathcal{T}| = k$. We obtain similar conclusions for Theorems 1(i) and 1(ii).

Moreover, in Theorem 1 the two initial states α and β are both k-UDA states. Here, by virtue of Lemma 2, we extend Theorem 1 to the case when α and β are k_1-UDA and k_2-UDA states, respectively, for different k_1 and k_2.

Corollary 1. Suppose α is a k_1-UDA state of the system $(A_1, \ldots, A_m)$ and β is a k_2-UDA state of the system $(B_1, \ldots, B_n)$. Let $k = \max\{k_1, k_2\}$. If one of α and β is a pure state, then (i) $\alpha \otimes \beta$ is an $(m+n)$-partite k-UDA state of the system $(A_1, , A_m, B_1, \ldots, B_n)$; (ii) $\alpha \otimes_K \beta$ is an ℓ-partite k-UDA state of the system $(C_1, \ldots, C_\ell)$, where $\ell = \max\{m, n\}$ and $C_i := A_iB_i$ for $i = 1, \ldots, \ell$; and (iii) $\alpha \otimes_{K_c} \beta$ is an $(m+n-\ell)$-partite k-UDA state of the system $(C_1, \ldots, C_\ell, A_{\ell+1}, \ldots, A_m, B_{\ell+1}, \ldots, n)$, where $\ell \leqslant \min\{m, n\}$ and $C_i := (A_iB_i)$ for $i = 1, \ldots, \ell$.

Next we consider the case when the two initial states are both mixed k-UDA states. We provide a characterization of the states that are compatible with all k-partite marginals of the composite state for different types of tensor product. Since the K_c product is a combination of the tensor product and the Kronecker product by applying the Kronecker product on a part of the subsystems, we will consider the two fundamental products for simplicity.

Lemma 4. Suppose that ρ and σ are both n-partite k-UDA states supported on the Hilbert spaces $\mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_n}$ and $\mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_n}$, respectively. Then (i) the states compatible with all k-partite marginals of $\rho \otimes \sigma$ admit an expansion as $\rho \otimes \sigma + \chi$, where the correlation matrix $\chi_{A_{[n]}B_{[n]}}$ has to be trace zero, and satisfy the conditions that (a) the reductions $\chi_{A_{[n]}}$ and $\chi_{B_{[n]}}$ are both zero and (b) any k-partite reduction of $\chi_{A_{[n]}B_{[n]}}$ is zero and (ii) the states compatible with all k-partite marginals of $\rho \otimes_K \sigma$ admit an expansion as $\rho \otimes_K \sigma + \gamma$, where the n-partite correlation matrix $\gamma_{(AB)_{[n]}}$ has to be trace zero, and satisfy that (a) the reductions $\gamma_{A_{[n]}}$ and $\gamma_{B_{[n]}}$ are both zero and (b) any k-partite reduction of $\gamma_{(AB)_{[n]}}$ is zero.

Proof. (i) Assume that α is a $2n$-partite state supported on the corresponding Hilbert space, which shares all the same k-partite marginals as $\rho \otimes \sigma$. It follows that

$$\alpha_{A_{\mathcal{S}}} := \mathrm{tr}_{A_{\mathcal{S}^c}}[\mathrm{tr}_{B_{[n]}}(\alpha)] \equiv \mathrm{tr}_{A_{\mathcal{S}^c}}[\mathrm{tr}_{B_{[n]}}(\rho \otimes \sigma)] = \rho_{A_{\mathcal{S}}},$$
$$\alpha_{B_{\mathcal{S}}} := \mathrm{tr}_{B_{\mathcal{S}^c}}[\mathrm{tr}_{A_{[n]}}(\alpha)] \equiv \mathrm{tr}_{B_{\mathcal{S}^c}}[\mathrm{tr}_{A_{[n]}}(\rho \otimes \sigma)] = \sigma_{B_{\mathcal{S}}} \tag{14}$$

for any subset $\mathcal{S} \subset [n]$ with $|\mathcal{S}| = k$. Specifically, the two sets of marginals $\{\alpha_{A_{\mathcal{S}}} \mid \forall\, |\mathcal{S}| = k\}$ and $\{\rho_{A_{\mathcal{S}}} \mid \forall\, |\mathcal{S}| = k\}$ are identical and the two sets of marginals $\{\alpha_{B_{\mathcal{S}}} \mid \forall\, |\mathcal{S}| = k\}$ and $\{\sigma_{B_{\mathcal{S}}} \mid \forall\, |\mathcal{S}| = k\}$ are identical. Since ρ and σ are both k-UDA states, it implies that the set of marginals $\{\alpha_{A_{\mathcal{S}}} \mid \forall\, |\mathcal{S}| = k\}$ is only compatible with the n-partite state ρ and the set of marginals $\{\alpha_{B_{\mathcal{S}}} \mid \forall\, |\mathcal{S}| = k\}$ is only compatible with the n-partite state σ. This means that

$$\alpha_{A_1,\ldots,A_n} := \mathrm{tr}_{B_1,\ldots,B_n}(\alpha) = \rho, \quad \alpha_{B_1,\ldots,B_n} := \mathrm{tr}_{A_1,\ldots,A_n}(\alpha) = \sigma. \tag{15}$$

Then we may regard α as a bipartite state of the system $(A_{[n]}, B_{[n]})$ whose two reduced density matrices are exactly ρ and σ, respectively. Thus, the generic expansion of α is $\rho \otimes \sigma + \chi$, where the correlation matrix χ has to be trace zero and satisfies that $\chi_{A_{[n]}} = \chi_{B_{[n]}} = 0$ [28]. Moreover, since α is compatible with all k-partite marginals of $\rho \otimes \sigma$, it requires that any k-partite reduction of χ is zero.

(ii) Assume that α is an n-partite state supported on the corresponding Hilbert space, which shares all the same k-partite marginals as $\rho \otimes_K \sigma$. It follows that

$$\alpha_{(AB)_{\mathcal{S}}} := \mathrm{tr}_{(AB)_{\mathcal{S}^c}}(\alpha) \equiv \mathrm{tr}_{(AB)_{\mathcal{S}^c}}(\rho \otimes_K \sigma) \tag{16}$$

for any subset $\mathcal{S} \subset [n]$ with $|\mathcal{S}| = k$. It then follows that

$$\alpha_{A_{\mathcal{S}}} := \mathrm{tr}_{B_{\mathcal{S}}}(\alpha_{(AB)_{\mathcal{S}}}) \equiv \mathrm{tr}_{B_{\mathcal{S}}}[\mathrm{tr}_{(AB)_{\mathcal{S}^c}}(\rho \otimes_K \sigma)] = \rho_{A_{\mathcal{S}}},$$
$$\alpha_{B_{\mathcal{S}}} := \mathrm{tr}_{A_{\mathcal{S}}}(\alpha_{(AB)_{\mathcal{S}}}) \equiv \mathrm{tr}_{A_{\mathcal{S}}}[\mathrm{tr}_{(AB)_{\mathcal{S}^c}}(\rho \otimes_K \sigma)] = \sigma_{B_{\mathcal{S}}} \tag{17}$$

for any subset $\mathcal{S} \subset [n]$ with $|\mathcal{S}| = k$. Since both ρ and σ are k-UDA states, it implies that the set of marginals $\{\alpha_{A_{\mathcal{S}}} \mid \forall\, |\mathcal{S}| = k\}$ is only compatible with the n-partite state ρ and the set of marginals $\{\alpha_{B_{\mathcal{S}}} \mid \forall\, |\mathcal{S}| = k\}$ is only compatible with the n-partite state σ. This means that

$$\alpha_{A_1,\dots,A_n} := \mathrm{tr}_{B_1,\dots,B_n}(\alpha) = \rho, \quad \alpha_{B_1,\dots,B_n} := \mathrm{tr}_{A_1,\dots,A_n}(\alpha) = \sigma. \tag{18}$$

Then we similarly obtain the generic expansion of α as $\rho \otimes_K \sigma + \gamma$, where the correlation matrix γ has to be trace zero and satisfies that $\gamma_{A_{[n]}} = \gamma_{B_{[n]}} = 0$. Moreover, since α shares all the same k-partite marginals as $\rho \otimes_K \sigma$, it implies that any k-partite marginal of γ is zero, i.e., $\gamma_{(AB)_\mathcal{S}} = 0$ for any $|\mathcal{S}| = k$.

This completes the proof. ∎

By observation of Lemma 4, the correlation matrices χ and γ are essential to characterize the states which are compatible with the marginals of $\rho \otimes \sigma$ and $\rho \otimes_K \sigma$, respectively. It is directly observed from Lemma 4 that $\rho \otimes \sigma$ and $\rho \otimes_K \sigma$ are k-UDA states if and only if the corresponding correlation matrices χ and γ under the constraints must be zero matrices. Hence, to determine the uniqueness, it is necessary to further study the existence of the two correlation matrices with required conditions. We specifically analyze the necessary condition that any k-partite reduction is zero for both multipartite correlation matrices χ and γ in Lemma 4. We find that this condition cannot restrict a multipartite Hermitian matrix of zero trace to be zero. For example, the following is a Hermitian matrix of zero trace acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$, each of whose single-body marginals is zero:

$$\begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ m_2^* & -m_1 & m_5 & -m_3 \\ m_3^* & m_5^* & -m_1 & -m_2 \\ m_4^* & -m_3^* & -m_2^* & m_1 \end{pmatrix} \tag{19}$$

for some real m_1 and complex m_2, m_3, m_4, m_5. This Hermitian matrix is nonzero if one of $m_1, \dots, m_5$ are nonzero. Next we show the general result on multipartite Hermitian matrices of zero trace.

Lemma 5. There exist infinitely many nonzero multipartite Hermitian matrices of zero trace whose k-partite reductions are all zero.

The proof of Lemma 5 is in the Appendix. The proof also shows a way to construct infinitely many nonzero multipartite Hermitian matrices of zero trace whose k-partite reductions are all zero. Lemma 5 indicates that there could be nonzero correlation matrices χ and γ satisfying the conditions given in Lemma 4 such that $\rho \otimes \sigma + \chi$ and $\rho \otimes_K \sigma + \gamma$ are positive semidefinite. Therefore, we conjecture there is generally no additivity when the two initial states are both mixed k-UDA states, for different composite types of tensor product.

IV. CONSTRUCTION OF UDA STATES WITH GENUINE MULTIPARTITE ENTANGLEMENT

The GME states are resourceful and have been widely used in various quantum information processing tasks. Thus, it is valuable to perform QST of GME states from an experimental perspective. In this view, it is necessary to consider the uniqueness issue for the GME states. It is also connected to detecting multipartite entanglement. Note that graph states are entangled pure states that exhibit complex structures of genuine multipartite entanglement. In Ref. [30] the authors considered detecting graph-state entanglement by measuring two-particle correlations only and concluded that this is impossible. For this reason, in this section we specifically study the 2-UDA states with genuine multipartite entanglement in systems with a large number of parties and high local dimensions. In other words, for such states, genuine multipartite entanglement can be detected by measuring two-particle correlations only.

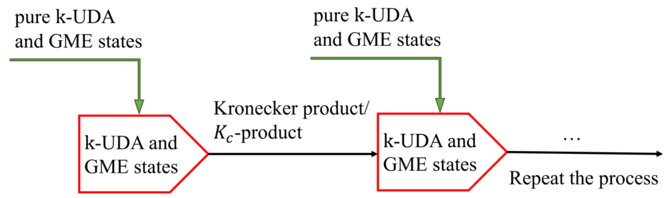

FIG. 4. Process of constructing genuinely entangled k-UDA states of a large number of parties. From Proposition 1, the output state of two genuinely entangled k-UDA states (one of which should be pure) via the Kronecker product or the K_c product is still a k-UDA state and genuinely entangled. By repeating the composition of a pure genuinely entangled k-UDA state and the output state, the number of parties and local dimensions can be continuously increased.

Inspired by the additivity of k-UDA states derived in Theorem 1 and the construction of GME states in Refs. [27,31], we find that it is effective to construct genuinely entangled k-UDA states via the Kronecker product and the K_c product given in Definitions 3(ii) and 3(iii), respectively. One can verify that the Kronecker product of two GME states is also genuinely entangled. For the K_c product, we have shown that if the range of one of the two initial states is not spanned by biproduct vectors, then the output state via the K_c product must be a GME state supported on the corresponding Hilbert space [27]. As a special but important case, the range of pure GME states cannot be spanned by biproduct vectors. Hence, if one of the two input states is a pure GME state, then the output state via the K_c product must be genuinely entangled by the above statement. Then combining such a construction with the additivity of k-UDA states, we derive an effective way to construct genuinely entangled k-UDA states as follows.

Proposition 1. Suppose that α and β are two k-UDA states of systems $(A_1, \dots, A_m)$ and $(B_1, \dots, B_n)$, respectively, and they are both genuinely entangled. If one of α and β is pure, then (i) $\alpha \otimes_K \beta$ is an ℓ-partite genuinely entangled k-UDA state, where $\ell = \max\{m, n\}$, and (ii) $\alpha \otimes_{K_c} \beta$ is an $(m + n - \ell)$-partite genuinely entangled k-UDA state for any $\ell < \min\{m, n\}$.

From Proposition 1 we can generate genuinely entangled k-UDA states in the systems of more parties and higher local dimensions using two genuinely entangled k-UDA states. By repeating the composite process, the number of parties and local dimensions can be continuously increased. We illustrate the construction process given by Proposition 1 in Fig. 4.

Next we propose a class of GME states which are 2-UDA states by virtue of Proposition 1. According to the construction process illustrated by Fig. 4, it is fundamental to discover two genuinely entangled 2-UDA states as the initial states. First, it is known that only three-qubit generalized GHZ states and their local unitary equivalents cannot be uniquely determined by the two-qubit marginal reductions [23]. Further, it was shown in Ref. [32] that there are exactly two locally inequivalent classes of genuinely entangled pure three-qubit states, namely, the class of GHZ type states and the class of

W-type states. Hence, the three-qubit W-type states are both genuinely entangled and 2-UDA states. Second, there is a mixed tripartite state constructed in Ref. [28] which is both genuinely entangled and a 2-UDA state. Based on the facts above, we propose the following class of states.

Example 1. The standard unique form of three-qubit-W type states was derived in Ref. [33] as

$$|\psi_W\rangle = \sqrt{a}|001\rangle + \sqrt{b}|010\rangle + \sqrt{c}|100\rangle + \sqrt{d}|000\rangle, \quad (20)$$

where $a, b, c > 0$ and $d \equiv 1 - (a + b + c) \geqslant 0$. It is also known from Ref. [33] that $|\psi_W\rangle$ is genuinely entangled and cannot be converted to the generalized GHZ states by invertible local operators and thus 2-UDA states.

Moreover, the following mixed states supported on $\mathcal{H}_{B_1B_2B_3} \cong \mathbb{C}^{d+1} \otimes \mathbb{C}^{d+1} \otimes \mathbb{C}^{d+1}$ are genuinely entangled and can be uniquely determined by their bipartite marginals [28]:

$$\beta_{B_1B_2B_3} = p_1\sigma_{B_1B_2B_3} + \sum_{m=2}^{d} p_m|mmm\rangle\langle mmm|. \quad (21)$$

Here $p_1 > 0$, $p_m \geqslant 0$, and

$$\sigma_{B_1B_2B_3} = \tfrac{2}{3}|\xi\rangle\langle\xi| + \tfrac{1}{3}|111\rangle\langle 111| \quad (22)$$

via $|\xi\rangle = \frac{1}{2}|010\rangle + \frac{1}{2}|110\rangle + \frac{1}{\sqrt{2}}|001\rangle$.

Let $\alpha^1_{A_1A_2A_3}$ and $\alpha^2_{A_1A_2A_3}$ be two three-qubit W-type states and $\beta_{B_1B_2B_3}$ be a state given by Eq. (21). It follows from Proposition 1 that $\alpha^1_{A_1A_2A_3} \otimes_K \alpha^2_{A_1A_2A_3}$ and $\alpha^1_{A_1A_2A_3} \otimes_{K_c} \alpha^2_{A_1A_2A_3}$ are pure genuinely entangled 2-UDA states and $\alpha^1_{A_1A_2A_3} \otimes_K \beta_{B_1B_2B_3}$ and $\alpha^1_{A_1A_2A_3} \otimes_{K_c} \beta_{B_1B_2B_3}$ are mixed genuinely entangled 2-UDA states. By repeating the composite process, we generate a class of genuinely entangled 2-UDA states from a three-qubit W-type state and a tripartite state given by Eq. (21).

Due to the uniqueness, the genuine multipartite entanglement in the states given by Example 1 can be detected by measuring two-particle correlations only, which is experimentally realizable. Note that the authors in Ref. [34] provided a construction of pure states which are genuinely entangled and 2-UDA states in the systems of any number of parties greater than 4. Differently, Proposition 1 also works for constructing mixed states that have the desired properties.

V. CONCLUSION

The pure states that can be completely determined by their marginals are essential to the efficient QST. In this paper we generalized the definition of k-UDA states to the context of arbitrary (whether pure or mixed) states, motivated by the efficient QST of low-rank states. Similar to pure k-UDA states, we have shown that for mixed k-UDA states, the UDA property is also maintained for a LU equivalence and "k-UDA" also implies "$(k+1)$-UDA." Due to the demand for k-UDA states with a large number of parties with a small number k, we considered the additivity of k-UDA states via three different composite types of tensor product. Two k-UDA states admit the additivity under each type of tensor product if the composite state for the corresponding product is still a k-UDA state. We have shown that for each type of tensor product, the additivity holds when one of the two initial k-UDA states is pure. Specifically, this implies that the additivity holds for the typical definition of pure states. We also proposed specific conditions to verify the additivity of two mixed k-UDA states. However, we conjectured there is generally no additivity for two mixed k-UDA states. Since one of the three composite types, namely, the K_c product, was adopted to construct GME states, we derived an operational and effective method to construct k-UDA states with genuine entanglement in systems with a large number of parties, by uniting the construction of GME states and the additivity of k-UDA states. Using this method, we constructed a class of concrete GME states which are uniquely determined by two-particle correlations only.

Future work is to reveal more interesting properties of mixed k-UDA states and study the detection of genuine multipartite entanglement based on mixed UDA states. Another interesting direction is to investigate the relation between the existence of mixed k-UDA states and the proportion of the rank to the whole dimension of the global system. This proportion cannot be high according to the discussion in Ref. [35] on the ranks of the global states compatible with a given set of marginal reductions. Nevertheless, the lack of a nontrivial upper bound on the ranks of possible k-UDA states needs further clarification.

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APPENDIX: PROOFS OF USEFUL LEMMAS

Proof of Lemma 1. We prove it by contradiction. Suppose that ρ is an n-partite k-UDA state. Next assume that σ is another n-partite state which is LU equivalent to ρ but not a k-UDA state. It follows that $\sigma = (U_1 \otimes \cdots \otimes U_n)\rho(U_1 \otimes \cdots \otimes U_n)^\dagger$ for some unitary operators $U_1, \ldots, U_n$. Due to the assumption, we conclude by definition that there exists an n-partite state α ($\neq \sigma$) all of whose k-partite marginals are the same as σ. Let $\tilde{\alpha} = (U_1 \otimes \cdots \otimes U_n)^\dagger \alpha (U_1 \otimes \cdots \otimes U_n)$. It is obvious that $\tilde{\alpha} \neq \rho$. However, one can verify that for any subset $\mathcal{S} \subset [n]$ with $|\mathcal{S}| = k$,

$$\begin{aligned}
\tilde{\alpha}_{\mathcal{S}} &:= \mathrm{tr}_{\mathcal{S}^c}(\tilde{\alpha}) \\
&= \mathrm{tr}_{\mathcal{S}^c}[(U_1 \otimes \cdots \otimes U_n)^\dagger \alpha (U_1 \otimes \cdots \otimes U_n)] \\
&= \left(\bigotimes_{j\in\mathcal{S}} U_j\right)^\dagger \mathrm{tr}_{\mathcal{S}^c}(\alpha) \left(\bigotimes_{j\in\mathcal{S}} U_j\right) \\
&= \left(\bigotimes_{j\in\mathcal{S}} U_j\right)^\dagger \mathrm{tr}_{\mathcal{S}^c}(\sigma) \left(\bigotimes_{j\in\mathcal{S}} U_j\right) \\
&= \mathrm{tr}_{\mathcal{S}^c}[(U_1 \otimes \cdots \otimes U_n)^\dagger \sigma (U_1 \otimes \cdots \otimes U_n)] \\
&= \mathrm{tr}_{\mathcal{S}^c}(\rho) = \rho_{\mathcal{S}}. \qquad (A1)
\end{aligned}$$

This implies that $\tilde{\alpha}$ shares all the same k-partite marginals as ρ, and we obtain a contradiction. Thus, we conclude that every state that is LU equivalent to ρ is also a k-UDA state. This completes the proof. ■

Proof of Lemma 2. We prove it by contradiction. Assume that ρ is an n-partite state which is a k-UDA state but not a $(k+1)$-UDA state. Due to the assumption, this implies by definition that there is a state σ ($\neq\rho$) having all the same $(k+1)$-partite marginals as ρ, that is, $\rho_{\mathcal{S}} = \sigma_{\mathcal{S}}$ for any subset $\mathcal{S} \subset [n]$ with $|\mathcal{S}| = k+1$. Further, any k-partite marginal can be generated from some $(k+1)$-partite marginal by tracing one more subsystem. It follows that

$$\mathrm{tr}_{j\in\mathcal{S}}(\rho_{\mathcal{S}}) = \mathrm{tr}_{j\in\mathcal{S}}(\sigma_{\mathcal{S}})\,\forall\,\mathcal{S}. \tag{A2}$$

According to Eq. (A2), we conclude that σ also has all the same k-partite marginals as ρ. This contradicts the assumption that ρ is a k-UDA state, and thus ρ has to be a $(k+1)$-UDA state. For the converse, one can verify it is incorrect by the following example. It is known that generic three-qubit pure states are 2-UDA states [10] but cannot be fixed by their single-body marginals. This completes the proof. ■

Proof of Lemma 5. Suppose that χ is an n-partite Hermitian matrix of zero trace acting on the Hilbert space $\mathcal{H}_{A_1}\otimes\cdots\otimes\mathcal{H}_{A_n}$ and each k-partite reduction of χ is zero. We formulate the matrix form of χ as

$$\chi = \sum_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}} m_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}} |i_1,\dots,i_n\rangle\langle j_1,\dots,j_n|, \tag{A3}$$

where both $\{|i_\ell\rangle : |0\rangle,\dots,|d_\ell-1\rangle\}$ and $\{|j_\ell\rangle : |0\rangle,\dots,|d_\ell-1\rangle\}$ are the computational basis of the lth subspace $\mathcal{H}_{A_\ell}$ for any $l=1,\dots,n$. Also the matrix elements satisfy that $m_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}} = m^*_{\substack{j_1,\dots,j_n\\ i_1,\dots,i_n}}$ because χ is Hermitian. For any subset $\mathcal{S}\subset[n]$ with $|\mathcal{S}|=k$, the k-partite reduction $\chi_{A_{\mathcal{S}}}$ can be calculated as

$$\begin{aligned}\chi_{A_{\mathcal{S}}} &:= \mathrm{tr}_{A_{\mathcal{S}^c}}(\chi)\\ &= \sum_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}} \langle j_{\mathcal{S}^c}|i_{\mathcal{S}^c}\rangle m_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}} \alpha_{ij}^{s_1,\dots,s_k}\\ &= \sum_{\substack{i_{s_1},\dots,i_{s_k}\\ j_{s_1},\dots,j_{s_k}}} \left(\sum_{i_{\mathcal{S}^c}=j_{\mathcal{S}^c}} m_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}}\right)\alpha_{ij}^{s_1,\dots,s_k},\end{aligned} \tag{A4}$$

where $\alpha_{ij}^{s_1,\dots,s_k} = |i_{s_1},\dots,i_{s_k}\rangle\langle j_{s_1},\dots,j_{s_k}|$, $s_1,\dots,s_k\in\mathcal{S}$, and $i_{\mathcal{S}^c}$ denotes a tuple $(i_{t_1},\dots,i_{t_{n-k}})$ for $t_1,\dots,t_{n-k}\in\mathcal{S}^c$ and similarly for $j_{\mathcal{S}^c}$. Due to the assumption that $\chi_{A_{\mathcal{S}}}=0$ for any subset $\mathcal{S}$, from Eq. (A4) we obtain that for any subset $\mathcal{S}$, each sum $\sum_{i_{\mathcal{S}^c}=j_{\mathcal{S}^c}} m_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}}$ is zero. This requirement cannot restrict Hermitian χ to be a zero matrix. For example, the following is such a nonzero Hermitian matrix:

$$\begin{aligned}\chi = \sum_{i_\ell\neq j_\ell\,\forall\,\ell} \big(&c_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}} |i_1,\dots,i_n\rangle\langle j_1,\dots,j_n|\\ &+ c^*_{\substack{i_1,\dots,i_n\\ j_1,\dots,j_n}} |j_1,\dots,j_n\rangle\langle i_1,\dots,i_n|\big).\end{aligned} \tag{A5}$$

One can verify that each $(n-1)$-partite reduction of the Hermitian χ in Eq. (A5) is zero, and thus each k-partite reduction is zero for any $k<n$. This completes the proof. ■

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