Coherence in multipath interference via quantum Fisher information

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Wave-particle duality, as complementarity displayed in interferometry, has been investigated extensively from both theoretical and experimental perspectives. In this work, we quantify coherence in multipath interference via quantum Fisher information and illuminate its basic properties. We prove that any rational quantifier of coherence satisfying monotonicity under incoherent operations is nonincreasing in the presence of path detectors. This quantitatively implies that increase on the path information will lead to the loss of coherence or interference (wave feature), which is consistent with wave-particle duality. Furthermore, we provide an operational illustration of wave feature as quantum Fisher information of phase shift parameters encoded in each interference path. By dividing the variance of a quantum state relative to a von Neumann measurement into quantum uncertainty and classical uncertainty, we establish a coherence-predictability-correlations triality relation, with coherence (wave feature) quantified by quantum Fisher information, predictability quantified by purity of a classical probability distribution, and correlations quantified by classical uncertainty relative to the von Neumann measurement determined by the interference paths. Finally, we make a comparison between this triality relation and the wave-particle-mixedness triality relation in the literature.

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I. INTRODUCTION

Wave-particle duality, as a manifestation of Bohr's complementarity, is one of the most intriguing features of quantum mechanics [1,2]. Its quantitative study was initiated by Wootters and Zurek [2]. Subsequently, lots of theoretical and experimental researches on wave-particle duality have emerged and a variety of complementarity relations have been established. The key to quantitatively formulate wave-particle duality is to find reasonable measures of the wave feature and the particle feature. In two-path interference, the wave feature is often quantified by the conventional fringe visibility [3]. In terms of this measure, a visibility-predictability duality relation was established in Ref. [4], and some visibility-distinguishability duality relations involving path detectors were derived in Refs. [5–7].

However, some problems of the conventional fringe visibility in quantifying wave feature in multipath interference were pointed out in Refs. [8–10] from both theoretical and experimental aspects, and numerous efforts have been made to find proper measures of the wave feature in multipath interference. Some criteria for bona fide measures of visibility and predictability were proposed in Refs. [11,12], some coherence-distinguishability duality relations involving the l_1 norm as a measure of the wave feature were established in Refs. [13,14]. Later, several measures of coherence, such as coherence in terms of the l_1 norm, the l_2 norm (Hilbert-Schmidt norm), relative entropy, and Wigner-Yanase skew information, have been employed to quantify the wave feature in multipath interference, and various wave-particle duality relations have been established [10,15–23].

Wave-particle duality has been further investigated and discussed in more general settings. For example, the studies on wave-particle duality relation have been extended from the symmetric beam interference to asymmetric cases [24–26], and from orthogonal pointers to nonorthogonal pointers and even nondistinguishable pointers [27,28]. Experimental investigations of wave-particle duality relations have also received much attention [29–35].

Most wave-particle duality relations are expressed in terms of inequalities, which suffer from the problem that simultaneous increase or decrease of both the wave feature and the particle feature may be possible. Indeed, such a phenomenon has been observed under certain conditions as verified experimentally in Ref. [32]. Thus it is desirable to complete the conventional wave-particle duality (in the form of inequalities) and obtain complete knowledge of complementarity. So far, some complete complementarity relations have been established, in which correlations with other systems play a vital role. For example, a visibility-predictability-entanglement triality relation in two-path interference was established in Ref. [36], with entanglement as a key entry for completing the wave-particle duality. By noticing that

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entanglement connects distinguishability with predictability, a coherence-predictability-entanglement triality relation in multipath interference was established in Ref. [37]. From the viewpoint of uncertainty decomposition, several triality relations involving quantum uncertainty, classical uncertainty, and predictability were obtained in Ref. [38], with quantum uncertainty quantified by coherence via Wigner-Yanase skew information, the l_1 norm, the l_2 norm, and relative entropy. Recently, an exact complementarity relation called the wave-particle-mixedness relation was derived in Ref. [39], in which the wave feature is measured by state uncertainty, the particle feature is measured by path certainty, and mixedness is measured by linear entropy.

In this work, we quantify coherence via quantum Fisher information [40] and then employ this coherence to quantify the wave feature in multipath interference. We explore its basic properties and operational meaning in metrology and further establish a coherence-predictability-correlations triality relation.

The remainder of the work is structured as follows. In Sec. II, we employ coherence based on quantum Fisher information as an interference visibility measure and illustrate its operational implication in metrology. In Sec. III, we introduce a quantifier of quantum correlations via coherence difference and establish a coherence-predictability-correlations triality relation. In Sec. IV, we make a comparison between the triality relation established in this work and the wave-particle-mixedness triality relation established in Ref. [39]. Finally, we conclude with a summary in Sec. V.

II. MULTIPATH INTERFERENCE VIA QUANTUM FISHER INFORMATION

In this section, we first recall the coherence measure via quantum Fisher information and its basic properties. Then we employ it to quantify the wave feature in multipath interference and explore its operational implication in quantum metrology.

A. Coherence via quantum Fisher information

For a quantum state ρ in an n-dimensional Hilbert space H with $\{|j\rangle : j = 1, 2, \dots, n\}$ being an orthornormal basis (computational basis, reference basis), a convenient measure of coherence of ρ relative to $\{|j\rangle : j = 1, 2, \dots, n\}$ (equivalently, the corresponding von Neumann measurement $\Pi = \{\Pi_j = |j\rangle\langle j| : j = 1, 2, \dots, n\}$) may be defined via quantum Fisher information as [40–42]

$$F(\rho, \Pi) = \sum_j F(\rho, |j\rangle\langle j|), \tag{1}$$

where

$$F(\rho, A) = \tfrac{1}{4}\mathrm{tr}(\rho L^2)$$

is the quantum Fisher information of ρ (relative to the observable A), and L is the symmetric logarithmic derivative determined by

$$i[\rho, A] = \tfrac{1}{2}(L\rho + \rho L).$$

Equation (1) is a natural extension of the coherence quantifier involving Wigner-Yanase skew information introduced in Refs. [43,44].

$F(\rho, \Pi)$ has the following properties [41,42].

(i) $0 \leqslant F(\rho, \Pi) \leqslant 1 - 1/n$. $F(\rho, \Pi) = 0$ if and only if ρ is a diagonal state (incoherent state) in the reference basis $\{|j\rangle : j = 1, 2, \dots, n\}$, i.e., $\sum_j \Pi_j \rho \Pi_j = \rho$. $F(\rho, \Pi) = 1 - 1/n$ if and only if $\rho = |\psi\rangle\langle\psi|$, with $|\psi\rangle = \sum_j e^{i\theta_j}|j\rangle/\sqrt{n}$ being a maximally superposing (coherent) state. Here θ_j are any real constants.

(ii) $F(U\rho U^\dagger, \Pi) = F(\rho, U^\dagger \Pi U)$ for any unitary operator U.

(iii) $F(\rho, \Pi)$ is nonincreasing under any incoherent operation in the sense that

$$F(\Phi(\rho), \Pi) \leqslant F(\rho, \Pi)$$

for any operation Φ mapping incoherent states to incoherent states.

(iv) $F(\rho, \Pi)$ is convex in ρ in the sense that

$$F\left(\sum_l p_l \rho_l, \Pi\right) \leqslant \sum_l p_l F(\rho_l, \Pi)$$

for any states ρ_l and numbers $p_l \geqslant 0$, $\sum_l p_l = 1$.

It is obvious that $F(\rho, \Pi)$ satisfies the criteria for a measure of quantum uncertainty as postulated in Ref. [45]. Consequently, $F(\rho, \Pi)$ can be naturally interpreted as quantifying quantum uncertainty of the von Neumann measurement Π in the state ρ.

Let $\rho = \sum_l \lambda_l |\phi_l\rangle\langle\phi_l|$ be the spectral decomposition of ρ, with λ_l being the eigenvalues and $|\phi_l\rangle$ being the corresponding normalized eigenvectors, and let $\Pi = \{\Pi_j = |j\rangle\langle j| : j = 1, 2, \dots, n\}$ be a von Neumann measurement, then by use of a result in Ref. [46], $F(\rho, \Pi)$ can be evaluated as

$$\begin{aligned} F(\rho, \Pi) &= \sum_{jll'} \frac{(\lambda_l - \lambda_{l'})^2}{2(\lambda_l + \lambda_{l'})} |\langle\phi_l|j\rangle|^2 |\langle\phi_{l'}|j\rangle|^2 \\ &= 1 - \sum_{jll'} \frac{2\lambda_l \lambda_{l'}}{\lambda_l + \lambda_{l'}} |\langle\phi_l|j\rangle|^2 |\langle\phi_{l'}|j\rangle|^2. \end{aligned} \tag{2}$$

In the next subsection, we use the coherence via quantum Fisher information to quantify the wave feature in multipath interference.

B. Interference via quantum Fisher information

Consider an n-path interference as depicted in Fig. 1. The interference paths can be described by the von Neumann measurement $\Pi = \{\Pi_j = |j\rangle\langle j| : j = 1, 2, \dots, n\}$. Let ρ be the state after the beam splitter. Various efforts have been devoted to quantifying the wave feature of the state ρ. Dürr suggested that any rational interference visibility (wave feature) quantifier $V(\rho|\Pi)$ should satisfy the following criteria [11].

(i) $V(\rho|\Pi)$ reaches its global minimum if the state $\rho = |j\rangle\langle j|$ for some j, and it reaches its global maximum if ρ is a pure state satisfying $\rho_{jj} = \langle j|\rho|j\rangle = 1/n$ for $j = 1, 2, \dots, n$.

(ii) $V(\rho|\Pi)$ is invariant under permutations of the path labels, i.e., $V(\rho|P\Pi P^\dagger) = V(\rho|\Pi)$ for any permutation matrix

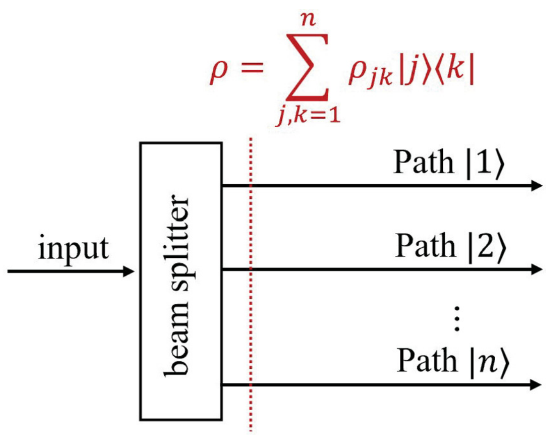

FIG. 1. Schematic of n-path interference without path detectors.

P (relative to the orthonormal basis $\{|j\rangle : j = 1, 2, \ldots, n\}$), and $P\Pi P^\dagger = \{P|j\rangle\langle j|P^\dagger : j = 1, 2, \ldots, n\}$.

(iii) $V(\rho|\Pi)$ is convex in ρ.

(iv) $V(\rho|\Pi)$ is a continuous function of the matrix elements of ρ.

From the general rationale of wave-particle duality, we expect that any reasonable quantifier of wave feature in multipath interference compatible with the complementary principle should not increase when one adds path detectors to the interference paths, which may acquire path information and thus increase path distinguishability (particle feature). This motivates us to suggest the following property for a quantifier of the wave feature.

(v) $V(\rho|\Pi)$ does not increase under the addition of path detectors.

Comparing the properties of $F(\rho, \Pi)$ with the criteria in Ref. [11], it is easy to verify that $F(\rho, \Pi)$ satisfies the above criteria (i)–(iv). In the following, we proceed to prove that criteria (v) is also satisfied for $F(\rho, \Pi)$. For this purpose, we first formulate multipath interference with path detectors.

Consider an n-path interference with a path detector D_j in the interference path $|j\rangle$, $j = 1, 2, \ldots, n$, as depicted in Fig. 2. Let H be the n-dimensional Hilbert space of the system S

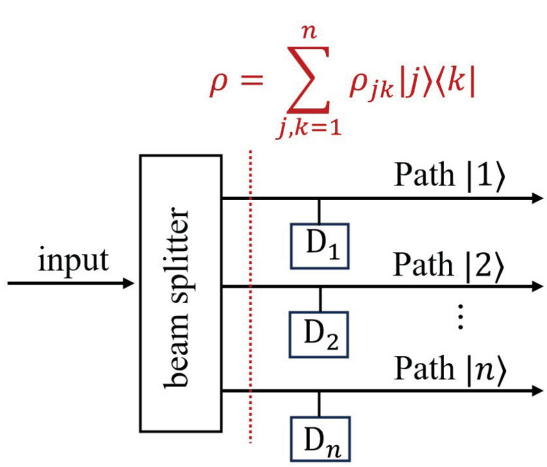

FIG. 2. Schematic of n-path interference with the path detector D_j in path $|j\rangle$.

spanned by the orthonormal basis $\{|j\rangle : j = 1, 2, \ldots, n\}$. Let H_{D_j} be the Hilbert space of the jth path detector D_j with dimension d_j, $j = 1, 2, \ldots, n$. Let ρ be the initial state of the system S and τ_j be the initial state of the path detector D_j associated with the path $|j\rangle$, $j = 1, 2, \ldots, n$. Then the initial state of the composite system $SD_1 \cdots D_n$ is

$$\rho_{SD} = \rho \otimes \tau_D,$$

with $\tau_D = \tau_1 \otimes \tau_2 \otimes \cdots \otimes \tau_n$ being the initial state of the n path detectors $D = D_1 D_2 \cdots D_n$. We model the interaction between the system S and the path detectors D as the controlled unitary operation

$$U = \sum_j |j\rangle\langle j| \otimes U_j,$$

with $U_j = \mathbf{1}_1 \otimes \cdots \otimes V_j \otimes \cdots \otimes \mathbf{1}_n$, V_j being a unitary operator, and $\mathbf{1}_j$ being the identity operator on H_{D_j}, $j = 1, 2, \ldots, n$.

The state of the composite system SD after interaction (with the path detectors) is

$$\tilde{\rho}_{SD} = U \rho_{SD} U^\dagger = \sum_{jk} \rho_{jk} |j\rangle\langle k| \otimes U_j \tau_D U_k^\dagger,$$

and the reduced state of the system S is

$$\tilde{\rho} = \mathrm{tr}_D(\tilde{\rho}_{SD}) = \sum_{jk} \rho_{jk} \mathrm{tr}(U_j \tau_D U_k^\dagger) |j\rangle\langle k|,$$

with $\rho_{jk} = \langle j|\rho|k\rangle$, $j, k = 1, 2, \ldots, n$.

Let $\{|\alpha_l\rangle : l = 1, 2, \ldots, d\}$ be an orthonormal basis of the Hilbert space $H_{D_1} \otimes H_{D_2} \otimes \cdots \otimes H_{D_n}$ associated with the n-path detectors with $d = d_1 d_2 \cdots d_n$, and let $\tau_D = \sum_m \mu_m |\psi_m\rangle\langle\psi_m|$ be the spectral decomposition of τ_D, with μ_m being its eigenvalues and $|\psi_m\rangle$ being its normalized eigenvectors. In terms of the above preparations, the quantum channel induced by the n-path detectors can be represented as

$$\mathcal{E}_D(\rho) = \tilde{\rho} = \sum_{lm} E_{lm} \rho E_{lm}^\dagger,$$

with the Kraus operators

$$E_{lm} = \sum_j \sqrt{\mu_m} \langle \alpha_l | U_j | \psi_m \rangle |j\rangle\langle j|.$$

It is obvious that $\mathcal{E}_D$ is an incoherent channel relative to the von Neumann measurement $\Pi = \{\Pi_j = |j\rangle\langle j| : j = 1, 2, \ldots, n\}$, which maps incoherent states to incoherent states. Based on this fact, we obtain the following result.

Proposition 1. Any coherence measure Q satisfying the monotonicity condition, i.e., $Q(\mathcal{E}(\rho), \Pi) \leqslant Q(\rho, \Pi)$ for any incoherent channel $\mathcal{E}$, is nonincreasing under the effect of n-path detectors, i.e.,

$$Q(\mathcal{E}_D(\rho), \Pi) \leqslant Q(\rho, \Pi).$$

In particular, for the coherence measure via quantum Fisher information, we have

$$F(\mathcal{E}_D(\rho), \Pi) \leqslant F(\rho, \Pi),$$

which shows that getting some knowledge of path information about the particle will decrease the wave feature of the particle. This is in line with Bohr's complementary principle.

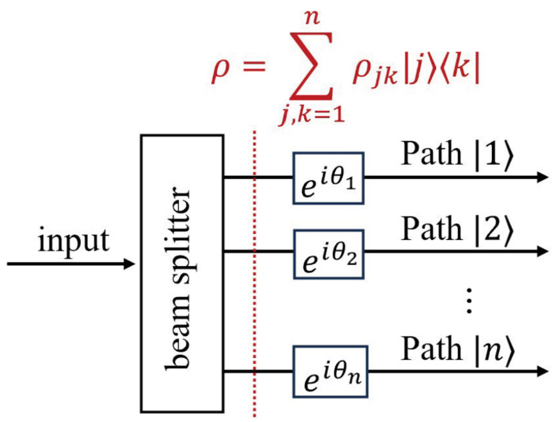

FIG. 3. Schematic of n-path interference with the phase shift $e^{i\theta_j}$ in path $|j\rangle$.

From the above discussions, we know that $F(\rho, \Pi)$ is a valid interference visibility (wave feature) quantifier.

C. An operational implication of interference in metrology

In this section, we provide an illustration of $F(\rho, \Pi)$ (wave feature via quantum Fisher information) in quantum metrology.

Consider an n-path interference with a phase shift in each interference path as depicted in Fig. 3. The interference path can be described by the von Neumann measurement $\Pi = \{\Pi_j = |j\rangle\langle j| : j = 1, 2, \ldots, n\}$ with each path undergoing a phase shift $e^{i\theta_j}$. Let ρ be the state after the beam splitter. Then the final state passing through the n paths is

$$\rho_{\boldsymbol{\theta}} = U_{\boldsymbol{\theta}} \rho U_{\boldsymbol{\theta}}^{\dagger}, \tag{3}$$

where

$$U_{\boldsymbol{\theta}} = \sum_j e^{i\theta_j} \Pi_j \tag{4}$$

and $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n) \in [0, 2\pi)^n$. Thus $\rho_{\boldsymbol{\theta}}$ encodes the information of the parameters θ_j. In order to explore the role of the phase shift parameters in quantifying wave features in multipath interference, it is desirable to first study the quantum Fisher information matrix $F = (F_{jk})$ of $\rho_{\boldsymbol{\theta}}$, which is defined as [47]

$$F_{jk} = \tfrac{1}{8}\text{tr}(\rho_{\boldsymbol{\theta}}(L_j L_k + L_k L_j)),$$

where the symmetric logarithmic derivatives L_j are defined as operator solutions to the equations

$$\frac{\partial}{\partial_{\theta_j}} \rho_{\boldsymbol{\theta}} = \frac{1}{2}(\rho_{\boldsymbol{\theta}} L_j + L_j \rho_{\boldsymbol{\theta}}), \quad j = 1, 2, \ldots, n.$$

Let $\rho = \sum_l \lambda_l |\phi_l\rangle\langle\phi_l|$ be the spectral decomposition of ρ, with λ_l being the eigenvalues and $|\phi_l\rangle$ being the corresponding normalized eigenvectors. Direct calculations show that the quantum Fisher information matrix of $\rho_{\boldsymbol{\theta}}$ can be evaluated as $F = (F_{jk})$, with

$$F_{jk} = \delta_{jk}\langle j|\rho|j\rangle - \sum_{ll'} \frac{2\lambda_l \lambda_{l'}}{\lambda_l + \lambda_{l'}} \text{Re}(\langle j|\phi_l\rangle\langle\phi_l|k\rangle\langle k|\phi_{l'}\rangle\langle\phi_{l'}|j\rangle)$$

and δ_{jk} being the Kronecker delta function. In particular, the quantum Fisher information of $\rho_{\boldsymbol{\theta}}$ for the parameter θ_j is

$$F_{jj} = \langle j|\rho|j\rangle - \sum_{ll'} \frac{2\lambda_l \lambda_{l'}}{\lambda_l + \lambda_{l'}} |\langle\phi_l|j\rangle|^2 |\langle\phi_{l'}|j\rangle|^2 = F(\rho, \Pi_j)$$

for $j = 1, 2, \ldots, n$, which implies that

$$F(\rho, \Pi) = \sum_j F_{jj}, \tag{5}$$

i.e., the wave feature of ρ relative to the interference path $\Pi = \{\Pi_j = |j\rangle\langle j| : j = 1, 2, \ldots, n\}$, as quantified by $F(\rho, \Pi)$, coincides with the sum of the quantum Fisher information of each phase shift parameter θ_j encoded in the state $\rho_{\boldsymbol{\theta}}$. Moreover, $F(\rho, \Pi)$ provides a lower bound to the total variance of unbiased estimation for the parameter $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$, as the following proposition shows.

Proposition 2. Let $\rho_{\boldsymbol{\theta}}$ be defined by Eq. (3) and $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n)$ be an unbiased estimator of $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$. Then

$$\sum_j \text{Var}(\rho_{\boldsymbol{\theta}}, \hat{\theta}_j) \geqslant \frac{n^2}{F(\rho, \Pi)}, \tag{6}$$

with $\text{Var}(\rho_{\boldsymbol{\theta}}, \hat{\theta}_j)$ being the variance of the estimator $\hat{\theta}_j$, $j = 1, 2, \ldots, n$.

Now we proceed to the proof of the above statement.

Let $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_n)$ be an unbiased estimation of $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$, then the covariance matrix $\text{Cov}(\rho_{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}})$ of $\hat{\boldsymbol{\theta}}$ in the state $\rho_{\boldsymbol{\theta}}$ satisfies the inequality [40,48]

$$\text{Cov}(\rho_{\boldsymbol{\theta}}, \hat{\boldsymbol{\theta}}) \geqslant F^{-1},$$

which implies that the total variance of the estimation $\hat{\boldsymbol{\theta}}$ is bounded as [49]

$$\sum_j \text{Var}(\rho_{\boldsymbol{\theta}}, \hat{\theta}_j) \geqslant \text{tr}F^{-1} \geqslant \sum_j \frac{1}{F_{jj}}.$$

By the inequality

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}} \leqslant \frac{x_1 + x_2 + \cdots + x_n}{n}$$

for any positive numbers x_j, we further obtain

$$\sum_j \text{Var}(\rho_{\boldsymbol{\theta}}, \hat{\theta}_j) \geqslant \frac{n^2}{\sum_j F_{jj}} = \frac{n^2}{F(\rho, \Pi)}.$$

III. COHERENCE-PREDICTABILITY-CORRELATIONS COMPLEMENTARITY

The wave-particle duality states that the wave feature and the particle feature are mutually exclusive. Quantitative studies of wave-particle duality relations have attracted much interest. By completing the wave-particle duality relations which are often expressed in terms of inequalities, several triality relations have been established from different perspectives, such as the predictability-coherence-entanglement complementary relation, the wave-particle-mixedness complementary relation, and so on [39].

In this section, by exploiting the coherence difference as a measure of correlations, we establish a new triality relation, i.e., coherence-predictability-correlations complementary relation, which provides an operational interpretation of correlations in completing wave-particle duality.

Let ρ be a state and $|\Psi\rangle\langle\Psi|$ be a purified state of ρ with an auxiliary system a, i.e., $\mathrm{tr}_a|\Psi\rangle\langle\Psi| = \rho$. A useful way to detect certain correlations encoded in $|\Psi\rangle\langle\Psi|$ has been proposed in Refs. [50,51] via the coherence difference induced by local channels. Following this line, a measure of correlations in $|\Psi\rangle\langle\Psi|$ detected by the local von Neumann measurement $\Pi = \{\Pi_j = |j\rangle\langle j| : j = 1, 2, \dots, n\}$ can be defined via quantum Fisher information as

$$C(\rho, \Pi) = F(|\Psi\rangle\langle\Psi|, \Pi \otimes \mathbf{1}_a) - F(\rho, \Pi), \tag{7}$$

with $\Pi \otimes \mathbf{1}_a = \{|j\rangle\langle j| \otimes \mathbf{1}_a : j = 1, 2, \dots, n\}$ being a Lüders measurement and $\mathbf{1}_a$ being the identity operator on the auxiliary system a. It turns out that

$$C(\rho, \Pi) = V(\rho, \Pi) - F(\rho, \Pi), \tag{8}$$

where

$$V(\rho, \Pi) = \sum_j V(\rho, \Pi_j) = 1 - \sum_j \langle j|\rho|j\rangle^2 \tag{9}$$

is the variance of ρ relative to the von Neumann measurement Π, which quantifies the total uncertainty of Π in ρ [52], and $V(\rho, X) = \mathrm{tr}\rho X_0^2$ is the variance of ρ relative to the observable X, with $X_0 = X - \mathrm{tr}\rho X$. This follows from the facts

$$V(|\Psi\rangle\langle\Psi|, X \otimes \mathbf{1}_a) = V(\rho, X),$$
$$V(|\Psi\rangle\langle\Psi|, X \otimes \mathbf{1}_a) = F(|\Psi\rangle\langle\Psi|, X \otimes \mathbf{1}_a),$$

for any observable X.

It is easy to verify that $C(\rho, \Pi)$ satisfies the following properties.

(i) $0 \leqslant C(\rho, \Pi) \leqslant V(\rho, \Pi)$, and $C(\rho, \Pi) = 0$ if ρ is a pure state, $C(\rho, \Pi) = V(\rho, \Pi)$ if and only if $[\rho, \Pi_j] = \rho\Pi_j - \Pi_j\rho = 0$ for any $j = 1, 2, \dots, n$, or equivalently, $\sum_j \Pi_j \rho \Pi_j = \rho$.

(ii) $C(\rho, \Pi)$ is concave in ρ.

Compared with the properties of measures of classical and quantum uncertainties postulated in Ref. [45], we know that $C(\rho, \Pi)$ and $F(\rho, \Pi)$ quantify the classical and quantum uncertainties of Π in ρ, respectively. Thus, the classical uncertainty of Π in ρ is essentially the correlations in the purified state, as detected locally by Π.

In the following, we employ the quantifier of correlations $C(\rho, \Pi)$ to complete wave-particle duality and thus also provide an operational interpretation of $C(\rho, \Pi)$.

For a state ρ and the von Neumann measurement $\Pi = \{\Pi_j = |j\rangle\langle j| : j = 1, 2, \dots, n\}$ associated with the n interference paths, the probability distribution induced by the von Neumann measurement Π is $\{p_j = \langle j|\rho|j\rangle : j = 1, 2, \dots, n\}$, which indicates the path information of particle. By employing the purity of this probability distribution [39]

$$P(\rho, \Pi) = \sum_{j=1}^{n} \langle j|\rho|j\rangle^2 \tag{10}$$

as a measure of path predictability (particle feature), we obtain the following coherence-predictability-correlations complementary relation.

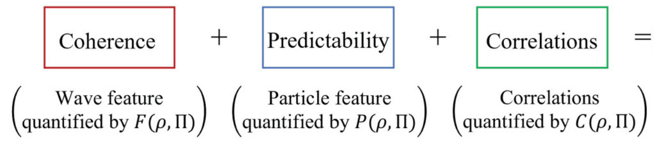

FIG. 4. Coherence-predictability-correlations triality as a resolution of unity.

Proposition 3. For a state ρ and the von Neumann measurement Π induced by the interference paths, we have

$$F(\rho, \Pi) + P(\rho, \Pi) + C(\rho, \Pi) = 1. \tag{11}$$

Proposition 3 follows readily from Eqs. (8)–(10).

This triality relation is illuminated in Fig. 4.

IV. COMPARISON

In this section, we make a comparison between the triality relation established in this work and the wave-particle-mixedness triality relation established in Ref. [39].

Let ρ be a state and $\Pi = \{\Pi_j = |j\rangle\langle j| : j = 1, 2, \dots, n\}$ be a von Neumann measurement. The coherence of ρ relative to Π via the Hilbert-Schmidt norm is defined as [13]

$$C_{\mathrm{HS}}(\rho, \Pi) = \|\rho - \Pi(\rho)\|^2 = \sum_{j \neq k} |\langle j|\rho|k\rangle|^2, \tag{12}$$

with $\|A\|^2 = \mathrm{tr}A^\dagger A$ being the squared Hilbert-Schmidt norm of operator A.

Due to the simple structure and important implication of the Hilbert-Schmidt norm, the coherence via Hilbert-Schmidt norm has been employed to quantify interference visibility [11,38,53]. For the convenience of subsequent comparison, it is desirable to explore the connections between the coherence via quantum Fisher information and that via the Hilbert-Schmidt norm. In fact, the coherence via the Hilbert-Schmidt norm provides a tight lower bound to the coherence via quantum Fisher information, as summarized in the following result.

Proposition 4. For a state ρ and a von Neumann measurement $\Pi = \{\Pi_j = |j\rangle\langle j| : j = 1, 2, \dots, n\}$, we have

$$C_{\mathrm{HS}}(\rho, \Pi) \leqslant F(\rho, \Pi).$$

The equality holds if and only if ρ satisfies one of the following conditions:

(a) ρ is a pure state.

(b) $\rho = \tau \oplus \mathbf{0}_{n-2}$ with τ an invertible state over the space spanned by some $\{|j_1\rangle, |j_2\rangle\} \subseteq \{|j\rangle : j = 1, 2, \dots, n\}$.

(c) ρ is an incoherent state, i.e., a diagonal matrix relative to the orthonormal basis $\{|j\rangle : j = 1, 2, \dots, n\}$.

This shows that the coherence via quantum Fisher information and that via the Hilbert-Schmidt norm are the same for all pure states, all qubit states, and all incoherent states. We give the proof of Proposition 4 in the Appendix.

Now we consider a simple case for which $C_{\mathrm{HS}}(\rho, \Pi) < F(\rho, \Pi)$. Consider the block-diagonal state $\rho = p_1\rho_1 \oplus p_2\rho_2$ in the orthonormal basis $\{|j\rangle : j = 1, 2, \dots, n\}$ related to Π with ρ_1 and ρ_2 being states on subspaces H_1 and H_2, respectively, and $p_1,\ p_2 \geqslant 0$ satisfying $p_1 + p_2 = 1$, then we have

$$C_{\mathrm{HS}}(p_1\rho_1 \oplus p_2\rho_2, \Pi) = p_1^2 C_{\mathrm{HS}}(\rho_1, \Pi) + p_2^2 C_{\mathrm{HS}}(\rho_2, \Pi),$$
$$F(p_1\rho_1 \oplus p_2\rho_2, \Pi) = p_1 F(\rho_1, \Pi) + p_2 F(\rho_2, \Pi),$$

where the ρ_l on the right side of each equality means the corresponding embedded state on the Hilbert space H_l, $l = 1$ and 2. Based on the above result, for the three-dimensional state

$$\rho = \frac{1}{12}\begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

we have

$$C_{\mathrm{HS}}(\rho, \Pi) = \tfrac{1}{72} < \tfrac{1}{60} = F(\rho, \Pi).$$

In the following, we derive a simple relation between coherence via quantum Fisher information and that via the l_1 norm by combining two existing results. It is known that

$$I(\rho, \Pi) \leqslant \frac{2}{n-1} C_{l_1}(\rho, \Pi)$$

for any invertible state ρ, with [13,43]

$$I(\rho, \Pi) = \frac{1}{2}\sum_j \|[\sqrt{\rho}, |j\rangle\langle j|]\|^2 = 1 - \sum_j \langle j|\sqrt{\rho}|j\rangle^2,$$
$$C_{l_1}(\rho, \Pi) = \sum_{j\neq k} |\langle j|\rho|k\rangle|,$$

respectively. Here $[X, Y] = XY - YX$ is the commutator between operators X and Y. By the relation

$$I(\rho, \Pi) \leqslant F(\rho, \Pi) \leqslant 2I(\rho, \Pi),$$

derived in Ref. [46], we further obtain

$$C_{\mathrm{HS}}(\rho, \Pi) \leqslant F(\rho, \Pi) \leqslant \frac{4}{n-1} C_{l_1}(\rho, \Pi)$$

for any invertible state $\rho > 0$.

The following wave-particle-mixedness triality relation

$$W(\rho, \Pi) + P(\rho, \Pi) + M(\rho) = 1 \tag{13}$$

was established in Ref. [39]. Here the measure of the wave feature $W(\rho, \Pi)$ is quantified by the uncertainty of the state ρ (relative to the von Neumann measurement Π) via variance, i.e.,

$$W(\rho, \Pi) = \sum_j V(|j\rangle\langle j|, \rho) = \sum_{j\neq k} |\langle j|\rho|k\rangle|^2. \tag{14}$$

Comparing Eqs. (12) and (14), one has

$$W(\rho, \Pi) = C_{\mathrm{HS}}(\rho, \Pi).$$

The measure of the particle feature $P(\rho, \Pi)$ is quantified by the path certainty in the state ρ, i.e.,

$$P(\rho, \Pi) = 1 - \sum_j V(\rho, |j\rangle\langle j|) = \sum_j |\langle j|\rho|j\rangle|^2. \tag{15}$$

The measure of mixedness $M(\rho)$ of the state ρ is quantified by the linear entropy, i.e.,

$$M(\rho) = 1 - \mathrm{tr}\rho^2. \tag{16}$$

Since the entropy of a state can be interpreted as the correlations in the corresponding purified state, the above triality relation can also be regarded as the coherence-predictability-correlations complementary relation.

By Proposition 4 and the complementary relations (11) and (13), we know that

$$F(\rho, \Pi) \geqslant C_{\mathrm{HS}}(\rho, \Pi), \quad C(\rho, \Pi) \leqslant M(\rho).$$

In particular, when ρ is a qubit state or a pure state, we have

$$F(\rho, \Pi) = C_{\mathrm{HS}}(\rho, \Pi), \quad C(\rho, \Pi) = M(\rho),$$

i.e., the two complementary relations coincide for these cases.

For the three-dimensional state

$$\rho = \frac{1}{12}\begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

we have

$$C_{\mathrm{HS}}(\rho, \Pi) = \tfrac{1}{72}, \quad P(\rho, \Pi) = \tfrac{3}{8}, \quad M(\rho) = \tfrac{11}{18},$$

and

$$F(\rho, \Pi) = \tfrac{1}{60}, \quad P(\rho, \Pi) = \tfrac{3}{8}, \quad C(\rho, \Pi) = \tfrac{73}{120}.$$

In the following, we further compare these complementary relations for the two-qubit Werner states and investigate the behavior of each information quantity relative to the state parameter. Consider the two-qubit Werner states

$$\mathbf{w} = \frac{1-\mu}{4}\mathbf{1}\otimes\mathbf{1} + \mu|\Psi^-\rangle\langle\Psi^-|,$$

with $-1/3 \leqslant \mu \leqslant 1$ and $|\Psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. For the von Neumann measurement $\Pi = \{|00\rangle\langle 00|, |01\rangle\langle 01|, |10\rangle\langle 10|, |11\rangle\langle 11|\}$, we have

$$F(\mathbf{w}, \Pi) = \frac{\mu^2}{1+\mu}, \quad P(\mathbf{w}, \Pi) = \frac{1+\mu^2}{4},$$
$$C(\mathbf{w}, \Pi) = \frac{3 + 3\mu - 5\mu^2 - \mu^3}{4(1+\mu)}, \quad C_{\mathrm{HS}}(\mathbf{w}, \Pi) = \frac{\mu^2}{2},$$
$$M(\mathbf{w}) = \frac{3}{4}(1-\mu^2),$$

with $-1/3 \leqslant \mu \leqslant 1$, respectively. It is interesting to compare the above information quantities, as depicted in Fig. 5.

From Fig. 5, we find the following facts. First, the coherence via quantum Fisher information is strictly larger than that via the Hilbert-Schmidt norm except for the maximally mixed state and the singlet state. Second, coherence via quantum Fisher information and the Hilbert-Schmidt norm, as well as path predictability, are decreasing with the parameter $\mu \in [-1/3, 0]$ and increasing with the parameter $\mu \in (0, 1]$. The quantifier of correlations in the purified state of $\mathbf{w}$ detected by the von Neumann measurement Π and the mixedness of $\mathbf{w}$ are both increasing with the parameter $\mu \in [-1/3, 0]$ and decreasing with the parameter $\mu \in (0, 1]$.

V. SUMMARY

We have investigated the coherence in multipath interference via quantum Fisher information and have proven that Hilbert-Schmidt norm coherence provides a tight lower

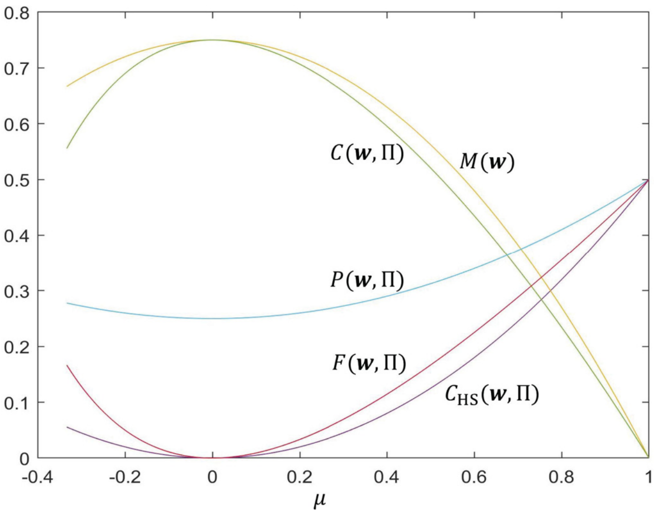

FIG. 5. Behaviors of $F(\mathbf{w},\Pi)$, $P(\mathbf{w},\Pi)$, $Q(\mathbf{w},\Pi)$, $C_{\rm HS}(\mathbf{w},\Pi)$, and $M(\mathbf{w})$ with the parameter μ.

bound for coherence via quantum Fisher information with the equality holding for pure states, rank-2 states under the corresponding reference basis, and incoherent states. We have revealed basic properties of the coherence quantifier and have shown that it is nonincreasing under the action of path detectors. Moreover, we have provided an operational illustration of the wave feature as quantum Fisher information of the phase shift parameters encoded in the interference paths.

In terms of the decomposition of variance into classical and quantum parts, and by use of coherence based on quantum Fisher information, we have derived a coherence-predictability-correlations complementary relation. This complementary relation can alternatively be interpreted as providing an operational interpretation of the quantifier of correlations defined via coherence difference. We have compared the triality relation derived in this work and the triality relation established in Ref. [39] for two-qubit Werner states.

We remark that the corresponding results can be extended to the coherence measure via metric-adjusted skew information, which includes quantum Fisher information based on the symmetric logarithmic derivative as a special case. In addition, compared with single particle interference, multiparticles can enhance interference effects and play a vital role in various quantum phenomena and applications. It is desirable to further investigate the implications of coherence and correlations in multipath interference for the multiparticle systems via quantum Fisher information and explore their applications in quantum metrology.

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APPENDIX

Here we present the detailed proof of Proposition 4.

Let $\rho=\sum_k\lambda_k|\phi_k\rangle\langle\phi_k|$ be the spectral decomposition of ρ, with $\{|\phi_k\rangle : k=1,2,\dots,n\}$ being an orthonormal basis, then

$$\mathrm{tr}\rho^2=\sum_{jkl}\frac{\lambda_k^2+\lambda_l^2}{2}|\langle\phi_k|j\rangle|^2|\langle\phi_l|j\rangle|^2$$

and

$$\sum_j\langle j|\rho|j\rangle^2=\sum_{jkl}\lambda_k\lambda_l|\langle\phi_k|j\rangle|^2|\langle\phi_l|j\rangle|^2.$$

Thus,

$$\begin{aligned}F(\rho,\Pi)-C_{\rm HS}(\rho,\Pi)&=\sum_{jkl}\frac{(\lambda_k-\lambda_l)^2}{2(\lambda_k+\lambda_l)}|\langle\phi_k|j\rangle|^2|\langle\phi_l|j\rangle|^2-\mathrm{tr}\rho^2+\sum_j\langle j|\rho|j\rangle^2\\&=\sum_{jkl}\left(\frac{(\lambda_k-\lambda_l)^2}{2(\lambda_k+\lambda_l)}-\frac{\lambda_k^2+\lambda_l^2}{2}+\lambda_k\lambda_l\right)|\langle\phi_k|j\rangle|^2|\langle\phi_l|j\rangle|^2\\&=\sum_{jkl}\frac{(\lambda_k-\lambda_l)^2(1-\lambda_k-\lambda_l)}{2(\lambda_k+\lambda_l)}|\langle\phi_k|j\rangle|^2|\langle\phi_l|j\rangle|^2\geqslant 0\end{aligned}$$

and $F(\rho,\Pi)=C_{\rm HS}(\rho,\Pi)$ if and only if, for any j,k, or l,

$$(\lambda_k-\lambda_l)^2(1-\lambda_k-\lambda_l)|\langle\phi_k|j\rangle|^2|\langle\phi_l|j\rangle|^2=0. \quad (A1)$$

This equation can be exhaustively analyzed as follows.

Case 1. Suppose that ρ is a pure state, then ρ has only one nonzero eigenvalue 1, and thus Eq. (A1) is satisfied.

Case 2. Suppose that ρ is a state of rank 2 with λ_1 and λ_2 being two nonzero eigenvalues. If 0 is not an eigenvalue of ρ, i.e., ρ is a qubit, then Eq. (A1) is satisfied. If $\lambda_3=\cdots=\lambda_n=0$ is an eigenvalue of ρ, then Eq. (A1) is equivalent to

$$|\langle\phi_k|j\rangle|^2|\langle\phi_l|j\rangle|^2=0$$

for all $k=3,\dots,n$, $l=1$ and 2, and any j. This means that the support over the basis $\{|j\rangle : j=1,2,\dots,n\}$ of any eigenvector associated with nonzero eigenvalues is orthogonal to that of any eigenvector associated with a zero eigenvalue. Thus, the state ρ satisfying Eq. (A1) has the form $\rho=\tau\oplus\mathbf{0}_{n-2}$.

Case 3. Suppose that ρ is a state of rank more than 2. Let $\lambda_1,\lambda_2,\dots,\lambda_t$ be all different eigenvalues of ρ with multiplicity $n_1,n_2,\dots,n_t$, respectively, and $|\phi_1\rangle,|\phi_2\rangle,\dots,|\phi_t\rangle$ be the corresponding eigenvectors, then $\sum_{k=1}^t\lambda_kn_k=1$ and $(\lambda_k-\lambda_l)^2(1-\lambda_k-\lambda_l)>0$ for any $k\neq l\in\{1,2,\dots,t\}$. In this case, the condition (A1) can be rewritten as

$$|\langle\phi_k|j\rangle|^2|\langle\phi_l|j\rangle|^2=0$$

for any $k\neq l=1,2,\dots,t$ and any j, which shows that the supports of eigenvectors associated with different eigenvalues are pairwise orthogonal. Thus any state satisfying Eq. (A1) is of the form

$$\rho=\lambda_1\gamma_1\oplus\lambda_2\gamma_2\oplus\cdots\oplus\lambda_t\gamma_t$$

with respect to the orthonormal basis $\{|j\rangle : j=1,2,\dots,n\}$. Here $\gamma_k\oplus\mathbf{0}_{n-n_k}$ is the sum of all eigenprojectors associated with the eigenvalue λ_k. Therefore, $\gamma_k\oplus\mathbf{0}_{n-n_k}$ must be the projection $\sum_{l=1}^{n_k} |k_l\rangle\langle k_l|$ with $\{k_l : l = 1, 2, \ldots, n_k\}$ being a subset of $\{1, 2, \ldots, n\}$, which further implies that ρ is a diagonal matrix relative to the basis $\{|j\rangle : j = 1, 2, \ldots, n\}$, i.e., ρ is an incoherent state.

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[1] N. Bohr, The quantum postulate and the recent development of atomic theory, Nature (London) **121**, 580 (1928).

[2] W. K. Wootters and W. H. Zurek, Complementarity in the double-slit experiment: Quantum nonseparability and a quantitative statement of Bohr's principle, Phys. Rev. D **19**, 473 (1979).

[3] M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge University, Cambridge, 1999).

[4] D. M. Greenberger and A. Yasin, Simultaneous wave and particle knowledge in a neutron interferometer, Phys. Lett. A **128**, 391 (1988).

[5] G. Jaeger, A. Shimony, and L. Vaidman, Two interferometric complementarities, Phys. Rev. A **51**, 54 (1995).

[6] B.-G. Englert, Fringe visibility and which-way information: An inequality, Phys. Rev. Lett. **77**, 2154 (1996).

[7] B. G. Englert and J. A. Bergou, Quantitative quantum erasure, Opt. Commun. **179**, 337 (2000).

[8] M. Mei and M. Weitz, Controlled decoherence in multiple beam Ramsey interference, Phys. Rev. Lett. **86**, 559 (2001).

[9] G. Bimonte and R. Musto, On interferometric duality in multibeam experiments, J. Phys. A **36**, 11481 (2003).

[10] T. Qureshi, Coherence, interference and visibility, Quanta **8**, 24 (2019).

[11] S. Dürr, Quantitative wave-particle duality in multibeam interferometers, Phys. Rev. A **64**, 042113 (2001).

[12] B.-G. Englert, D. Kaszlikowski, L. C. Kwek, and W. H. Chee, Wave-particle duality in multi-path interferometers: General concepts and three-path interferometers, Int. J. Quantum Inf. **06**, 129 (2008).

[13] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying coherence, Phys. Rev. Lett. **113**, 140401 (2014).

[14] M. N. Bera, T. Qureshi, M. A. Siddiqui, and A. K. Pati, Duality of quantum coherence and path distinguishability, Phys. Rev. A **92**, 012118 (2015).

[15] E. Bagan, J. A. Bergou, S. S. Cottrell, and M. Hillery, Relations between coherence and path information, Phys. Rev. Lett. **116**, 160406 (2016).

[16] T. Biswas, M. G. Díaz, and A. Winter, Interferometric visibility and coherence, Proc. R. Soc. A **473**, 20170170 (2017).

[17] T. Paul and T. Qureshi, Measuring quantum coherence in multislit interference, Phys. Rev. A **95**, 042110 (2017).

[18] C. Xiong, A. Kumar, and J. Wu, Family of coherence measures and duality between quantum coherence and path distinguishability, Phys. Rev. A **98**, 032324 (2018).

[19] K. Bu, L. Li, J. Wu, and S. M. Fei, Duality relation between coherence and path information in the presence of quantum memory, J. Phys. A **51**, 085304 (2018).

[20] P. Roy and T. Qureshi, Path predictability and quantum coherence in multi-slit interference, Phys. Scr. **94**, 095004 (2019).

[21] S. Mishra, A. Venugopalan, and T. Qureshi, Decoherence and visibility enhancement in multi-path interference, Phys. Rev. A **100**, 042122 (2019).

[22] T. Qureshi, Interference visibility and wave-particle duality in multipath interference, Phys. Rev. A **100**, 042105 (2019).

[23] Y. Sun and S. Luo, Quantifying interference via coherence, Ann. Phys. (Berlin, Ger.) **533**, 2100303 (2021).

[24] L. Li, N. L. Liu, and S. Yu, Duality relations in a two-path interferometer with an asymmetric beam splitter, Phys. Rev. A **85**, 054101 (2012).

[25] K. M. Keerthy and T. Qureshi, Wave-particle duality in asymmetric beam interference, Phys. Rev. A **98**, 022130 (2018).

[26] Y. Liu, J. Lu, Z. Peng, L. Zhou, and D. Zheng, Fringe visibility and distinguishability in two-path interferometer with an asymmetric beam splitter, Chin. Phys. B **28**, 030303 (2019).

[27] S. Das, C. Mukhopadhyay, S. S. Roy, S. Bhattacharya, A. Sen(De), and U. Sen, Wave-particle duality employing quantum coherence in superposition with non-orthogonal pointers, J. Phys. A **53**, 115301 (2020).

[28] C. Srivastava, S. Das, and U. Sen, Resource theory of quantum coherence with probabilistically nondistinguishable pointers and corresponding wave-particle duality, Phys. Rev. A **103**, 022417 (2021).

[29] X. Y. Zou, L. J. Wang, and L. Mandel, Induced coherence and indistinguishability in optical interference, Phys. Rev. Lett. **67**, 318 (1991).

[30] P. Bertet, S. Osnaghi, A. Rauschenbeutel, G. Nogues, A. Auffeves, M. Brune, J. M. Raimond, and S. Haroche, A complementarity experiment with an interferometer at the quantum-classical boundary, Nature (London) **411**, 166 (2001).

[31] R. Auccaise, R. M. Serra, J. G. Filgueiras, R. S. Sarthour, I. S. Oliveira, and L. S. Céleri, Experimental analysis of the quantum complementarity principle, Phys. Rev. A **85**, 032121 (2012).

[32] X.-F. Qian, K. Konthasinghe, S. K. Manikandan, D. Spiecker, A. N. Vamivakas, and J. H. Eberly, Turning off quantum duality, Phys. Rev. Res. **2**, 012016(R) (2020).

[33] M. B. Pozzobom, M. L. W. Basso, and J. Maziero, Experimental tests of the density matrix's property-based complementarity relations, Phys. Rev. A **103**, 022212 (2021).

[34] D.-Xu Chen, Y. Zhang, J.-L. Zhao, Q.-C. Wu, Y.-L. Fang, C.-P. Yang, and F. Nori, Experimental investigation of wave-particle duality relations in asymmetric beam interference, npj Quantum Inf. **8**, 101 (2022).

[35] X. J. Huang *et al.*, Entanglement-interference complementarity and experimental demonstration in a superconducting circuit, npj Quantum Inf. **9**, 43 (2023).

[36] M. Jakob and J. A. Bergou, Quantitative complementarity relations in bipartite systems: Entanglement as a physical reality, Opt. Commun. **283**, 827 (2010).

[37] T. Qureshi, Predictability, distinguishability, and entanglement, Opt. Lett. **46**, 492 (2021).

[38] M. L. W. Basso and J. Maziero, An uncertainty view on complementarity and a complementarity view on uncertainty, Quantum Inf. Process. **20**, 201 (2021).

[39] S. Fu and S. Luo, From wave-particle duality to wave-particle-mixedness triality: An uncertainty approach, Commun. Theor. Phys. **74**, 035103 (2022).

[40] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Springer, New York, 2011).

[41] L. Li, Q.-W. Wang, S.-Q. Shen, and M. Li, Quantum coherence measures based on Fisher information with applications, Phys. Rev A **103**, 012401 (2021).
[42] J. Xu, Quantifying coherence with respect to general measurements via quantum Fisher information, arXiv:2107.10615.
[43] C.-s. Yu, Quantum coherence via skew information and its polygamy, Phys. Rev. A **95**, 042337 (2017).
[44] S. Luo and Y. Sun, Partial coherence with application to the monotonicity problem of coherence involving skew information, Phys. Rev. A **96**, 022136 (2017).
[45] S. Luo, Quantum versus classical uncertainty, Theor. Math. Phys. **143**, 681 (2005).
[46] S. Luo, Wigner-Yanase skew information vs. quantum Fisher information, Proc. Amer. Math. Soc. **132**, 885 (2004).
[47] D. Šadfránek, Simple expression for the quantum Fisher information matrix, Phys. Rev. A **97**, 042322 (2018).
[48] J. P. Dowling and K. P. Seshadreesan, Quantum optical technologies for metrology, sensing, and imaging, J. Lightwave Technol. **33**, 2359 (2015).
[49] J. Liu, H. Yuan, X.-M. Lu, and X. Wang, Quantum Fisher information matrix and multiparameter estimation, J. Phys. A **53**, 023001 (2020).
[50] Y. Sun, Y. Mao, and S. Luo, From quantum coherence to quantum correlations, Europhys. Lett. **118**, 60007 (2017).
[51] N. Li, S. Luo, and Y. Sun, Quantifying correlations via local channels, Phys. Rev. A **105**, 032436 (2022).
[52] Y. Sun and N. Li, The uncertainty of quantum channels in terms of variance, Quantum Inf. Process. **20**, 25 (2021).
[53] A. K. Roy, N. Pathania, N. K. Chandra, P. K. Panigrahi, and T. Qureshi, Coherence, path predictability, and I concurrence: A triality, Phys. Rev. A **105**, 032209 (2022).