Strongest nonlocal sets with small sizes

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A set of orthogonal states is the strongest nonlocal set if it is locally stable in every bipartition, which shows the strong quantum nonlocality proposed by Halder *et al.* [Phys. Rev. Lett. **122**, 040403 (2019)]. The existence of the strongest nonlocal sets with the minimum size is an open question. In this work, we partially solve this question by constructing the strongest nonlocal sets with the minimum size in $2\otimes 2\otimes 2\otimes 2$ and $2\otimes d_2\otimes d_3$, where $2\leqslant d_2\leqslant d_3$. Moreover, we also give the strongest nonlocal sets with size $d_2d_3+d_1-1$ in $d_1\otimes d_2\otimes d_3$ and the strongest nonlocal sets with size d^3+d-1 in $d\otimes d\otimes d\otimes d$. All the sizes of the strongest nonlocal sets are close to the minimum size and are smaller than all previously known constructions. As an application, our strongest nonlocal sets can be used to construct partially genuinely entangled subspaces in $d_1\otimes d_2\otimes d_3$ when $2\leqslant d_1\leqslant d_2\leqslant d_3$ and $d_2\geqslant 3$.

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I. INTRODUCTION

Quantum state discrimination is a fundamental problem in quantum information theory. If a set of orthogonal states cannot be perfectly distinguished under local operation and classical communication, then it is called locally indistinguishable. Locally indistinguishable sets have a wide range of applications such as quantum information hiding [1–3] and quantum secret sharing [4–6]. Locally indistinguishable sets also exhibit quantum nonlocality, which is different from Bell nonlocality [7]. This is because Bell nonlocality appears on only entangled states, while the nonlocality based on local indistinguishability can appear on product states. Bennett *et al.* [8] first presented a locally indistinguishable orthogonal product basis in $3\otimes 3$. Since then, the construction of locally indistinguishable orthogonal product sets and orthogonal entangled sets has received much attention [9–38].

Halder *et al.* proposed the concepts of local irreducibility and strong quantum nonlocality [39]. Local irreducibility is a stronger concept than local indistinguishability. A set of orthogonal states is called locally irreducible if it is not possible to eliminate one or more states from the set by orthogonality-preserving local measurements, and a set of orthogonal states is called strongly nonlocal if it is locally irreducible in every bipartition. Halder *et al.* also showed two strongly nonlocal orthogonal product bases in $3\otimes 3\otimes 3$ and $4\otimes 4\otimes 4$, respectively. Strongly nonlocal orthogonal product sets and orthogonal entangled sets have also been widely constructed [37,40–48]. See Table I for a summary.

The primary method of demonstrating local irreducibility of a set of orthogonal states is by proving that the only orthogonality-preserving measurements on each subsystem are trivial, and any set with this property is called locally stable [37]. Moreover, a set of orthogonal states is called "strongest nonlocal" if it is locally stable in every bipartition [44]. A lower bound on the sizes of the strongest nonlocal sets exists; that is, for any strongest nonlocal set $\mathcal{C}$ in $d_1\otimes d_2\otimes\cdots\otimes d_n$ with $d_i\geqslant 2$, we have $|\mathcal{C}|\geqslant(\prod_{i=1}^n d_i/d_{\min})+1$, where $d_{\min}=\min\{d_1,d_2,\ldots,d_n\}$ [37] (see Table I). However, the existence of strongest nonlocal sets with the minimum size is an open question.

In this paper, we partially solve this open question by constructing the strongest nonlocal sets with the minimum size in $2\otimes d_2\otimes d_3$ and $2\otimes 2\otimes 2\otimes 2$. We also present the strongest nonlocal sets with size $d_2d_3+d_1-1$ in $d_1\otimes d_2\otimes d_3$ for $2\leqslant d_1\leqslant d_2\leqslant d_3$. For four-partite systems, we show the strongest nonlocal sets with size d^3+d-1 in $d\otimes d\otimes d\otimes d$ for $d\geqslant 2$. All of our constructions are close to the lower bound in [37]. As an application, we can construct some genuinely entangled mixed states from our strongest nonlocal sets.

Constructing locally indistinguishable sets with minimal sizes is a focal point in the exploration of local indistinguishability [25,28,29,32,34–38]. Since local indistinguishability can be used for quantum information hiding [1–3] and quantum secret sharing [4–6], locally indistinguishable sets with small sizes are beneficial for experimental implementation. For example, if one wants to realize quantum secret sharing in a two-qubit system through a locally indistinguishable set, then one needs to prepare only three orthogonal Bell states [10]. Recently, quantum data hiding and quantum secret sharing based on strong quantum nonlocality were also proposed [44,47]. Thus, the strongest nonlocal sets with small

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TABLE I. Results for the cardinality of strongly nonlocal sets. Here we assume that $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$.

Systems	Lower bound	Sizes of known constructions	Ref.
$d \otimes d \otimes d$ $(d \geqslant 3)$	d^2+1	$6(d-1)^2$	[41]
$d \otimes d \otimes (d+1)$ $(d \geqslant 3)$	d^2+1	$6d^2-8d+4$	[41]
$d \otimes d \otimes d$ $(d \geqslant 3)$	d^2+1	d^3-d (d is odd), d^3-d-6 (d is even)	[40]
$d \otimes d \otimes d$ $(d \geqslant 3)$	d^2+1	d^3-4d+4 (d is odd), d^3-4d+8 (d is even)	[44]
$d \otimes d \otimes d$ $(d \geqslant 3)$	d^2+1	$d^3-(d-2)^3$ (d is odd), $d^3-(d-2)^3+2$ (d is even)	[42]
$d_1 \otimes d_2 \otimes d_3$ $(d_i \geqslant 3)$	d_2d_3+1	$2((d_1d_2+d_2d_3+d_1d_3)-3(d_1+d_2+d_3)+12)$	[46]
$d_1 \otimes d_2 \otimes d_3$ $(d_i \geqslant 3)$	d_2d_3+1	$d_1d_2d_3-8(n+1)$, $(0 \leqslant n \leqslant \lfloor \frac{d_1-3}{2} \rfloor)$	[43]
$d_1 \otimes d_2 \otimes d_3 \otimes d_4$	$d_2d_3d_4+1$	$d_1d_2d_3d_4-(d_1-2)(d_2-2)(d_3-2)(d_4-2)-2$	[46]
$d^{\otimes n}$ $(n \geqslant 3, d \geqslant 2)$	$d^{n-1}+1$	$d^n-(d-1)^n+1$	[47]
$d_1 \otimes d_2 \otimes \cdots \otimes d_n$ $(n \geqslant 3, n$ is odd, $d_i \geqslant 3)$	$d_2d_3\cdots d_n+1$	$d_1d_2\cdots d_n-(d_1-2)(d_2-2)\cdots(d_n-2)$	[45]
$d_1 \otimes d_2 \otimes \cdots \otimes d_n$ $(n > 3, n$ is even, $d_i \geqslant 3)$	$d_2d_3\cdots d_n+1$	$d_1d_2\cdots d_n-(d_1-2)(d_2-2)\cdots(d_n-2)$	[48]
$d_1 \otimes d_2 \otimes \cdots \otimes d_n$ $(n \geqslant 3, d_i \geqslant 2)$	$d_2d_3\cdots d_n+1$	$d_1d_2\cdots d_n-(d_1-1)(d_2-1)\cdots(d_n-1)$	[37]
$d \otimes d \otimes d$ $(d \geqslant 2)$	d^2+1	d^2+d-1	This work
$d_1 \otimes d_2 \otimes d_3$ $(d_i \geqslant 2)$	d_2d_3+1	$d_2d_3+d_1-1$	This work
$d \otimes d \otimes d \otimes d$ $(d \geqslant 2)$	d^3+1	d^3+d-1	This work

sizes are important for experiments with strong quantum nonlocality.

The rest of this paper is organized as follows. In Sec. II, we introduce some notations and preliminary knowledge. In Sec. III, we construct the strongest nonlocal sets with size $d_2d_3+d_1-1$ in $d_1 \otimes d_2 \otimes d_3$ for $2 \leqslant d_1 \leqslant d_2 \leqslant d_3$. In Sec. IV, we present the strongest nonlocal sets with size d^3+d-1 in $d \otimes d \otimes d \otimes d$ for $d \geqslant 2$. The proofs of Propositions 2 and 3 are given in the Supplemental Material [49]. In Sec. V, we construct some partially genuinely entangled subspaces from our strongest nonlocal sets. Finally, we summarize in Sec. VI.

II. PRELIMINARIES

Throughout this paper, we do not normalize states for simplicity. We denote $\mathbb{Z}_d = \{0, 1, \dots, d-1\}$, and $w_n = e^{\frac{2\pi i}{n}}$. Let $\{|i\rangle\}_{i\in\mathbb{Z}_d}$ be the computational basis of the d-dimensional Hilbert space $\mathcal{H}$. For $(i_1, i_2, \dots, i_N) \in \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_N}$, we denote $wt(i_1, i_2, \dots, i_N)$ as the number of nonzeros i_k for $1 \leqslant k \leqslant N$. And we define $S(d_1, d_2, \dots, d_n) = d_2d_3\cdots d_n + d_1 - 1 (2 \leqslant d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n)$, which will be the sizes of our strongest nonlocal sets.

A set of positive-semidefinite operators $\{E_m\}$ on $\mathcal{H}$ is a positive operator-valued measure (POVM) if it satisfies $\sum E_m = \mathbb{I}_{\mathcal{H}}$, where $\mathbb{I}_{\mathcal{H}}$ is the identity operator of $\mathcal{H}$. Each E_m is called a POVM element. A measurement is trivial if each POVM element is proportional to the identity operator; otherwise, it is nontrivial. For a set of orthogonal states $\{|\psi_i\rangle\}_{i=1}^k$ in $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$, a measurement $\{E_m\}$ performed on A_i is called an orthogonality-preserving local measurement (OPLM) if the postmeasurement states are mutually orthogonal, i.e.,

$$\langle \psi_i | \mathbb{I}_{A_1} \otimes \cdots \mathbb{I}_{A_{i-1}} \otimes E_m \otimes \mathbb{I}_{A_{i+1}} \otimes \cdots \otimes \mathbb{I}_{A_N} | \psi_j \rangle = 0,$$

where $1 \leqslant i \neq j \leqslant N$.

Definition 1. A set of orthogonal states in $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$ is *locally irreducible* if it is not possible to eliminate one or more states from the set by OPLMs; a set of orthogonal states in $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$ is *strongly nonlocal* if it is locally irreducible in every bipartition [39].

Definition 2. A set of orthogonal states in $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$ is *locally stable* if for each party A_i the only OPLMs on each party A_i are trivial; a set of orthogonal states in $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$ is *strongest nonlocal* if it is locally stable in every bipartition [37,44].

From Definitions 1 and 2, we know that a locally stable set must be a locally irreducible set. However, a locally irreducible set is not necessarily a locally stable set. For example, the Bell basis $\{|00\rangle \pm |11\rangle, |01\rangle \pm |10\rangle\}$ in $2 \otimes 2$ is locally irreducible [39]. The Bell basis can be seen as a set of orthogonal states in $2 \otimes 3$, and it is still locally irreducible in $2 \otimes 3$. Since A_2 can perform a nontrivial OPLM $\{|0\rangle\langle 0| + |1\rangle\langle 1|, |2\rangle\langle 2|\}$, the Bell basis is not locally stable in $2 \otimes 3$. Similarly, a strongest nonlocal set must be a strongly nonlocal set, but the converse is not necessarily true. Note that all the previous locally irreducible sets (strongly nonlocal sets) are also locally stable sets (strongest nonlocal sets) [37,40–48]. If we do not embed the small space into a large space, Definitions 1 and 2 could be equivalent.

A lower bound on the size of the strongest nonlocal set $\mathcal{C}$ in $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$ exists, where $\mathcal{H}_{A_i}$ has dimension d_i,

$$|\mathcal{C}| \geqslant \left(\prod_{i=1}^{n} d_i / d_{\min} \right) + 1, \quad d_{\min} = \min\{d_1, d_2, \dots, d_n\}. \tag{1}$$

The strongest nonlocal set has the minimum size if it reaches the above lower bound. The existence of such sets is unknown [37], and we will focus on this open question. In order to show that $\mathcal{C}$ is strongest nonlocal, we need to show only that any OPLM performed on $\hat{A}_i$ is trivial for $1 \leqslant i \leqslant N$, where $\hat{A}_i = \{A_1, A_2, \dots, A_N\} \setminus \{A_i\}$ [43].

Next, we introduce two useful tools for showing that an OPLM is trivial. Let $\{|\alpha_i\rangle\}_{i\in\mathbb{Z}_s}$ be a set of orthogonal states; then a set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_s}$ is spanned by $\{|\alpha_i\rangle\}_{i\in\mathbb{Z}_s}$ if each state $|\psi_i\rangle$ is a linear combination of the states from $\{|\alpha_i\rangle\}_{i\in\mathbb{Z}_s}$.

Lemma 1. Block-zero lemma [43]. Let E be a $d\times d$ matrix and $\{|\alpha_i\rangle\}_{i\in\mathbb{Z}_s}$ and $\{|\beta_j\rangle\}_{j\in\mathbb{Z}_t}$ be two sets of orthogonal states in $\mathcal{H}$ with $\dim(\mathcal{H})=d$. Assume that $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_s}$ and $\{|\phi_j\rangle\}_{j\in\mathbb{Z}_t}$ are two sets of orthogonal states spanned by $\{|\alpha_i\rangle\}_{i\in\mathbb{Z}_s}$ and $\{|\beta_j\rangle\}_{j\in\mathbb{Z}_t}$, respectively. If $\langle\psi_i|E|\phi_j\rangle=0$ for any $i\in\mathbb{Z}_s$, $j\in\mathbb{Z}_t$, then $\langle\alpha_i|E|\beta_j\rangle=0$ for any $i\in\mathbb{Z}_s$, $j\in\mathbb{Z}_t$.

Lemma 2. Block-trivial lemma [43]. Let E be a $d\times d$ matrix and $\{|\alpha_i\rangle\}_{i\in\mathbb{Z}_s}$ be a set of orthogonal states in $\mathcal{H}$ with $\dim(\mathcal{H})=d$. Assume that $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_s}$ is a set of orthogonal states spanned by $\{|\alpha_i\rangle\}_{i\in\mathbb{Z}_s}$ and $\langle\psi_i|E|\psi_j\rangle=0$ for any $i\neq j\in\mathbb{Z}_s$. If a state $|\alpha_t\rangle$ exists such that $\langle\alpha_t|E|\alpha_r\rangle=0$ for any $r\neq t\in\mathbb{Z}_s$ and $\langle\alpha_t|\psi_j\rangle\neq 0$ for any $j\in\mathbb{Z}_s$, then $\langle\alpha_i|E|\alpha_j\rangle=0$ for any $i\neq j\in\mathbb{Z}_s$, and $\langle\alpha_i|E|\alpha_i\rangle=\langle\alpha_j|E|\alpha_j\rangle$ for any $i\neq j\in\mathbb{Z}_s$.

III. STRONGEST NONLOCAL SETS IN TRIPARTITE SYSTEMS

In this section, we first give the strongest nonlocal set with a minimum size of 5 in $2\otimes2\otimes2$. Then we generalize it to $d\otimes d\otimes d$ and $d_1\otimes d_2\otimes d_3$.

Example 1. In $2\otimes2\otimes2$, the set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_5}$ is strongest nonlocal, where

$$\begin{aligned}
|\psi_0\rangle &= |000\rangle,\\
|\psi_1\rangle &= |100\rangle+|010\rangle+|001\rangle\\
&\quad+\frac{1}{\sqrt{3}}(|011\rangle+|110\rangle+|101\rangle),\\
|\psi_2\rangle &= |100\rangle+w_4|010\rangle+w_4^2|001\rangle\\
&\quad+w_4^3\frac{1}{\sqrt{3}}(|011\rangle+|110\rangle+|101\rangle),\\
|\psi_3\rangle &= |100\rangle+w_4^2|010\rangle+w_4^4|001\rangle\\
&\quad+w_4^6\frac{1}{\sqrt{3}}(|011\rangle+|110\rangle+|101\rangle),\\
|\psi_4\rangle &= |100\rangle+w_4^3|010\rangle+w_4^6|001\rangle\\
&\quad+w_4^9\frac{1}{\sqrt{3}}(|011\rangle+|110\rangle+|101\rangle).
\end{aligned}\tag{2}$$

Proof. Note that $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_5}$ has a similar structure under the cyclic permutation of the parties $\{A_1,A_2,A_3\}$, so we need to show only that any OPLM performed on A_2A_3 is trivial. We assume that A_2A_3 performs an OPLM $\{E\}$; then we have $\langle\psi_i|\mathbb{I}_{A_1}\otimes E|\psi_j\rangle=0$ for $i\neq j\in\mathbb{Z}_5$.

Let $|\alpha_0\rangle=|000\rangle$, $|\alpha_1\rangle=|100\rangle$, $|\alpha_2\rangle=|010\rangle$, $|\alpha_3\rangle=|001\rangle$, and $|\alpha_4\rangle=\frac{1}{\sqrt{3}}(|011\rangle+|110\rangle+|101\rangle)$. This means that $\{|\psi_0\rangle\}$ is spanned by $\{|\alpha_0\rangle\}$, and $\{|\psi_i\rangle\}_{i=1}^4$ is spanned by $\{|\alpha_i\rangle\}_{i=1}^4$. Since $\langle\psi_0|\mathbb{I}_{A_1}\otimes E|\psi_i\rangle=0$ for $1\leqslant i\leqslant 4$, we have $\langle\alpha_0|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle=0$ for $1\leqslant i\leqslant 4$ from Lemma 1. This implies $\langle 00|E|01\rangle=\langle 00|E|10\rangle=\langle 00|E|11\rangle=\langle 01|E|00\rangle=\langle 10|E|00\rangle=\langle 11|E|00\rangle=0$. Then we have

$$\begin{aligned}
&\langle\alpha_1|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle=0, && 2\leqslant i\leqslant 4;\\
&\langle\alpha_1|\psi_i\rangle\neq 0, && 1\leqslant i\leqslant 4.
\end{aligned}$$

Applying Lemma 2 to $\{|\psi_i\rangle\}_{i=1}^4$, we have $\langle\alpha_i|\mathbb{I}\otimes E|\alpha_j\rangle=0$ for $1\leqslant i\neq j\leqslant 4$ and $\langle\alpha_i|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle=\langle\alpha_j|\mathbb{I}_{A_1}\otimes E|\alpha_j\rangle$ for $1\leqslant i\neq j\leqslant 4$.

Since $\langle\alpha_2|\mathbb{I}\otimes E|\alpha_3\rangle=\langle\alpha_3|\mathbb{I}\otimes E|\alpha_2\rangle=0$, we have $\langle 10|E|01\rangle=\langle 01|E|10\rangle=0$. As $\langle\alpha_2|\mathbb{I}\otimes E|\alpha_4\rangle=\langle\alpha_4|\mathbb{I}\otimes E|\alpha_2\rangle=0$, we obtain $\langle 10|E|11\rangle=\langle 11|E|10\rangle=0$. Moreover, since $\langle\alpha_3|\mathbb{I}\otimes E|\alpha_4\rangle=\langle\alpha_4|\mathbb{I}\otimes E|\alpha_3\rangle=0$, we have $\langle 01|E|11\rangle=\langle 11|E|01\rangle=0$. This means that the off-diagonal elements of E are all zeros. Next, we consider the diagonal elements of E. Since $\langle\alpha_1|\mathbb{I}_{A_1}\otimes E|\alpha_1\rangle=\langle\alpha_2|\mathbb{I}_{A_1}\otimes E|\alpha_2\rangle=\langle\alpha_3|\mathbb{I}_{A_1}\otimes E|\alpha_3\rangle$, we have $\langle 00|E|00\rangle=\langle 10|E|10\rangle=\langle 01|E|01\rangle$. Furthermore, since $\langle\alpha_4|\mathbb{I}\otimes E|\alpha_4\rangle=\langle\alpha_1|\mathbb{I}\otimes E|\alpha_1\rangle$, we have $\frac{1}{3}(\langle 11|E|11\rangle+\langle 10|E|10\rangle+\langle 01|E|01\rangle)=\langle 00|E|00\rangle$, which implies $\langle 11|E|11\rangle=\langle 00|E|00\rangle$. Thus, $E\propto\mathbb{I}$. This completes the proof. ■

Next, we give the general construction of strongest nonlocal sets in $d\otimes d\otimes d$. Let

$$\begin{aligned}
\mathcal{A}_0 &:= \{|000\rangle\},\\
\mathcal{A}_1 &:= \{|i00\rangle,|0i0\rangle,|00i\rangle\}_{i=1}^{d-1},\\
\mathcal{A}_2 &:= \left\{\frac{1}{\sqrt{3}}(|0ij\rangle+|j0i\rangle+|ij0\rangle):1\leqslant i,j\leqslant d-1\right\},
\end{aligned}\tag{3}$$

where $|\mathcal{A}_0|=1$, $|\mathcal{A}_1|=3(d-1)$, $|\mathcal{A}_2|=(d-1)^2$, and $|\mathcal{A}_0|+|\mathcal{A}_1|+|\mathcal{A}_2|=d^2+d-1=S(d,d,d)$. We denote $|\alpha_0\rangle=|000\rangle$ and $\{|\alpha_i\rangle\}_{i=1}^{S(d,d,d)-1}=\mathcal{A}_1\bigcup\mathcal{A}_2$. Then we can construct a set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d,d,d)}}$, where

$$\begin{aligned}
|\psi_0\rangle &= |\alpha_0\rangle,\\
|\psi_{i+1}\rangle &= \sum_{j\in\mathbb{Z}_{S(d,d,d)-1}} w_{S(d,d,d)-1}^{ij}|\alpha_{j+1}\rangle,\quad i\in\mathbb{Z}_{S(d,d,d)-1}.
\end{aligned}\tag{4}$$

Proposition 1. In $d\otimes d\otimes d$, $d\geqslant 2$, the set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d,d,d)}}$ given by Eq. (4) is strongest nonlocal.

Proof. Since $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d,d,d)}}$ has a similar structure under the cyclic permutation of the parties $\{A_1,A_2,A_3\}$, we need to show only that any OPLM performed on A_2A_3 is trivial. We assume that A_2A_3 performs an OPLM $\{E\}$; then we have $\langle\psi_i|\mathbb{I}_{A_1}\otimes E|\psi_j\rangle=0$ for $i\neq j\in\mathbb{Z}_{S(d,d,d)}$.

Note that $\{|\psi_0\rangle\}$ is spanned by $\{|\alpha_0\rangle\}$, and $\{|\psi_i\rangle\}_{i=1}^{S(d,d,d)-1}$ is spanned by $\{|\alpha_i\rangle\}_{i=1}^{S(d,d,d)-1}$. Since $\langle\psi_0|\mathbb{I}_{A_1}\otimes E|\psi_i\rangle=0$ for $1\leqslant i\leqslant S(d,d,d)-1$, we have $\langle\alpha_0|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle=0$ for $1\leqslant i\leqslant S(d,d,d)-1$ from Lemma 1. This implies that

$$\langle 00|E|ij\rangle=\langle ij|E|00\rangle=0,\quad (i,j)\in\mathbb{Z}_d\times\mathbb{Z}_d\setminus\{(0,0)\}.\tag{5}$$

Without loss of generality, we assume $|\alpha_1\rangle=|100\rangle$. Then we have

$$\begin{aligned}
&\langle\alpha_1|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle=0, && 2\leqslant i\leqslant S(d,d,d)-1;\\
&\langle\alpha_1|\psi_i\rangle\neq 0, && 1\leqslant i\leqslant S(d,d,d)-1.
\end{aligned}$$

Applying Lemma 2 to $\{|\psi_i\rangle\}_{i=1}^{S(d,d,d)-1}$, we obtain

$$\begin{aligned}
&\langle\alpha_i|\mathbb{I}_{A_1}\otimes E|\alpha_j\rangle=0,\quad 1\leqslant i\neq j\leqslant S(d,d,d)-1;\\
&\langle\alpha_i|\mathbb{I}_{A_1}\otimes E|\alpha_i\rangle=\langle\alpha_j|\mathbb{I}_{A_1}\otimes E|\alpha_j\rangle,\\
&\qquad 1\leqslant i\neq j\leqslant S(d,d,d)-1.
\end{aligned}\tag{6}$$

If $|\alpha_i\rangle, |\alpha_j\rangle \in \mathcal{A}_1$, then we have

$$\langle i_1 j_1|E|i_2 j_2\rangle = 0,\quad wt(i_1, j_1) = wt(i_2, j_2) = 1,\ (i_1, j_1) \neq (i_2, j_2) \in \mathbb{Z}_d \times \mathbb{Z}_d. \tag{7}$$

If $|\alpha_i\rangle \in \mathcal{A}_1$ and $|\alpha_j\rangle \in \mathcal{A}_2$, then we have

$$\begin{aligned}0 &= \langle 00i_1|\mathbb{I}_{A_1} \otimes E \frac{1}{\sqrt{3}}(|0i_2j_2\rangle + |j_20i_2\rangle + |i_2j_20\rangle)\\ &= \frac{1}{\sqrt{3}}(\langle 0i_2j_2| + \langle j_20i_2| + \langle i_2j_20|)\mathbb{I}_{A_1} \otimes E|00i_1\rangle\\ &= \langle 0i_1|E|i_2j_2\rangle = \langle i_2j_2|E|0i_1\rangle, \quad i_1, i_2, j_2 \in \mathbb{Z}_d \setminus \{0\},\\ 0 &= \langle 0i_10|\mathbb{I}_{A_1} \otimes E \frac{1}{\sqrt{3}}(|0i_2j_2\rangle + |j_20i_2\rangle + |i_2j_20\rangle)\\ &= \frac{1}{\sqrt{3}}(\langle 0i_2j_2| + \langle j_20i_2| + \langle i_2j_20|)\mathbb{I}_{A_1} \otimes E|0i_10\rangle\\ &= \langle i_10|E|i_2j_2\rangle = \langle i_2j_2|E|i_10\rangle, \quad i_1, i_2, j_2 \in \mathbb{Z}_d \setminus \{0\}.\end{aligned}$$

This means that

$$\langle i_1j_1|E|i_2j_2\rangle = \langle i_2j_2|E|i_1j_1\rangle = 0,\quad wt(i_1, j_1) = 1, wt(i_2, j_2) = 2,\quad (i_1, j_1) \neq (i_2, j_2) \in \mathbb{Z}_d \times \mathbb{Z}_d. \tag{8}$$

If $|\alpha_i\rangle, |\alpha_j\rangle \in \mathcal{A}_2$, then we have

$$\begin{aligned}0 &= \frac{1}{\sqrt{3}}(\langle 0i_1j_1| + \langle j_10i_1| + \langle i_1j_10|)\mathbb{I}_{A_1} \otimes E\\ &\quad \times \frac{1}{\sqrt{3}}(|0i_2j_2\rangle + |j_20i_2\rangle + |i_2j_20\rangle)\\ &= \frac{1}{3}(\langle i_1j_1|E|i_2j_2\rangle + \langle j_1|j_2\rangle\langle 0i_1|E|0i_2\rangle\\ &\quad + \langle j_1|i_2\rangle\langle 0i_1|E|j_20\rangle + \langle i_1|j_2\rangle\langle j_10|E|0i_2\rangle\\ &\quad + \langle i_1|i_2\rangle\langle j_10|E|j_20\rangle)\\ &= \frac{1}{3}(\langle i_1j_1|E|i_2j_2\rangle + \langle j_1|j_2\rangle\langle 0i_1|E|0i_2\rangle\\ &\quad + \langle i_1|i_2\rangle\langle j_10|E|j_20\rangle)\\ &= \frac{1}{3}(\langle i_1j_1|E|i_2j_2\rangle),\ i_1, j_1, i_2, j_2 \in \mathbb{Z}_d \setminus \{0\},\\ &(i_1, j_1) \neq (i_2, j_2).\end{aligned}$$

That is,

$$\langle i_1j_1|E|i_2j_2\rangle = 0,\quad wt(i_1, j_1) = wt(i_2, j_2) = 2,\quad (i_1, j_1) \neq (i_2, j_2) \in \mathbb{Z}_d \times \mathbb{Z}_d. \tag{9}$$

From Eqs. (5), (7), (8), and (9), we find that the off-diagonal elements of E are all zeros.

Next, we consider the diagonal elements of E. If $|\alpha_i\rangle \in \mathcal{A}_1$, then we have

$$\begin{aligned}\langle 00|E|00\rangle &= \langle \alpha_1|\mathbb{I}_{A_1} \otimes E|\alpha_1\rangle = \langle \alpha_i|\mathbb{I}_{A_1} \otimes E|\alpha_i\rangle\\ &= \langle i_1j_1|E|i_1j_1\rangle,\\ wt(i_1, j_1) &= 1,\ (i_1, j_1) \in \mathbb{Z}_d \times \mathbb{Z}_d.\end{aligned} \tag{10}$$

If $|\alpha_i\rangle \in \mathcal{A}_2$, then we have

$$\begin{aligned}\langle 00|E|00\rangle &= \langle \alpha_1|\mathbb{I}_{A_1} \otimes E|\alpha_1\rangle = \langle \alpha_i|\mathbb{I}_{A_1} \otimes E|\alpha_i\rangle\\ &= \tfrac{1}{3}(\langle i_1j_1|E|i_1j_1\rangle + \langle 0i_1|E|0i_1\rangle + \langle j_10|E|j_10\rangle),\\ wt(i_1, j_1) &= 2,\ (i_1, j_1) \in \mathbb{Z}_d \times \mathbb{Z}_d.\end{aligned}$$

That is,

$$\langle 00|E|00\rangle = \langle i_1j_1|E|i_1j_1\rangle,\quad wt(i_1, j_1) = 2,\ (i_1, j_1) \in \mathbb{Z}_d \times \mathbb{Z}_d. \tag{11}$$

From Eqs. (10) and (11), we find that the diagonal elements of E are all equal. Thus, $E \propto \mathbb{I}$. This completes the proof. ■

We can also generalize the strongest nonlocal sets in $d \otimes d \otimes d$ to the system $d_1 \otimes d_2 \otimes d_3$, where $2 \leqslant d_1 \leqslant d_2 \leqslant d_3$. Let

$$\begin{aligned}\mathcal{A}_0 &:= \{|000\rangle\},\\ \mathcal{A}_1 &:= \{|i00\rangle : 1 \leqslant i \leqslant d_1 - 1\} \bigcup \{|0i0\rangle : 1 \leqslant i \leqslant d_2 - 1\}\\ &\quad \bigcup \{|00i\rangle : 1 \leqslant i \leqslant d_3 - 1\},\\ \mathcal{A}_2 &:= \left\{\frac{1}{\sqrt{3}}(|0ij\rangle + |j0i\rangle + |ij0\rangle) : 1 \leqslant i, j \leqslant d_1 - 1\right\}\\ &\quad \bigcup \left\{\frac{1}{\sqrt{2}}(|0ij\rangle + |j0i\rangle) :\ d_1 \leqslant i \leqslant d_2 - 1, 1 \leqslant j \leqslant d_1 - 1\right\}\\ &\quad \bigcup \left\{\frac{1}{\sqrt{2}}(|0ij\rangle + |ij0\rangle) :\ 1 \leqslant i \leqslant d_1 - 1,\ d_1 \leqslant j \leqslant d_2 - 1\right\}\\ &\quad \bigcup \left\{|0ij\rangle : d_1 \leqslant i, j \leqslant d_2 - 1\right\}\\ &\quad \bigcup \left\{\frac{1}{\sqrt{2}}(|0ij\rangle + |i0j\rangle) :\ 1 \leqslant i \leqslant d_1 - 1,\ d_2 \leqslant j \leqslant d_3 - 1\right\}\\ &\quad \bigcup \left\{|0ij\rangle : d_1 \leqslant i \leqslant d_2 - 1,\ d_2 \leqslant j \leqslant d_3 - 1\right\},\end{aligned} \tag{12}$$

where $|\mathcal{A}_0| = 1$, $|\mathcal{A}_1| = d_1 + d_2 + d_3 - 3$, $|\mathcal{A}_2| = (d_2 - 1)(d_3 - 1)$, and $|\mathcal{A}_0| + |\mathcal{A}_1| + |\mathcal{A}_2| = d_2d_3 + d_1 - 1 = S(d_1, d_2, d_3)$. We denote $|\alpha_0\rangle = |000\rangle$ and $\{|\alpha_i\rangle\}_{i=1}^{S(d_1,d_2,d_3)-1} = \mathcal{A}_1 \bigcup \mathcal{A}_2$. Then we can construct a set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d_1,d_2,d_3)}}$, where

$$\begin{aligned}|\psi_0\rangle &= |\alpha_0\rangle,\\ |\psi_{i+1}\rangle &= \sum_{j\in\mathbb{Z}_{S(d_1,d_2,d_3)-1}} w_{S(d_1,d_2,d_3)-1}^{ij}|\alpha_{j+1}\rangle,\end{aligned} \tag{13}$$

where $i \in \mathbb{Z}_{S(d_1,d_2,d_3)-1}$.

Proposition 2. In $d_1 \otimes d_2 \otimes d_3$, $2 \leqslant d_1 \leqslant d_2 \leqslant d_3$, the set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d_1,d_2,d_3)}}$ given by Eq. (13) is strongest nonlocal.

The proof of Proposition 2 is given in the Supplemental Material [49]. Our strongest nonlocal set in $d_1 \otimes d_2 \otimes d_3$ $(2 \leqslant d_1 \leqslant d_2 \leqslant d_3)$ has size $S(d_1, d_2, d_3)$, and it is close to the minimum size $d_2 d_3 + 1$ given in Eq. (1), which is smaller than all known constructions. When $d_1 = 2$, our strongest nonlocal set has the minimum size. Note that $|\psi_0\rangle = |000\rangle$ is a product state, while other states $\{|\psi_i\rangle\}_{i=1}^{S(d_1,d_2,d_3)-1}$ are genuinely entangled states.

IV. STRONGEST NONLOCAL SETS IN FOUR-PARTITE SYSTEMS

In this section, we construct the strongest nonlocal orthogonal states in $d \otimes d \otimes d \otimes d$. Let

$$\begin{aligned}
\mathcal{A}_0 &:= \{|0000\rangle\},\\
\mathcal{A}_1 &:= \{|i000\rangle, |0i00\rangle, |00i0\rangle, |000i\rangle\}_{i=1}^{d-1},\\
\mathcal{A}_2 &:= \left\{\frac{1}{\sqrt{2}}(|00ij\rangle + |ij00\rangle) : 1 \leqslant i, j \leqslant d-1\right\}\\
&\quad \bigcup \left\{\frac{1}{\sqrt{2}}(|0ij0\rangle + |j00i\rangle) : 1 \leqslant i, j \leqslant d-1\right\}\\
&\quad \bigcup \left\{\frac{1}{\sqrt{2}}(|0i0j\rangle + |i0j0\rangle) : 1 \leqslant i, j \leqslant d-1\right\},\\
\mathcal{A}_3 &:= \left\{\frac{1}{2}(|0ijk\rangle + |k0ij\rangle + |jk0i\rangle + |ijk0\rangle) :\right.\\
&\quad \left. 1 \leqslant i, j, k \leqslant d-1\right\},
\end{aligned}$$

where $|\mathcal{A}_0| = 1$, $|\mathcal{A}_1| = 4(d-1)$, $|\mathcal{A}_2| = 3(d-1)^2$, $|\mathcal{A}_3| = (d-1)^3$, and $|\mathcal{A}_0| + |\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{A}_3| = d^3 + d - 1 = S(d, d, d, d)$.

We denote $|\alpha_0\rangle = |0000\rangle$ and $\{|\alpha_i\rangle\}_{i=1}^{S(d,d,d,d)-1} = \mathcal{A}_1 \bigcup \mathcal{A}_2 \bigcup \mathcal{A}_3$. Then we can construct a set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d,d,d,d)}}$, where

$$\begin{aligned}
|\psi_0\rangle &= |\alpha_0\rangle,\\
|\psi_{i+1}\rangle &= \sum_{j\in\mathbb{Z}_{S(d,d,d,d)-1}} w_{S(d,d,d,d)-1}^{ij} |\alpha_{j+1}\rangle,
\end{aligned} \tag{14}$$

where $i \in \mathbb{Z}_{S(d,d,d,d)-1}$.

Proposition 3. In $d \otimes d \otimes d \otimes d$, $d \geqslant 2$, the set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d,d,d,d)}}$ given by Eq. (14) is strongest nonlocal.

The proof of Proposition 3 is given in the Supplemental Material [49]. Note that we construct a strongest nonlocal set with size $S(d, d, d, d)$ in $d \otimes d \otimes d \otimes d$, which is also close to the minimum size $d^3 + 1$ given in Eq. (1), which is smaller than all known constructions. When $d = 2$, our construction has the minimum size of 9. Note that $|\psi_0\rangle = |0000\rangle$ is a product state, while other states $\{|\psi_i\rangle\}_{i=1}^{S(d,d,d,d)-1}$ are genuinely entangled states.

Example 2. In $2 \otimes 2 \otimes 2 \otimes 2$, let

$$\begin{aligned}
\mathcal{A}_0 &:= \{|0000\rangle\} = \{|\alpha_0\rangle\},\\
\mathcal{A}_1 &:= \{|0001\rangle, |0010\rangle, |0100\rangle, |1000\rangle\}\\
&= \{|\alpha_1\rangle, |\alpha_2\rangle, |\alpha_3\rangle, |\alpha_4\rangle\},\\
\mathcal{A}_2 &:= \left\{\frac{1}{\sqrt{2}}(|0011\rangle + |1100\rangle), \frac{1}{\sqrt{2}}(|0110\rangle + |1001\rangle),\right.\\
&\quad \left.\frac{1}{\sqrt{2}}(|1010\rangle + |0101\rangle)\right\} = \{|\alpha_5\rangle, |\alpha_6\rangle, |\alpha_7\rangle\},\\
\mathcal{A}_3 &:= \left\{\frac{1}{2}(|0111\rangle + |1011\rangle + |1101\rangle + |1110\rangle)\right\} = \{|\alpha_8\rangle\}.
\end{aligned}$$

Then the set of orthogonal states $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_9}$ is strongest nonlocal with the minimum size, where

$$\begin{aligned}
|\psi_0\rangle &= |\alpha_0\rangle,\\
|\psi_1\rangle &= |\alpha_1\rangle + |\alpha_2\rangle + |\alpha_3\rangle + |\alpha_4\rangle\\
&\quad + |\alpha_5\rangle + |\alpha_6\rangle + |\alpha_7\rangle + |\alpha_8\rangle,\\
|\psi_2\rangle &= |\alpha_1\rangle + w_8|\alpha_2\rangle + w_8^2|\alpha_3\rangle + w_8^3|\alpha_4\rangle\\
&\quad + w_8^4|\alpha_5\rangle + w_8^5|\alpha_6\rangle + w_8^6|\alpha_7\rangle + w_8^7|\alpha_8\rangle,\\
|\psi_3\rangle &= |\alpha_1\rangle + w_8^2|\alpha_2\rangle + w_8^4|\alpha_3\rangle + w_8^6|\alpha_4\rangle\\
&\quad + w_8^8|\alpha_5\rangle + w_8^{10}|\alpha_6\rangle + w_8^{12}|\alpha_7\rangle + w_8^{14}|\alpha_8\rangle,\\
|\psi_4\rangle &= |\alpha_1\rangle + w_8^3|\alpha_2\rangle + w_8^6|\alpha_3\rangle + w_8^9|\alpha_4\rangle\\
&\quad + w_8^{12}|\alpha_5\rangle + w_8^{15}|\alpha_6\rangle + w_8^{18}|\alpha_7\rangle + w_8^{21}|\alpha_8\rangle,\\
|\psi_5\rangle &= |\alpha_1\rangle + w_8^4|\alpha_2\rangle + w_8^8|\alpha_3\rangle + w_8^{12}|\alpha_4\rangle\\
&\quad + w_8^{16}|\alpha_5\rangle + w_8^{20}|\alpha_6\rangle + w_8^{24}|\alpha_7\rangle + w_8^{28}|\alpha_8\rangle,\\
|\psi_6\rangle &= |\alpha_1\rangle + w_8^5|\alpha_2\rangle + w_8^{10}|\alpha_3\rangle + w_8^{15}|\alpha_4\rangle\\
&\quad + w_8^{20}|\alpha_5\rangle + w_8^{25}|\alpha_6\rangle + w_8^{30}|\alpha_7\rangle + w_8^{35}|\alpha_8\rangle,\\
|\psi_7\rangle &= |\alpha_1\rangle + w_8^6|\alpha_2\rangle + w_8^{12}|\alpha_3\rangle + w_8^{18}|\alpha_4\rangle\\
&\quad + w_8^{24}|\alpha_5\rangle + w_8^{30}|\alpha_6\rangle + w_8^{36}|\alpha_7\rangle + w_8^{42}|\alpha_8\rangle,\\
|\psi_8\rangle &= |\alpha_1\rangle + w_8^7|\alpha_2\rangle + w_8^{14}|\alpha_3\rangle + w_8^{21}|\alpha_4\rangle\\
&\quad + w_8^{28}|\alpha_5\rangle + w_8^{35}|\alpha_6\rangle + w_8^{42}|\alpha_7\rangle + w_8^{49}|\alpha_8\rangle.
\end{aligned}$$

V. APPLICATION: PARTIALLY GENUINELY ENTANGLED SUBSPACES FROM THE STRONGEST NONLOCAL SETS

In this section, we construct some genuinely entangled mixed states from our strongest nonlocal sets. First, let us recall some concepts. A bipartite pure state $|\psi\rangle \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ with dimension $m \otimes n$ can be written as

$$|\psi\rangle = \sum_{i\in\mathbb{Z}_m, j\in\mathbb{Z}_n} a_{i,j}|ij\rangle,$$

where $\{|i\rangle_{i\in\mathbb{Z}_m}\}$ and $\{|j\rangle_{j\in\mathbb{Z}_n}\}$ are computational bases of $\mathcal{H}_{A_1}$. Then $|\psi\rangle$ corresponds to an $m \times n$ matrix $M = (a_{i,j})_{i\in\mathbb{Z}_m, j\in\mathbb{Z}_n}$, and $|\psi\rangle$ is a product state if and only if $\text{rank}(M) = 1$. For a pure state $|\psi\rangle \in \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$, $|\psi\rangle$ is biproduct if a bipartition $S|\overline{S}$ (where $S \cup \overline{S} = \{A_1, A_2, \ldots, A_n\}$) exists such that $|\psi\rangle = |\phi\rangle_S \otimes |\varphi\rangle_{\overline{S}}$. The state $|\psi\rangle$ is genuinely entangled if it is not a biproduct state. A mixed state ρ is biseparable if it can be written as

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|, \quad p_i \geqslant 0,\ \sum_i p_i = 1,$$

where $|\psi_i\rangle$ is a biproduct state for each i. A mixed state is a genuinely entangled state if it is not a biseperable state. Now, we give the definition of a partially genuinely entangled subspace.

Definition 3. Given a subspace $\mathcal{S}$ of $\mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}\otimes\cdots\otimes\mathcal{H}_{A_n}$, we call $\mathcal{S}$ *partially genuinely entangled* if all the biproduct states in $\mathcal{S}$ cannot span $\mathcal{S}$.

By the range criterion [50], if the range of a mixed state ρ is contained in a partially genuinely entangled subspace $\mathcal{S}$ and $\text{rank}(\rho)=\dim(\mathcal{S})$, then ρ is genuinely entangled. For example, the normalized projector on a partially genuinely entangled subspace must be a genuinely entangled state.

For a subspace $\mathcal{B}_0$ of $\mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}\otimes\cdots\otimes\mathcal{H}_{A_n}$, we can write $|\phi\rangle=|\phi_1\rangle+|\phi_2\rangle$, where $|\phi_1\rangle\in\mathcal{B}_0$ and $|\phi_2\rangle\in\mathcal{B}_0^{\perp}$. Then $|\phi_1\rangle$ is called the projection of $|\phi\rangle$ on $\mathcal{B}_0$. Assume $\mathcal{B}$ is another subspace of $\mathcal{H}_{A_1}\otimes\mathcal{H}_{A_2}\otimes\cdots\otimes\mathcal{H}_{A_n}$; then the projection of $\mathcal{B}$ on $\mathcal{B}_0$ is a subspace of $\mathcal{B}_0$ which consists of the projections of all states in $\mathcal{B}$ on $\mathcal{B}_0$. Note that if $\mathcal{B}_0\subseteq\mathcal{B}$, then the projection of $\mathcal{B}$ on $\mathcal{B}_0$ is $\mathcal{B}_0$.

Proposition 4. In $d_1\otimes d_2\otimes d_3$, $2\leqslant d_1\leqslant d_2\leqslant d_3$, and $d_2\geqslant 3$, let $\mathcal{S}$ be the subspace spanned by $\{|\psi_i\rangle\}_{i\in\mathbb{Z}_{S(d_1,d_2,d_3)}}$ given in Eq. (13); the complementary subspace $\mathcal{S}^{\perp}$ of $\mathcal{S}$ is partially genuinely entangled with $\dim(\mathcal{S}^{\perp})=d_1d_2d_3-S(d_1,d_2,d_3)$.

Proof. Note that $\mathcal{S}$ is also spanned by the states in $\mathcal{A}_0\cup\mathcal{A}_1\cup\mathcal{A}_2$ given in Eq. (12). There are two cases.

Case 1. $d_1=2$. Suppose all the biproduct states in $\mathcal{S}^{\perp}$ span a subspace $\mathcal{B}$. Our aim is to prove $\mathcal{B}\subsetneq\mathcal{S}^{\perp}$. Obviously, $|021\rangle-|102\rangle\in\mathcal{S}^{\perp}$. If $|021\rangle-|102\rangle\notin\mathcal{B}$, then $\mathcal{B}\subsetneq\mathcal{S}^{\perp}$. We need to show only $|021\rangle-|102\rangle\notin\mathcal{B}$.

If $|021\rangle-|102\rangle\in\mathcal{B}$, then a set of biproduct states $\{|\phi_i\rangle\}$ of $\mathcal{B}$ exists such that $|021\rangle-|102\rangle=\sum_i a_i|\phi_i\rangle$. Then a biproduct state $|\phi\rangle$ of $\{|\phi_i\rangle\}$ must exist such that $|\phi\rangle=\sum_{ijk}\lambda_{ijk}|ijk\rangle$, where $\lambda_{021}\neq 0$. Since $|\phi\rangle$ is orthogonal to $|021\rangle+|102\rangle\in\mathcal{A}_2$, we have $\lambda_{021}=-\lambda_{102}$. Note that $|\phi\rangle$ is orthogonal to any state in $\mathcal{A}_1$, which implies $\lambda_{ijk}=0$ for $wt(i,j,k)=1$.

If $|\phi\rangle$ is a biproduct across the bipartition $A_1|A_2A_3$, then $|\phi\rangle_{A_1|A_2A_3}$ corresponds to a $d_1\times d_2d_3$ matrix M_1 with $\text{rank}(M_1)=1$. Consider the submatrix of M_1,

$$\begin{array}{c} \\ 0\\ 1\end{array}\begin{array}{c} \begin{array}{cc}21 & 02\end{array}\\ \begin{pmatrix}\lambda_{021} & 0\\ \lambda_{121} & \lambda_{102}\end{pmatrix}\end{array}.$$

Since this submatrix has rank 2, $|\phi\rangle$ is not a biproduct across the bipartition $A_1|A_2A_3$.

If $|\phi\rangle$ is a biproduct state across the bipartition $A_2|A_1A_3$, then $|\phi\rangle_{A_2|A_1A_3}$ corresponds to a $d_2\times d_1d_3$ matrix M_2 with $\text{rank}(M_2)=1$. Consider the submatrix of M_2,

$$\begin{array}{c} \\ 0\\ 2\end{array}\begin{array}{c} \begin{array}{cc}01 & 12\end{array}\\ \begin{pmatrix}0 & \lambda_{102}\\ \lambda_{021} & \lambda_{122}\end{pmatrix}\end{array}.$$

Since this submatrix has rank 2, $|\phi\rangle$ is not a biproduct state across the bipartition $A_2|A_1A_3$.

If $|\phi\rangle$ is a biproduct state across the bipartition $A_3|A_1A_2$, then $|\phi\rangle_{A_3|A_1A_2}$ corresponds to a $d_3\times d_1d_2$ matrix M_3 with $\text{rank}(M_3)=1$. Consider the submatrix of M_3,

$$\begin{array}{c} \\ 0\\ 1\\ 2\end{array}\begin{array}{c} \begin{array}{ccccc}01 & 02 & 10 & 11 & 12\end{array}\\ \begin{pmatrix}0 & 0 & 0 & \lambda_{110} & \lambda_{120}\\ \lambda_{011} & \lambda_{021} & \lambda_{101} & \lambda_{111} & \lambda_{121}\\ \lambda_{012} & \lambda_{022} & \lambda_{102} & \lambda_{112} & \lambda_{122}\end{pmatrix}\end{array}.$$

Since $\lambda_{021}=-\lambda_{102}\neq 0$, we have $\lambda_{101}\neq 0$, $\lambda_{022}\neq 0$, and $\lambda_{120}=\lambda_{110}=0$. Since $|\phi\rangle$ is orthogonal to $|101\rangle+|110\rangle+|011\rangle$, we have $\lambda_{011}\neq 0$ and $\lambda_{012}\neq 0$. Since $|\phi\rangle$ is orthogonal to $|012\rangle+|120\rangle$, we have $\lambda_{120}=-\lambda_{012}\neq 0$, which is impossible. Thus, $|\phi\rangle$ is not a biproduct state across the bipartition $A_3|A_1A_2$.

Above all, $|021\rangle-|102\rangle\notin\mathcal{B}$.

Case 2. $d_2\geqslant 3$. Suppose all the biproduct states in $\mathcal{S}^{\perp}$ span a subspace $\mathcal{B}$. Our aim is to prove $\mathcal{B}\subsetneq\mathcal{S}^{\perp}$. Since $|012\rangle+|201\rangle+|120\rangle$, $|021\rangle+|102\rangle+|210\rangle\in\mathcal{A}_2$, we have the four states $|012\rangle-|201\rangle$, $|201\rangle-|120\rangle$, $|021\rangle-|102\rangle$, and $|102\rangle-|210\rangle\in\mathcal{S}^{\perp}$, which span a subspace $\mathcal{B}_0$. If $\mathcal{B}_0\nsubseteq\mathcal{B}$, then $\mathcal{B}\subsetneq\mathcal{S}^{\perp}$. We need to show only $\mathcal{B}_0\nsubseteq\mathcal{B}$.

For any biproduct state $|\phi\rangle=\sum_{ijk}\lambda_{ijk}|ijk\rangle\in\mathcal{B}$, we have $\lambda_{ijk}=0$ for $wt(i,j,k)=1$. If $|\phi\rangle$ is a biproduct state across the bipartition $A_1|A_2A_3$, then $|\phi\rangle_{A_1|A_2A_3}$ corresponds to a $d_1\times d_2d_3$ matrix M with $\text{rank}(M)=1$. Consider the submatrix of M,

$$\begin{array}{c} \\ 0\\ 1\\ 2\end{array}\begin{array}{c} \begin{array}{cccccccc}01 & 02 & 10 & 11 & 12 & 20 & 21 & 22\end{array}\\ \begin{pmatrix}0 & 0 & 0 & \lambda_{011} & \lambda_{012} & 0 & \lambda_{021} & \lambda_{022}\\ \lambda_{101} & \lambda_{102} & \lambda_{110} & \lambda_{111} & \lambda_{112} & \lambda_{120} & \lambda_{121} & \lambda_{122}\\ \lambda_{201} & \lambda_{202} & \lambda_{210} & \lambda_{211} & \lambda_{212} & \lambda_{220} & \lambda_{221} & \lambda_{222}\end{pmatrix}\end{array}.$$

Note that $|\phi\rangle$ is orthogonal to $|011\rangle+|101\rangle+|110\rangle$, $|022\rangle+|202\rangle+|220\rangle$, $|012\rangle+|201\rangle+|120\rangle$, and $|021\rangle+|102\rangle+|201\rangle$. If $\lambda_{011}\neq 0$, then $\lambda_{101}\neq 0$ or $\lambda_{110}\neq 0$, which contradicts $\text{rank}(M)=1$. Then $\lambda_{011}=0$. Similarly, we also have $\lambda_{012}=\lambda_{021}=\lambda_{022}=0$. Then $\lambda_{101}=-\lambda_{110}$, $\lambda_{202}=-\lambda_{220}$, $\lambda_{201}=-\lambda_{120}$, and $\lambda_{102}=-\lambda_{201}$. Now we will consider the projection of $|\phi\rangle$ in $\mathcal{B}_0$. If $\lambda_{201}=\lambda_{210}=\lambda_{102}=\lambda_{120}=0$, then the projection of $|\phi\rangle$ on $\mathcal{B}_0$ is zero. Assume there is at least one nonzero coefficient of $\{\lambda_{201},\lambda_{210},\lambda_{102},\lambda_{120}\}$; without loss of generality, $\lambda_{201}\neq 0$ (other cases are the same).

Since $\lambda_{201}\neq 0$, we have $\lambda_{201}=-\lambda_{120}\neq 0$, which implies λ_{101}, $\lambda_{220}\neq 0$. Then we have λ_{110}, $\lambda_{202}\neq 0$ and λ_{210}, $\lambda_{102}\neq 0$. Consider the following submatrix of M:

$$\begin{array}{c} \\ 1\\ 2\end{array}\begin{array}{c} \begin{array}{cc}01 & 10\end{array}\\ \begin{pmatrix}\lambda_{101} & \lambda_{110}\\ \lambda_{201} & \lambda_{210}\end{pmatrix}\end{array}.$$

Then we have $\lambda_{201}=-\lambda_{210}$. Therefore, we can conclude $\lambda_{201}=-\lambda_{210}=\lambda_{102}=-\lambda_{120}$. So the projection of $|\phi\rangle$ on $\mathcal{B}_0$ is $\lambda_{201}(|201\rangle-|210\rangle+|102\rangle-|120\rangle)$. Similarly, if $|\phi\rangle$ is a biproduct state across bipartition $A_2|A_1A_3$, then the projection of $|\phi\rangle$ on $\mathcal{B}_0$ is $\lambda_{021}(|021\rangle-|120\rangle+|012\rangle-|210\rangle)$. If $|\phi\rangle$ is a biproduct state across bipartition $A_2|A_1A_3$, then the projection of $|\phi\rangle$ on $\mathcal{B}_0$ is $\lambda_{012}(|012\rangle-|102\rangle+|021\rangle-|201\rangle)$. For any $|\psi\rangle=\sum_i a_i|\phi_i\rangle\in\mathcal{B}$, each $|\phi_i\rangle$ is a biproduct state. Note that the nonzero projection of each $|\phi_i\rangle$ on $\mathcal{B}_0$ can have only three forms; then the projection of $|\phi\rangle$ on $\mathcal{B}_0$ belongs to a three-dimensional space. This means that the projection of $\mathcal{B}$ on

$\mathcal{B}_0$ has dimension 3. Since $\mathcal{B}_0$ has dimension 4, $\mathcal{B}_0 \nsubseteq \mathcal{B}$. This completes this proof. ■

In $2 \otimes 2 \otimes d_3$, the subspace $\mathcal{S}^{\perp}$ is not partially genuinely entangled since the biproduct states $\{|111\rangle - |110\rangle + |101\rangle$, $|111\rangle - |101\rangle + |011\rangle$, $|111\rangle - |011\rangle + |110\rangle$, $|012\rangle - |102\rangle$, $|013\rangle - |103\rangle, \ldots, |01d_3-1\rangle - |10d_3-1\rangle$, $|112\rangle, \ldots$, $|11d_3-1\rangle\} = \{|\phi_i\rangle\}_{i=0}^{2d_3-2}$ in $\mathcal{S}^{\perp}$ span $\mathcal{S}^{\perp}$.

Now we have constructed some genuinely entangled mixed states from our strongest nonlocal sets. For example, let

$$\rho = \frac{1}{d_1 d_2 d_3 - S(d_1, d_2, d_3)}\left(\mathbb{I} - \sum_{i \in \mathbb{Z}_{S(d_1,d_2,d_3)}} |\widetilde{\psi}_i\rangle\langle\widetilde{\psi}_i|\right),$$

where $|\widetilde{\psi}_i\rangle$ is the normalized state of $|\psi_i\rangle$ for $i \in \mathbb{Z}_{S(d_1,d_2,d_3)}$. Then ρ is genuinely entangled when $d_2 \geqslant 3$. For $d_2 = 2$, ρ is biseparable, but one can check that ρ is genuinely entangled across every bipartition.

VI. CONCLUSION AND DISCUSSION

In this paper, we constructed the strongest nonlocal sets in three-partite and four-partite systems. The sizes of our strongest nonlocal sets are close to the minimum size. Especially, our strongest nonlocal sets have the minimum size in $2 \otimes d_2 \otimes d_3$ and $2 \otimes 2 \otimes 2 \otimes 2$. All of our strongest nonlocal sets consist of genuinely entangled states except one product state. Moreover, in the same systems, our strongest nonlocal sets have smaller sizes than those of all the strongest nonlocal sets in [37,40–48] (see Table I). We also constructed some partially genuinely entangled subspaces from our strongest nonlocal sets.

There are some questions left. For example, how do we construct the strongest nonlocal sets with the minimum size in general $d_1 \otimes d_2 \otimes \cdots \otimes d_n$? Can we improve the lower bound on the sizes of the strongest nonlocal sets in [37].

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