Generalized iterative formula for Bell inequalities

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Bell inequality is a vital tool to detect the nonlocal correlations, but the construction of it for multipartite systems is still a complicated problem. In this work, inspired via a decomposition of $(n+1)$-partite Bell inequalities into n-partite ones, we present a generalized iterative formula to construct nontrivial $(n+1)$-partite ones from the n-partite ones. Our iterative formulas recover the well-known Mermin-Ardehali-Belinskĭ-Klyshko (MABK) and other families in the literature as special cases. Moreover, a family of "dual-use" Bell inequalities is proposed, in the sense that for the generalized Greenberger-Horne-Zeilinger states these inequalities lead to the same quantum violation as the MABK family and, at the same time, the inequalities are able to detect the nonlocality in the entire entangled region. Furthermore, we present a generalization of the I3322 inequality to any n-partite case which is still tight, and of the 46 Śliwa's inequalities to the four-partite tight ones, by applying our iteration method to each inequality and its equivalence class.

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I. INTRODUCTION

The fact that quantum mechanical correlations are incompatible with local hidden variable (LHV) models [1] is known as Bell nonlocality [2], which is usually revealed by the violation of Bell inequalities [3]. As an extraordinary nonclassical phenomenon, Bell nonlocality plays an important role both on the theoretical level and in applications [4]. On the one hand, Bell nonlocality is related to many other quantum correlations like quantum entanglement [5], quantum steering [6], and quantum contextuality [7]. On the other hand, Bell nonlocality is crucial in quantum applications like quantum key distribution [8], quantum randomness [9,10], and quantum computation [11].

Regarding the importance of Bell nonlocality, the construction of Bell inequalities is always one central topic in quantum information theory. The first Bell inequality was discovered by Bell in a bipartite scenario to tackle the long-standing paradox proposed by Einstein, Podolsky, and Rosen [12] from the statistical aspect. However, the original Bell inequality is hard to test experimentally. A revised version of the original Bell inequality, the Clause-Horne-Shimony-Holt (CHSH) inequality [13], is more friendly for experiments. Employing the CHSH inequality, three loophole-free Bell experiments succeeded finally [14–16]. The development of Bell inequalities has many directions, for example, Bell inequalities which are more robust against imperfectness in experiments [17,18], and the generalization of Bell inequalities into multipartite scenarios; see, e.g., [19–21].

A given LHV model can be described by a polytope, whose facets define the tight Bell inequalities [22–24]. In turn, all the tight Bell inequalities characterize the corresponding LHV model completely. However, the full characterization is usually a hard problem. This has only been done for simple (n, m, d) scenarios, i.e., the (2,2,2) [13], (2,2,3) [25], (2,3,2) [26], and (3,2,2) [20] scenarios, where n, m, and d stand for the number of parties, measurements per party, and outcomes per measurement, respectively. For more complicated scenarios, all Bell inequalities can sometimes be characterized, if additional symmetries are considered [27,28]. Iterative formulas have then been a powerful alternative approach to find Bell inequalities of interest, especially in the multipartite case [29–35]. Mermin *et al.* presented an iteration method, which has successfully been generalized the CHSH inequality to the Mermin-Ardehali-Belinskĭ-Klyshko (MABK) inequalities for arbitrary n-partite scenario [30–32]. As it turned out, the MABK inequalities are the strongest ones to detect the multiqubit Greenberger-Horne-Zeilinger (GHZ) state [36], in the sense that the quantum-classical ratio is the highest [29,37]. To detect genuine multipartite entanglement and nonlocality, Svetlichny constructed a tripartite inequality [38], which was then generalized to the multipartite scenario through iteration [33,34]. Chen-Albeverio-Fei (CAF) inequalities [35] are other

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relevant examples, which can probe the generalized GHZ state $\cos\theta|000\rangle + \sin\theta|111\rangle$ in the whole entangled region.

However, all those iterative formulas are very specific and shed little light on new constructions. In this work, we first observe that any Bell inequality in the $(n+1, m, 2)$ scenario always has a decomposition into n-partite ones, which can be obtained by enumerating all the extremal classical assignments for the measurements of the last party. Based on this observation, we propose a generalized iterative formula to construct nontrivial $(n+1)$-partite Bell inequalities from a set of n-partite ones. Our iterative formula can recover several families in the literature as special cases, like the MABK inequalities and CAF inequalities. Moreover, we come up with additional families of Bell inequalities for specific purposes. For example, a family of "dual-use" inequalities is proposed. On the one hand, those inequalities have the quantum violation for the full entangled range of the generalized GHZ state as the CAF inequalities. On the other hand, the maximal quantum violation is as large as the one for MABK inequalities. Furthermore, we simplify our iterative formula by introducing more linear or symmetric constraints among the n-partite Bell inequalities. The 46 Śliwa's inequalities [20] for the (3,2,2) scenario and the I3322 inequality [26] for the (2,3,2) scenario are taken as examples.

II. DECOMPOSITION OF BELL INEQUALITIES AND THE GENERALIZED ITERATIVE FORMULA

First, we introduce the notation of a Bell function; e.g., any $(n+1)$-partite Bell function can be written as $\mathbb{B}_{n+1} = \sum_{i\cdots jk} \alpha_{i\cdots jk} A_i \cdots B_j C_k$, where A_i, B_j, C_k are random variables for different parties. Then $\langle\mathbb{B}_{n+1}\rangle \leqslant l$ defines one Bell inequality, where l is the maximal expectation $\langle\mathbb{B}_{n+1}\rangle_{\max}$ for any probability distribution coming from a LHV model. Furthermore, we call a Bell inequality normalized if its LHV upper bound equals $l = 1$. Since $\langle\mathbb{B}_{n+1}\rangle$ is a linear function of $\mathbb{B}_{n+1}$ either in the LHV model or in quantum theory, and we care about only the maximum of $\langle\mathbb{B}_{n+1}\rangle$ in each case, we do not distinguish $\langle\mathbb{B}_{n+1}\rangle$ and $\mathbb{B}_{n+1}$ in the following if there is no risk of confusion. Then we can formulate the following.

Observation 1. Consider a Bell function $\mathbb{B}_{n+1} = \sum_{i\cdots jk} \alpha_{i\cdots jk} A_i \cdots B_j C_k$ in the $(n+1, m, 2)$ scenario, where $(A_i, \ldots, B_j, C_k)$ have outcomes $\{-1, 1\}$. Then $\mathbb{B}_{n+1} = \sum_{\boldsymbol{s}} \mathbb{B}_n^{\boldsymbol{s}} \prod_k [(1 + s_k C_k)/2]$, where $s_k \in \{-1, 1\}$, $\boldsymbol{s} = (s_1, \ldots, s_m)$ denotes a vector labeling all possible measurement results for the last party and consequently $\mathbb{B}_n^{\boldsymbol{s}} = \sum_{i\cdots jk} \alpha_{i\cdots jk}\, s_k A_i \cdots B_j$.

Proof. The direct expansion shows that

$$\sum_{\boldsymbol{s}} s_k \prod_{k'} \left[\frac{(1 + s_{k'} C_{k'})}{2}\right] = \sum_{\boldsymbol{s}} \left[\left(\frac{s_k + C_k}{2}\right) \prod_{k' \neq k} \left(\frac{1 + s_{k'} C_{k'}}{2}\right)\right] = C_k. \quad (1)$$

The last equality follows from the fact that $\sum_{s_k} [(s_k + C_k)/2] = C_k$ and $\sum_{s_{k'}} [(1 + s_{k'} C_{k'})/2] = 1$ for each $k' \neq k$.

By changing the order of summation, we have

$$\sum_{\boldsymbol{s}} \mathbb{B}_n^{\boldsymbol{s}} \prod_k \left[\frac{(1 + s_k C_k)}{2}\right] = \sum_{i\cdots jk} \alpha_{i\cdots jk} A_i \cdots B_j \left[\sum_{\boldsymbol{s}} s_k \prod_{k'} \left(\frac{1 + s_{k'} C_{k'}}{2}\right)\right],$$

which leads to $\mathbb{B}_{n+1}$ by inserting Eq. (1). ■

Since $\mathbb{B}_n^{\boldsymbol{s}}$ is obtained by assigning a specific value s_k to each C_k in $\mathbb{B}_{n+1}$, the maximal LHV value of $\mathbb{B}_n^{\boldsymbol{s}}$ cannot exceed the one of $\mathbb{B}_{n+1}$, i.e., $\mathbb{B}_{n+1} \leqslant l$ implies $\mathbb{B}_n^{\boldsymbol{s}} \leqslant l$ for any LHV model. However, the exact LHV bound of $\mathbb{B}_n^{\boldsymbol{s}}$ for a given $\boldsymbol{s}$ might be strictly smaller than l.

Notice that the $\mathbb{B}_n^{\boldsymbol{s}}$'s are not independent of each other. They satisfy extra conditions such that the higher-order terms like $C_k C_{k'}$ are eliminated. Conversely, this inspires a general formula to build $(n+1)$-partite Bell inequalities from the n-partite ones.

Observation 2. For a given set of n-partite normalized Bell inequalities $\{\mathbb{B}_n^{\boldsymbol{s}} \leqslant 1\}_{\boldsymbol{s}}$, a $(n+1)$-partite normalized Bell inequality can be constructed as

$$\mathbb{B}_{n+1} = \frac{1}{2^m} \sum_{\boldsymbol{s}} \mathbb{B}_n^{\boldsymbol{s}} \left(1 + \sum_k s_k C_k\right) \leqslant 1, \quad (2)$$

if for any subset K of $\{1, \ldots, m\}$, which contains at least two elements, we have

$$\sum\nolimits_{\boldsymbol{s}} \mathbb{B}_n^{\boldsymbol{s}} \left(\prod\nolimits_{k\in K} s_k\right) = 0. \quad (3)$$

Proof. The conditions in Eq. (3) imply that

$$\sum_{\boldsymbol{s}} \mathbb{B}_n^{\boldsymbol{s}} \prod_k (1 + s_k C_k) = \sum_{\boldsymbol{s},K} \mathbb{B}_n^{\boldsymbol{s}} \prod_{k\in K} (s_k C_k) = \sum_{\boldsymbol{s},|K|\leqslant 1} \mathbb{B}_n^{\boldsymbol{s}} \prod_{k\in K} (s_k C_k),$$

where $\prod_{k\in K}(s_k C_k) = 1$ if $K = \emptyset$. Consequently, $\mathbb{B}_{n+1} = \sum_{\boldsymbol{s}} \mathbb{B}_n^{\boldsymbol{s}} \prod_k [(1 + s_k C_k)/2]$. From this and the condition in Eq. (3) one can directly calculate that if one starts with $\mathbb{B}_{n+1}$ and computes the expression $\tilde{\mathbb{B}}_n^{\boldsymbol{s}} \equiv \sum_{i\cdots jk} \alpha_{i\cdots jk}\, s_k A_i \cdots B_j$ as in Observation 1, one finds that $\tilde{\mathbb{B}}_n^{\boldsymbol{s}} = \mathbb{B}_n^{\boldsymbol{s}}$. Consequently, as the maximal LHV value of any of $\mathbb{B}_n^{\boldsymbol{s}}$ is 1 and they are obtained from $\mathbb{B}_{n+1}$ by fixing the values of the C_k, it follows that the one of $\mathbb{B}_{n+1}$ is also 1, i.e., $\mathbb{B}_{n+1}$ is also normalized.

Similarly, the maximal quantum value of $\mathbb{B}_{n+1}$ is no less than the one of any $\mathbb{B}_n^{\boldsymbol{s}}$, since we can always recover the quantum operator $\mathbb{B}_n^{\boldsymbol{s}}$ by setting $C_k = s_k \mathbb{1}$, where $\mathbb{1}$ signifies the identity operator. ■

Note that $\mathbb{B}_{n+1}$ constructed in Observation 2 is nontrivial in the sense that its quantum value is strictly larger than 1, once at least one of $\mathbb{B}_n$ are nontrivial in the same sense. This fails for the property of tightness (the property of being facets on the local polytope); namely, if all $\mathbb{B}_n$ are tight, the $\mathbb{B}_{n+1}$ is not necessarily tight (a counterexample is listed in Appendix A).

The inequality in Eq. (2) is the generalized iterative formula under the constraints in Eq. (3). In the $(n+1, 2, 2)$ scenario, the only candidate of K is $\{1, 2\}$, then the resulting extra constraint is $\mathbb{B}_n^{(++)} + \mathbb{B}_n^{(--)} = \mathbb{B}_n^{(+-)} + \mathbb{B}_n^{(-+)}$. Consequently, there are still three $\mathbb{B}_n^{(\pm\pm)}$'s independent of each other, and under this circumstance,

$$\mathbb{B}_{n+1} = \frac{1}{2}[\mathbb{B}_n^{(++)}(C_1 + C_2) + \mathbb{B}_n^{(+-)}(1 - C_2) + \mathbb{B}_n^{(-+)}(1 - C_1)] \leqslant 1. \quad (4)$$

We remark that the iterative formula in Observation 2 cannot recover all the $(n+1)$-partite Bell inequalities, since we have employed only normalized $\mathbb{B}_n^s$'s. However, by imposing other linear or symmetric constraints, we can recover well-known iterative Bell inequalities in the literature [20,26,30–32,35,39,40].

III. RECOVERING PREVIOUS ITERATIVE FORMULAS AND BEYOND

One special solution to the constraint in Eq. (4) for the $(n+1, 2, 2)$ scenario is that $\mathbb{B}_n^{(--)} = -\mathbb{B}_n^{(++)}$ and $\mathbb{B}_n^{(-+)} = -\mathbb{B}_n^{(+-)}$. In this case, the iterative formula in Observation 2 becomes

$$\mathbb{B}_{n+1} = \frac{1}{2}[\mathbb{B}_n^{(++)}(C_1 + C_2) + \mathbb{B}_n^{(+-)}(C_1 - C_2)] \leqslant 1. \quad (5)$$

Starting with $\mathbb{B}_2^{(++)} \leqslant 1$ being the CHSH inequality

$$\text{CHSH} = \frac{1}{2}(A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2) \leqslant 1, \quad (6)$$

the iterative formula of the MABK inequalities [30–32] can be recovered by taking $\mathbb{B}_n^{(++)}$ to be the MABK inequality for the $(n, 2, 2)$ scenario, and $\mathbb{B}_n^{(+-)}$ to be obtained from $\mathbb{B}_n^{(++)}$ by swapping the two measurements for each party, e.g., $A_1 \leftrightarrow A_2$. Similarly, if we take $\mathbb{B}_n^{(++)} \leqslant 1$ to be the n-partite MABK inequality and $\mathbb{B}_n^{(+-)} = \mathbb{1}$, we obtain the CAF iterative formula [35].

Moreover, the generalized GHZ state $|\Phi(\theta)\rangle = \cos(\theta)|000\rangle + \sin(\theta)|111\rangle$ is a good test bed to observe the quantum behaviors with a given Bell inequality. The n-partite MABK inequality reaches its maximal quantum violation $2^{(n-1)/2}$ [29] when $\theta = \pi/4$. However, for CAF inequalities, the maximal violation is only $2^{(n-2)/2}$ [35]. The advantage of CAF inequalities is that they are always violated for the whole entangled region $\theta \in (0, \pi/2)$. In comparison, there exist some ranges of θ without quantum violation of the MABK inequalities; see Fig. 1 for more details of the tripartite case. As we have seen, the MABK inequalities and the CAF inequalities have their own advantages, which are not shared by each other. This intrigues one important question: Can one find some "dual-use" Bell inequalities, in the sense that the following two properties are satisfied simultaneously: (1) their maximal quantum violations are not weaker than the ones of MABK inequalities; and (2) they have the quantum violation in the whole entangled range of θ?

IV. "DUAL-USE" BELL INEQUALITIES

Bell inequalities with more measurements can reveal more entanglement usually. We consider the iterative formula in Eq. (5) in the (3,3,2) scenario, which leads to one "dual-use" inequality. More explicitly, we take $\mathbb{B}_n^{(++)} \leqslant 1$ to be the standard CHSH inequality in Eq. (6), and $\mathbb{B}_n^{(+-)}$ is obtained

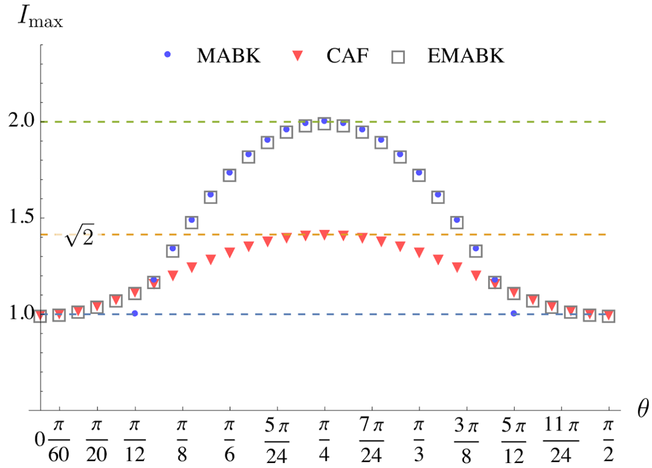

FIG. 1. Maximal violations of the tripartite MABK, CAF, and EMABK inequalities byusing the state $\cos(\theta)|000\rangle + \sin(\theta)|111\rangle$ with $\theta \in [0, \pi/2]$. Note that the MABK inequality (blue dot) shows the maximal quantum violations 2, but it is violated only in the region $\theta \in (\pi/12, 5\pi/12)$; the CAF inequality (red triangle) is violated in the whole entangled region; however, its maximal quantum violation is only $\sqrt{2}$; In comparison, the tripartite EMABK inequality (gray square) is a "dual-use" inequality, in the sense that it has the advantages of the MABK and CAF inequalities simultaneously.

from $\mathbb{B}_n^{(++)}$ by changing A_1 to A_3 and A_2 to A_t ($t = 1$ if n is odd, and $t = 4$ otherwise), and the same for other parties. For convenience, we name this inequality the extended-MABK (EMABK) inequality.

Furthermore, the tripartite EMABK inequality can be generalized into one iterative formula where the $(n+1)$-partite inequality is always of "dual use."

Observation 3. Let $\mathbb{B}_n^{(++)} \leqslant 1$ be the standard n-partite MABK inequality, and $\mathbb{B}_n^{(+-)}$ be obtained from $\mathbb{B}_n^{(++)}$ by $M_1^{(i)} \rightarrow M_3^{(i)}$ and $M_2^{(i)} \rightarrow M_t^{(i)}$, where $M_k^{(i)}$ is the kth measurement for the ith party, $t = 1$ if n is odd, and $t = 4$ otherwise. Then the $(n+1)$-partite Bell inequality defined in Eq. (5) has the "dual-use" property.

The proof of Observation 3 is provided in Appendix B. As we have verified numerically, the n-partite EMABK inequality is tight for $n = 3, 4, 5$. We have one remark. The tripartite EMABK inequality is not the unique "dual-use" inequality if we allow more measurements for the third party, for example, the Wieśniak-Badziag-Żukowski (WBZ) inequality [40].

V. THREE MEASUREMENTS

First, we can simplify the iterative formula in Observation 2 with the linear constraints in Eq. (3), which result in the following solution:

$$\mathbb{B}_n^{(+--)} = -\mathbb{B}_n^{(+++)} + \mathbb{B}_n^{(++-)} + \mathbb{B}_n^{(+-+)},$$
$$\mathbb{B}_n^{(-++)} = 2\,\mathbb{B}_n^{(+++)} - \mathbb{B}_n^{(++-)} - \mathbb{B}_n^{(+-+)} + \mathbb{B}_n^{(---)},$$
$$\mathbb{B}_n^{(-+-)} = \mathbb{B}_n^{(+++)} - \mathbb{B}_n^{(+-+)} + \mathbb{B}_n^{(---)},$$
$$\mathbb{B}_n^{(--+)} = \mathbb{B}_n^{(+++)} - \mathbb{B}_n^{(++-)} + \mathbb{B}_n^{(---)}.$$

By inserting this solution into Eq. (2), we have the simplified general iterative formula:

$$\mathbb{B}_{n+1} = \tfrac{1}{2}[\mathbb{B}_n^{(+++)}(1 - C_1 + C_2 + C_3) + \mathbb{B}_n^{(++-)}(C_1 - C_3) + \mathbb{B}_n^{(+-+)}(C_1 - C_2) + \mathbb{B}_n^{(---)}(1 - C_1)] \leqslant 1. \quad (7)$$

Notice that, even if all $\mathbb{B}_n^{(\pm\pm\pm)}$'s contain only n-partite correlations terms, there could still be terms in $\mathbb{B}_{n+1}$ which are not n-partite correlations. This issue can be overcome by introducing one extra condition that $\mathbb{B}_n^{(---)} = -\mathbb{B}_n^{(+++)}$. Then Eq. (7) reduces to a simpler iterative formula as follows:

$$\mathbb{B}_{n+1} = \tfrac{1}{2}[\mathbb{B}_n^{(+++)}(C_2 + C_3) + \mathbb{B}_n^{(++-)}(C_1 - C_3) + \mathbb{B}_n^{(+-+)}(C_1 - C_2)] \leqslant 1.$$

For example, as a tripartite inequality with n-partite correlations terms, the WBZ inequality is recovered with $\mathbb{B}_2^{(+++)}$ being the standard CHSH inequality in Eq. (6), $\mathbb{B}_2^{(+-+)}$ obtained from $\mathbb{B}_2^{(+++)}$ by $B_1 \to B_3$, and then $A \leftrightarrow B$, $\mathbb{B}_2^{(++-)}$ by $A_2 \to A_3$ and $B_1 \to B_3$.

All the previous examples are based on the CHSH inequality, which contains only two measurements per party, whereas the CHSH inequality is not the only relevant inequality in the (2,3,2) scenario. The additional tight and relevant Bell inequality in the (2,3,2) scenario is the I3322 inequality [20,26], which can probe the nonlocality of some two-qubit states out of the capability of the CHSH inequality [41]. Another interesting point of I3322 inequality is that its maximal quantum violation increases together with the dimension of the tested quantum system [42], which can identify the dimension of a quantum system device independently. Here we develop an iterative formula based on the I3322 inequality. For convenience, denote $\mathbb{B}_n^{3322}$ the n-partite I3322-Bell function according to this iterative formula, where $\mathbb{B}_2^{3322} \leqslant 1$ is the standard normalized I3322 inequality [26]:

$$\mathbb{B}_2^{3322} = \tfrac{1}{4}[A_1 - A_2 + B_1 - B_2 - (A_1 - A_2)(B_1 - B_2) + (A_1 + A_2)B_3 + A_3(B_1 + B_2)] \leqslant 1.$$

Observation 4. Let $\mathbb{B}_n^{(+++)}$ be the standard $\mathbb{B}_n^{3322}$ in the iterative formula, from which we obtain $\mathbb{B}_n^{(++-)}$ by $B_3 \to -B_3$, $\mathbb{B}_n^{(+-+)}$ by $A_3 \to -A_3$, and $\mathbb{B}_n^{(---)}$ by $A_1 \to -A_1$, $A_2 \to -A_2$ and $A_3 \to -A_3$, and then we obtain the $(n+1)$-partite Bell inequality $\mathbb{B}_{n+1}^{3322}$ as in Eq. (7).

The explicit expression of the tripartite Bell inequality in this iteration reads

$$\mathbb{B}_3^{3322} = \tfrac{1}{4}[(A_1 - A_2)C_1 + B_1 - B_2 - (A_1 - A_2)(B_1 - B_2)C_1 + (A_1 + A_2)B_3C_3 + A_3(B_1 + B_2)C_2] \leqslant 1.$$

By replacing C_i with C_iD_i, it results in $\mathbb{B}_4^{3322}$. In the same way, we obtain $\mathbb{B}_n^{3322}$ for $n \geqslant 5$. As it turns out, $\mathbb{B}_n^{3322} \leqslant 1$ is still tight. Note that $\mathbb{B}_n^{3322}$ is not invariant under the permutation of parties anymore, which is different from the generalizations of I3322 in Refs. [27,28,43].

VI. EXTENDING ŚLIWA'S INEQUALITIES

As we have seen in the previous sections, it is already powerful enough to use one $\mathbb{B}_n \leqslant 1$ and its equivalent Bell inequalities as generators for $(n+1)$-partite Bell inequalities. For a given normalized $\mathbb{B}_n \leqslant 1$, the general procedure is as follows. First, we generate the equivalence class of $\mathbb{B}_n$ by permutations of parties, measurements and outcomes. Then we choose all $\mathbb{B}_n^s$ from this class and test the conditions in Eq. (3). If all the conditions are fulfilled, we construct $(n+1)$-partite Bell inequality as in Eq. (2).

In Appendix C 2, we have employed Śliwa's inequalities in such a way to generate four-partite tight ones. As an example, we present the first inequality in Śliwa's family, i.e., $\text{Śliwa}_1 = A_1 + B_1 - A_1B_1 + C_1 - A_1C_1 - B_1C_1 + A_1B_1C_1 \leqslant 1$, which is trivial from the perspective of quantum violation. Notwithstanding, let $\mathbb{B}_3^{(++)}$ be the standard Śliwa_1 in the iterative formula, from which we obtain $\mathbb{B}_3^{(+-)}$ by $C_1 \leftrightarrow C_2$, $\mathbb{B}_3^{(-+)}$ by $C_1 \leftrightarrow C_2$, then $C_2 \to -C_2$, and $\mathbb{B}_3^{(--)}$ by $C_1 \to -C_1$, and finally we obtain the four-partite generalization of Śliwa_1 as in Observation 2 with the measurements D_1, D_2 for the last party. More explicitly, the four-partite inequality reads

$$\mathbb{B}_4 = \tfrac{1}{2}[\mathbb{B}_3^{(++)}(D_1 + D_2) + \mathbb{B}_3^{(+-)}(1 - D_2) + \mathbb{B}_3^{(-+)}(1 - D_1)] \leqslant 1,$$

whose maximal quantum violation is $4\sqrt{2} - 3$.

VII. CONCLUSION AND DISCUSSION

In this work we have developed a very general iterative formula to generate multipartite Bell inequalities from few-partite ones. By imposing extra linear conditions and symmetries on the n-partite ones, the iterative formula can also be simplified. Based on this observation and starting from the CHSH inequality, we have not only recovered famous families of multipartite Bell inequalities, like the MABK inequalities and the CAF inequalities, but also found the additional EMABK inequalities to combine the advantages of the aforementioned two families. Starting with I3322 and the inequalities discovered by Śliwa, additional inequalities with the properties like tightness have also been constructed. Hence our general iterative formula is powerful in the field of Bell nonlocality.

We have concerned ourselves mainly with the Bell inequalities in the $(n, m, 2)$-scenarios here, while the generalization to the (n, m, d)-scenarios or other concepts like network nonlocality is yet to be done. An effective algorithm to implement the iterative formula starting from a given Bell inequality and its equivalence is also desired. Finally, as Bell inequalities are closely connected to information processing tasks, it would be very interesting to clarify in which sense our iteration method allows one to identify novel quantum resources.

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APPENDIX A: A COUNTEREXAMPLE OF NONTIGHT $\mathbb{B}_4$ FROM TIGHT $\mathbb{B}_3$

Let $\mathbb{B}_3^{(++)}$ be the Śliwa's first inequality [20],

$$\mathbb{B}_3^{(++)}=A_1+B_1+C_1-A_1C_1-B_1C_1-A_1B_1 +A_1B_1C_1\leqslant 1,$$

and through the symmetric transformations

$$\mathbb{B}_3^{(+-)}=\mathbb{B}_3^{(++)},\quad \mathbb{B}_3^{(--)}=\mathbb{B}_3^{(-+)}=\mathbb{B}_3^{(++)}(C_1\to -C_1), \tag{A1}$$

then the iterative formula

$$\mathbb{B}_4=\tfrac{1}{4}[\mathbb{B}_3^{(++)}(1+D_1)(1+D_2)+\mathbb{B}_3^{(+-)}(1+D_1)(1-D_2) +\mathbb{B}_3^{(-+)}(1-D_1)(1+D_2)+\mathbb{B}_3^{(--)}(1-D_1)(1-D_2)] \leqslant 1$$

tells us

$$\mathbb{B}_4=A_1+B_1+C_1D_1-A_1C_1D_1-B_1C_1D_1-A_1B_1 +A_1B_1C_1D_1\leqslant 1. \tag{A2}$$

The inequality (A2) is not tight from numerical check, while its split forms in Eq. (A1) are all tight.

APPENDIX B: PROOF OF OBSERVATION 3

Given a standard Clause-Horne-Shimony-Holt (CHSH) inequality [13] $(A_1B_1+A_1B_2+A_2B_1-A_2B_2)/2\leqslant 1$, a complete group of its equivalent partners can be generated under the three kinds of transformations (permutations of parties, measurements, or outcomes):

$$\begin{aligned}\mathcal{S}_{\rm CHSH}:=\big\{&\tfrac{1}{2}(A_1B_1-A_1B_2-A_2B_1-A_2B_2)\leqslant 1,\ \tfrac{1}{2}(-A_1B_1-A_1B_2+A_2B_1-A_2B_2)\leqslant 1,\\ &\tfrac{1}{2}(-A_1B_1+A_1B_2-A_2B_1-A_2B_2)\leqslant 1,\ \tfrac{1}{2}(A_1B_1+A_1B_2+A_2B_1-A_2B_2)\leqslant 1,\\ &\tfrac{1}{2}(-A_1B_1-A_1B_2-A_2B_1+A_2B_2)\leqslant 1,\ \tfrac{1}{2}(A_1B_1-A_1B_2+A_2B_1+A_2B_2)\leqslant 1,\\ &\tfrac{1}{2}(A_1B_1+A_1B_2-A_2B_1+A_2B_2)\leqslant 1,\ \tfrac{1}{2}(-A_1B_1+A_1B_2+A_2B_1+A_2B_2)\leqslant 1\big\}.\end{aligned} \tag{B1}$$

For convenience, let $\mathcal{S}_{\rm CHSH}[i]$ denote the ith entity in the set $\mathcal{S}_{\rm CHSH}$, $i\in\{1,2,\dots,8\}$, thus $\mathcal{S}_{\rm CHSH}[4]$ refers to the standard CHSH inequality. Arbitrary $(n+1)$-partite Mermin-Ardehali-Belinskĭ-Klyshko (MABK) inequalities [30–32] are constructed as follows. (Commonly) choose $\mathbb{B}_2^{(++)}=$ CHSH as the standard one in (B1) (i.e., CHSH $=\mathcal{S}_{\rm CHSH}[4]$) and $\mathbb{B}_2^{(+-)}$ to be obtained from $\mathbb{B}_2^{(++)}$ by swapping the two measurements for parties A and B, i.e., $A_1\leftrightarrow A_2$ and $B_1\leftrightarrow B_2$. Through

$$\mathbb{B}_3=\tfrac{1}{2}[\mathbb{B}_2^{(++)}(C_1+C_2)+\mathbb{B}_2^{(+-)}(C_1-C_2)]\leqslant 1,$$

we get the tripartite MABK inequality. Similarly, let $\mathbb{B}_n^{(++)}$ be the n-partite MABK polynomial, $\mathbb{B}_n^{(+-)}$ to be obtained from $\mathbb{B}_n^{(++)}$ by swapping the two measurements for each party. Finally using Eq. (5) completes the iteration.

Observation 3 in the main text reads as follows.

Let $\mathbb{B}_n^{(++)}\leqslant 1$ be the standard n-partite MABK inequality, and $\mathbb{B}_n^{(+-)}$ be obtained from $\mathbb{B}_n^{(++)}$ by $M_1^{(i)}\to M_3^{(i)}$ and $M_2^{(i)}\to M_t^{(i)}$ where $M_k^{(i)}$ is the kth measurement for the ith party, $t=1$ if n is odd, and $t=4$ otherwise, then the $(n+1)$-partite extended MABK (EMABK) inequality defined via

$$\mathbb{B}_{n+1}=\tfrac{1}{2}[\mathbb{B}_n^{(++)}(C_1+C_2)+\mathbb{B}_n^{(+-)}(C_1-C_2)]\leqslant 1$$

is "dual-use."

1. Predefinitions and analyses

Quantum mechanically, for the observables A_l, B_l, and C_l with two measurement outcomes,

$$A_l\equiv\vec{\sigma}\cdot(\sin\theta_{al}\cos\varphi_{al},\sin\theta_{al}\sin\varphi_{al},\cos\theta_{al}) =\begin{bmatrix}\cos\theta_{al} & \sin\theta_{al}\,\mathrm{e}^{-\mathrm{i}\varphi_{al}}\\ \sin\theta_{al}\,\mathrm{e}^{\mathrm{i}\varphi_{al}} & -\cos\theta_{al}\end{bmatrix},$$

so do B_l and C_l, $l\in\{1,2,3\}$. Without loss of generality, we fix all Bloch vectors relating to the observations shown in a typical Bell inequality on xz plane, which means that $\varphi_{\rm m}=0$, and here "m" expresses the parameter $\varphi_{\rm m}$ involving "measurements,"

$$A_l=\begin{bmatrix}\cos\theta_{al} & \sin\theta_{al}\\ \sin\theta_{al} & -\cos\theta_{al}\end{bmatrix}, \tag{B2}$$

and so do B_l and C_l, $l\in\{1,2,3\}$. Of course, there is another way to simplify calculation, namely, setting $\theta_{\rm m}=\pi/2$ (xy plane), then

$$A_l=\begin{bmatrix}0 & \mathrm{e}^{-\mathrm{i}\varphi_{al}}\\ \mathrm{e}^{\mathrm{i}\varphi_{al}} & 0\end{bmatrix}, \tag{B3}$$

and so do B_l and C_l, $l\in\{1,2,3\}$.

To accomplish the demonstration, we need to prove the following two lemmas for even and odd n, respectively.

Lemma 1. The maximal quantum violation of EMABK inequality $\mathbb{B}_n \leqslant 1$ is as strong as that of the n-partite MABK inequality, i.e., $\mathbb{B}_n^{\max} = 2^{(n-1)/2}$.

Lemma 2. The n-partite EMABK inequality $\mathbb{B}_n \leqslant 1$ is violated in the whole entangled region $\theta \in (0, \pi/2)$.

We have known that the maximal quantum violation of the (normalized) n-qubit MABK inequality is $2^{(n-1)/2}$ [29], iff the system is at the n-qubit Greenberger-Horne-Zeilinger (GHZ) state [36] or its unitary equivalent partners [37]. First, we try to expain this fact using the generalized GHZ state $|\Psi_{\text{GGHZ}}(\theta)\rangle = \cos(\theta)|00\cdots0\rangle + \sin(\theta)|11\cdots1\rangle$, $\theta \in (0, \pi/2)$, (when $\theta = \pi/4$, that is the n-qubit GHZ state).

For simplicity, we denote the antidiagonal elements e_j of a given square matrix in column j as $\text{Adiag}(e_1, e_2, \ldots, e_{2^n-1}, e_{2^n})$. Likewise, define $\text{Diag}(e_1, e_2, \ldots, e_{2^n-1}, e_{2^n})$ as the diagonal elements of a matrix. Since we care about the elements posed on four corners in a specific matrix only, the corner or (anti)diagonal elements can be used to mark a matrix thereafter.

Select $\theta_{\text{m}} = \pi/2$, and

$$\varphi_{a1} = 0, \quad \varphi_{a2} = \frac{\pi}{2},$$
$$\varphi_{b1} = \varphi_{c1} = \cdots = \varphi_{d1} = -\frac{\pi}{4},$$
$$\varphi_{b2} = \varphi_{c2} = \cdots = \varphi_{d2} = \frac{\pi}{4},$$

which indicates

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \text{Adiag}(1, 1), \quad A_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} = \mathrm{i}\,\text{Adiag}(1, -1),$$

$$B_1 = C_1 = \cdots = D_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1+\mathrm{i} \\ 1-\mathrm{i} & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\text{Adiag}(1-\mathrm{i}, 1+\mathrm{i}),$$

$$B_2 = C_2 = \cdots = D_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1-\mathrm{i} \\ 1+\mathrm{i} & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\text{Adiag}(1+\mathrm{i}, 1-\mathrm{i}).$$

For example, when $n = 2$,

$$\text{CHSH} \equiv \tfrac{1}{2}[A_1(B_1 + B_2) + A_2(B_1 - B_2)] = \sqrt{2}\,\text{Adiag}(1, 0, 0, 1), \tag{B4}$$

so does

$$\text{CHSH}' \equiv \tfrac{1}{2}[A_2(B_1 + B_2) - A_1(B_1 - B_2)] = \sqrt{2}\,\mathrm{i}\,\text{Adiag}(1, 0, 0, -1),$$

up to a setting permutation; cf. Eq. (B4). Recursively, we write the tripartite MABK operator,

$$\mathbb{B}_3^{\text{MABK}} \equiv \tfrac{1}{2}[\text{CHSH}(C_1 + C_2) + \text{CHSH}'(C_1 - C_2)] = 2\,\text{Adiag}(1, 0, 0, 0, 0, 0, 0, 1).$$

Through the iteration above, the n-qubit 2^n-dimensional MABK operator reads

$$\mathbb{B}_n^{\text{MABK}} \equiv 2^{(n-1)/2}\,\text{Adiag}(1, 0, \ldots, 0, 1),$$

which can be maximized to $2^{(n-1)/2}$ for the n-qubit GHZ state $(|00\cdots0\rangle + |11\cdots1\rangle)/\sqrt{2}$.

2. Proof for even n

a. Lemma 1

Proof. For Lemma 1, we find that under the assumption of (B3), $\langle\mathbb{B}_n\rangle \leqslant 2^{(n-1)/2}$, and if

$$\varphi_{a1} = 0, \quad \varphi_{a2} = \varphi_{a3} = \frac{\pi}{2},$$
$$\varphi_{b1} = \varphi_{c1} = \cdots = \varphi_{d1} = -\frac{\pi}{4},$$
$$\varphi_{b2} = \varphi_{b3} = \varphi_{c2} = \varphi_{c3} = \cdots = \varphi_{d2} = \varphi_{d3} = \frac{\pi}{4}, \tag{B5}$$

the maximal violation is gained.

Since we have known that in MABK inequality

$$\mathbb{B}_n^{\text{MABK}} = \tfrac{1}{2}\big[\mathbb{B}_{n-1}^{\text{MABK}}(D_1 + D_2) + \big(\mathbb{B}_{n-1}^{\text{MABK}}\big)'(D_1 - D_2)\big] \leqslant 1,$$

and $(\mathbb{B}_{n-1}^{\text{MABK}})'$ is obtained from $\mathbb{B}_{n-1}^{\text{MABK}}$ via a permutation for measurement settings. Although $\mathbb{B}_{n-1}^{(+-)} = \mathbb{B}_{n-1}^{(++)}(A_1 \to A_3, A_2 \to A_1, B_1 \to B_3, B_2 \to B_1, \ldots, D_1 \to D_3, D_2 \to D_1)$, the matrix forms of the operators $(\mathbb{B}_{n-1}^{\text{MABK}})'$ and $\mathbb{B}_{n-1}^{(+-)}$ in Observation 3 are uniform under the constraint (B5). Hence $\langle \mathbb{B}_n^{\rm EMABK}\rangle = \langle \mathbb{B}_n^{\rm MABK}\rangle$ for the n-qubit GHZ state $(|00\cdots 0\rangle + |11\cdots 1\rangle)/\sqrt{2}$. This ends the proof of Lemma 1 for even n. ■

b. Lemma 2

As for Lemma 2, we notice once condition (B2) holds, together with

$$\begin{aligned}&\theta_{a1} = \theta_{a2} = \frac{\pi}{2}, \quad \theta_{a3} = 0,\\ &\theta_{b2} = \cdots = \theta_{c2} = \frac{\pi}{2},\ \theta_{b1} = \cdots = \theta_{c1} = \theta_{b3} = \cdots = \theta_{c3} = 0,\\ &\theta_{d2} = \pi - \theta_{d1},\end{aligned}$$

namely,

$$\begin{aligned}&A_1 = A_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad A_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\\ &B_2 = \cdots = C_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad B_1 = \cdots = C_1 = B_3 = \cdots = C_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\\ &D_1 = \begin{bmatrix}\cos\theta_{d1} & \sin\theta_{d1}\\ \sin\theta_{d1} & -\cos\theta_{d1}\end{bmatrix}, \quad D_2 = \begin{bmatrix}-\cos\theta_{d1} & \sin\theta_{d1}\\ \sin\theta_{d1} & \cos\theta_{d1}\end{bmatrix},\end{aligned} \tag{B6}$$

then $\langle \mathbb{B}_n\rangle > 1$ is satisfied in the whole entangled region $\theta \in (0, \pi/2)$.

Proof. When $n = 2$, for $\theta_{a1} = \pi/2$, $\theta_{a3} = 0$, and $\theta_{b2} = \pi - \theta_{b1}$, we obtain

$$A_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad A_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad B_1 = \begin{bmatrix}\cos\theta_{b1} & \sin\theta_{b1}\\ \sin\theta_{b1} & -\cos\theta_{b1}\end{bmatrix}, \quad B_2 = \begin{bmatrix}-\cos\theta_{b1} & \sin\theta_{b1}\\ \sin\theta_{b1} & \cos\theta_{b1}\end{bmatrix},$$

which implies

$$\mathrm{CHSH} = \frac{1}{2}[A_1(B_1 + B_2) + A_3(B_1 - B_2)] = \begin{bmatrix}\cos\theta_{b1} & 0 & 0 & \sin\theta_{b1}\\ 0 & -\cos\theta_{b1} & \sin\theta_{b1} & 0\\ 0 & \sin\theta_{b1} & -\cos\theta_{b1} & 0\\ \sin\theta_{b1} & 0 & 0 & \cos\theta_{b1}\end{bmatrix}.$$

After that, the mean value of CHSH operator for the state $|\Psi(\theta)\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$ reads

$$\langle \mathrm{CHSH}\rangle = \cos\theta_{b1} + \sin(2\theta)\sin\theta_{b1} = \sqrt{1+\sin^2(2\theta)}\cos(\theta_{b1} - \eta),$$

where $\tan\eta = \sin(2\theta)$. In this way, set $\theta_{b1} = \eta$, then $\forall\,\theta \in (0, \pi/2)$, $\langle\mathrm{CHSH}\rangle > 1$ all the time. Notice for every n (e.g., $n = 2$ for CHSH), the two measurement settings of the last party are completely the same, and therefore we may need to construct $\mathbb{B}_{n-1}^{(++)}$ with the nonzero antidiagonal elements k_1 in both column 1 and 2^{n-1}, and similarly for the nonzero diagonal elements of $\mathbb{B}_{n-1}^{(+-)}$ in column 1 valued k_2 and column 2^{n-1} valued $-k_2$,

$$\mathbb{B}_{n-1}^{(++)} = k_1\,\mathrm{Adiag}(1, \ldots, 1),\ \mathbb{B}_{n-1}^{(+-)} = k_2\,\mathrm{Diag}(1, \ldots, -1), \tag{B7}$$

with k_1, k_2 implying two constants, inspired via the expressions of A_1 and A_3.

In the case of $n = 4$, select the following measurements:

$$\theta_{a1} = \theta_{a2} = \frac{\pi}{2}, \quad \theta_{a3} = 0, \quad \theta_{b2} = \theta_{c2} = \frac{\pi}{2}, \quad \theta_{b1} = \theta_{c1} = \theta_{b3} = \theta_{c3} = 0, \quad \theta_{d2} = \pi - \theta_{d1},$$

then

$$\begin{aligned}&A_1 = A_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad A_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\\ &B_2 = C_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad B_1 = B_3 = C_1 = C_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\\ &D_1 = \begin{bmatrix}\cos\theta_{d1} & \sin\theta_{d1}\\ \sin\theta_{d1} & -\cos\theta_{d1}\end{bmatrix}, D_2 = \begin{bmatrix}-\cos\theta_{d1} & \sin\theta_{d1}\\ \sin\theta_{d1} & \cos\theta_{d1}\end{bmatrix}.\end{aligned}$$

Further we obtain

$$\mathbb{B}_2^{(++)} = \mathrm{CHSH} = \tfrac{1}{2}[A_1(B_1 + B_2) + A_2(B_1 - B_2)] = A_1 B_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},$$

$$\mathbb{B}_2^{(+-)} = \frac{1}{2}[A_2(B_1+B_2)+A_1(B_2-B_1)] = A_2B_2 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \otimes \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = \text{Adiag}(1,1,1,1),$$

and

$$\mathbb{B}_2^{(++)'} := \mathbb{B}_2^{(++)}(A_{1\to3}, A_{2\to1}, B_{1\to3}, B_{2\to1}) = \frac{1}{2}[A_3(B_3+B_1)+A_2(B_3-B_1)] = A_3B_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$
$$= \text{Diag}(1,-1,-1,1),$$
$$\mathbb{B}_2^{(+-)'} := \mathbb{B}_2^{(+-)}(A_{1\to3}, A_{2\to1}, B_{1\to3}, B_{2\to1}) = \frac{1}{2}[A_2(B_3+B_1)+A_3(B_1-B_3)] = A_2B_3 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},$$
$$C_3+C_1 = 2\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad \mathbb{B}_3^{(+-)} := \frac{1}{2}[\mathbb{B}_2^{(++)'}(C_3+C_1)+\mathbb{B}_2^{(+-)'}(C_3-C_1)] = \mathbb{B}_2^{(++)'} \otimes \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

After that, the matrix forms of $\mathbb{B}_3^{(++)}$, and $\mathbb{B}_3^{(+-)}$ are as follows:

$$\mathbb{B}_3^{(++)} := \mathbb{B}_3^{\text{MABK}} = \frac{1}{2}[\mathbb{B}_2^{(++)}(C_1+C_2)+\mathbb{B}_2^{(+-)}(C_1-C_2)] = -\frac{1}{2}\begin{bmatrix}0 & M\\ M & 0\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}0 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 0\end{bmatrix}_{8\times8}, \tag{B8}$$

with

$$M \equiv \begin{bmatrix}1 & 1 & 1 & -1\\ 1 & -1 & -1 & -1\\ 1 & -1 & -1 & -1\\ -1 & -1 & -1 & 1\end{bmatrix}$$

and

$$\mathbb{B}_3^{(+-)} := \mathbb{B}_3^{(++)}(A_{1\to3}, A_{2\to1}, B_{1\to3}, B_{2\to1}, C_{1\to3}, C_{2\to1}) = \text{Diag}(1,-1,-1,1,-1,1,1,-1). \tag{B9}$$

In view of the fact that we concern ourselves with the (anti)diagonal elements in Eqs. (B8) and (B9), thus they match Eq. (B7) with $k_1 = -1/2$ and $k_2 = 1$. Further, we discover

$$\mathbb{B}_{n-1}^{(++)} := \mathbb{B}_{n-1}^{\text{MABK}} = \tfrac{1}{2}[\mathbb{B}_{n-2}^{(++)}(C_1+C_2)+\mathbb{B}_{n-2}^{(+-)}(C_1-C_2)],$$

and $C_1 - C_2 = [\begin{smallmatrix}1 & -1\\ -1 & -1\end{smallmatrix}]$ holds for any even n. Hence according to the iterative formula of MABK inequality,

$$\mathbb{B}_{n-1}^{(++)} = -\frac{1}{2^{(n-2)/2}}\begin{bmatrix}0 & \cdots & 1\\ \vdots & \ddots & \vdots\\ 1 & \cdots & 0\end{bmatrix}_{2^{n-1}\times2^{n-1}}.$$

Likewise,

$$\mathbb{B}_{n-1}^{(+-)} := \mathbb{B}_{n-1}^{\text{MABK}}(A_1\to A_3, A_2\to A_1, \dots, B_1\to B_3, B_2\to B_1)$$
$$= \tfrac{1}{2}[\mathbb{B}_{n-2}^{(++)'}(C_3+C_1)+\mathbb{B}_{n-2}^{(+-)'}(C_3-C_1)],$$

and $C_3 + C_1 = 2[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}]$ is ascertained for any even n, then recursively,

$$\mathbb{B}_{n-1}^{(+-)} = \begin{bmatrix}1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & -1\end{bmatrix}_{2^{n-1}\times2^{n-1}},$$

which implies that

$$\mathbb{B}_n^{\text{EMABK}} = \frac{1}{2}[\mathbb{B}_{n-1}^{(++)}(D_1+D_2)+\mathbb{B}_{n-1}^{(+-)}(D_1-D_2)] = \begin{bmatrix}\cos\theta_{d1} & \cdots & -\frac{1}{2^{(n-2)/2}}\sin\theta_{d1}\\ \vdots & \ddots & \vdots\\ -\frac{1}{2^{(n-2)/2}}\sin\theta_{d1} & \cdots & \cos\theta_{d1}\end{bmatrix}_{2^n\times2^n},$$

and the expectation value of $\mathbb{B}_n^{\text{EMABK}}$ for the n-qubit generalized GHZ state $\cos\theta\,|00\cdots0\rangle + \sin\theta\,|11\cdots1\rangle$, reads

$$\langle\mathbb{B}_n^{\text{EMABK}}\rangle = \cos\theta_{d1} - \frac{\sin(2\theta)}{2^{(n-2)/2}}\sin\theta_{d1},$$

$n > 2$, and n is even.

In summary, $\forall$ even n, the preceding measurement settings (B6) reduce $\langle \mathbb{B}_n^{\mathrm{EMABK}} \rangle$ to

$$\langle \mathbb{B}_n^{\mathrm{EMABK}} \rangle = \cos\theta_{d1} - \frac{\sin(2\,\theta)}{2^{(n-2)/2}}\sin\theta_{d1}$$
$$= \sqrt{1+\frac{\sin^2(2\theta)}{2^{n-2}}}\cos(\theta_{d1}+\xi),$$

where $\tan\xi := \sin(2\theta)/[2^{(n-2)/2}]$. $\langle \mathbb{B}_n^{\mathrm{EMABK}} \rangle$ is greater than 1 $\forall\theta \in (0, \pi/2)$, after designating θ_{d1} as some typical values, e.g., $-\xi$. ■

3. Proof for odd *n*

a. Lemma 1

In the case of odd n, we consider the constraints in (B3), then $\langle \mathbb{B}_n^{\mathrm{EMABK}} \rangle \leqslant 2^{(n-1)/2}$. Once

$$\begin{aligned}
&\varphi_{a1} = \varphi_{a4} = 0, \quad \varphi_{a2} = \varphi_{a3} = \frac{\pi}{2},\\
&\varphi_{b1} = \varphi_{b4} = \varphi_{c1} = \varphi_{c4} = \cdots = \varphi_{d1} = \varphi_{d4} = \varphi_{e1} = -\frac{\pi}{4},\\
&\varphi_{b2} = \varphi_{b3} = \varphi_{c2} = \varphi_{c3} = \cdots = \varphi_{d2} = \varphi_{d3} = \varphi_{e2} = \frac{\pi}{4},
\end{aligned} \tag{B10}$$

the maximal violations are attained.

Proof. In comparison to the transformation rule of $\mathbb{B}^{(+-)}$ between odd and even n, namely,

$$\mathbb{B}_{n-1}^{(+-)} = \begin{cases} \mathbb{B}_{n-1}^{(++)}(A_1 \to A_3, A_2 \to A_1, B_1 \to B_3, B_2 \to B_1, \ldots, C_1 \to C_3, C_2 \to C_1), & \text{even } n,\\ \mathbb{B}_{n-1}^{(++)}(A_1 \to A_3, A_2 \to A_4, B_1 \to B_3, B_2 \to B_4, \ldots, D_1 \to D_3, D_2 \to D_4), & \text{odd } n. \end{cases}$$

We conclude the permutation process for even n can be recovered as long as $A_4 = A_1$, $B_4 = B_1$,...,$C_4 = C_1$ under the circumstance of odd n. Thus set $\varphi_{a4} = \varphi_{a1} = 0$, and $B_4 = B_1 = \cdots = C_4 = C_1 = -\pi/4$, then the matrix forms of $(\mathbb{B}_{n-1}^{\mathrm{MABK}})'$ and $\mathbb{B}_{n-1}^{(+-)}$ for odd n are the same under the constraint (B10). Therefore, $\langle \mathbb{B}_n^{\mathrm{EMABK}} \rangle = \langle \mathbb{B}_n^{\mathrm{MABK}} \rangle$ for the n-qubit GHZ state $(|00\cdots0\rangle + |11\cdots1\rangle)/\sqrt{2}$. This ends the proof of Lemma 1 for odd n. ■

b. Lemma 2

As for Lemma 2, we observe that once the following condition (B11) holds, $\langle \mathbb{B}_n^{\mathrm{EMABK}} \rangle > 1$ is satisfied in the whole entangled region $\theta \in (0, \pi/2)$:

$$\begin{aligned}
&\theta_{a1} = \theta_{a2} = \theta_{b1} = \theta_{b2} = \cdots = \theta_{d1} = \theta_{d2} = 0; \quad \theta_{a3} = \theta_{a4} = \theta_{b3} = \theta_{b4} = \cdots = \theta_{d3} = \theta_{d4} = \frac{\pi}{2},\\
&\varphi_{a1} = \varphi_{a2} = \varphi_{a3} = \varphi_{b1} = \varphi_{b2} = \varphi_{b3} = \cdots = \varphi_{d1} = \varphi_{d2} = \varphi_{d3} = 0; \quad \varphi_{a4} = \varphi_{b4} = \cdots = \varphi_{d4} = \frac{\pi}{2},\\
&\theta_{e2} = -\theta_{e1},\ \varphi_{e2} = \varphi_{e1} = [(-1)^{(n-1)/2}]\frac{\pi}{4},
\end{aligned} \tag{B11}$$

namely,

$$\begin{aligned}
&A_1 = A_2 = B_1 = B_2 = \cdots = D_1 = D_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_3 = B_3 = \cdots = D_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\\
&A_4 = B_4 = \cdots = D_4 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix},\\
&E_1 = \begin{bmatrix} \cos\theta_{e1} & \frac{1}{\sqrt{2}}\sin\theta_{e1}[1-\mathrm{i}\,(-1)^{(n-1)/2}] \\ \frac{1}{\sqrt{2}}\sin\theta_{e1}[1+\mathrm{i}\,(-1)^{(n-1)/2}] & -\cos\theta_{e1} \end{bmatrix},\\
&E_2 = -\begin{bmatrix} -\cos\theta_{e1} & \frac{1}{\sqrt{2}}\sin\theta_{e1}\{1-[(-1)^{(n-1)/2}]\mathrm{i}\} \\ \frac{1}{\sqrt{2}}\sin\theta_{e1}\{1+[(-1)^{(n-1)/2}]\mathrm{i}\} & \cos\theta_{e1} \end{bmatrix}.
\end{aligned} \tag{B12}$$

Proof. If $n=3$,

$$A_1=A_2=B_1=B_2=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad A_3=B_3=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad A_4=B_4=\begin{bmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{bmatrix},$$
$$C_1=\begin{bmatrix}\cos\theta_{c1}&\frac{\sin\theta_{c1}(1+\mathrm{i})}{\sqrt{2}}\\\frac{\sin\theta_{c1}(1-\mathrm{i})}{\sqrt{2}}&-\cos\theta_{c1}\end{bmatrix},\quad C_2=-\begin{bmatrix}-\cos\theta_{c1}&\frac{\sin\theta_{c1}(1+\mathrm{i})}{\sqrt{2}}\\\frac{\sin\theta_{c1}(1-\mathrm{i})}{\sqrt{2}}&\cos\theta_{c1}\end{bmatrix},$$

then

$$C_1+C_2=2\begin{bmatrix}\cos\theta_{c1}&0\\0&-\cos\theta_{c1}\end{bmatrix},\quad C_1-C_2=\sqrt{2}\begin{bmatrix}0&\sin\theta_{c1}(1+\mathrm{i})\\\sin\theta_{c1}(1-\mathrm{i})&0\end{bmatrix},$$

and

$$\mathbb{B}_2^{(++)}=\mathrm{CHSH}:=\tfrac{1}{2}[A_1(B_1+B_2)+A_2(B_1-B_2)]=\mathrm{Diag}(1,-1,-1,1),$$

and, similarly,

$$\mathbb{B}_2^{(+-)}:=\tfrac{1}{2}[A_3(B_3+B_4)+A_4(B_3-B_4)]=\mathrm{Adiag}(1-\mathrm{i},0,0,1+\mathrm{i}),$$

which signifies that

$$\begin{aligned}\mathbb{B}_3&:=\frac{1}{2}[\mathbb{B}_2^{(++)}(C_1+C_2)+\mathbb{B}_2^{(+-)}(C_1-C_2)]\\&=\mathrm{Diag}(1,-1,-1,1)\otimes\mathrm{Diag}(\cos\theta_{c1},-\cos\theta_{c1})+\frac{\sin\theta_{c1}}{\sqrt{2}}\mathrm{Adiag}(1-\mathrm{i},0,0,1+\mathrm{i})\otimes\mathrm{Adiag}(1+\mathrm{i},0,0,1-\mathrm{i})\\&=\cos\theta_{c1}\,\mathrm{Diag}(1,\ldots,-1)+\sqrt{2}\,\sin\theta_{c1}\,\mathrm{Adiag}(1,\ldots,1)\\&=\begin{bmatrix}\cos\theta_{c1}&\cdots&\sqrt{2}\,\sin\theta_{c1}\\\vdots&\ddots&\vdots\\\sqrt{2}\,\sin\theta_{c1}&\cdots&-\cos\theta_{c1}\end{bmatrix}_{8\times8}.\end{aligned}$$

Further, the mean value of $\mathbb{B}_3$ for the three-qubit generalized GHZ state $\cos\theta\,|000\rangle+\sin\theta\,|111\rangle$ reads

$$\langle\mathbb{B}_3\rangle=\cos(2\theta)\cos\theta_{c1}+\sqrt{2}\,\sin(2\theta)\sin\theta_{c1}.$$

Once $n=5$, select the following measurement:

$$A_1=A_2=B_1=B_2=C_1=C_2=D_1=D_2=\begin{bmatrix}1&0\\0&-1\end{bmatrix},$$
$$A_3=B_3=C_3=D_3=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad A_4=B_4=C_3=D_3=\begin{bmatrix}0&-\mathrm{i}\\\mathrm{i}&0\end{bmatrix},$$
$$E_1=\begin{bmatrix}\cos\theta_{e1}&\frac{1}{\sqrt{2}}\sin\theta_{e1}(1-\mathrm{i})\\\frac{1}{\sqrt{2}}\sin\theta_{e1}(1+\mathrm{i})&-\cos\theta_{e1}\end{bmatrix},\quad E_2=-\begin{bmatrix}-\cos\theta_{e1}&\frac{1}{\sqrt{2}}\sin\theta_{e1}(1-\mathrm{i})\\\frac{1}{\sqrt{2}}\sin\theta_{e1}(1+\mathrm{i})&\cos\theta_{e1}\end{bmatrix},$$

then we have

$$\langle\mathbb{B}_5\rangle=\cos(2\theta)\cos\theta_{e1}-2\sqrt{2}\,\sin(2\theta)\sin\theta_{e1}.$$

For arbitrary odd n ($n>5$), via iteration,

$$\mathbb{B}_2^{(++)}=\frac{1}{2}[A_1(B_1+B_2)+A_2(B_1-B_2)]=A_1B_1=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\otimes\begin{bmatrix}1&0\\0&-1\end{bmatrix}=\begin{bmatrix}1&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&1\end{bmatrix}_{4\times4},$$
$$\mathbb{B}_4^{(++)}=\frac{1}{2}[\mathbb{B}_3^{(++)}(D_1+D_2)+\mathbb{B}_3^{(+-)}(D_1-D_2)]=\mathbb{B}_3^{(++)}D_1=\mathrm{Diag}(1,\ldots,-1)_{8\times8}\otimes\mathrm{Diag}(1,-1)=\mathrm{Diag}(1,\ldots,1)_{16\times16},$$

which means that

$$\mathbb{B}_{n-1}^{(++)}=\mathbb{B}_{n-1}^{\mathrm{MABK}}=\mathrm{Diag}(1,\ldots,1)_{2^{n-1}\times2^{n-1}},$$

for odd n under the constraints of (B12).

Likewise,

$$\mathbb{B}_2^{(+-)} = \tfrac{1}{2}[A_3(B_3+B_4)+A_4(B_3-B_4)] = \mathrm{Adiag}(1-\mathrm{i},0,0,1+\mathrm{i})$$
$$\mathbb{B}_4^{(+-)} = \mathbb{B}_4^{(++)}(A_1\to A_3, A_2\to A_4,\ldots,D_1\to D_3, D_2\to D_4)$$
$$= 2\,\mathrm{Adiag}(-1-\mathrm{i},\ldots,-1+\mathrm{i})_{16\times16},$$
$$\ldots,$$

which indicates that

$$\mathbb{B}_{n-1}^{(+-)} = 2^{(n-3)/2}\times\begin{cases}\mathrm{Adiag}(1-\mathrm{i},\ldots,1+\mathrm{i})_{2^{n-1}\times2^{n-1}}, & \left(\frac{n-1}{2}\right)\text{ is odd},\\ \mathrm{Adiag}(-1-\mathrm{i},\ldots,-1+\mathrm{i})_{2^{n-1}\times2^{n-1}}, & \left(\frac{n-1}{2}\right)\text{ is even}.\end{cases}$$

After that, we obtain

$$\mathbb{B}_n^{\mathrm{EMABK}} = \frac{1}{2}[\mathbb{B}_{n-1}^{(++)}(E_1+E_2)+\mathbb{B}_{n-1}^{(+-)}(E_1-E_2)]$$
$$=\begin{cases}\begin{bmatrix}\cos\theta_{e1} & \cdots & 2^{(n-2)/2}\sin\theta_{e1}\\ \vdots & \ddots & \vdots\\ 2^{(n-2)/2}\sin\theta_{e1} & \cdots & -\cos\theta_{e1}\end{bmatrix}_{2^n\times2^n}, & \left(\frac{n-1}{2}\right)\text{ is odd},\\[2ex] \begin{bmatrix}\cos\theta_{e1} & \cdots & -2^{(n-2)/2}\sin\theta_{e1}\\ \vdots & \ddots & \vdots\\ -2^{(n-2)/2}\sin\theta_{e1} & \cdots & -\cos\theta_{e1}\end{bmatrix}_{2^n\times2^n}, & \left(\frac{n-1}{2}\right)\text{ is even},\end{cases}$$

and the expectation value of $\mathbb{B}_n^{\mathrm{EMABK}}$ for the n-qubit generalized GHZ state $\cos\theta|00\cdots0\rangle+\sin\theta|11\cdots1\rangle$, reads

$$\langle\mathbb{B}_n^{\mathrm{EMABK}}\rangle = \cos(2\theta)\cos\theta_{e1}+[(-1)^{(n+1)/2}]2^{(n-2)/2}\sin(2\theta)\sin\theta_{e1},$$

$n>2$, and n is odd.

In conclusion, $\forall$ odd n, $n\geqslant3$, under the condition of (B12), the expectation value of $\mathbb{B}_n^{\mathrm{EMABK}}$ reads

$$\langle\mathbb{B}_n^{\mathrm{EMABK}}\rangle = \cos(2\theta)\cos\theta_{e1}+[(-1)^{(n+1)/2}]2^{(n-2)/2}\sin(2\theta)\sin\theta_{e1}$$
$$=\sqrt{\cos^2(2\theta)+2^{(n-2)}\sin^2(2\theta)}\cos[\theta_{e1}+(-1)^{(n-1)/2}\zeta],$$

where $\tan\zeta := [(-1)^{(n+1)/2}]\tan(2\theta)$. Obviously, $\forall\theta\in(0,\pi/2)$, $\langle\mathbb{B}_n^{\mathrm{EMABK}}\rangle>1$ for some typical valued θ_{e1}, such as $[(-1)^{(n+1)/2}]\zeta$. ■

APPENDIX C: ŚLIWA'S 46 INEQUALITIES AND THEIR TIGHT GENERALIZATIONS

1. Decomposition of Śliwa's 46 inequalities

In the following Table I, the 46 Śliwa's inequalities [20] are rewritten via the iterative formula

$$\mathbb{B}_3 = \tfrac{1}{2}[\mathbb{B}_2^{(++)}(C_1+C_2)+\mathbb{B}_2^{(+-)}(1-C_2)+\mathbb{B}_2^{(-+)}(1-C_1)]\leqslant1,$$

under the condition

$$\mathbb{B}_2^{(--)} = \mathbb{B}_2^{(+-)}+\mathbb{B}_2^{(-+)}-\mathbb{B}_2^{(++)}.$$

2. A family of (4,2,2) inequalities

Next we will start from Śliwa's 46 tight (3,2,2) inequalities [20], adopting the equivalent transformations and iterative formula

$$\mathbb{B}_4 = \tfrac{1}{2}[\mathbb{B}_3^{(++)}(D_1+D_2)+\mathbb{B}_3^{(+-)}(1-D_2)+\mathbb{B}_3^{(-+)}(1-D_1)]\leqslant1,$$

under the condition

$$\mathbb{B}_3^{(--)} = \mathbb{B}_3^{(+-)}+\mathbb{B}_3^{(-+)}-\mathbb{B}_3^{(++)},$$

to establish tight (4,2,2) Bell inequalities. For simplicity, all $\mathbb{B}_3^{(++)}$'s are set to associated Śliwa's inequalities after normalization, which are positioned in the first row of Tables II–XLVII below. Note that all additionally generated inequalities are normalized and $\mathcal{Q}$ refers to the numerical quantum upper bound of corresponding inequality, whereas the additional (4,2,2) inequalities are too cumbersome to be listed completely after Śliwa$_5$, so we present two of them only and list them entirely in [44], which is open source.

TABLE I. 46 Normalized Śliwa's inequalities. Śliwa$_l$ indicates the lth Śliwa's inequality.

Number	Bell inequalities		
Śliwa$_1$	$A_1+B_1+C_1-A_1C_1-B_1C_1-A_1B_1+A_1B_1C_1\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	1	1	$-1+2A_1+2B_1-2A_1B_1$
Śliwa$_2$	$\frac{1}{2}(A_1B_1C_1+A_2B_2C_1+A_2B_1C_2-A_1B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{2}(A_1B_1+A_2B_1-A_1B_2+A_2B_2)$	$\frac{1}{2}(A_1B_1-A_2B_1+A_1B_2+A_2B_2)$	$-\mathbb{B}_2^{(+-)}$
Śliwa$_3$	$\frac{1}{2}(A_1B_1C_1+A_2B_1C_1+A_1B_2C_2-A_2B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{2}(A_1B_1+A_2B_1+A_1B_2-A_2B_2)$	$\frac{1}{2}(A_1B_1+A_2B_1-A_1B_2+A_2B_2)$	$-\mathbb{B}_2^{(+-)}$
Śliwa$_4$	$\frac{1}{2}(-A_1B_1C_1-A_1B_2C_1-A_1B_1C_2+A_1B_2C_2+2A_1+B_1C_1+B_2C_1+B_1C_2-B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{2}(-2A_1B_1+2A_1+2B_1)$	$\frac{1}{2}(-2A_1B_2+2A_1+2B_2)$	$\frac{1}{2}(2A_1B_2+2A_1-2B_2)$
Śliwa$_5$	$\frac{1}{3}(-A_1B_1C_1-A_1B_2C_1-A_1B_1C_2-A_2B_1C_1+A_2B_2C_2+A_1B_2+A_2B_1-A_2B_2+A_1C_2$ $+A_2C_1-A_2C_2+A_1+B_2C_1+B_1C_2-B_2C_2+B_1+C_1)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{3}(-2A_1B_1+2A_1+2B_1+1)$	$\frac{1}{3}(-2A_2B_2+2A_2+2B_2+1)$	$\frac{1}{3}(2A_1B_2+2A_2B_1+2A_1-2A_2$ $+2B_1-2B_2-1)$
Śliwa$_6$	$\frac{1}{3}(-A_1B_1C_1-A_1B_2C_1-A_1B_2C_2-A_2B_1C_1+A_2B_1C_2+A_1B_1+A_1C_2+A_2C_1-A_2C_2$ $+A_1+B_2C_1-B_1C_2+B_2C_2+B_1+C_1)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{3}(-2A_1B_2+2A_1+2B_2+1)$	$\frac{1}{3}(-2A_2B_1+2A_2+2B_1+1)$	$\frac{1}{3}(2A_1B_1+2A_2B_1+2A_1-2A_2-1)$
Śliwa$_7$	$\frac{1}{4}(3A_1B_1C_1+A_2B_1C_1+A_1B_2C_1-A_2B_2C_1+A_1B_1C_2-A_2B_1C_2-A_1B_2C_2+A_2B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	A_1B_1	$\frac{1}{2}(A_1B_1+A_1B_2+A_2B_1-A_2B_2)$	$-\mathbb{B}_2^{(+-)}$
Śliwa$_8$	$\frac{1}{4}(2A_1B_1C_1+A_1B_1C_2-A_2B_1C_2-2A_2B_2C_1-A_1B_2C_2+A_2B_2C_2+A_1B_1+A_2B_1+A_1B_2+A_2B_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	A_1B_1	$\frac{1}{2}(A_1B_1+A_1B_2+A_2B_1-A_2B_2)$	A_2B_2
Śliwa$_9$	$\frac{1}{4}(2A_1B_1C_1+A_1B_1C_2-A_2B_1C_2-2A_1B_2C_1+A_1B_2C_2-A_2B_2C_2+A_1B_1+A_2B_1+A_1B_2+A_2B_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	A_1B_1	$\frac{1}{2}(A_1B_1-A_1B_2+A_2B_1+A_2B_2)$	A_1B_2
Śliwa$_{10}$	$\frac{1}{4}(A_1B_1+A_2B_1+A_1B_2+A_2B_2+A_1C_1-A_2C_1+A_1C_2-A_2C_2+B_1C_1-B_1C_2-B_2C_1+B_2C_2$ $+A_1B_1C_1+A_2B_1C_2-A_2B_2C_1-A_1B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{2}(A_1-A_2+A_1B_1+A_2B_1)$	$\frac{1}{2}(B_1-B_2+A_1B_1+A_1B_2)$	$\frac{1}{2}(-B_1+B_2+A_2B_1+A_2B_2)$

3. Some (5,2,2) Inequalities

In this subsection we will start from the third (4,2,2) inequality shown in Table IV, with the help of equivalent transformations and iterative formula

$$\mathbb{B}_5=\frac{1}{2}[\mathbb{B}_4^{(++)}(E_1+E_2)+\mathbb{B}_4^{(+-)}(1-E_2)+\mathbb{B}_4^{(-+)}(1-E_1)]\leqslant 1,$$

under the condition

$$\mathbb{B}_4^{(--)}=\mathbb{B}_4^{(+-)}+\mathbb{B}_4^{(-+)}-\mathbb{B}_4^{(++)},$$

to build a family of tight (5,2,2) Bell inequalities. Akin to the circumstance of last subsection, the $\mathbb{B}_4^{(++)}$ is set to the third (4,2,2) inequality in Table IV, which is listed in the first row of Table XLVIII. Note that all additionally generated inequalities are normalized and Q refers to the numerical quantum upper bound of corresponding inequality. Likewise, the additionally established (5,2,2) inequalities are too cumbersome to be presented completely, so we print two of them only and list them entirely in [44].

TABLE I. (*Continued*.)

Number	Bell inequalities		
Śliwa$_{11}$	$\frac{1}{4}(A_1B_1C_1+A_2B_1C_1+A_1B_1C_2-A_2B_1C_2-A_1B_2C_1-A_2B_2C_1+A_1B_2C_2-A_2B_2C_2+2A_1B_1+2A_2B_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	A_1B_1	$\frac{1}{2}(A_1B_1+A_2B_1-A_1B_2+A_2B_2)$	$\frac{1}{2}(A_1B_1-A_2B_1+A_1B_2+A_2B_2)$
Śliwa$_{12}$	$\frac{1}{4}(2A_1B_1+2A_2B_2+A_1C_1+A_2C_1-B_1C_1-B_2C_1+A_1C_2+A_2C_2-B_1C_2-B_2C_2$ $+A_2B_1C_1-A_1B_2C_1-A_2B_1C_2+A_1B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{4}(A_1+A_2-B_1-B_2+A_1B_1+A_2B_2)$	$\frac{1}{4}(A_1B_1-A_1B_2+A_2B_1+A_2B_2)$	$\frac{1}{4}(A_1B_1+A_1B_2-A_2B_1+A_2B_2)$
Śliwa$_{13}$	$\frac{1}{4}(A_1B_1C_1-A_2B_1C_1+A_1B_1C_2-A_2B_1C_2+A_1B_2C_1-A_2B_2C_1-A_1B_2C_2+A_2B_2C_2+2A_1B_1+2A_2B_1)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	A_1B_1	$\frac{1}{2}(A_1B_1+A_1B_2+A_2B_1-A_2B_2)$	$\frac{1}{2}(A_1B_1-A_1B_2+A_2B_1+A_2B_2)$
Śliwa$_{14}$	$\frac{1}{4}(A_1B_2C_1-A_2B_2C_1-A_1B_2C_2+A_2B_2C_2+2A_1B_1+2A_2B_1+A_1C_1-A_2C_1+A_1C_2-A_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{2}(A_1-A_2+A_1B_1+A_2B_1)$	$\frac{1}{2}(A_1B_1+A_2B_1+A_1B_2-A_2B_2)$	$\frac{1}{2}(A_1B_1+A_2B_1-A_1B_2+A_2B_2)$
Śliwa$_{15}$	$\frac{1}{4}(2A_1B_1+2A_2B_1+A_1C_1+A_2C_1-2B_1C_1+A_1C_2+A_2C_2-2B_1C_2$ $+A_1B_2C_1-A_1B_2C_2-A_2B_2C_1+A_2B_2C_2)\leqslant 1$		
	$\frac{1}{4}(A_1+A_2-2B_1+A_1B_1+A_2B_1)$	$\frac{1}{4}(A_1B_1+A_1B_2+A_2B_1-A_2B_2)$	$\frac{1}{4}(A_1B_1-A_1B_2+A_2B_1+A_2B_2)$
Śliwa$_{16}$	$\frac{1}{4}(A_1B_2C_1+A_1B_1C_2-A_1B_2C_2-2A_2B_1C_1-A_2B_2C_1-A_2B_1C_2$ $+A_2B_2C_2+A_1B_1+A_2B_1+A_1C_1+A_2C_1+A_1+A_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{2}(A_1+A_2+A_1B_1-A_2B_1)$	$\frac{1}{2}(A_1+A_2+A_1B_2-A_2B_2)$	$\frac{1}{2}(A_1B_1+A_2B_1-A_1B_2+A_2B_2)$
Śliwa$_{17}$	$\frac{1}{4}(-A_1B_1C_1+2A_1B_2C_2-A_2B_1C_1-2A_2B_2C_2+A_1B_1+A_2B_1+A_1C_1+A_2C_1+A_1+A_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{2}(A_1+A_2+A_1B_2-A_2B_2)$	$\frac{1}{2}(A_1+A_2-A_1B_2+A_2B_2)$	$\frac{1}{2}(A_1B_1+A_2B_1+A_1B_2-A_2B_2)$
Śliwa$_{18}$	$\frac{1}{4}(A_1B_2C_1+A_1B_1C_2-A_1B_2C_2-A_2B_2C_1-A_2B_1C_2-A_2B_2C_2$ $+A_1B_1+A_2B_1+A_1C_1+A_2C_1+A_1+A_2-2B_1C_1+2B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{2}(A_1+A_2-B_1+A_1B_1+B_2-A_2B_2)$	$\frac{1}{2}(A_1+A_2-B_1+A_2B_1-B_2+A_1B_2)$	$\frac{1}{2}(B_1+B_2+A_1B_1-A_1B_2)$
Śliwa$_{19}$	$\frac{1}{4}(-A_1B_2C_1+A_1B_1C_2+A_1B_2C_2-A_2B_2C_1-A_2B_1C_2$ $-A_2B_2C_2+A_1B_1+A_2B_1+A_1C_1+A_2C_1+A_1+A_2-2B_1C_1+2B_2C_1)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{2}(A_1+A_2-B_1+A_1B_1+B_2-A_2B_2)$	$\frac{1}{2}(A_1+A_2-B_1+A_2B_1+B_2-A_1B_2)$	$\frac{1}{2}(B_1-B_2+A_1B_1+A_1B_2)$
Śliwa$_{20}$	$\frac{1}{4}(A_1B_1C_1+A_1B_2C_1-A_1B_1C_2+A_1B_2C_2+A_2B_1C_1+A_2B_2C_1-A_2B_1C_2$ $+A_2B_2C_2+A_1B_1+A_1B_2-A_2B_1-A_2B_2+A_1C_1-A_2C_1+A_1+A_2-B_1C_1-B_2C_1+B_1C_2-B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{4}(A_1B_1+3A_1B_2-A_2B_1+A_2B_2$ $+2A_1-2B_2)$	$\frac{1}{4}(3A_1B_1+A_1B_2+A_2B_1-A_2B_2+2A_1$ $-2B_1)$	$\frac{1}{4}(-3A_2B_1-A_2B_2-A_1B_1+A_1B_2$ $+2A_2+2B_1)$
Śliwa$_{21}$	$\frac{1}{4}(-2A_1B_1C_1-A_1B_2C_1+A_1B_1C_2-A_1B_2C_2-A_2B_1C_1-A_2B_1C_2+A_2B_2C_2+A_1B_1$ $-A_2B_2+A_1C_1+A_2C_1+A_1+A_2+B_1C_1+B_2C_1+B_1+B_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{4}(-A_1B_2-A_2B_1+A_1+A_2+B_1$ $+B_2)$	$\frac{1}{4}(-A_1B_1-A_2B_2+A_1+A_2+B_1+B_2)$	A_1B_1
Śliwa$_{22}$	$\frac{1}{4}(A_1+A_2+B_1+B_2+C_1+C_2+A_1B_1-A_2B_2+A_1C_1-A_2C_2+B_1C_1-B_2C_2$ $-2A_1B_1C_1-A_1B_2C_1-A_1B_1C_2+A_1B_2C_2-A_2B_1C_1+A_2B_2C_1+A_2B_1C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{2}(1+A_1+B_1-A_1B_1)$	$\frac{1}{2}(1+A_1+A_2+B_1+B_2-A_1B_2-A_2B_1)$	$\frac{1}{2}(A_1B_1+A_1B_2+A_2B_1-A_2B_2)$
Śliwa$_{23}$	$\frac{1}{4}(A_1+A_2+B_1+B_2-A_1B_1-A_1B_2-A_2B_1-A_2B_2+A_1C_1-A_2C_1+B_1C_2-B_2C_2$ $-A_1B_1C_1-A_1B_2C_1-A_1B_1C_2+A_1B_2C_2+A_2B_1C_1+A_2B_2C_1-A_2B_1C_2+A_2B_2C_2)\leqslant 1$		

TABLE I. (*Continued*.)

Number	Bell inequalities		
Split form	$\mathbb{B}_2^{(++)}$ $\frac{1}{4}(2A_1+2B_1-3A_1B_1-A_1B_2-A_2B_1+A_2B_2)$	$\mathbb{B}_2^{(+-)}$ $\frac{1}{4}(2A_1+2B_2-A_1B_1-3A_1B_2+A_2B_1-A_2B_2)$	$\mathbb{B}_2^{(-+)}$ $\frac{1}{4}(2A_2+2B_1-A_1B_1+A_1B_2-3A_2B_1-A_2B_2)$
Śliwa$_{24}$	$\frac{1}{5}(A_1+B_1+C_1+A_1B_2+A_2B_1+A_2B_2+A_1C_2+A_2C_1+A_2C_2-B_1C_1+2A_1B_1C_1-A_1B_2C_2-A_2B_1C_1-2A_2B_2C_1-2A_2B_1C_2+A_2B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$ $\frac{1}{5}(1+2A_1+2A_2+2A_1B_1-2A_2B_1)$	$\mathbb{B}_2^{(+-)}$ $\frac{1}{5}(1+2A_1B_1+2A_1B_2+2A_2B_1-2A_2B_2)$	$\mathbb{B}_2^{(-+)}$ $\frac{1}{5}(-1+2A_1+2B_1-2A_1B_1+4A_2B_2)$
Śliwa$_{25}$	$\frac{1}{5}(A_1+B_1+C_1+A_1B_2+A_2B_1+A_2B_2+A_1C_2+A_2C_1+A_2C_2-B_1C_1+2A_1B_1C_1-2A_1B_2C_1+A_1B_2C_2-A_2B_1C_1-2A_2B_1C_2-A_2B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$ $\frac{1}{5}(1+2A_1+2A_2+2A_1B_2-A_2B_2)$	$\mathbb{B}_2^{(+-)}$ $\frac{1}{5}(1+4A_1B_1)$	$\mathbb{B}_2^{(-+)}$ $\frac{1}{5}(-1+2A_1+2B_1-4A_1B_1+2A_2B_1+2A_1B_2+2A_2B_2)$
Śliwa$_{26}$	$\frac{1}{5}(A_1+B_1+C_1+A_1B_1+2A_2B_2+A_1C_1+2A_2C_2+B_1C_1-2B_2C_2-A_1B_1C_1+2A_1B_2C_2-2A_2B_2C_1-2A_2B_1C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$ $\frac{1}{5}(2A_1B_2-2A_2B_1+2A_1+2A_2+2B_1-2B_2+1)$	$\mathbb{B}_2^{(+-)}$ $\frac{1}{5}(-2A_1B_2+2A_2B_1+2A_1-2A_2+2B_1+2B_2+1)$	$\mathbb{B}_2^{(-+)}$ $\frac{1}{5}(1+A_2-B_2+A_1B_1-A_2B_1+A_1B_2+2A_2B_2)$
Śliwa$_{27}$	$\frac{1}{5}(2A_1+A_2+B_1+C_1-A_1B_1+A_1B_2+A_2B_2-A_1C_1+A_1C_2+A_2C_2+B_2C_1+B_1C_2+B_2C_2+2A_1B_1C_1-A_1B_2C_1-A_1B_1C_2-2A_1B_2C_2-2A_2B_1C_1-A_2B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$ $\frac{1}{5}(-2A_1B_2-2A_2B_1+2A_1+2A_2+2B_1+2B_2+1)$	$\mathbb{B}_2^{(+-)}$ $\frac{1}{5}(2A_1B_1-2A_2B_1+2A_1B_2+2A_2B_2+1)$	$\mathbb{B}_2^{(-+)}$ $\frac{1}{5}(-4A_1B_1+2A_2B_1+4A_1+2A_2+2B_1-1)$
Śliwa$_{28}$	$\frac{1}{6}(2A_1B_1C_1-A_1B_2C_1-A_1B_1C_2-3A_1B_2C_2+A_2B_1C_1-2A_2B_2C_1-2A_2B_1C_2+A_1B_1-A_2B_1+A_1C_1-A_2C_1+A_1+A_2-B_1C_1+B_2C_1+B_1C_2+B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$ $\frac{1}{3}(A_1B_1-2A_1B_2-A_2B_1-A_2B_2+A_1+B_2)$	$\mathbb{B}_2^{(+-)}$ $\frac{1}{3}(2A_1B_1+A_1B_2+A_2B_1-A_2B_2+A_1-B_1)$	$\mathbb{B}_2^{(-+)}$ $\frac{1}{3}(-2A_2B_1+A_2B_2-A_1B_1-A_1B_2+A_2+B_1)$
Śliwa$_{29}$	$\frac{1}{6}(2A_1B_1C_1-A_1B_2C_1-3A_1B_1C_2-A_1B_2C_2+A_2B_1C_1-2A_2B_2C_1-2A_2B_2C_2+A_1B_1-A_2B_1+A_1C_1-A_2C_1+A_1+A_2-B_1C_1+B_2C_1+B_1C_2+B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$ $\frac{1}{3}(-A_1B_2-2A_2B_2+A_1+B_2)$	$\mathbb{B}_2^{(+-)}$ $\frac{1}{3}(3A_1B_1+A_1-B_1)$	$\mathbb{B}_2^{(-+)}$ $\frac{1}{3}(-A_2B_1-2A_1B_1+A_2+B_1)$
Śliwa$_{30}$	$\frac{1}{6}(2A_1B_1C_1+A_1B_2C_1-2A_1B_1C_2+2A_1B_2C_2+A_2B_1C_1+2A_2B_2C_1-A_2B_1C_2+A_2B_2C_2+2A_1B_1+A_1B_2-2A_2B_1-A_2B_2+A_1C_1-A_2C_1+A_1+A_2-B_1C_1-B_2C_1+B_1C_2-B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$ $\frac{1}{3}(A_1B_1+2A_1B_2-A_2B_1+A_2B_2+A_1-B_2)$	$\mathbb{B}_2^{(+-)}$ $\frac{1}{3}(3A_1B_1+A_1-B_1)$	$\mathbb{B}_2^{(-+)}$ $\frac{1}{3}(-2A_2B_1-A_2B_2-A_1B_1+A_1B_2+A_2+B_1)$
Śliwa$_{31}$	$\frac{1}{6}(-A_1B_2C_1-2A_1B_1C_2+2A_1B_2C_2+2A_2B_1C_1+3A_2B_2C_1-A_2B_1C_2+A_2B_2C_2-A_1B_2-A_2B_1+A_1C_1-A_2C_1+A_1+A_2+B_1C_2-B_2C_2+B_1+B_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$ $\frac{1}{3}(-A_1B_1+2A_2B_2+A_1+B_1)$	$\mathbb{B}_2^{(+-)}$ $\frac{1}{3}(A_1B_1-2A_1B_2+A_2B_1+A_2B_2+A_1+B_2)$	$\mathbb{B}_2^{(-+)}$ $\frac{1}{3}(-2A_2B_1-A_2B_2-A_1B_1+A_1B_2+A_2+B_1)$
Śliwa$_{32}$	$\frac{1}{6}(2A_1B_1C_2-A_1B_2C_2+2A_2B_1C_1+2A_2B_2C_1+A_2B_1C_2-2A_2B_2C_2-A_1B_2-A_2B_1+2A_1C_1+A_1C_2-2A_2C_1-A_2C_2+A_1+A_2-B_1C_2+B_2C_2+B_1+B_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$ $\frac{1}{3}(A_1B_1-A_1B_2+A_2B_1+2A_1-A_2+B_2)$	$\mathbb{B}_2^{(+-)}$ $\frac{1}{3}(-A_1B_1+2A_2B_2+A_1+B_1)$	$\mathbb{B}_2^{(-+)}$ $\frac{1}{3}(-A_2B_1-2A_2B_2+A_1B_1-A_1B_2+A_2+B_2)$
Śliwa$_{33}$	$\frac{1}{6}(A_1+A_2+B_1+B_2+C_1+C_2-A_1B_2-A_2B_1-A_1C_2-A_2C_1-B_2C_1-B_1C_2+2A_1B_2C_1+2A_1B_1C_2+A_1B_2C_2+2A_2B_1C_1+A_2B_2C_1+A_2B_1C_2-3A_2B_2C_2)\leqslant 1$		

TABLE I. (*Continued*.)

Number	Bell inequalities		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{3}(1+A_1B_1+A_1B_2+A_2B_1-A_2B_2)$	$\frac{1}{3}(A_1+B_1-A_1B_1+2A_2B_2)$	$\frac{1}{3}(A_2+B_2+A_1B_1-A_1B_2-A_2B_1-2A_2B_2)$
Śliwa$_{34}$	$\frac{1}{6}(-2A_1B_1C_1+2A_1B_2C_1+A_1B_2C_2+A_2B_1C_1+2A_2B_2C_1-2A_2B_2C_2$ $-A_1B_2-A_2B_1-A_1C_2-A_2C_1+A_1+A_2-B_1C_1-2B_2C_1-2B_1C_2-B_2C_2+B_1+B_2+C_1+C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{3}(-A_1B_1+A_1B_2-B_1-B_2+1)$	$\frac{1}{3}(-A_1B_1+2A_2B_2+A_1+B_1)$	$\frac{1}{3}(-A_2B_1-2A_2B_2+A_1B_1-A_1B_2+A_2+B_2)$
Śliwa$_{35}$	$\frac{1}{6}(A_1+A_2+B_1+B_2-A_1B_1-2A_1B_2-2A_2B_1-A_2B_2+A_1C_1-A_2C_1+B_1C_2-B_2C_2$ $-A_1B_1C_1-2A_1B_2C_1-2A_1B_1C_2+2A_1B_2C_2+A_2B_1C_1+2A_2B_2C_1-A_2B_1C_2+A_2B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{3}(A_1+B_1-2A_1B_1-A_1B_2-A_2B_1+A_2B_2)$	$\frac{1}{3}(A_1+B_2-3A_1B_2)$	$\frac{1}{3}(A_2+B_1-A_1B_1+A_1B_2-2A_2B_1-A_2B_2)$
Śliwa$_{36}$	$\frac{1}{6}(-2A_1B_1C_1+A_1B_2C_1+A_1B_1C_2+2A_1B_2C_2+A_2B_1C_1-2A_2B_2C_1-2A_2B_1C_2-A_2B_2C_2+A_1B_1$ $+A_1B_2+A_2B_1+A_2B_2+A_1C_1+A_1C_2+A_2C_1+A_2C_2+2A_1+B_1C_1-B_2C_1-B_1C_2-B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{3}(2A_1B_2-A_2B_2+2A_1+A_2-B_2)$	$\frac{1}{3}(-A_1B_1+2A_2B_1+A_1+B_1)$	$\frac{1}{3}(2A_1B_1+A_1B_2-A_2B_1+A_2B_2+A_1-B_1)$
Śliwa$_{37}$	$\frac{1}{6}(-3A_1B_1C_1+2A_1B_2C_1+2A_1B_1C_2+A_1B_2C_2-A_2B_2C_1-A_2B_1C_2-2A_2B_2C_2+A_1B_1$ $+A_1B_2+A_2B_1+A_2B_2+A_1C_1+A_1C_2+A_2C_1+A_2C_2+2A_1+B_1C_1-B_2C_1-B_1C_2-B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{3}(2A_1B_2-A_2B_2+2A_1+A_2-B_2)$	$\frac{1}{3}(-2A_1B_1+A_1B_2+A_2B_1+A_2B_2+A_1+B_1)$	$\frac{1}{3}(3A_1B_1+A_1-B_1)$
Śliwa$_{38}$	$\frac{1}{6}(A_1B_1C_1-2A_1B_2C_1+A_1B_1C_2+2A_1B_2C_2-2A_2B_1C_1+A_2B_2C_1-2A_2B_1C_2-A_2B_2C_2+2A_1B_1$ $+2A_2B_1+A_1C_1+A_1C_2+A_2C_1+A_2C_2+2A_1-B_1C_1+B_2C_1-B_1C_2-B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{3}(2A_1B_1-A_2B_1+2A_1+A_2-B_1)$	$\frac{1}{3}(A_1B_1-2A_1B_2+A_2B_1+A_2B_2+A_1+B_2)$	$\frac{1}{3}(A_1B_1+2A_1B_2+A_2B_1-A_2B_2+A_1-B_2)$
Śliwa$_{39}$	$\frac{1}{6}(2A_1+2B_1+2C_1-A_1B_1+A_1B_2+A_2B_1+A_2B_2-A_1C_1+A_1C_2+A_2C_1+A_2C_2-B_1C_1+B_2C_1+B_1C_2+B_2C_2$ $+2A_1B_1C_1-A_1B_2C_1-A_1B_1C_2-2A_1B_2C_2-A_2B_1C_1-2A_2B_2C_1-2A_2B_1C_2+A_2B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{3}(1+A_1+A_2+B_1+B_2-A_1B_2-A_2B_1)$	$\frac{1}{3}(1+A_1B_1+A_1B_2+A_2B_1-A_2B_2)$	$\frac{1}{3}(-1+2A_1+2B_1-2A_1B_1+2A_2B_2)$
Śliwa$_{40}$	$\frac{1}{6}(2A_1+2A_2+2B_1-A_1B_1C_1-2A_1B_2C_1-2A_1B_1C_2+A_1B_2C_2-A_2B_1C_1-2A_2B_2C_1+2A_2B_1C_2$ $-A_2B_2C_2-A_1B_1+A_1B_2-A_2B_1+A_2B_2+A_1C_1+A_1C_2+A_2C_1-A_2C_2+2B_1C_1+2B_2C_1)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{3}(-2A_1B_1-A_2B_2+2A_1+A_2+2B_1+B_2)$	$\frac{1}{3}(-A_1B_2-2A_2B_1+A_1+2A_2+2B_1+B_2)$	$\frac{1}{3}(-A_1B_1+2A_1B_2+A_2B_1+A_2B_2+A_1-B_2)$
Śliwa$_{41}$	$\frac{1}{7}(-3A_1B_1C_1-A_1B_2C_1-4A_1B_1C_2+A_1B_2C_2-A_2B_1C_1-2A_2B_2C_1+A_2B_1C_2$ $+2A_2B_2C_2+A_1B_1+A_1C_2+A_2C_1-A_2C_2+A_1+B_2C_1+B_1C_2-B_2C_2+B_1+C_1)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{7}(-6A_1B_1+2A_1+2B_1+1)$	$\frac{1}{7}(-2A_2B_1-4A_2B_2+2A_1B_1-2A_1B_2+2A_2+2B_2+1)$	$\frac{1}{7}(2A_1B_2+2A_2B_1+4A_2B_2+2A_1-2A_2+2B_1-2B_2-1)$
Śliwa$_{42}$	$\frac{1}{8}(A_1+A_2+B_1+B_2+A_1B_1-A_2B_2+A_1C_1-A_2C_1+2A_2C_2+B_1C_1-B_2C_1+2B_2C_2$ $-2A_1B_1C_1-A_1B_2C_1-A_1B_1C_2-3A_1B_2C_2-A_2B_1C_1+4A_2B_2C_1-3A_2B_1C_2-A_2B_2C_2)\leqslant 1$		

TABLE I. (*Continued*.)

Number	Bell inequalities		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{4}(A_1+A_2+B_1+B_2-A_1B_1-2A_1B_2-2A_2B_1+A_2B_2)$	$\frac{1}{4}(A_1-A_2+B_1-B_2+A_1B_2+A_2B_1+2A_2B_2)$	$\frac{1}{4}(2A_2+2B_2+A_1B_1-A_1B_2-A_2B_1-3A_2B_2)$
Śliwa$_{43}$	$\frac{1}{8}(2A_1+B_2C_2+2B_1-A_1B_1+A_1B_2+A_2B_1-A_2B_2+A_1C_1+A_1C_2+A_2C_1-A_2C_2+B_1C_1-B_2C_1+B_1C_2$ $-2A_1B_1C_1+A_1B_2C_1-3A_1B_1C_2-4A_1B_2C_2-3A_2B_1C_1+2A_2B_2C_1+A_2B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{4}(2A_1+2B_1-3A_1B_1-A_1B_2-A_2B_1+A_2B_2)$	$\frac{1}{4}(A_1+A_2+B_1-B_2-A_2B_1+3A_1B_2)$	$\frac{1}{4}(A_1-A_2+B_1+B_2-A_1B_1-2A_1B_2+2A_2B_1-A_2B_2)$
Śliwa$_{44}$	$\frac{1}{8}(2A_1B_1C_1-A_1B_2C_1+2A_1B_1C_2+3A_1B_2C_2+2A_2B_1C_1-3A_2B_2C_1+2A_2B_1C_2+A_2B_2C_2$ $+2A_1B_1-2A_2B_1+A_1C_1+A_1C_2-A_2C_1-A_2C_2+2A_1+2A_2-2B_1C_1+2B_2C_1-2B_1C_2-2B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{4}(3A_1B_1+A_1B_2+A_2B_1-A_2B_2+2A_1-2B_1)$	$\frac{1}{4}(A_1B_1-2A_1B_2-A_2B_1-2A_2B_2+A_1+A_2+2B_2)$	$\frac{1}{4}(A_1B_1+2A_1B_2-A_2B_1+2A_2B_2+A_1+A_2-2B_2)$
Śliwa$_{45}$	$\frac{1}{8}(3A_1+A_2 2A_1B_1C_1+2A_1B_2C_1+2A_1B_1C_2-3A_1B_2C_2+2A_2B_1C_1+2A_2B_2C_1+2A_2B_1C_2-A_2B_2C_2$ $+2A_1B_1+A_1B_2-2A_2B_1-A_2B_2+2A_1C_1+A_1C_2-2A_2C_1-A_2C_2-2B_1C_1-2B_2C_1-2B_1C_2+2B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{4}(3A_1B_1+A_2B_1+3A_1-A_2-2B_1)$	$\frac{1}{4}(A_1B_1+3A_1B_2-A_2B_1+A_2B_2+2A_1-2B_2)$	$\frac{1}{4}(A_1B_1-2A_1B_2-A_2B_1-2A_2B_2+A_1+A_2+2B_2)$
Śliwa$_{46}$	$\frac{1}{10}(3A_1+A_2+3B_1+B_2-2A_1B_1-A_1B_2-A_2B_1-2A_2B_2+2A_1C_1+A_1C_2-2A_2C_1+A_2C_2+B_1C_1+B_2C_1+2B_1C_2-2B_2C_2$ $-3A_1B_1C_1-A_1B_2C_1-3A_1B_1C_2+4A_1B_2C_2+4A_2B_1C_1+2A_2B_2C_1-A_2B_1C_2+2A_2B_2C_2)\leqslant 1$		
Split form	$\mathbb{B}_2^{(++)}$	$\mathbb{B}_2^{(+-)}$	$\mathbb{B}_2^{(-+)}$
	$\frac{1}{5}(3A_1+3B_1-4A_1B_1+A_1B_2+A_2B_1+A_2B_2)$	$\frac{1}{5}(2A_1-A_2+B_1+2B_2-A_1B_1-3A_1B_2+2A_2B_1-A_2B_2)$	$\frac{1}{5}(A_1+2A_2+2B_1-B_2-A_1B_1+2A_1B_2-3A_2B_1-A_2B_2)$

TABLE II. The (4,2,2) tight inequality generated from Śliwa$_1$.

Śliwa$_1$	$A_1+B_1-A_1B_1+C_1-A_1C_1-B_1C_1+A_1B_1C_1\leqslant 1$	$\mathcal{Q}=1$
Number	New tight inequality	Remarks
1	$\frac{1}{2}(A_1B_1C_1D_1+A_1B_1C_2D_1+A_1B_1C_1D_2-A_1B_1C_2D_2-2A_1B_1-A_1C_1D_1-A_1C_2D_1-A_1C_1D_2+A_1C_2D_2+2A_1-B_1C_1D_1-B_1C_2D_1-B_1C_1D_2+B_1C_2D_2+2B_1+C_1D_1+C_2D_1+C_1D_2-C_2D_2)\leqslant 1$	$\mathcal{Q}=4\sqrt{2}-3$
$\mathbb{B}_3^{(+-)}$	$A_1+B_1-A_1B_1+C_2-A_1C_2-B_1C_2+A_1B_1C_2\leqslant 1$	$\mathbb{B}_3^{(++)}\overset{C_{1\leftrightarrow 2}}{\longleftrightarrow}\mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$A_1+B_1-A_1B_1-C_2+A_1C_2+B_1C_2-A_1B_1C_2\leqslant 1$	$\mathbb{B}_3^{(++)}\overset{C_{1\leftrightarrow 2},C_2\to -C_2}{\longleftrightarrow}\mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$A_1+B_1-A_1B_1-C_1+A_1C_1+B_1C_1-A_1B_1C_1\leqslant 1$	$\mathbb{B}_3^{(++)}\overset{C_1\to -C_1}{\longleftrightarrow}\mathbb{B}_3^{(--)}$

TABLE III. The (4,2,2) tight inequalities generated from Śliwa$_2$.

Śliwa$_2$	$\frac{1}{2}(A_1B_1C_1+A_2B_2C_1+A_2B_1C_2-A_1B_2C_2)\leqslant 1$	$\mathcal{Q}=2$
Number	New tight inequalities	Remarks
1	$\mathbb{B}_3^{(++)}\overset{A_1\to -A_1}{\longleftrightarrow}\mathbb{B}_3^{(+-)},\ \mathbb{B}_3^{(++)}\overset{A_2\to -A_2}{\longleftrightarrow}\mathbb{B}_3^{(-+)},\ \mathbb{B}_3^{(++)}\overset{A\leftrightarrow C,B_1\to -B_1}{\longleftrightarrow}\mathbb{B}_3^{(--)}$	$\mathbb{B}_3^{(-+)}=-\mathbb{B}_3^{(+-)}$, $\mathcal{Q}=2$
2	$\mathbb{B}_3^{(++)}\overset{C_{1\leftrightarrow 2}}{\longleftrightarrow}\mathbb{B}_3^{(+-)},\ \mathbb{B}_3^{(++)}\overset{A_{1\leftrightarrow 2},B_1\to -B_1}{\longleftrightarrow}\mathbb{B}_3^{(-+)},\ \mathbb{B}_3^{(++)}\overset{A\leftrightarrow C,B_1\to -B_1}{\longleftrightarrow}\mathbb{B}_3^{(--)}$	$\mathbb{B}_3^{(-+)}=-\mathbb{B}_3^{(+-)}$, $\mathcal{Q}=2$
3	$\mathbb{B}_3^{(++)}\overset{A_{1\leftrightarrow 2},B_{1\leftrightarrow 2},C_{1\leftrightarrow 2}}{\longleftrightarrow}\mathbb{B}_3^{(+-)},\ \mathbb{B}_3^{(++)}\overset{A_2\to -A_2,A_{1\leftrightarrow 2}}{\longleftrightarrow}\mathbb{B}_3^{(-+)},\ \mathbb{B}_3^{(++)}\overset{A\leftrightarrow C,B_1\to -B_1}{\longleftrightarrow}\mathbb{B}_3^{(--)}$	$\mathbb{B}_3^{(-+)}=-\mathbb{B}_3^{(+-)}$, $\mathcal{Q}=2\sqrt{2}$, $\mathbb{B}_4^{\mathrm{MABK}}$

TABLE IV. The (4,2,2) tight inequalities generated from Śliwa$_3$.

Śliwa$_3$	$\frac{1}{2}(A_1B_1C_1 + A_2B_1C_1 + A_1B_2C_2 - A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = \sqrt{2}$
Number	New tight inequalities	Remarks
1	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1, B_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, C_2\to -C_2} \mathbb{B}_3^{(--)}$	$\mathbb{B}_3^{(-+)} = -\mathbb{B}_3^{(+-)}$, $\mathcal{Q} = \sqrt{2}$
2	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1, B_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1, B_{1\leftrightarrow 2}, A\leftrightarrow B} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \sqrt{2}$
3	$\frac{1}{4}(A_1B_1C_1D_1 + A_2B_1C_1D_1 + A_1B_2C_1D_1 + A_2B_2C_1D_1 + A_2B_1C_1D_2$ $- A_1B_2C_1D_2 - A_1B_1C_2D_1 + A_2B_1C_2D_1 + A_1B_2C_2D_1 - A_2B_2C_2D_1$ $- A_2B_1C_2D_2 + A_1B_2C_2D_2 + A_1B_1C_1 - A_2B_2C_1 + A_1B_1C_2 - A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 2$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{2}(A_1B_1C_1 + A_1B_2C_1 + A_2B_1C_2 - A_2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{2}(-A_1B_2C_1 - A_2B_2C_1 + A_1B_1C_2 - A_2B_1C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}, B_{1\leftrightarrow 2}, C\to -C} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{2}(-A_2B_1C_1 - A_2B_2C_1 + A_1B_1C_2 - A_1B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, B_2\to -B_2, A\leftrightarrow B} \mathbb{B}_3^{(--)}$
4	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B, A_{1\leftrightarrow 2}, B_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_2\to -C_2, B_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \sqrt{2}$
5	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 2$
6	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_2\to -A_2, B_2\to -B_2, A\leftrightarrow B} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, A_1\to -A_1} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \sqrt{2}$
7	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B, B_2\to -B_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B, B_1\to -B_1} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, C_2\to -C_2} \mathbb{B}_3^{(--)}$	$\mathbb{B}_3^{(-+)} = -\mathbb{B}_3^{(+-)}$, $\mathcal{Q} = \sqrt{3}$
8	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1, A\leftrightarrow B, B_2\to -B_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_2\to -A_2, B_2\to -B_2, A\leftrightarrow B} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, C_2\to -C_2} \mathbb{B}_3^{(--)}$	$\mathbb{B}_3^{(-+)} = -\mathbb{B}_3^{(+-)}$, $\mathcal{Q} = 2$
9	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_2\to -B_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, C_2\to -C_2} \mathbb{B}_3^{(--)}$	$\mathbb{B}_3^{(-+)} = -\mathbb{B}_3^{(+-)}$, $\mathcal{Q} = \sqrt{2}$
10	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_2\to -A_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, C_2\to -C_2} \mathbb{B}_3^{(--)}$	$\mathbb{B}_3^{(-+)} = -\mathbb{B}_3^{(+-)}$, $\mathcal{Q} = \sqrt{2}$
11	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, A_2\to -A_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1, A\leftrightarrow B, B_2\to -B_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 2$
12	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, A_2\to -A_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, A_1\to -A_1} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, C_2\to -C_2} \mathbb{B}_3^{(--)}$	$\mathbb{B}_3^{(-+)} = -\mathbb{B}_3^{(+-)}$, $\mathcal{Q} = 2$
13	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_2\to -B_2, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, C_2\to -C_2} \mathbb{B}_3^{(--)}$	$\mathbb{B}_3^{(-+)} = -\mathbb{B}_3^{(+-)}$, $\mathcal{Q} = 2$

TABLE V. The (4,2,2) tight inequalities generated from Śliwa$_4$.

Śliwa$_4$	$\frac{1}{2}(2A_1 + B_1C_1 - A_1B_1C_1 + B_2C_1 - A_1B_2C_1 + B_1C_2 - A_1B_1C_2 - B_2C_2 + A_1B_2C_2) \leqslant 1$	$\mathcal{Q} = 2\sqrt{2} - 1$
Number	New tight inequalities	Remarks
1	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 2\sqrt{2} - 1$
2	$\frac{1}{2}(-A_1B_1C_1D_1 + A_1B_2C_2D_1 - A_1B_2C_1D_2 - A_1B_1C_2D_2 + 2A_1 + B_1C_1D_1 - B_2C_2D_1$ $+ B_2C_1D_2 + B_1C_2D_2) \leqslant 1$	$\mathcal{Q} = 3$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{2}(2A_1 + B_1C_1 - A_1B_1C_1 - B_2C_1 + A_1B_2C_1 - B_1C_2 + A_1B_1C_2 - B_2C_2 + A_1B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, B_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{2}(2A_1 - B_1C_1 + A_1B_1C_1 + B_2C_1 - A_1B_2C_1 + B_1C_2 - A_1B_1C_2 + B_2C_2 - A_1B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{2}(2A_1 - B_1C_1 + A_1B_1C_1 - B_2C_1 + A_1B_2C_1 - B_1C_2 + A_1B_1C_2 + B_2C_2 + A_1B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$
3	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}, A_2\to -A_2} \mathbb{B}_3$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 2$

TABLE VI. The (4,2,2) tight inequalities generated from Śliwa$_5$.

Śliwa$_5$	$\frac{1}{3}(A_1 + B_1 + A_2B_1 + A_1B_2 - A_2B_2 + C_1 + A_2C_1 - A_1B_1C_1 - A_2B_1C_1 + B_2C_1$ $- A_1B_2C_1 + A_1C_2 - A_2C_2 + B_1C_2 - A_1B_1C_2 - B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = \frac{8\sqrt{5}-13}{3}$
Number	New tight inequalities	Remarks
1	$\frac{1}{6}(-2A_1B_1C_1D_1 - A_1B_1C_2D_1 + A_2B_1C_2D_1 - 2A_2B_1C_1D_2 - A_1B_1C_2D_2 - A_2B_1C_2D_2$ $- A_1B_2C_1D_1 + A_2B_2C_1D_1 + A_1B_2C_2D_1 + A_2B_2C_2D_1 - A_1B_2C_1D_2 - A_2B_2C_1D_2$ $- A_1B_2C_2D_2 + A_2B_2C_2D_2 + A_1B_1D_1 + A_2B_1D_1 - A_1B_1D_2 + A_2B_1D_2 - 2A_2B_2D_1$ $+ 2A_1B_2D_2 + A_1C_1D_1 + A_2C_1D_1 - 2A_2C_2D_1 - A_1C_1D_2 + A_2C_1D_2 + 2A_1C_2D_2$ $+ A_1D_1 - A_2D_1 + A_1D_2 + A_2D_2 + 2B_1C_2 + 2B_2C_1 - 2B_2C_2 + 2B_1 + 2C_1) \leqslant 1$	$\mathcal{Q} = 2.41$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{3}(-A_2 + B_1 + A_1B_1 - A_1B_2 - A_2B_2 + C_1 + A_1C_1 - A_1B_1C_1 + A_2B_1C_1$ $+ B_2C_1 + A_2B_2C_1 - A_1C_2 - A_2C_2 + B_1C_2 + A_2B_1C_2 - B_2C_2 + A_1B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_2\to -A_2, B_1\to -B_1} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{3}(A_2 + B_1 - A_1B_1 + A_1B_2 + A_2B_2 + C_1 - A_1C_1 + A_1B_1C_1 - A_2B_1C_1 + B_2C_1 - A_2B_2C_1$ $+ A_1C_2 + A_2C_2 + B_1C_2 - A_2B_1C_2 - B_2C_2 - A_1B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{D_{1\leftrightarrow 2}, B_2\to -B_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{3}(-A_1 + B_1 - A_2B_1 - A_1B_2 + A_2B_2 + C_1 - A_2C_1 + A_1B_1C_1 + A_2B_1C_1 + B_2C_1$ $+ A_1B_2C_1 - A_1C_2 + A_2C_2 + B_1C_2 + A_1B_1C_2 + B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$
2	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1, C_2\to -C_2, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_2\to -C_2, C_1\to -C_1} \mathbb{B}_3^{(--)}$	
3	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_2\to -C_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}, A_{1\leftrightarrow 2}, B_{1\leftrightarrow 2}, C_2\to -C_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, A_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$	
4	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2},\ B_{1\leftrightarrow 2}, C_1\to -C_1, C_2\to -C_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_2\to -C_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, C_1\to -C_1, A_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{8\sqrt{5}-13}{3}$
5	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_2\to -A_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1, A_2\to -A_2} \mathbb{B}_3^{(--)}$	
6	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1, B_2\to -B_2, A_2\to -A_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1, A_2\to -A_2, B_1\to -B_1, B_2\to -B_2, C_2\to -C_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, C_2\to -C_2, C_1\to -C_1, A_1\to -A_1} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{2\sqrt{5}+1}{3}$

TABLE VII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_6$.

Śliwa$_6$	$\frac{1}{3}(A_1 + B_1 + A_1B_1 + C_1 + A_2C_1 - A_1B_1C_1 - A_2B_1C_1 + B_2C_1 - A_1B_2C_1$ $+ A_1C_2 - A_2C_2 - B_1C_2 + A_2B_1C_2 + B_2C_2 - A_1B_2C_2) \leqslant 1$	$\mathcal{Q} = \frac{4\sqrt{2}-1}{3}$
Number	New tight inequalities	Remarks
1	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow C, B_1\to -B_1, B_2\to -B_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow C} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, B_2\to -B_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{4\sqrt{2}-1}{3}$
2	$\frac{1}{6}(-2A_1B_1C_1D_1 - 2A_1B_2C_1D_1 - A_1B_1C_2D_1 - A_1B_2C_2D_1 + A_1B_1C_2D_2$ $- A_1B_2C_2D_2 - A_2B_1C_1D_1 - A_2B_2C_1D_1 + 2A_2B_1C_2D_1 - A_2B_1C_1D_2$ $+ A_2B_2C_1D_2 + A_1B_1D_1 + A_1B_2D_1 + A_1B_1D_2 - A_1B_2D_2 - A_2B_1D_1$ $+ A_2B_2D_1 + A_2B_1D_2 - A_2B_2D_2 + 2A_1C_2 + 2A_2C_1 - 2A_2C_2 + 2A_1$ $+ B_1C_1D_1 + B_2C_1D_1 - B_1C_2D_1 + B_2C_2D_1 - B_1C_1D_2 + B_2C_1D_2 - B_1C_2D_2$ $+ B_2C_2D_2 + 2B_1D_1 + 2C_1) \leqslant 1$	$\mathcal{Q} = 2.41$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{3}(-A_1B_1C_1 - A_1B_2C_1 - A_1B_1C_2 - A_2B_2C_1 + A_2B_1C_2 + A_1B_2 - A_2B_1 + A_2B_2$ $+ A_1C_2 + A_2C_1 - A_2C_2 + A_1 + B_1C_1 + B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow C} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{3}(A_1B_1C_1 + A_1B_2C_1 + A_1B_1C_2 + A_2B_2C_1 - A_2B_1C_2 - A_1B_2 + A_2B_1 - A_2B_2$ $+ A_1C_2 + A_2C_1 - A_2C_2 + A_1 - B_1C_1 - B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow C, B_1\to -B_1, B_2\to -B_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{3}(A_1B_1C_1 + A_1B_2C_1 + A_1B_2C_2 + A_2B_1C_1 - A_2B_1C_2 - A_1B_1 + A_1C_2 + A_2C_1$ $- A_2C_2 + A_1 - B_2C_1 + B_1C_2 - B_2C_2 - B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, B_2\to -B_2} \mathbb{B}_3^{(--)}$

TABLE VIII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_7$.

Śliwa$_7$	$\frac{1}{4}(3A_1B_1C_1+A_2B_1C_1+A_1B_2C_1-A_2B_2C_1+A_1B_1C_2-A_2B_1C_2-A_1B_2C_2+A_2B_2C_2)\leqslant 1$	$\mathcal{Q}=\frac{5}{3}$
Number	New tight inequalities	Remarks
1	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},B_1\to -B_1,C_1\to -C_1}\mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{B_{1\leftrightarrow 2},C_{1\leftrightarrow 2},B_1\to -B_1,C_2\to -C_2}\mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{A_2\to -A_2,B_{1\leftrightarrow 2},C_{1\leftrightarrow 2},A_1\to -A_1,A_{1\leftrightarrow 2}}\mathbb{B}_3^{(--)}$	$\mathcal{Q}=\frac{5}{3}$
2	$\frac{1}{4}(A_1B_1C_1D_1-A_2B_2C_1D_1+A_1B_1C_1D_2+A_2B_1C_1D_2+A_1B_2C_1D_2+A_2B_2C_1D_2-A_1B_1C_2D_1+A_2B_2C_2D_1+A_1B_1C_2D_2-A_2B_1C_2D_2-A_1B_2C_2D_2+A_2B_2C_2D_2+A_1B_1C_1-A_2B_2C_1+A_1B_1C_2-A_2B_2C_2)\leqslant 1$	$\mathcal{Q}=2$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(A_1B_1C_1-A_2B_1C_1-A_1B_2C_1-3A_2B_2C_1-A_1B_1C_2+A_2B_1C_2+A_1B_2C_2-A_2B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},B_{1\leftrightarrow 2},A_1\to -A_1,A_2\to -A_2}\mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(A_1B_1C_1+A_2B_1C_1+A_1B_2C_1+A_2B_2C_1+3A_1B_1C_2-A_2B_1C_2-A_1B_2C_2-A_2B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{C_{1\leftrightarrow 2},A_1\to -A_1,B_1\to -B_1}\mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(-A_1B_1C_1-A_2B_1C_1-A_1B_2C_1-A_2B_2C_1+A_1B_1C_2+A_2B_1C_2+A_1B_2C_2-3A_2B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},B_2\to -B_2,C_{1\leftrightarrow 2},B_{1\leftrightarrow 2},A_2\to -A_2}\mathbb{B}_3^{(--)}$

TABLE IX. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_8$.

Śliwa$_8$	$\frac{1}{4}(A_1B_1+A_2B_1+A_1B_2+A_2B_2+2A_1B_1C_1-2A_2B_2C_1+A_1B_1C_2-A_2B_1C_2-A_1B_2C_2+A_2B_2C_2)\leqslant 1$	$\mathcal{Q}=\frac{5}{3}$
Number	New tight inequalities	Remarks
1	$\mathbb{B}_3^{(++)}\xleftrightarrow{B_{1\leftrightarrow 2},C_{1\leftrightarrow 2}}\mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{B\leftrightarrow C,A_{1\leftrightarrow 2},B_{1\leftrightarrow 2},C_1\to -C_1}\mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{B\leftrightarrow C,B_2\to -B_2,C_1\to -C_1}\mathbb{B}_3^{(--)}$	$\mathcal{Q}=1.67$
3	$\frac{1}{8}(3A_1B_1C_1D_1-A_2B_1C_1D_1+A_1B_1C_2D_1-3A_2B_1C_2D_1+A_1B_1C_1D_2+A_2B_1C_1D_2+A_1B_1C_2D_2+A_2B_1C_2D_2-A_1B_2C_1D_1-A_2B_2C_1D_1+A_1B_2C_2D_1+A_2B_2C_2D_1+A_1B_2C_1D_2-3A_2B_2C_1D_2-3A_1B_2C_2D_2+A_2B_2C_2D_2+2A_1B_1+2A_2B_1+2A_1B_2+2A_2B_2)\leqslant 1$	$\mathcal{Q}=2$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(A_1B_1C_1-A_2B_1C_1-2A_2B_1C_2-A_1B_2C_1+A_2B_2C_1+2A_1B_2C_2+A_1B_1+A_2B_1+A_1B_2+A_2B_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{B_{1\leftrightarrow 2},C_{1\leftrightarrow 2},C_1\to -C_1}\mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(-A_1B_1C_1+A_2B_1C_1+2A_2B_1C_2+A_1B_2C_1-A_2B_2C_1-2A_1B_2C_2+A_1B_1+A_2B_1+A_1B_2+A_2B_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},C_{1\leftrightarrow 2}}\mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(-2A_1B_1C_1-A_1B_1C_2+A_2B_1C_2+2A_2B_2C_1+A_1B_2C_2-A_2B_2C_2+A_1B_1+A_2B_1+A_1B_2+A_2B_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{C_1\to -C_1,C_2\to -C_2}\mathbb{B}_3^{(--)}$

TABLE X. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_9$.

Śliwa$_9$	$\frac{1}{4}(A_1B_1+A_2B_1+A_1B_2+A_2B_2+2A_1B_1C_1-2A_1B_2C_1+A_1B_1C_2-A_2B_1C_2+A_1B_2C_2-A_2B_2C_2)\leqslant 1$	$\mathcal{Q}=\sqrt{2}$
Number	New tight inequalities	Remarks
3	$\frac{1}{8}(A_1B_1C_1D_1+A_2B_1C_1D_1-3A_1B_2C_1D_1+A_2B_2C_1D_1+A_1B_1C_1D_2-A_2B_1C_1D_2-A_1B_2C_1D_2+A_2B_2C_1D_2+A_1B_1C_2D_1-3A_2B_1C_2D_1+A_1B_2C_2D_1+A_2B_2C_2D_1-A_1B_1C_2D_2+A_2B_1C_2D_2+A_1B_2C_2D_2-A_2B_2C_2D_2+2A_1B_1C_1-2A_2B_2C_1+2A_1B_1C_2-2A_2B_2C_2+2A_1B_1D_2+2A_2B_1D_2+2A_1B_2D_2+2A_2B_2D_2)\leqslant 1$	$\mathcal{Q}=2$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(A_1B_1C_1+A_2B_1C_1+2A_1B_1C_2-2A_2B_1C_2-A_1B_2C_1-A_2B_2C_1-A_1B_1-A_2B_1-A_1B_2-A_2B_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A\leftrightarrow B,C_2\to -C_2,A_2\to -A_2,C_{1\leftrightarrow 2},A_{1\leftrightarrow 2},A_2\to -A_2}\mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(A_1B_1C_1-A_2B_1C_1+2A_2B_1C_2+A_1B_2C_1-A_2B_2C_1-2A_2B_2C_2+A_1B_1+A_2B_1+A_1B_2+A_2B_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},C_{1\leftrightarrow 2},C_1\to -C_1}\mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(A_1B_1C_2+A_2B_1C_2+2A_1B_2C_1-2A_2B_2C_1-A_1B_2C_2-A_2B_2C_2-A_1B_1-A_2B_1-A_1B_2-A_2B_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{B_{1\leftrightarrow 2},A_2\to -A_2,A_{1\leftrightarrow 2},A_2\to -A_2,A\leftrightarrow B}\mathbb{B}_3^{(--)}$
4	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_2\to -A_2,A_1\to -A_1,A\leftrightarrow B,C_2\to -C_2,C_{1\leftrightarrow 2},A_{1\leftrightarrow 2}}\mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{A\leftrightarrow B,C_2\to -C_2}\mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},C_{1\leftrightarrow 2},A_1\to -A_1,A_2\to -A_2}\mathbb{B}_3^{(--)}$	$\mathcal{Q}=\frac{5}{3}$

TABLE XI. Two cases of the (4,2,2) tight inequalities generated from Śliwa_{10}.

Śliwa_{10}	$\frac{1}{4}(A_1B_1+A_2B_1+A_1B_2+A_2B_2+A_1C_1-A_2C_1+B_1C_1+A_1B_1C_1-B_2C_1-A_2B_2C_1+A_1C_2 -A_2C_2-B_1C_2+A_2B_1C_2+B_2C_2-A_1B_2C_2)\leqslant 1$	$\mathcal{Q}=1$
Number	New tight inequalities	Remarks
1	$\mathbb{B}_3^{(++)}\xleftrightarrow{B_{1\leftrightarrow 2},C_{1\leftrightarrow 2}}\mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{A\leftrightarrow B,A\leftrightarrow C,B_1\to -B_1,C_1\to -C_1}\mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{B\leftrightarrow C,A_1\to -A_1}\mathbb{B}_3^{(--)}$	
2	$\frac{1}{4}(A_1B_1C_1D_2+A_2B_1C_2D_2-A_2B_2C_1D_2-A_1B_2C_2D_2+A_1B_1D_1+A_2B_2D_1+A_2B_1+A_1B_2 +A_1C_1D_1-A_2C_2D_1-A_2C_1+A_1C_2-B_1C_2D_1-B_2C_1D_1+B_1C_1+B_2C_2)\leqslant 1$	$\mathcal{Q}=\sqrt{2}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(-A_1B_1C_1-A_2B_1C_2+A_2B_2C_1+A_1B_2C_2+A_1B_1+A_2B_1+A_1B_2+A_2B_2+A_1C_1-A_2C_1+A_1C_2 -A_2C_2+B_1C_1-B_1C_2-B_2C_1+B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{B_{1\leftrightarrow 2},C_{1\leftrightarrow 2}}\mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(A_1B_1C_1+A_2B_1C_2-A_2B_2C_1-A_1B_2C_2-A_1B_1+A_2B_1+A_1B_2-A_2B_2-A_1C_1-A_2C_1+A_1C_2 +A_2C_2+B_1C_1+B_1C_2+B_2C_1+B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A\leftrightarrow C,A_{1\leftrightarrow 2},C_{1\leftrightarrow 2}}\mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(-A_1B_1C_1-A_2B_1C_2+A_2B_2C_1+A_1B_2C_2-A_1B_1+A_2B_1+A_1B_2-A_2B_2-A_1C_1-A_2C_1+A_1C_2 +A_2C_2+B_1C_1+B_1C_2+B_2C_1+B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A\leftrightarrow B,A\leftrightarrow C,A_1\to -A_1}\mathbb{B}_3^{(--)}$

TABLE XII. Two cases of the (4,2,2) tight inequalities generated from Śliwa_{11}.

Śliwa_{11}	$\frac{1}{4}(2A_1B_1+2A_2B_2+A_1B_1C_1+A_2B_1C_1-A_1B_2C_1-A_2B_2C_1+A_1B_1C_2-A_2B_1C_2+A_1B_2C_2 -A_2B_2C_2)\leqslant 1$	$\mathcal{Q}=\sqrt{2}$
Number	New tight inequalities	Remarks
1	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},C_2\to -C_2}\mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},A_1\to -A_1,A_2\to -A_2,C_1\to -C_1}\mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},B_{1\leftrightarrow 2},A_1\to -A_1,A_2\to -A_2}\mathbb{B}_3^{(--)}$	$\mathcal{Q}=\sqrt{2}$
2	$\frac{1}{4}(A_2B_1C_1D_1-A_1B_2C_1D_1-A_2B_1C_2D_1+A_1B_2C_2D_1+A_1B_1C_1-A_2B_2C_1+A_1B_1C_2-A_2B_2C_2 +2A_1B_1D_2+2A_2B_2D_2)\leqslant 1$	$\mathcal{Q}=2$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(A_1B_1C_1+A_2B_1C_1+A_1B_1C_2-A_2B_1C_2-A_1B_2C_1-A_2B_2C_1+A_1B_2C_2-A_2B_2C_2-2A_1B_1 -2A_2B_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},B_{1\leftrightarrow 2},A_1\to -A_1,A_2\to -A_2}\mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(A_1B_1C_1-A_2B_1C_1+A_1B_1C_2+A_2B_1C_2+A_1B_2C_1-A_2B_2C_1-A_1B_2C_2-A_2B_2C_2+2A_1B_1 +2A_2B_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A\leftrightarrow B}\mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(A_1B_1C_1-A_2B_1C_1+A_1B_1C_2+A_2B_1C_2+A_1B_2C_1-A_2B_2C_1-A_1B_2C_2-A_2B_2C_2-2A_1B_1 -2A_2B_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},B_{1\leftrightarrow 2},A_1\to -A_1,B_2\to -B_2}\mathbb{B}_3^{(--)}$

TABLE XIII. Two cases of the (4,2,2) tight inequalities generated from Śliwa_{12}.

Śliwa_{12}	$\frac{1}{4}(2A_1B_1+2A_2B_2+A_1C_1+A_2C_1-B_1C_1+A_2B_1C_1-B_2C_1-A_1B_2C_1+A_1C_2+A_2C_2-B_1C_2 -A_2B_1C_2-B_2C_2+A_1B_2C_2)\leqslant 1$	$\mathcal{Q}=\sqrt{2}$
Number	New tight inequalities	Remarks
1	$\mathbb{B}_3^{(++)}\xleftrightarrow{C_1\to -C_1}\mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{C_2\to -C_2}\mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{A\leftrightarrow B}\mathbb{B}_3^{(--)}$	$\mathcal{Q}=\sqrt{2}$
2	$\frac{1}{4}(-A_1B_2C_1D_1+A_2B_1C_1D_2+A_1B_2C_2D_1-A_2B_1C_2D_2+2A_2B_2D_1+2A_1B_1D_2+A_2C_1D_1+A_1C_1D_2 +A_2C_2D_1+A_1C_2D_2-B_1C_1-B_2C_1-B_1C_2-B_2C_2)\leqslant 1$	$\mathcal{Q}=2$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(-A_2B_1C_1+A_2B_1C_2-A_1B_2C_1+A_1B_2C_2-2A_1B_1+2A_2B_2-A_1C_1+A_2C_1-A_1C_2+A_2C_2-B_1C_1 -B_1C_2-B_2C_1-B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{C_{1\leftrightarrow 2},A_1\to -A_1}\mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(A_2B_1C_1-A_2B_1C_2+A_1B_2C_1-A_1B_2C_2+2A_1B_1-2A_2B_2+A_1C_1-A_2C_1+A_1C_2-A_2C_2-B_1C_1 -B_1C_2-B_2C_1-B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{C_{1\leftrightarrow 2},A_2\to -A_2}\mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(-A_2B_1C_1+A_2B_1C_2+A_1B_2C_1-A_1B_2C_2-2A_1B_1-2A_2B_2-A_1C_1-A_2C_1-A_1C_2-A_2C_2 -B_1C_1-B_1C_2-B_2C_1-B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_1\to -A_1,A_2\to -A_2}\mathbb{B}_3^{(--)}$

TABLE XIV. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{13}$.

Śliwa$_{13}$	$\frac{1}{4}(2A_1B_1 + 2A_2B_1 + A_1B_1C_1 - A_2B_1C_1 + A_1B_2C_1 - A_2B_2C_1 + A_1B_1C_2 - A_2B_1C_2 - A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = \sqrt{2}$
Number	New tight inequalities	Remarks
3	$\frac{1}{4}(A_1B_2C_1D_2 - A_2B_2C_1D_2 - A_1B_2C_2D_2 + A_2B_2C_2D_2 + A_1B_1C_1 - A_2B_1C_1 + A_1B_1C_2 - A_2B_1C_2 + A_1B_1D_1 + A_2B_1D_1 - A_1B_2D_1 - A_2B_2D_1 + A_1B_1 + A_2B_1 + A_1B_2 + A_2B_2) \leqslant 1$	$\mathcal{Q} = 1.58$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(A_1B_1C_1 - A_2B_1C_1 + A_1B_1C_2 - A_2B_1C_2 - A_1B_2C_1 + A_2B_2C_1 + A_1B_2C_2 - A_2B_2C_2 + 2A_1B_1 + 2A_2B_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(A_1B_2C_1 - A_2B_2C_1 - A_1B_2C_2 + A_2B_2C_2 + A_1B_1C_1 - A_2B_1C_1 + A_1B_1C_2 - A_2B_1C_2 + 2A_1B_2 + 2A_2B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, C_2 \to -C_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(-A_1B_2C_1 + A_2B_2C_1 + A_1B_2C_2 - A_2B_2C_2 + A_1B_1C_1 - A_2B_1C_1 + A_1B_1C_2 - A_2B_1C_2 + 2A_1B_2 + 2A_2B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, C_{1\leftrightarrow 2}, C_1 \to -C_1} \mathbb{B}_3^{(--)}$
5	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow C} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow C, A_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \sqrt{2}$

TABLE XV. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{14}$.

Śliwa$_{14}$	$\frac{1}{4}(2A_1B_1 + 2A_2B_1 + A_1C_1 - A_2C_1 + A_1B_2C_1 - A_2B_2C_1 + A_1C_2 - A_2C_2 - A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = \sqrt{2}$
Number	New tight inequalities	Remarks
1	$\frac{1}{8}(A_1B_1C_1D_1 - A_2B_1C_1D_1 + A_1B_2C_1D_1 - A_2B_2C_1D_1 - A_1B_1C_1D_2 + A_2B_1C_1D_2 + A_1B_2C_1D_2 - A_2B_2C_1D_2 - A_1B_1C_2D_1 + A_2B_1C_2D_1 - A_1B_2C_2D_1 + A_2B_2C_2D_1 + A_1B_1C_2D_2 - A_2B_1C_2D_2 - A_1B_2C_2D_2 + A_2B_2C_2D_2 + 2A_1B_1D_1 + 2A_2B_1D_1 - 2A_1B_2D_1 - 2A_2B_2D_1 + 2A_1B_1D_2 + 2A_2B_1D_2 + 2A_1B_2D_2 + 2A_2B_2D_2 + 2A_1C_1 - 2A_2C_1 + 2A_1C_2 - 2A_2C_2) \leqslant 1$	$\mathcal{Q} = 2$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(A_1B_1C_1 - A_2B_1C_1 - A_1B_1C_2 + A_2B_1C_2 - 2A_1B_2 - 2A_2B_2 + A_1C_1 - A_2C_1 + A_1C_2 - A_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, B_2 \to -B_2} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(-A_1B_1C_1 + A_2B_1C_1 + A_1B_1C_2 - A_2B_1C_2 + 2A_1B_2 + 2A_2B_2 + A_1C_1 - A_2C_1 + A_1C_2 - A_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(-A_1B_2C_1 + A_2B_2C_1 + A_1B_2C_2 - A_2B_2C_2 - 2A_1B_1 - 2A_2B_1 + A_1C_1 - A_2C_1 + A_1C_2 - A_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}, B_1 \to -B_1} \mathbb{B}_3^{(--)}$
3	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, C_{1\leftrightarrow 2}, B_2 \to -B_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}, B_1 \to -B_1} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \sqrt{2}$

TABLE XVI. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{15}$.

Śliwa$_{15}$	$\frac{1}{4}(2A_1B_1 + 2A_2B_1 + A_1C_1 + A_2C_1 - 2B_1C_1 + A_1B_2C_1 - A_2B_2C_1 + A_1C_2 + A_2C_2 - 2B_1C_2 - A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.5$
Number	New tight inequalities	Remarks
1	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow C, A_{1\leftrightarrow 2}, B_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow C, A_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 2\sqrt{2} - 1$
2	$\frac{1}{8}(A_1B_1C_1D_1 - A_2B_1C_1D_1 + A_1B_2C_1D_1 - A_2B_2C_1D_1 - A_1B_1C_1D_2 + A_2B_1C_1D_2 + A_1B_2C_1D_2 - A_2B_2C_1D_2 - A_1B_1C_2D_1 + A_2B_1C_2D_1 - A_1B_2C_2D_1 + A_2B_2C_2D_1 + A_1B_1C_2D_2 - A_2B_1C_2D_2 - A_1B_2C_2D_2 + A_2B_2C_2D_2 + 2A_1B_1D_1 + 2A_2B_1D_1 - 2A_1B_2D_1 - 2A_2B_2D_1 + 2A_1B_1D_2 + 2A_2B_1D_2 + 2A_1B_2D_2 + 2A_2B_2D_2 + 2A_1C_1 + 2A_2C_1 + 2A_1C_2 + 2A_2C_2 - 2B_1C_1D_1 + 2B_2C_1D_1 - 2B_1C_1D_2 - 2B_2C_1D_2 - 2B_1C_2D_1 + 2B_2C_2D_1 - 2B_1C_2D_2 - 2B_2C_2D_2) \leqslant 1$	$\mathcal{Q} = 3\sqrt{2} - 2$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(A_1B_1C_1 - A_2B_1C_1 - A_1B_1C_2 + A_2B_1C_2 - 2A_1B_2 - 2A_2B_2 + A_1C_1 + A_2C_1 + A_1C_2 + A_2C_2 + 2B_2C_1 + 2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow C, B_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(-A_1B_1C_1 + A_2B_1C_1 + A_1B_1C_2 - A_2B_1C_2 + 2A_1B_2 + 2A_2B_2 + A_1C_1 + A_2C_1 + A_1C_2 + A_2C_2 - 2B_2C_1 - 2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}, B_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(-A_1B_2C_1 + A_2B_2C_1 + A_1B_2C_2 - A_2B_2C_2 - 2A_1B_1 - 2A_2B_1 + A_1C_1 + A_2C_1 + A_1C_2 + A_2C_2 + 2B_1C_1 + 2B_1C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow C, A_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$

TABLE XVII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{16}$.

Śliwa$_{16}$	$\frac{1}{4}(A_1+A_2+A_1B_1+A_2B_1+A_1C_1+A_2C_1-2A_2B_1C_1+A_1B_2C_1 - A_2B_2C_1+A_1B_1C_2-A_2B_1C_2-A_1B_2C_2+A_2B_2C_2)\leqslant 1$	$\mathcal{Q}=1.53$
Number	New tight inequalities	Remarks
3	$\frac{1}{8}(-A_1B_1C_1D_1-A_2B_1C_1D_1+2A_1B_2C_1D_1-2A_2B_2C_1D_1-A_1B_1C_1D_2 - A_2B_1C_1D_2-3A_1B_2C_2D_1+A_2B_2C_2D_1+A_1B_2C_2D_2+A_2B_2C_2D_2 + 2A_1B_1C_1-2A_2B_1C_1+2A_1B_1C_2-2A_2B_1C_2+A_1B_1D_1+A_2B_1D_1 - A_1B_2D_1-A_2B_2D_1+A_1B_1D_2+A_2B_1D_2+A_1B_2D_2+A_2B_2D_2 + A_1C_1D_1+A_2C_1D_1+A_1C_1D_2+A_2C_1D_2-A_1C_2D_1-A_2C_2D_1 + A_1C_2D_2+A_2C_2D_2+2A_1D_1+2A_2D_1)\leqslant 1$	$\mathcal{Q}=\frac{5}{3}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(A_1B_1C_1+A_1B_2C_1+A_1B_1C_2-2A_1B_2C_2-A_2B_1C_1-A_2B_2C_1 - A_2B_1C_2-A_1B_2-A_2B_2-A_1C_2-A_2C_2+A_1+A_2)\leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1,B_{1\leftrightarrow 2},C_1\to -C_1,A_{1\leftrightarrow 2},C_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(A_1B_1C_1-A_1B_2C_1+A_1B_1C_2+2A_1B_2C_2-A_2B_1C_1+A_2B_2C_1 - A_2B_1C_2+A_1B_2+A_2B_2+A_1C_2+A_2C_2-A_1-A_2)\leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_2\to -B_2,C_1\to -C_1,A_1\to -A_1,A_{1\leftrightarrow 2}A_1\to -A_1,B_{1\leftrightarrow 2},C_{1\leftrightarrow 2},B_2\to -B_2}$ $\mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(2A_1B_1C_1-A_1B_2C_1+A_1B_1C_2+A_1B_2C_2+A_2B_2C_1-A_2B_1C_2 - A_2B_2C_2-A_1B_1-A_2B_1-A_1C_1-A_2C_1-A_1-A_2)\leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2},A_1\to -A_1,A_2\to -A_2,B_2\to -B_2} \mathbb{B}_3^{(--)}$
5	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1,C_{1\leftrightarrow 2},B_{1\leftrightarrow 2},B_2\to -B_2,A_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2},C_{1\leftrightarrow 2},B_2\to -B_2,C_2\to -C_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q}=3\sqrt{\frac{3}{11}}$

TABLE XVIII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{17}$.

Śliwa$_{17}$	$\frac{1}{4}(A_1+A_2+A_1B_1+A_2B_1+A_1C_1+A_2C_1-A_1B_1C_1-A_2B_1C_1+2A_1B_2C_2-2A_2B_2C_2)\leqslant 1$	$\mathcal{Q}=\sqrt{2}$
Number	New tight inequalities	Remarks
2	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2},C_{1\leftrightarrow 2},C_2\to -C_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2},C_1\to -C_1} \mathbb{B}_3^{(--)}$	$\mathcal{Q}=1.54$
5	$\frac{1}{8}(-A_1B_1C_1D_1-A_1B_2C_1D_1-2A_1B_1C_2D_1+2A_1B_2C_2D_1-A_1B_1C_1D_2+A_1B_2C_1D_2+2A_1B_1C_2D_2 + 2A_1B_2C_2D_2-A_2B_1C_1D_1-A_2B_2C_1D_1+2A_2B_1C_2D_1-2A_2B_2C_2D_1-A_2B_1C_1D_2+A_2B_2C_1D_2 - 2A_2B_1C_2D_2-2A_2B_2C_2D_2+A_1B_1D_1+A_1B_2D_1+A_1B_1D_2-A_1B_2D_2+A_2B_1D_1+A_2B_2D_1 + A_2B_1D_2-A_2B_2D_2+2A_1C_1+2A_2C_1+2A_1+2A_2)\leqslant 1$	$\mathcal{Q}=3\sqrt{2}-2$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(-A_1B_2C_1-2A_1B_1C_2-A_2B_2C_1+2A_2B_1C_2+A_1B_2+A_2B_2+A_1C_1+A_2C_1+A_1+A_2)\leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2},B_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(A_1B_2C_1+2A_1B_1C_2+A_2B_2C_1-2A_2B_1C_2-A_1B_2-A_2B_2+A_1C_1+A_2C_1+A_1+A_2)\leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2},B_2\to -B_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(A_1B_1C_1-2A_1B_2C_2+A_2B_1C_1+2A_2B_2C_2-A_1B_1-A_2B_1+A_1C_1+A_2C_1+A_1+A_2)\leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2},B_1\to -B_1} \mathbb{B}_3^{(--)}$

TABLE XIX. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{18}$.

Śliwa$_{18}$	$\frac{1}{4}(A_1+A_2+A_1B_1+A_2B_1+A_1C_1+A_2C_1-2B_1C_1+A_1B_2C_1-A_2B_2C_1+A_1B_1C_2-A_2B_1C_2 + 2B_2C_2-A_1B_2C_2-A_2B_2C_2)\leqslant 1$	$\mathcal{Q}=\frac{7-\sqrt{17}}{2}$
Number	New tight inequalities	Remarks
8	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2},B_1\to -B_1,C_1\to -C_1} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_2\to -B_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1,C_1\to -C_1,C_2\to -C_2} \mathbb{B}_3^{(--)}$	
11	$\frac{1}{4}(A_1B_2C_1D_1-A_1B_2C_2D_1+A_1B_1C_2D_2-A_2B_2C_1D_1-A_2B_2C_2D_1-A_2B_1C_2D_2+A_1B_1+A_2B_1 + A_1C_1D_2+A_2C_1D_2+A_1+A_2+2B_2C_2D_1-2B_1C_1D_2)\leqslant 1$	$\mathcal{Q}=1.5$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(A_1B_2C_1-A_1B_1C_2-A_1B_2C_2-A_2B_2C_1+A_2B_1C_2-A_2B_2C_2+A_1B_1+A_2B_1-A_1C_1-A_2C_1+A_1 + A_2+2B_1C_1+2B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2},C_1\to -C_1} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(-A_1B_2C_1+A_1B_1C_2+A_1B_2C_2+A_2B_2C_1-A_2B_1C_2+A_2B_2C_2+A_1B_1+A_2B_1+A_1C_1+A_2C_1+A_1 + A_2-2B_1C_1-2B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_2\to -B_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(-A_1B_2C_1-A_1B_1C_2+A_1B_2C_2+A_2B_2C_1+A_2B_1C_2+A_2B_2C_2+A_1B_1+A_2B_1-A_1C_1-A_2C_1+A_1 + A_2+2B_1C_1-2B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1,C_2\to -C_2} \mathbb{B}_3^{(--)}$

TABLE XX. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{19}$.

Śliwa$_{19}$	$\frac{1}{4}(A_1 + A_2 + A_1B_1 + A_2B_1 + A_1C_1 + A_2C_1 - 2B_1C_1 + 2B_2C_1 - A_1B_2C_1 - A_2B_2C_1 + A_1B_1C_2 - A_2B_1C_2 + A_1B_2C_2 - A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.45$
Number	New tight inequalities	Remarks
4	$\frac{1}{4}(A_1B_2C_1D_1 - A_1B_2C_2D_1 + A_1B_1C_2D_2 - A_2B_2C_1D_1 - A_2B_2C_2D_1 - A_2B_1C_2D_2 + A_1B_1 + A_2B_1 + A_1C_1D_2 + A_2C_1D_2 + A_1 + A_2 + 2B_2C_2D_1 - 2B_1C_1D_2) \leqslant 1$	$\mathcal{Q} = 1.5$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(A_1B_2C_1 - A_1B_1C_2 - A_1B_2C_2 - A_2B_2C_1 + A_2B_1C_2 - A_2B_2C_2 + A_1B_1 + A_2B_1 - A_1C_1 - A_2C_1 + A_1 + A_2 + 2B_1C_1 + 2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B\leftrightarrow C, B_{1\leftrightarrow 2}, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(-A_1B_2C_1 + A_1B_1C_2 + A_1B_2C_2 + A_2B_2C_1 - A_2B_1C_2 + A_2B_2C_2 + A_1B_1 + A_2B_1 + A_1C_1 + A_2C_1 + A_1 + A_2 - 2B_1C_1 - 2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B\leftrightarrow C, A_{1\leftrightarrow 2}, C_2\rightarrow -C_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(-A_1B_2C_1 - A_1B_1C_2 + A_1B_2C_2 + A_2B_2C_1 + A_2B_1C_2 + A_2B_2C_2 + A_1B_1 + A_2B_1 - A_1C_1 - A_2C_1 + A_1 + A_2 + 2B_1C_1 - 2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}, B_{1\leftrightarrow 2}, C_{1\leftrightarrow 2}, B_1\rightarrow -B_1} \mathbb{B}_3^{(--)}$
17	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, A_2\rightarrow -A_2, B_1\rightarrow -B_1, B_2\rightarrow -B_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, A_1\rightarrow -A_1, B_1\rightarrow -B_1, B_2\rightarrow -B_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\rightarrow -A_1, A_2\rightarrow -A_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{5}{3}$

TABLE XXI. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{20}$.

Śliwa$_{20}$	$\frac{1}{4}(A_1 + A_2 + A_1B_1 - A_2B_1 + A_1B_2 - A_2B_2 + A_1C_1 - A_2C_1 - B_1C_1 + A_1B_1C_1 + A_2B_1C_1 - B_2C_1 + A_1B_2C_1 + A_2B_2C_1 + B_1C_2 - A_1B_1C_2 - A_2B_1C_2 - B_2C_2 + A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = \frac{3\sqrt{2}-1}{2}$
Number	New tight inequalities	Remarks
4	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\rightarrow -A_1} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_2\rightarrow -A_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\rightarrow -A_1, A_2\rightarrow -A_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{3\sqrt{2}-1}{2}$
5	$\frac{1}{4}(A_2B_1C_1D_1 + A_2B_2C_1D_1 + A_1B_1C_1D_2 + A_1B_2C_1D_2 - A_2B_1C_2D_1 + A_2B_2C_2D_1 - A_1B_1C_2D_2 + A_1B_2C_2D_2 + A_1B_1D_1 + A_1B_2D_1 - A_2B_1D_2 - A_2B_2D_2 + A_1C_1D_1 - A_2C_1D_2 + A_2D_1 + A_1D_2 - B_1C_1 - B_2C_1 + B_1C_2 - B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.84$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(-A_1B_1C_1 - A_1B_2C_1 + A_1B_1C_2 - A_1B_2C_2 + A_2B_1C_1 + A_2B_2C_1 - A_2B_1C_2 + A_2B_2C_2 + A_1B_1 + A_1B_2 + A_2B_1 + A_2B_2 + A_1C_1 + A_2C_1 - A_1 + A_2 - B_1C_1 - B_2C_1 + B_1C_2 - B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}, A_1\rightarrow -A_1} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(A_1B_1C_1 + A_1B_2C_1 - A_1B_1C_2 + A_1B_2C_2 - A_2B_1C_1 - A_2B_2C_1 + A_2B_1C_2 - A_2B_2C_2 - A_1B_1 - A_1B_2 - A_2B_1 - A_2B_2 - A_1C_1 - A_2C_1 + A_1 - A_2 - B_1C_1 - B_2C_1 + B_1C_2 - B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}, A_2\rightarrow -A_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(-A_1B_1C_1 - A_1B_2C_1 + A_1B_1C_2 - A_1B_2C_2 - A_2B_1C_1 - A_2B_2C_1 + A_2B_1C_2 - A_2B_2C_2 - A_1B_1 - A_1B_2 + A_2B_1 + A_2B_2 - A_1C_1 + A_2C_1 - A_1 - A_2 - B_1C_1 - B_2C_1 + B_1C_2 - B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\rightarrow -A_1, A_2\rightarrow -A_2} \mathbb{B}_3^{(--)}$

TABLE XXII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{21}$.

Śliwa$_{21}$	$\frac{1}{4}(A_1 + A_2 + B_1 + A_1B_1 + B_2 - A_2B_2 + A_1C_1 + A_2C_1 + B_1C_1 - 2A_1B_1C_1 - A_2B_1C_1 + B_2C_1 - A_1B_2C_1 + A_1B_1C_2 - A_2B_1C_2 - A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.49$
Number	New tight inequalities	Remarks
2	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}, C_1\rightarrow -C_1, C_2\rightarrow -C_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\rightarrow -C_1, C_2\rightarrow -C_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 2\sqrt{2} - 1$
3	$\frac{1}{8}(-A_1B_1C_1D_1 - 2A_1B_2C_1D_1 + 3A_1B_1C_2D_1 - 3A_1B_1C_1D_2 - A_1B_1C_2D_2 - 2A_1B_2C_2D_2 - 2A_2B_1C_1D_1 + A_2B_2C_1D_1 + A_2B_2C_2D_1 - A_2B_2C_1D_2 - 2A_2B_1C_2D_2 + A_2B_2C_2D_2 + 2A_1B_1 - 2A_2B_2 + A_1C_1D_1 - A_1C_2D_1 + A_1C_1D_2 + A_1C_2D_2 + A_2C_1D_1 - A_2C_2D_1 + A_2C_1D_2 + A_2C_2D_2 + 2A_1 + 2A_2 + B_1C_1D_1 + B_2C_1D_1 - B_1C_2D_1 - B_2C_2D_1 + B_1C_1D_2 + B_2C_1D_2 + B_1C_2D_2 + B_2C_2D_2 + 2B_1 + 2B_2) \leqslant 1$	$\mathcal{Q} = 2.03$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(A_1B_1C_1 - A_1B_2C_1 + 2A_1B_1C_2 + A_1B_2C_2 - A_2B_1C_1 + A_2B_2C_1 + A_2B_1C_2 + A_1B_1 - A_2B_2 - A_1C_2 - A_2C_2 + A_1 + A_2 - B_1C_2 - B_2C_2 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}, C_2\rightarrow -C_2} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(-A_1B_1C_1 + A_1B_2C_1 - 2A_1B_1C_2 - A_1B_2C_2 + A_2B_1C_1 - A_2B_2C_1 - A_2B_1C_2 + A_1B_1 - A_2B_2 + A_1C_2 + A_2C_2 + A_1 + A_2 + B_1C_2 + B_2C_2 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}, C_1\rightarrow -C_1} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(2A_1B_1C_1 + A_1B_2C_1 - A_1B_1C_2 + A_1B_2C_2 + A_2B_1C_1 + A_2B_1C_2 - A_2B_2C_2 + A_1B_1 - A_2B_2 - A_1C_1 - A_2C_1 + A_1 + A_2 - B_1C_1 - B_2C_1 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\rightarrow -C_1, C_2\rightarrow -C_2} \mathbb{B}_3^{(--)}$

TABLE XXIII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{22}$.

Śliwa$_{22}$	$\frac{1}{4}(A_1 + A_2 + B_1 + A_1B_1 + B_2 - A_2B_2 + C_1 + A_1C_1 + B_1C_1 - 2A_1B_1C_1 - A_2B_1C_1 - A_1B_2C_1 + A_2B_2C_1 + C_2 - A_2C_2 - A_1B_1C_2 + A_2B_1C_2 - B_2C_2 + A_1B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.55$
Number	New tight inequalities	Remarks
2	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1 \to -C_1} \mathbb{B}_3^{(+-)}, \quad \mathbb{B}_3^{(++)} \xleftrightarrow{C_2 \to -C_2} \mathbb{B}_3^{(-+)}, \quad \mathbb{B}_3^{(++)} \xleftrightarrow{C_1 \to -C_1, C_2 \to -C_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 1.55$
5	$\frac{1}{8}(-A_1B_1C_1D_1 - 2A_1B_2C_1D_1 - 3A_1B_1C_2D_1 - 3A_1B_1C_1D_2 + A_1B_1C_2D_2 + 2A_1B_2C_2D_2 - 2A_2B_1C_1D_1 + A_2B_2C_1D_1 + A_2B_2C_2D_1 + A_2B_2C_1D_2 + 2A_2B_1C_2D_2 - A_2B_2C_2D_2 + 2A_1B_1 - 2A_2B_2 + A_1C_1D_1 + A_1C_2D_1 + A_1C_1D_2 - A_1C_2D_2 + A_2C_1D_1 - A_2C_2D_1 - A_2C_1D_2 - A_2C_2D_2 + 2A_1 + 2A_2 + B_1C_1D_1 + B_2C_1D_1 + B_1C_2D_1 - B_2C_2D_1 + B_1C_1D_2 - B_2C_1D_2 - B_1C_2D_2 - B_2C_2D_2 + 2B_1 + 2B_2 + 2C_2D_1 + 2C_1D_2) \leqslant 1$	$\mathcal{Q} = 2.22$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(A_1B_1C_1 - A_1B_2C_1 - 2A_1B_1C_2 - A_1B_2C_2 - A_2B_1C_1 - A_2B_1C_2 + A_2B_2C_2 + A_1B_1 - A_2B_2 + A_1C_2 + A_2C_1 + A_1 + A_2 + B_2C_1 + B_1C_2 + B_1 + B_2 - C_1 + C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}, C_1 \to -C_1} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(-A_1B_1C_1 + A_1B_2C_1 + 2A_1B_1C_2 + A_1B_2C_2 + A_2B_1C_1 + A_2B_1C_2 - A_2B_2C_2 + A_1B_1 - A_2B_2 - A_1C_2 - A_2C_1 + A_1 + A_2 - B_2C_1 - B_1C_2 + B_1 + B_2 + C_1 - C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}, C_2 \to -C_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(2A_1B_1C_1 + A_1B_2C_1 + A_1B_1C_2 - A_1B_2C_2 + A_2B_1C_1 - A_2B_2C_1 - A_2B_1C_2 + A_1B_1 - A_2B_2 - A_1C_1 + A_2C_2 + A_1 + A_2 - B_1C_1 + B_2C_2 + B_1 + B_2 - C_1 - C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1 \to -C_1, C_2 \to -C_2} \mathbb{B}_3^{(--)}$

TABLE XXIV. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{23}$.

Śliwa$_{23}$	$\frac{1}{4}(A_1 + A_2 + B_1 - A_1B_1 - A_2B_1 + B_2 - A_1B_2 - A_2B_2 + A_1C_1 - A_2C_1 - A_1B_1C_1 + A_2B_1C_1 - A_1B_2C_1 + A_2B_2C_1 + B_1C_2 - A_1B_1C_2 - A_2B_1C_2 - B_2C_2 + A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = \frac{3\sqrt{17}-3}{8}$
Number	New tight inequalities	Remarks
4	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}, \mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}, \mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}, B_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{3\sqrt{17}-3}{8}$
5	$\frac{1}{4}(-A_1B_1C_1D_1 + A_1B_2C_2D_1 - A_1B_2C_1D_2 - A_1B_1C_2D_2 + A_2B_1C_1D_1 + A_2B_2C_2D_1 + A_2B_2C_1D_2 - A_2B_1C_2D_2 - A_1B_1D_1 - A_1B_2D_2 - A_2B_1D_1 - A_2B_2D_2 + A_1C_1 - A_2C_1 + A_1 + A_2 - B_2C_2D_1 + B_1C_2D_2 + B_1D_1 + B_2D_2) \leqslant 1$	$\mathcal{Q} = \frac{\sqrt{10}+7}{6}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{4}(-A_1B_1C_1 + A_1B_2C_1 + A_1B_1C_2 + A_1B_2C_2 + A_2B_1C_1 - A_2B_2C_1 + A_2B_1C_2 + A_2B_2C_2 - A_1B_1 + A_1B_2 - A_2B_1 + A_2B_2 + A_1C_1 - A_2C_1 + A_1 + A_2 - B_1C_2 - B_2C_2 + B_1 - B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, B_2 \to -B_2} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{4}(A_1B_1C_1 - A_1B_2C_1 - A_1B_1C_2 - A_1B_2C_2 - A_2B_1C_1 + A_2B_2C_1 - A_2B_1C_2 - A_2B_2C_2 + A_1B_1 - A_1B_2 + A_2B_1 - A_2B_2 + A_1C_1 - A_2C_1 + A_1 + A_2 + B_1C_2 + B_2C_2 - B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, B_1 \to -B_1} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{4}(A_1B_1C_1 + A_1B_2C_1 + A_1B_1C_2 - A_1B_2C_2 - A_2B_1C_1 - A_2B_2C_1 + A_2B_1C_2 - A_2B_2C_2 + A_1B_1 + A_1B_2 + A_2B_1 + A_2B_2 + A_1C_1 - A_2C_1 + A_1 + A_2 - B_1C_2 + B_2C_2 - B_1 - B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1 \to -B_1, B_2 \to -B_2} \mathbb{B}_3^{(--)}$

TABLE XXV. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{24}$.

Śliwa$_{24}$	$\frac{1}{5}(A_1 + B_1 + A_2B_1 + A_1B_2 + A_2B_2 + C_1 + A_2C_1 - B_1C_1 + 2A_1B_1C_1 - A_2B_1C_1 - 2A_2B_2C_1 + A_1C_2 + A_2C_2 - 2A_2B_1C_2 - A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.588$
Number	New tight inequalities	Remarks
4	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}, \mathbb{B}_3^{(++)} \xleftrightarrow{A \leftrightarrow B} \mathbb{B}_3^{(-+)}, \mathbb{B}_3^{(++)} \xleftrightarrow{A \leftrightarrow B, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 1.588$
11	$\frac{1}{5}(-A_2B_1C_1D_1 + 2A_1B_1C_1D_2 - 2A_2B_1C_2D_2 - 2A_2B_2C_1D_1 - A_1B_2C_2D_1 + A_2B_2C_2D_2 + A_2B_1D_1 + A_2B_2D_1 + A_1B_2D_2 + A_2C_1D_1 + A_1C_2D_1 + A_2C_2D_2 + A_1D_2 - B_1C_1 + B_1 + C_1) \leqslant 1$	$\mathcal{Q} = \frac{9}{5}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{5}(-2A_1B_1C_1 - A_1B_2C_2 - A_2B_1C_1 - 2A_2B_2C_1 + 2A_2B_1C_2 - A_2B_2C_2 - A_1B_2 + A_2B_1 + A_2B_2 + A_1C_2 + A_2C_1 - A_2C_2 - A_1 - B_1C_1 + B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1 \to -A_1, C_2 \to -C_2} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{5}(2A_1B_1C_1 + A_1B_2C_2 + A_2B_1C_1 + 2A_2B_2C_1 - 2A_2B_1C_2 + A_2B_2C_2 + A_1B_2 - A_2B_1 - A_2B_2 - A_1C_2 - A_2C_1 + A_2C_2 + A_1 - B_1C_1 + B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_2 \to -A_2, C_2 \to -C_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{5}(-2A_1B_1C_1 + A_1B_2C_2 + A_2B_1C_1 + 2A_2B_2C_1 + 2A_2B_1C_2 - A_2B_2C_2 - A_1B_2 - A_2B_1 - A_2B_2 - A_1C_2 - A_2C_1 - A_2C_2 - A_1 - B_1C_1 + B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1 \to -A_1, A_2 \to -A_2} \mathbb{B}_3^{(--)}$

TABLE XXVI. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{25}$.

Śliwa$_{25}$	$\frac{1}{5}(A_1 + B_1 + A_2B_1 + A_1B_2 + A_2B_2 + C_1 + A_2C_1 - B_1C_1 + 2A_1B_1C_1 - A_2B_1C_1 - 2A_2B_2C_1 + A_1C_2 + A_2C_2 - 2A_1B_1C_2 + A_1B_2C_2 - A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.36$
Number	New tight inequalities	Remarks
2	$\frac{1}{10}(2A_1B_1C_1D_1 + 2A_1B_2C_1D_1 - A_1B_1C_2D_1 - A_1B_2C_2D_1 + 2A_1B_1C_1D_2 - 2A_1B_2C_1D_2 - 3A_1B_1C_2D_2 + 3A_1B_2C_2D_2 - 3A_2B_1C_1D_1 - 3A_2B_2C_1D_1 - A_2B_1C_2D_1 - A_2B_2C_2D_1 + A_2B_1C_1D_2 - A_2B_2C_1D_2 + A_2B_1C_2D_2 - A_2B_2C_2D_2 + A_1B_1D_1 + A_1B_2D_1 - A_1B_1D_2 + A_1B_2D_2 + 2A_2B_1D_1 + 2A_2B_2D_1 + 2A_1C_2 + 2A_2C_1 + 2A_2C_2 + 2A_1 - B_1C_1D_1 - B_2C_1D_1 - B_1C_1D_2 + B_2C_1D_2 + B_1D_1 + B_2D_1 + B_1D_2 - B_2D_2 + 2C_1) \leqslant 1$	$\mathcal{Q} = \frac{8\sqrt{2}-3}{5}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{5}(2A_1B_2C_1 + A_1B_1C_2 - 2A_1B_2C_2 - 2A_2B_1C_1 - A_2B_2C_1 - A_2B_1C_2 + A_1B_1 + A_2B_1 + A_2B_2 + A_1C_2 + A_2C_1 + A_2C_2 + A_1 - B_2C_1 + B_2 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{5}(-2A_1B_2C_1 - A_1B_1C_2 + 2A_1B_2C_2 + 2A_2B_1C_1 + A_2B_2C_1 + A_2B_1C_2 - A_1B_1 - A_2B_1 - A_2B_2 + A_1C_2 + A_2C_1 + A_2C_2 + A_1 + B_2C_1 - B_2 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, B_1 \to -B_1, B_2 \to -B_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{5}(-2A_1B_1C_1 + 2A_1B_1C_2 - A_1B_2C_2 + A_2B_1C_1 + 2A_2B_2C_1 + A_2B_2C_2 - A_1B_2 - A_2B_1 - A_2B_2 + A_1C_2 + A_2C_1 + A_2C_2 + A_1 + B_1C_1 - B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1 \to -B_1, B_2 \to -B_2} \mathbb{B}_3^{(--)}$
3	$\mathbb{B}_3^{(++)} \xleftrightarrow{B\leftrightarrow C, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B\leftrightarrow C, C_{1\leftrightarrow 2}, C_1 \to -C_1, C_2 \to -C_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_1 \to -C_1, C_2 \to -C_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{15}{23}$

TABLE XXVII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{26}$.

Śliwa$_{26}$	$\frac{1}{5}(A_1 + B_1 + A_1B_1 + 2A_2B_2 + C_1 + A_1C_1 + B_1C_1 - A_1B_1C_1 - 2A_2B_2C_1 + 2A_2C_2 - 2A_2B_1C_2 - 2B_2C_2 + 2A_1B_2C_2) \leqslant 1$	$\mathcal{Q} = \frac{4\sqrt{3}+1}{5}$
Number	New tight inequalities	Remarks
1	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1 \to -B_1} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow C} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow C, B_1 \to -B_1} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{4\sqrt{3}+1}{5}$
3	$\frac{1}{10}(-A_1B_1C_1D_1 - A_2B_1C_1D_1 + 2A_1B_1C_2D_1 - 2A_2B_1C_2D_1 - A_1B_1C_1D_2 + A_2B_1C_1D_2 - 2A_1B_1C_2D_2 - 2A_2B_1C_2D_2 + 2A_1B_2C_1D_1 - 2A_2B_2C_1D_1 + 2A_1B_2C_2D_1 + 2A_2B_2C_2D_1 - 2A_1B_2C_1D_2 - 2A_2B_2C_1D_2 + 2A_1B_2C_2D_2 - 2A_2B_2C_2D_2 + A_1B_1D_1 + A_2B_1D_1 + A_1B_1D_2 - A_2B_1D_2 - 2A_1B_2D_1 + 2A_2B_2D_1 + 2A_1B_2D_2 + 2A_2B_2D_2 + A_1C_1D_1 + A_2C_1D_1 - 2A_1C_2D_1 + 2A_2C_2D_1 + A_1C_1D_2 - A_2C_1D_2 + 2A_1C_2D_2 + 2A_2C_2D_2 + A_1D_1 + A_2D_1 + A_1D_2 - A_2D_2 + 2B_1C_1 - 4B_2C_2 + 2B_1 + 2C_1) \leqslant 1$	$\mathcal{Q} = \frac{8\sqrt{2}-3}{5}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{5}(-A_2B_1C_1 + 2A_2B_2C_2 + 2A_1B_2C_1 + 2A_1B_1C_2 + A_2B_1 - 2A_1B_2 + A_2C_1 - 2A_1C_2 + A_2 + B_1C_1 - 2B_2C_2 + B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}, A_1 \to -A_1} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{5}(A_2B_1C_1 - 2A_2B_2C_2 - 2A_1B_2C_1 - 2A_1B_1C_2 - A_2B_1 + 2A_1B_2 - A_2C_1 + 2A_1C_2 - A_2 + B_1C_1 - 2B_2C_2 + B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}, A_2 \to -A_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{5}(A_1B_1C_1 - 2A_1B_2C_2 + 2A_2B_2C_1 + 2A_2B_1C_2 - A_1B_1 - 2A_2B_2 - A_1C_1 - 2A_2C_2 - A_1 + B_1C_1 - 2B_2C_2 + B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1 \to -A_1, A_2 \to -A_2} \mathbb{B}_3^{(--)}$

TABLE XXVIII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{27}$.

Śliwa$_{27}$	$\frac{1}{5}(2A_1 + A_2 + B_1 - A_1B_1 + A_1B_2 + A_2B_2 + C_1 - A_1C_1 + 2A_1B_1C_1 - 2A_2B_1C_1 + B_2C_1 - A_1B_2C_1 + A_1C_2 + A_2C_2 + B_1C_2 - A_1B_1C_2 + B_2C_2 - 2A_1B_2C_2 - A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.39$
Number	New tight inequalities	Remarks
5	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}, C_1 \to -C_1, C_2 \to -C_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_1 \to -C_1, C_2 \to -C_2} \mathbb{B}_3^{(--)}$	
6	$\frac{1}{10}(A_1B_1C_1D_1 + A_1B_2C_1D_1 - 3A_1B_1C_2D_1 - 3A_1B_2C_2D_1 + 3A_1B_1C_1D_2 - 3A_1B_2C_1D_2 + A_1B_1C_2D_2 - A_1B_2C_2D_2 - 2A_2B_1C_1D_1 - 2A_2B_2C_1D_1 - A_2B_1C_2D_1 - A_2B_2C_2D_1 - 2A_2B_1C_1D_2 + 2A_2B_2C_1D_2 + A_2B_1C_2D_2 - A_2B_2C_2D_2 - 2A_1B_1D_2 + 2A_1B_2D_2 + A_2B_1D_1 + A_2B_2D_1 - A_2B_1D_2 + A_2B_2D_2 - 2A_1C_1 + 2A_1C_2 + 2A_2C_2 + 4A_1 + 2A_2 + B_1C_1D_1 + B_2C_1D_1 + 2B_1C_2D_1 + 2B_2C_2D_1 - B_1C_1D_2 + B_2C_1D_2 + B_1D_1 + B_2D_1 + B_1D_2 - B_2D_2 + 2C_1) \leqslant 1$	$\mathcal{Q} = \frac{8\sqrt{2}-3}{5}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{5}(-A_1B_1C_1 + 2A_1B_2C_1 - 2A_1B_1C_2 - A_1B_2C_2 - 2A_2B_2C_1 - A_2B_1C_2 + A_1B_1 - A_1B_2 + A_2B_1 - A_1C_1 + A_1C_2 + A_2C_2 + 2A_1 + A_2 + B_1C_1 + B_1C_2 + B_2C_2 + B_2 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{5}(A_1B_1C_1 - 2A_1B_2C_1 + 2A_1B_1C_2 + A_1B_2C_2 + 2A_2B_2C_1 + A_2B_1C_2 - A_1B_1 + A_1B_2 - A_2B_1 - A_1C_1 + A_1C_2 + A_2C_2 + 2A_1 + A_2 - B_1C_1 - B_1C_2 - B_2C_2 - B_2 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, B_1 \to -B_1, B_2 \to -B_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{5}(-2A_1B_1C_1 + A_1B_2C_1 + A_1B_1C_2 + 2A_1B_2C_2 + 2A_2B_1C_1 + A_2B_2C_2 + A_1B_1 - A_1B_2 - A_2B_2 - A_1C_1 + A_1C_2 + A_2C_2 + 2A_1 + A_2 - B_2C_1 - B_1C_2 - B_2C_2 - B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1 \to -B_1, B_2 \to -B_2} \mathbb{B}_3^{(--)}$

TABLE XXIX. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{28}$.

Śliwa$_{28}$	$\frac{1}{6}(A_1+A_2+A_1B_1-A_2B_1+A_1C_1-A_2C_1-B_1C_1+2A_1B_1C_1+A_2B_1C_1+B_2C_1-A_1B_2C_1-2A_2B_2C_1+B_1C_2-A_1B_1C_2-2A_2B_1C_2+B_2C_2-3A_1B_2C_2)\leqslant 1$	$\mathcal{Q}=1.65$
Number	New tight inequalities	Remarks
10	$\mathbb{B}_3^{(++)}\xleftrightarrow{B_1\to -B_1,B_2\to -B_2,C_2\to -C_2}\mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{C_2\to -C_2}\mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{B_1\to -B_1,B_2\to -B_2}\mathbb{B}_3^{(--)}$	$\mathcal{Q}=\frac{\sqrt{65}+13}{12}$
14	$\frac{1}{6}(2A_1B_1C_1D_1-3A_1B_2C_2D_1-A_1B_2C_1D_2-A_1B_1C_2D_2+A_2B_1C_1D_1-2A_2B_2C_1D_2-2A_2B_1C_2D_2+A_1B_1-A_2B_1+A_1C_1D_1-A_2C_1D_1+A_1+A_2-B_1C_1D_1+B_2C_2D_1+B_2C_1D_2+B_1C_2D_2)\leqslant 1$	$\mathcal{Q}=\frac{7}{3}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(2A_1B_1C_1D_1-3A_1B_2C_2D_1-A_1B_2C_1D_2-A_1B_1C_2D_2+A_2B_1C_1D_1-2A_2B_2C_1D_2-2A_2B_1C_2D_2+A_1B_1-A_2B_1+A_1C_1D_1-A_2C_1D_1+A_1+A_2-B_1C_1D_1+B_2C_2D_1+B_2C_1D_2+B_1C_2D_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{B_2\to -B_2,C_2\to -C_2}\mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(-2A_1B_1C_1-A_1B_2C_1-A_1B_1C_2+3A_1B_2C_2-A_2B_1C_1-2A_2B_2C_1-2A_2B_1C_2+A_1B_1-A_2B_1-A_1C_1+A_2C_1+A_1+A_2+B_1C_1+B_2C_1+B_1C_2-B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{B_2\to -B_2,C_1\to -C_1}\mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(-2A_1B_1C_1+A_1B_2C_1+A_1B_1C_2+3A_1B_2C_2-A_2B_1C_1+2A_2B_2C_1+2A_2B_1C_2+A_1B_1-A_2B_1-A_1C_1+A_2C_1+A_1+A_2+B_1C_1-B_2C_1-B_1C_2-B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{C_1\to -C_1,C_2\to -C_2}\mathbb{B}_3^{(--)}$

TABLE XXX. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{29}$.

Śliwa$_{29}$	$\frac{1}{6}(A_1+A_2+A_1B_1-A_2B_1+A_1C_1-A_2C_1-B_1C_1+2A_1B_1C_1+A_2B_1C_1+B_2C_1-A_1B_2C_1-2A_2B_2C_1+B_1C_2-3A_1B_1C_2+B_2C_2-A_1B_2C_2-2A_2B_2C_2)\leqslant 1$	$\mathcal{Q}=\frac{4\sqrt{2}-1}{3}$
Number	New tight inequalities	Remarks
16	$\mathbb{B}_3^{(++)}\xleftrightarrow{B_{1\leftrightarrow 2},C_1\to -C_1}\mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},B_{1\leftrightarrow 2},C_1\to -C_1}\mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2}}\mathbb{B}_3^{(--)}$	
18	$\frac{1}{12}(A_1B_1C_1D_1+A_1B_2C_1D_1-4A_1B_1C_2D_1-4A_1B_2C_2D_1+3A_1B_1C_1D_2-3A_1B_2C_1D_2-2A_1B_1C_2D_2+2A_1B_2C_2D_2-A_2B_1C_1D_1-A_2B_2C_1D_1-2A_2B_1C_2D_1-2A_2B_2C_2D_1+3A_2B_1C_1D_2-3A_2B_2C_1D_2+2A_2B_1C_2D_2-2A_2B_2C_2D_2+A_1B_1D_1+A_1B_2D_1+A_1B_1D_2-A_1B_2D_2-A_2B_1D_1-A_2B_2D_1-A_2B_1D_2+A_2B_2D_2+2A_1C_1-2A_2C_1+2A_1+2A_2+2B_1C_2D_1+2B_2C_2D_1-2B_1C_1D_2+2B_2C_1D_2)\leqslant 1$	$\mathcal{Q}=\frac{4\sqrt{2}-1}{3}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(-A_1B_1C_1+2A_1B_2C_1-A_1B_1C_2-3A_1B_2C_2-2A_2B_1C_1+A_2B_2C_1-2A_2B_1C_2+A_1B_2-A_2B_2+A_1C_1-A_2C_1+A_1+A_2+B_1C_1-B_2C_1+B_1C_2+B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{B_{1\leftrightarrow 2}}\mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(A_1B_1C_1-2A_1B_2C_1+A_1B_1C_2+3A_1B_2C_2+2A_2B_1C_1-A_2B_2C_1+2A_2B_1C_2-A_1B_2+A_2B_2+A_1C_1-A_2C_1+A_1+A_2-B_1C_1+B_2C_1-B_1C_2-B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{B_{1\leftrightarrow 2},B_1\to -B_1,B_2\to -B_2}\mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(-2A_1B_1C_1+A_1B_2C_1+3A_1B_1C_2+A_1B_2C_2-A_2B_1C_1+2A_2B_2C_1+2A_2B_2C_2-A_1B_1+A_2B_1+A_1C_1-A_2C_1+A_1+A_2+B_1C_1-B_2C_1-B_1C_2-B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{B_1\to -B_1,B_2\to -B_2}\mathbb{B}_3^{(--)}$

TABLE XXXI. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{30}$.

Śliwa$_{30}$	$\frac{1}{6}(A_1+A_2+2A_1B_1-2A_2B_1+A_1B_2-A_2B_2+A_1C_1-A_2C_1-B_1C_1+2A_1B_1C_1+A_2B_1C_1-B_2C_1+A_1B_2C_1+2A_2B_2C_1+B_1C_2-2A_1B_1C_2-A_2B_1C_2-B_2C_2+2A_1B_2C_2+A_2B_2C_2)\leqslant 1$	$\mathcal{Q}=\frac{4\sqrt{2}-1}{3}$
Number	New tight inequalities	Remarks
40	$\mathbb{B}_3^{(++)}\xleftrightarrow{C_{1\leftrightarrow 2},A_1\to -A_1,A_2\to -A_2,B_1\to -B_1}\mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{C_{1\leftrightarrow 2},B_1\to -B_1}\mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)}\xleftrightarrow{A_1\to -A_1,A_2\to -A_2}\mathbb{B}_3^{(--)}$	$\mathcal{Q}=\frac{4\sqrt{2}-1}{3}$
53	$\frac{1}{12}(A_1B_1C_1D_1-A_2B_1C_1D_1-A_1B_2C_1D_1+A_2B_2C_1D_1+3A_1B_1C_1D_2+3A_2B_1C_1D_2+3A_1B_2C_1D_2+3A_2B_2C_1D_2-A_1B_1C_2D_1+A_2B_1C_2D_1+A_1B_2C_2D_1-A_2B_2C_2D_1-3A_1B_1C_2D_2-3A_2B_1C_2D_2+3A_1B_2C_2D_2+3A_2B_2C_2D_2+4A_1B_1D_1-4A_2B_1D_1+2A_1B_2D_1-2A_2B_2D_1+2A_1C_1D_1-2A_2C_1D_1+2A_1D_2+2A_2D_2-2B_1C_1-2B_2C_1+2B_1C_2-2B_2C_2)\leqslant 1$	$\mathcal{Q}=1.59$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(-A_1B_1C_1-2A_1B_2C_1+A_1B_1C_2-A_1B_2C_2-2A_2B_1C_1-A_2B_2C_1+2A_2B_1C_2-2A_2B_2C_2+2A_1B_1+A_1B_2-2A_2B_1-A_2B_2+A_1C_1-A_2C_1-A_1-A_2-B_1C_1-B_2C_1+B_1C_2-B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2},A_1\to -A_1,A_2\to -A_2}\mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(A_1B_1C_1+2A_1B_2C_1-A_1B_1C_2+A_1B_2C_2+2A_2B_1C_1+A_2B_2C_1-2A_2B_1C_2+2A_2B_2C_2-2A_1B_1-A_1B_2+2A_2B_1+A_2B_2-A_1C_1+A_2C_1+A_1+A_2-B_1C_1-B_2C_1+B_1C_2-B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_{1\leftrightarrow 2}}\mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(-2A_1B_1C_1-A_1B_2C_1+2A_1B_1C_2-2A_1B_2C_2-A_2B_1C_1-2A_2B_2C_1+A_2B_1C_2-A_2B_2C_2-2A_1B_1-A_1B_2+2A_2B_1+A_2B_2-A_1C_1+A_2C_1-A_1-A_2-B_1C_1-B_2C_1+B_1C_2-B_2C_2)\leqslant 1$	$\mathbb{B}_3^{(++)}\xleftrightarrow{A_1\to -A_1,A_2\to -A_2}\mathbb{B}_3^{(--)}$

TABLE XXXII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{31}$.

Śliwa$_{31}$	$\frac{1}{6}(A_1 + A_2 + B_1 - A_2B_1 + B_2 - A_1B_2 + A_1C_1 - A_2C_1 + 2A_1B_1C_1 - A_1B_2C_1 + 3A_2B_2C_1 + B_1C_2 - 2A_1B_1C_2 - A_2B_1C_2 - B_2C_2 + 2A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = \frac{2\sqrt{2}+1}{3}$
Number	New tight inequalities	Remarks
162	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2},B_1\to -B_1,B_2\to -B_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B,C_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B,B_1\to -B_1,B_2\to -B_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{2\sqrt{2}+1}{3}$
236	$\frac{1}{12}(-3A_1B_2C_1D_1 + 3A_1B_2C_2D_1 - 2A_1B_1C_1D_2 - A_1B_2C_1D_2 - 3A_1B_1C_2D_2 + 3A_2B_1C_1D_1 - 3A_2B_1C_2D_1 - A_2B_1C_1D_2 + 4A_2B_2C_1D_2 + 3A_2B_2C_2D_2 + 2A_1B_1C_1 + 2A_1B_2C_1 - A_1B_1C_2 + A_1B_2C_2 + 2A_2B_1C_1 + 2A_2B_2C_1 + A_2B_1C_2 - A_2B_2C_2 + A_1B_1D_2 - A_1B_2D_2 - A_2B_1D_2 + A_2B_2D_2 - A_1B_1 - A_1B_2 - A_2B_1 - A_2B_2 + A_1C_1D_1 - A_1C_2D_1 + A_1C_1D_2 + A_1C_2D_2 - A_2C_1D_1 + A_2C_2D_1 - A_2C_1D_2 - A_2C_2D_2 + 2A_1 + 2A_2 - B_1C_1D_1 + B_2C_1D_1 + B_1C_2D_1 - B_2C_2D_1 + B_1C_1D_2 - B_2C_1D_2 + B_1C_2D_2 - B_2C_2D_2 + 2B_1 + 2B_2) \leqslant 1$	$\mathcal{Q} = 1.55$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(2A_1B_1C_1 + A_1B_1C_2 + 2A_1B_2C_2 + 3A_2B_1C_1 - A_2B_2C_1 - A_2B_1C_2 - 2A_2B_2C_2 - A_1B_1 - A_2B_2 - A_1C_2 + A_2C_2 + A_1 + A_2 - B_1C_1 + B_2C_1 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B,B_{1\leftrightarrow 2},C_2\to -C_2} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(2A_1B_2C_1 - 2A_1B_1C_2 - A_1B_2C_2 - A_2B_1C_1 + 3A_2B_2C_1 + 2A_2B_1C_2 + A_2B_2C_2 - A_1B_2 - A_2B_1 + A_1C_2 - A_2C_2 + A_1 + A_2 + B_1C_1 - B_2C_1 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A\leftrightarrow B} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(2A_1B_1C_1 + 3A_1B_2C_1 + A_1B_1C_2 - A_1B_2C_2 - A_2B_2C_1 + 2A_2B_1C_2 - 2A_2B_2C_2 - A_1B_1 - A_2B_2 - A_1C_1 + A_2C_1 + A_1 + A_2 - B_1C_2 + B_2C_2 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2},C_{1\leftrightarrow 2}} \mathbb{B}_3^{(--)}$

TABLE XXXIII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{32}$.

Śliwa$_{32}$	$\frac{1}{6}(A_1 + A_2 + B_1 - A_2B_1 + B_2 - A_1B_2 + 2A_1C_1 - 2A_2C_1 + 2A_2B_1C_1 + 2A_2B_2C_1 + A_1C_2 - A_2C_2 - B_1C_2 + 2A_1B_1C_2 + A_2B_1C_2 + B_2C_2 - A_1B_2C_2 - 2A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.36$
Number	New tight inequalities	Remarks
12	$\frac{1}{12}(2A_1B_1C_1D_1 - A_1B_2C_1D_1 + 2A_1B_1C_2D_1 - A_1B_2C_2D_1 - 2A_1B_1C_1D_2 + A_1B_2C_1D_2 + 2A_1B_1C_2D_2 - A_1B_2C_2D_2 + 3A_2B_1C_1D_1 - A_2B_1C_2D_1 - 4A_2B_2C_2D_1 + A_2B_1C_1D_2 + 4A_2B_2C_1D_2 + 3A_2B_1C_2D_2 - 2A_1B_2 - 2A_2B_1 + 3A_1C_1D_1 - A_1C_2D_1 + A_1C_1D_2 + 3A_1C_2D_2 - 3A_2C_1D_1 + A_2C_2D_1 - A_2C_1D_2 - 3A_2C_2D_2 + 2A_1 + 2A_2 - B_1C_1D_1 + B_2C_1D_1 - B_1C_2D_1 + B_2C_2D_1 + B_1C_1D_2 - B_2C_1D_2 - B_1C_2D_2 + B_2C_2D_2 + 2B_1 + 2B_2) \leqslant 1$	$\mathcal{Q} = 1.87$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(2A_1B_1C_1 - A_1B_2C_1 + A_2B_1C_1 - 2A_2B_2C_1 - 2A_2B_1C_2 - 2A_2B_2C_2 - A_1B_2 - A_2B_1 + A_1C_1 - 2A_1C_2 - A_2C_1 + 2A_2C_2 + A_1 + A_2 - B_1C_1 + B_2C_1 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2},C_2\to -C_2} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(-2A_1B_1C_1 + A_1B_2C_1 - A_2B_1C_1 + 2A_2B_2C_1 + 2A_2B_1C_2 + 2A_2B_2C_2 - A_1B_2 - A_2B_1 - A_1C_1 + 2A_1C_2 + A_2C_1 - 2A_2C_2 + A_1 + A_2 + B_1C_1 - B_2C_1 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2},C_1\to -C_1} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(-2A_1B_1C_2 + A_1B_2C_2 - 2A_2B_1C_1 - 2A_2B_2C_1 - A_2B_1C_2 + 2A_2B_2C_2 - A_1B_2 - A_2B_1 - 2A_1C_1 - A_1C_2 + 2A_2C_1 + A_2C_2 + A_1 + A_2 + B_1C_2 - B_2C_2 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1,C_2\to -C_2} \mathbb{B}_3^{(--)}$
13	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2},C_1\to -C_1,C_2\to -C_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1,C_2\to -C_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{4\sqrt{2}-1}{3}$

TABLE XXXIV. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{33}$.

Śliwa$_{33}$	$\frac{1}{6}(A_1 + A_2 + B_1 - A_2B_1 + B_2 - A_1B_2 + C_1 - 2A_2C_1 + 2A_2B_1C_1 - B_2C_1 + 2A_1B_2C_1 + A_2B_2C_1 + C_2 - A_1C_2 - B_1C_2 + 2A_1B_1C_2 + A_2B_1C_2 + A_1B_2C_2 - 3A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.63$
Number	New tight inequalities	Remarks
8	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2},,C_1\to -C_1,C_2\to -C_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1,C_2\to -C_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 1.65$
9	$\frac{1}{12}(-2A_1B_1C_1D_1 + A_1B_2C_1D_1 + 2A_1B_1C_2D_1 + 3A_1B_2C_2D_1 + 2A_1B_1C_1D_2 + 3A_1B_2C_1D_2 + 2A_1B_1C_2D_2 - A_1B_2C_2D_2 + A_2B_1C_1D_1 + 4A_2B_2C_1D_1 + 3A_2B_1C_2D_1 - 2A_2B_2C_2D_1 + 3A_2B_1C_1D_2 - 2A_2B_2C_1D_2 - A_2B_1C_2D_2 - 4A_2B_2C_2D_2 - 2A_1B_2 - 2A_2B_1 + A_1C_1D_1 - A_1C_2D_1 - A_1C_1D_2 - A_1C_2D_2 - A_2C_1D_1 - A_2C_2D_1 - A_2C_1D_2 + A_2C_2D_2 + 2A_1 + 2A_2 + B_1C_1D_1 - B_2C_1D_1 - B_1C_2D_1 - B_2C_2D_1 - B_1C_1D_2 - B_2C_1D_2 - B_1C_2D_2 + B_2C_2D_2 + 2B_1 + 2B_2 + 2C_2D_1 + 2C_1D_2) \leqslant 1$	$\mathcal{Q} = 2.32$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(-2A_1B_1C_1 - A_1B_2C_1 + 2A_1B_2C_2 - A_2B_1C_1 + 3A_2B_2C_1 + 2A_2B_1C_2 + A_2B_2C_2 - A_1B_2 - A_2B_1 + A_1C_1 - A_2C_2 + A_1 + A_2 + B_1C_1 - B_2C_2 + B_1 + B_2 - C_1 + C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2},C_1\to -C_1} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(2A_1B_1C_1 + A_1B_2C_1 - 2A_1B_2C_2 + A_2B_1C_1 - 3A_2B_2C_1 - 2A_2B_1C_2 - A_2B_2C_2 - A_1B_2 - A_2B_1 - A_1C_1 + A_2C_2 + A_1 + A_2 - B_1C_1 + B_2C_2 + B_1 + B_2 + C_1 - C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2},C_2\to -C_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(-2A_1B_2C_1 - 2A_1B_1C_2 - A_1B_2C_2 - 2A_2B_1C_1 - A_2B_2C_1 - A_2B_1C_2 + 3A_2B_2C_2 - A_1B_2 - A_2B_1 + A_1C_2 + A_2C_1 + A_1 + A_2 + B_2C_1 + B_1C_2 + B_1 + B_2 - C_1 - C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1,C_2\to -C_2} \mathbb{B}_3^{(--)}$

TABLE XXXV. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{34}$.

Śliwa$_{34}$	$\frac{1}{6}(A_1 + A_2 + B_1 - A_2B_1 + B_2 - A_1B_2 + C_1 - A_2C_1 - B_1C_1 - 2A_2B_1C_1 + A_2B_1C_1 - 2B_2C_1 + 2A_1B_2C_1 + 2A_2B_2C_1 + C_2 - A_1C_2 - 2B_1C_2 - B_2C_2 + A_1B_2C_2 - 2A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.38$
Number	New tight inequalities	Remarks
17	$\frac{1}{12}(-2A_1B_1C_1D_1 + 3A_1B_2C_1D_1 + 2A_1B_1C_2D_1 - A_1B_2C_2D_1 - 2A_1B_1C_1D_2 + A_1B_2C_1D_2 - 2A_1B_1C_2D_2 + 3A_1B_2C_2D_2 + A_2B_1C_1D_1 - A_2B_1C_2D_1 - 4A_2B_2C_2D_1 + A_2B_1C_1D_2 + 4A_2B_2C_1D_2 + A_2B_1C_2D_2 - 2A_1B_2 - 2A_2B_1 - A_1C_1D_1 - A_1C_2D_1 + A_1C_1D_2 - A_1C_2D_2 - A_2C_1D_1 + A_2C_2D_1 - A_2C_1D_2 - A_2C_2D_2 + 2A_1 + 2A_2 - 3B_1C_1D_1 - 3B_2C_1D_1 - B_1C_2D_1 + B_2C_2D_1 + B_1C_1D_2 - B_2C_1D_2 - 3B_1C_2D_2 - 3B_2C_2D_2 + 2B_1 + 2B_2 + 2C_1D_1 + 2C_2D_2) \leqslant 1$	$\mathcal{Q} = 1.98$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(A_1B_2C_1 + 2A_1B_1C_2 - 2A_1B_2C_2 - 2A_2B_2C_1 - A_2B_1C_2 - 2A_2B_2C_2 - A_1B_2 - A_2B_1 - A_1C_1 + A_2C_2 + A_1 + A_2 - 2B_1C_1 - B_2C_1 + B_1C_2 + 2B_2C_2 + B_1 + B_2 + C_1 - C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\leftrightarrow 2, C_2\rightarrow -C_2} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(-A_1B_2C_1 - 2A_1B_1C_2 + 2A_1B_2C_2 + 2A_2B_2C_1 + A_2B_1C_2 + 2A_2B_2C_2 - A_1B_2 - A_2B_1 + A_1C_1 - A_2C_2 + A_1 + A_2 + 2B_1C_1 + B_2C_1 - B_1C_2 - 2B_2C_2 + B_1 + B_2 - C_1 + C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\leftrightarrow 2, C_1\rightarrow -C_1} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(2A_1B_1C_1 - 2A_1B_2C_1 - A_1B_2C_2 - A_2B_1C_1 - 2A_2B_2C_1 + 2A_2B_2C_2 - A_1B_2 - A_2B_1 + A_1C_2 + A_2C_1 + A_1 + A_2 + B_1C_1 + 2B_2C_1 + 2B_1C_2 + B_2C_2 + B_1 + B_2 - C_1 - C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\rightarrow -C_1, C_2\rightarrow -C_2} \mathbb{B}_3^{(--)}$
19	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\rightarrow -A_1} \mathbb{B}_3^{(+-)}, \mathbb{B}_3^{(++)} \xleftrightarrow{A_2\rightarrow -A_2} \mathbb{B}_3^{(-+)}, \mathbb{B}_3^{(++)} \xleftrightarrow{A_1\rightarrow -A_1, A_2\rightarrow -A_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 1.38$

TABLE XXXVI. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{35}$.

Śliwa$_{35}$	$\frac{1}{6}(A_1 + A_2 + B_1 - A_1B_1 - 2A_2B_1 + B_2 - 2A_1B_2 - 2A_2B_2 + A_1C_1 - A_2C_1 - A_1B_1C_1 + A_2B_1C_1 - 2A_1B_2C_1 + 2A_2B_2C_1 + B_1C_2 - 2A_1B_1C_2 - A_2B_1C_2 - B_2C_2 + 2A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.31$
Number	New tight inequalities	Remarks
40	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_2\rightarrow -C_2} \mathbb{B}_3^{(+-)}, \mathbb{B}_3^{(++)} \xleftrightarrow{B_1\leftrightarrow 2, C_2\rightarrow -C_2} \mathbb{B}_3^{(-+)}, \mathbb{B}_3^{(++)} \xleftrightarrow{B_1\leftrightarrow 2} \mathbb{B}_3^{(--)}$	
53	$\frac{1}{6}(A_2B_1C_1D_1 - A_2B_1C_2D_1 - A_1B_1C_1D_2 - 2A_1B_1C_2D_2 + 2A_2B_2C_1D_1 + A_2B_2C_2D_1 - 2A_1B_2C_1D_2 + 2A_1B_2C_2D_2 - 2A_2B_1D_1 - A_1B_1D_2 - A_2B_2D_1 - 2A_1B_2D_2 - A_2C_1D_1 + A_1C_1D_2 + A_2D_1 + A_1D_2 + B_1C_2 - B_2C_2 + B_1 + B_2) \leqslant 1$	$\mathcal{Q} = 1.31$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(A_1B_1C_1 + 2A_1B_2C_1 + 2A_1B_1C_2 - 2A_1B_2C_2 + A_2B_1C_1 + 2A_2B_2C_1 - A_2B_1C_2 + A_2B_2C_2 + A_1B_1 + 2A_1B_2 - 2A_2B_1 - A_2B_2 - A_1C_1 - A_2C_1 - A_1 + A_2 + B_1C_2 - B_2C_2 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\rightarrow -A_1} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(-A_1B_1C_1 - 2A_1B_2C_1 - 2A_1B_1C_2 + 2A_1B_2C_2 - A_2B_1C_1 - 2A_2B_2C_1 + A_2B_1C_2 - A_2B_2C_2 - A_1B_1 - 2A_1B_2 + 2A_2B_1 + A_2B_2 + A_1C_1 + A_2C_1 + A_1 - A_2 + B_1C_2 - B_2C_2 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_2\rightarrow -A_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(A_1B_1C_1 + 2A_1B_2C_1 + 2A_1B_1C_2 - 2A_1B_2C_2 - A_2B_1C_1 - 2A_2B_2C_1 + A_2B_1C_2 - A_2B_2C_2 + A_1B_1 + 2A_1B_2 + 2A_2B_1 + A_2B_2 - A_1C_1 + A_2C_1 - A_1 - A_2 + B_1C_2 - B_2C_2 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\rightarrow -A_1, A_2\rightarrow -A_2} \mathbb{B}_3^{(--)}$

TABLE XXXVII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{36}$.

Śliwa$_{36}$	$\frac{1}{6}(2A_1 + A_1B_1 + A_2B_1 + A_1B_2 + A_2B_2 + A_1C_1 + A_2C_1 + B_1C_1 - 2A_1B_1C_1 + A_2B_1C_1 - B_2C_1 + A_1B_2C_1 - 2A_2B_2C_1 + A_1C_2 + A_2C_2 - B_1C_2 + A_1B_1C_2 - 2A_2B_1C_2 - B_2C_2 + 2A_1B_2C_2 - A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.58$
Number	New tight inequalities	Remarks
164	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_2\rightarrow -C_2} \mathbb{B}_3^{(+-)}, \mathbb{B}_3^{(++)} \xleftrightarrow{C_1\rightarrow -C_1} \mathbb{B}_3^{(-+)}, \mathbb{B}_3^{(++)} \xleftrightarrow{C_1\rightarrow -C_1, C_2\rightarrow -C_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 1.58$
212	$\frac{1}{12}(-A_1B_1C_1D_1 + 3A_1B_2C_1D_1 + 3A_1B_1C_2D_1 + A_1B_2C_2D_1 - 3A_1B_1C_1D_2 - A_1B_2C_1D_2 - A_1B_1C_2D_2 + 3A_1B_2C_2D_2 - A_2B_1C_1D_1 - 3A_2B_2C_1D_1 - 3A_2B_1C_2D_1 + A_2B_2C_2D_1 + 3A_2B_1C_1D_2 - A_2B_2C_1D_2 - A_2B_1C_2D_2 - 3A_2B_2C_2D_2 + 2A_1B_1D_1 + 2A_1B_2D_2 + 2A_2B_1D_1 + 2A_2B_2D_2 + 2A_1C_1 + 2A_1C_2 + 2A_2C_1 + 2A_2C_2 + 4A_1 - 2B_2C_1D_1 - 2B_1C_2D_1 + 2B_1C_1D_2 - 2B_2C_2D_2) \leqslant 1$	$\mathcal{Q} = \frac{7}{3}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(A_1B_1C_1 + 2A_1B_2C_1 + 2A_1B_1C_2 - A_1B_2C_2 - 2A_2B_1C_1 - A_2B_2C_1 - A_2B_1C_2 + 2A_2B_2C_2 + A_1B_1 - A_1B_2 + A_2B_1 - A_2B_2 + A_1C_1 + A_1C_2 + A_2C_1 + A_2C_2 + 2A_1 - B_1C_1 - B_2C_1 - B_1C_2 + B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\leftrightarrow 2, B_2\rightarrow -B_2} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(-A_1B_1C_1 - 2A_1B_2C_1 - 2A_1B_1C_2 + A_1B_2C_2 + 2A_2B_1C_1 + A_2B_2C_1 + A_2B_1C_2 - 2A_2B_2C_2 - A_1B_1 + A_1B_2 - A_2B_1 + A_2B_2 + A_1C_1 + A_1C_2 + A_2C_1 + A_2C_2 + 2A_1 + B_1C_1 + B_2C_1 + B_1C_2 - B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\leftrightarrow 2, B_1\rightarrow -B_1} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(2A_1B_1C_1 - A_1B_2C_1 - A_1B_1C_2 - 2A_1B_2C_2 - A_2B_1C_1 + 2A_2B_2C_1 + 2A_2B_1C_2 + A_2B_2C_2 - A_1B_1 - A_1B_2 - A_2B_1 - A_2B_2 + A_1C_1 + A_1C_2 + A_2C_1 + A_2C_2 + 2A_1 - B_1C_1 + B_2C_1 + B_1C_2 + B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\rightarrow -B_1, B_2\rightarrow -B_2} \mathbb{B}_3^{(--)}$

TABLE XXXVIII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{37}$.

Śliwa$_{37}$	$\frac{1}{6}(2A_1 + A_1B_1 + A_2B_1 + A_1B_2 + A_2B_2 + A_1C_1 + A_2C_1 + B_1C_1 - 3A_1B_1C_1 - B_2C_1 + 2A_1B_2C_1 - A_2B_2C_1 + A_1C_2 + A_2C_2 - B_1C_2 + 2A_1B_1C_2 - A_2B_1C_2 - B_2C_2 + A_1B_2C_2 - 2A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = \frac{4\sqrt{2}-1}{3}$
Number	New tight inequalities	Remarks
35	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1} \mathbb{B}_3^{(+-)}, \mathbb{B}_3^{(++)} \xleftrightarrow{B_2\to -B_2} \mathbb{B}_3^{(-+)}, \mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, B_2\to -B_2} \mathbb{B}_3^{(--)}$	
44	$\frac{1}{12}(-5A_1B_1C_1D_1 + 3A_1B_2C_1D_1 + 5A_1B_1C_2D_1 + 3A_1B_2C_2D_1 - A_1B_1C_1D_2 + A_1B_2C_1D_2 - A_1B_1C_2D_2 - A_1B_2C_2D_2 + A_2B_1C_1D_1 - 3A_2B_2C_1D_1 - A_2B_1C_2D_1 - 3A_2B_2C_2D_1 - A_2B_1C_1D_2 + A_2B_2C_1D_2 - A_2B_1C_2D_2 - A_2B_2C_2D_2 + 2A_1B_2D_1 + 2A_1B_1D_2 + 2A_2B_2D_1 + 2A_2B_1D_2 + 2A_1C_1 + 2A_1C_2 + 2A_2C_1 + 2A_2C_2 + 4A_1 + 2B_1C_1D_1 - 2B_2C_1D_1 - 2B_1C_2D_1 - 2B_2C_2D_1) \leqslant 1$	$\mathcal{Q} = \frac{4\sqrt{2}-1}{3}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(-2A_1B_1C_1 + A_1B_2C_1 + 3A_1B_1C_2 + 2A_1B_2C_2 + A_2B_1C_1 - 2A_2B_2C_1 - A_2B_2C_2 - A_1B_1 + A_1B_2 - A_2B_1 + A_2B_2 + A_1C_1 + A_1C_2 + A_2C_1 + A_2C_2 + 2A_1 + B_1C_1 - B_2C_1 - B_1C_2 - B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}, B_1\to -B_1} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(`2A_1B_1C_1 - A_1B_2C_1 - 3A_1B_1C_2 - 2A_1B_2C_2 - A_2B_1C_1 + 2A_2B_2C_1 + A_2B_2C_2 + A_1B_1 - A_1B_2 + A_2B_1 - A_2B_2 + A_1C_1 + A_1C_2 + A_2C_1 + A_2C_2 + 2A_1 - B_1C_1 + B_2C_1 + B_1C_2 + B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}, B_2\to -B_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(3A_1B_1C_1 - 2A_1B_2C_1 - 2A_1B_1C_2 - A_1B_2C_2 + A_2B_2C_1 + A_2B_1C_2 + 2A_2B_2C_2 - A_1B_1 - A_1B_2 - A_2B_1 - A_2B_2 + A_1C_1 + A_1C_2 + A_2C_1 + A_2C_2 + 2A_1 - B_1C_1 + B_2C_1 + B_1C_2 + B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, B_2\to -B_2} \mathbb{B}_3^{(--)}$

TABLE XXXIX. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{38}$.

Śliwa$_{38}$	$\frac{1}{6}(2A_1 + 2A_1B_1 + 2A_2B_1 + A_1C_1 + A_2C_1 - B_1C_1 + A_1B_1C_1 - 2A_2B_1C_1 + B_2C_1 - 2A_1B_2C_1 + A_2B_2C_1 + A_1C_2 + A_2C_2 - B_1C_2 + A_1B_1C_2 - 2A_2B_1C_2 - B_2C_2 + 2A_1B_2C_2 - A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = \frac{4\sqrt{2}-1}{3}$
Number	New tight inequalities	Remarks
23	$\frac{1}{12}(3A_1B_1C_1D_1 - A_1B_2C_1D_1 - A_1B_1C_2D_1 + 3A_1B_2C_2D_1 - A_1B_1C_1D_2 - 3A_1B_2C_1D_2 + 3A_1B_1C_2D_2 + A_1B_2C_2D_2 - 3A_2B_1C_1D_1 - A_2B_2C_1D_1 - A_2B_1C_2D_1 - 3A_2B_2C_2D_1 - A_2B_1C_1D_2 + 3A_2B_2C_1D_2 - 3A_2B_1C_2D_2 + A_2B_2C_2D_2 + 2A_1B_1D_1 + 2A_1B_2D_1 + 2A_1B_1D_2 - 2A_1B_2D_2 + 2A_2B_1D_1 + 2A_2B_2D_1 + 2A_2B_1D_2 - 2A_2B_2D_2 + 2A_1C_1 + 2A_1C_2 + 2A_2C_1 + 2A_2C_2 + 4A_1 - 2B_1C_1D_1 - 2B_2C_2D_1 + 2B_2C_1D_2 - 2B_1C_2D_2) \leqslant 1$	$\mathcal{Q} = \frac{7}{3}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(2A_1B_1C_1 + A_1B_2C_1 - 2A_1B_1C_2 + A_1B_2C_2 - A_2B_1C_1 - 2A_2B_2C_1 + A_2B_1C_2 - 2A_2B_2C_2 + 2A_1B_2 + 2A_2B_2 + A_1C_1 + A_1C_2 + A_2C_1 + A_2C_2 + 2A_1 - B_1C_1 - B_2C_1 + B_1C_2 - B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, C_{1\leftrightarrow 2}} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(-2A_1B_1C_1 - A_1B_2C_1 + 2A_1B_1C_2 - A_1B_2C_2 + A_2B_1C_1 + 2A_2B_2C_1 - A_2B_1C_2 + 2A_2B_2C_2 - 2A_1B_2 - 2A_2B_2 + A_1C_1 + A_1C_2 + A_2C_1 + A_2C_2 + 2A_1 + B_1C_1 + B_2C_1 - B_1C_2 + B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_{1\leftrightarrow 2}, B_2\to -B_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(-A_1B_1C_1 + 2A_1B_2C_1 - A_1B_1C_2 - 2A_1B_2C_2 + 2A_2B_1C_1 - A_2B_2C_1 + 2A_2B_1C_2 + A_2B_2C_2 - 2A_1B_1 - 2A_2B_1 + A_1C_1 + A_1C_2 + A_2C_1 + A_2C_2 + 2A_1 + B_1C_1 - B_2C_1 + B_1C_2 + B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}, B_1\to -B_1} \mathbb{B}_3^{(--)}$
55	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1} \mathbb{B}_3^{(+-)}, \mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}} \mathbb{B}_3^{(-+)}, \mathbb{B}_3^{(++)} \xleftrightarrow{C_{1\leftrightarrow 2}, B_1\to -B_1} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{4\sqrt{2}-1}{3}$

TABLE XL. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{39}$.

Śliwa$_{39}$	$\frac{1}{6}(2A_1 + 2B_1 - A_1B_1 + A_2B_1 + A_1B_2 + A_2B_2 + 2C_1 - A_1C_1 + A_2C_1 - B_1C_1 + 2A_1B_1C_1 - A_2B_1C_1 + B_2C_1 - A_1B_2C_1 - 2A_2B_2C_1 + A_1C_2 + A_2C_2 + B_1C_2 - A_1B_1C_2 - 2A_2B_1C_2 + B_2C_2 - 2A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.55$
Number	New tight inequalities	Remarks
5	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1} \mathbb{B}_3^{(+-)}, \mathbb{B}_3^{(++)} \xleftrightarrow{A_2\to -A_2} \mathbb{B}_3^{(-+)}, \mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1, A_2\to -A_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 1.55$
13	$\frac{1}{12}(A_1B_1C_1D_1 - 3A_2B_1C_1D_1 - 3A_1B_1C_2D_1 - A_2B_1C_2D_1 + 3A_1B_1C_1D_2 + A_2B_1C_1D_2 + A_1B_1C_2D_2 - 3A_2B_1C_2D_2 - 3A_1B_2C_1D_1 - A_2B_2C_1D_1 - A_1B_2C_2D_1 + 3A_2B_2C_2D_1 + A_1B_2C_1D_2 - 3A_2B_2C_1D_2 - 3A_1B_2C_2D_2 - A_2B_2C_2D_2 + 2A_2B_1D_1 - 2A_1B_1D_2 + 2A_1B_2D_1 + 2A_2B_2D_2 + 2A_2C_1D_1 + 2A_1C_2D_1 - 2A_1C_1D_2 + 2A_2C_2D_2 + 2A_1D_1 - 2A_2D_1 + 2A_1D_2 + 2A_2D_2 - 2B_1C_1 + 2B_1C_2 + 2B_2C_1 + 2B_2C_2 + 4B_1 + 4C_1) \leqslant 1$	$\mathcal{Q} = 2.24$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(-2A_2B_1C_1 + A_2B_2C_1 + A_2B_1C_2 + 2A_2B_2C_2 - A_1B_1C_1 - 2A_1B_2C_1 - 2A_1B_1C_2 + A_1B_2C_2 + A_2B_1 - A_2B_2 + A_1B_1 + A_1B_2 + A_2C_1 - A_2C_2 + A_1C_1 + A_1C_2 - 2A_2 - B_1C_1 + B_2C_1 + B_1C_2 + B_2C_2 + 2B_1 + 2C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}, A_2\to -A_2} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(2A_2B_1C_1 - A_2B_2C_1 - A_2B_1C_2 - 2A_2B_2C_2 + A_1B_1C_1 + 2A_1B_2C_1 + 2A_1B_1C_2 - A_1B_2C_2 - A_2B_1 + A_2B_2 - A_1B_1 - A_1B_2 - A_2C_1 + A_2C_2 - A_1C_1 - A_1C_2 + 2A_2 - B_1C_1 + B_2C_1 + B_1C_2 + B_2C_2 + 2B_1 + 2C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_{1\leftrightarrow 2}, A_1\to -A_1} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(-2A_1B_1C_1 + A_1B_2C_1 + A_1B_1C_2 + 2A_1B_2C_2 + A_2B_1C_1 + 2A_2B_2C_1 + 2A_2B_1C_2 - A_2B_2C_2 + A_1B_1 - A_1B_2 - A_2B_1 - A_2B_2 + A_1C_1 - A_1C_2 - A_2C_1 - A_2C_2 - 2A_1 - B_1C_1 + B_2C_1 + B_1C_2 + B_2C_2 + 2B_1 + 2C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1, A_2\to -A_2} \mathbb{B}_3^{(--)}$

TABLE XLI. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{40}$.

Śliwa$_{40}$	$\frac{1}{6}(2A_1 + 2A_2 + 2B_1 - A_1B_1 - A_2B_1 + A_1B_2 + A_2B_2 + A_1C_1 + A_2C_1 + 2B_1C_1 - A_1B_1C_1 - A_2B_1C_1 + 2B_2C_1 - 2A_1B_2C_1 - 2A_2B_2C_1 + A_1C_2 - A_2C_2 - 2A_1B_1C_2 + 2A_2B_1C_2 + A_1B_2C_2 - A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.35$
Number	New tight inequalities	Remarks
13	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\leftrightarrow 2, A_1\to -A_1} \mathbb{B}_3^{(+-)}, \mathbb{B}_3^{(++)} \xleftrightarrow{A_1\leftrightarrow 2, A_2\to -A_2} \mathbb{B}_3^{(-+)}, \mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1, A_2\to -A_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{13+\sqrt{65}}{12}$
23	$\frac{1}{12}(-3A_1B_1C_1D_1 - A_1B_2C_1D_1 - A_1B_1C_2D_1 + 3A_1B_2C_2D_1 + A_1B_1C_1D_2 - 3A_1B_2C_1D_2 - 3A_1B_1C_2D_2 - A_1B_2C_2D_2 + A_2B_1C_1D_1 - 3A_2B_2C_1D_1 + 3A_2B_1C_2D_1 + A_2B_2C_2D_1 - 3A_2B_1C_1D_2 - A_2B_2C_1D_2 + A_2B_1C_2D_2 - 3A_2B_2C_2D_2 - 2A_1B_1 + 2A_1B_2 - 2A_2B_1 + 2A_2B_2 + 2A_1C_1D_1 + 2A_1C_2D_2 - 2A_2C_2D_1 + 2A_2C_1D_2 + 4A_1 + 4A_2 + 2B_1C_1D_1 + 2B_2C_1D_1 - 2B_1C_2D_1 - 2B_2C_2D_1 + 2B_1C_1D_2 + 2B_2C_1D_2 + 2B_1C_2D_2 + 2B_2C_2D_2 + 4B_1) \leqslant 1$	$\mathcal{Q} = 2\sqrt{6} - 3$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{6}(-2A_1B_1C_1 + A_1B_2C_1 + A_1B_1C_2 + 2A_1B_2C_2 + 2A_2B_1C_1 - A_2B_2C_1 + A_2B_1C_2 + 2A_2B_2C_2 - A_1B_1 + A_1B_2 - A_2B_1 + A_2B_2 + A_1C_1 - A_1C_2 - A_2C_1 - A_2C_2 + 2A_1 + 2A_2 - 2B_1C_2 - 2B_2C_2 + 2B_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\leftrightarrow 2, C_2\to -C_2} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{6}(2A_1B_1C_1 - A_1B_2C_1 - A_1B_1C_2 - 2A_1B_2C_2 - 2A_2B_1C_1 + A_2B_2C_1 - A_2B_1C_2 - 2A_2B_2C_2 - A_1B_1 + A_1B_2 - A_2B_1 + A_2B_2 - A_1C_1 + A_1C_2 + A_2C_1 + A_2C_2 + 2A_1 + 2A_2 + 2B_1C_2 + 2B_2C_2 + 2B_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\leftrightarrow 2, C_1\leftrightarrow 2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{6}(A_1B_1C_1 + 2A_1B_2C_1 + 2A_1B_1C_2 - A_1B_2C_2 + A_2B_1C_1 + 2A_2B_2C_1 - 2A_2B_1C_2 + A_2B_2C_2 - A_1B_1 + A_1B_2 - A_2B_1 + A_2B_2 - A_1C_1 - A_1C_2 - A_2C_1 + A_2C_2 + 2A_1 + 2A_2 - 2B_1C_1 - 2B_2C_1 + 2B_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\leftrightarrow 2, C_1\to -C_1} \mathbb{B}_3^{(--)}$

TABLE XLII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{41}$.

Śliwa$_{41}$	$\frac{1}{7}(A_1 + B_1 + A_1B_1 + C_1 + A_2C_1 - 3A_1B_1C_1 - A_2B_1C_1 + B_2C_1 - A_1B_2C_1 - 2A_2B_2C_1 + A_1C_2 - A_2C_2 + B_1C_2 - 4A_1B_1C_2 + A_2B_1C_2 - B_2C_2 + A_1B_2C_2 + 2A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.48$
Number	New tight inequalities	Remarks
4	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_2\to -C_2} \mathbb{B}_3^{(+-)}, \mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1} \mathbb{B}_3^{(-+)}, \mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1, C_2\to -C_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 1.48$
17	$\frac{1}{14}(-2A_1B_1C_1D_1 - 4A_2B_1C_1D_1 - 5A_1B_1C_2D_1 - 3A_2B_1C_2D_1 - 4A_1B_1C_1D_2 + 2A_2B_1C_1D_2 - 3A_1B_1C_2D_2 + 5A_2B_1C_2D_2 + A_1B_2C_1D_1 - 3A_2B_2C_1D_1 - A_1B_2C_2D_1 + 3A_2B_2C_2D_1 - 3A_1B_2C_1D_2 - A_2B_2C_1D_2 + 3A_1B_2C_2D_2 + A_2B_2C_2D_2 + A_1B_1D_1 + A_2B_1D_1 + A_1B_1D_2 - A_2B_1D_2 - A_1C_1D_1 + A_2C_1D_1 + 2A_1C_2D_1 + A_1C_1D_2 + A_2C_1D_2 - 2A_2C_2D_2 + A_1D_1 + A_2D_1 + A_1D_2 - A_2D_2 + 2B_1C_2 + 2B_2C_1 - 2B_2C_2 + 2B_1 + 2C_1) \leqslant 1$	$\mathcal{Q} = \frac{15}{7}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{7}(-3A_2B_1C_1 - A_2B_2C_1 - 4A_2B_1C_2 + A_2B_2C_2 + A_1B_1C_1 + 2A_1B_2C_1 - A_1B_1C_2 - 2A_1B_2C_2 + A_2B_1 + A_2C_2 - A_1C_1 + A_1C_2 + A_2 + B_2C_1 + B_1C_2 - B_2C_2 + B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\leftrightarrow 2, A_1\to -A_1} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{7}(3A_2B_1C_1 + A_2B_2C_1 + 4A_2B_1C_2 - A_2B_2C_2 - A_1B_1C_1 - 2A_1B_2C_1 + A_1B_1C_2 + 2A_1B_2C_2 - A_2B_1 - A_2C_2 + A_1C_1 - A_1C_2 - A_2 + B_2C_1 + B_1C_2 - B_2C_2 + B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\leftrightarrow 2, A_2\to -A_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{7}(3A_1B_1C_1 + A_1B_2C_1 + 4A_1B_1C_2 - A_1B_2C_2 + A_2B_1C_1 + 2A_2B_2C_1 - A_2B_1C_2 - 2A_2B_2C_2 - A_1B_1 - A_1C_2 - A_2C_1 + A_2C_2 - A_1 + B_2C_1 + B_1C_2 - B_2C_2 + B_1 + C_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1, A_2\to -A_2} \mathbb{B}_3^{(--)}$

TABLE XLIII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{42}$.

Śliwa$_{42}$	$\frac{1}{8}(A_1 + A_2 + B_1 + A_1B_1 + B_2 - A_2B_2 + A_1C_1 - A_2C_1 + B_1C_1 - 2A_1B_1C_1 - A_2B_1C_1 - B_2C_1 - A_1B_2C_1 + 4A_2B_2C_1 + 2A_2C_2 - A_1B_1C_2 - 3A_2B_1C_2 + 2B_2C_2 - 3A_1B_2C_2 - A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.63$
Number	New tight inequalities	Remarks
10	$\frac{1}{16}(7A_2B_2C_1D_1 - 7A_2B_1C_2D_1 - 4A_1B_1C_1D_2 - 2A_1B_2C_1D_2 - 2A_1B_1C_2D_2 - 6A_1B_2C_2D_2 - 2A_2B_1C_1 + A_2B_2C_1 + A_2B_1C_2 - 2A_2B_2C_2 + A_2B_1D_1 - 3A_2B_2D_1 + 2A_1B_1D_2 - A_2B_1 + A_2B_2 - A_2C_1D_1 + 3A_2C_2D_1 + 2A_1C_1D_2 - A_2C_1 + A_2C_2 + 2A_1D_2 + 2A_2 - B_2C_1D_1 + B_1C_2D_1 + 2B_1C_1 - B_2C_1 - B_1C_2 + 4B_2C_2 + B_1D_1 + B_2D_1 + B_1 + B_2 - C_1D_1 - C_2D_1 + C_1 + C_2) \leqslant 1$	$\mathcal{Q} = 1.66$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{8}(2A_1B_1C_1 + A_1B_2C_1 + A_1B_1C_2 + 3A_1B_2C_2 - A_2B_1C_1 + 4A_2B_2C_1 - 3A_2B_1C_2 - A_2B_2C_2 - A_1B_1 - A_2B_2 - A_1C_1 - A_2C_1 + 2A_2C_2 - A_1 + A_2 + B_1C_1 - B_2C_1 + 2B_2C_2 + B_1 + B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{8}(-2A_1B_1C_1 - A_1B_2C_1 - A_1B_1C_2 - 3A_1B_2C_2 - A_2B_1C_1 - 3A_2B_2C_1 + 4A_2B_1C_2 - A_2B_2C_2 + A_1B_1 - A_2B_1 + 2A_2B_2 + A_1C_1 - A_2C_2 + A_1 + A_2 + B_1C_1 - B_1C_2 + 2B_2C_2 + C_1 + C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B\leftrightarrow C} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{8}(2A_1B_1C_1 + A_1B_2C_1 + A_1B_1C_2 + 3A_1B_2C_2 - A_2B_1C_1 - 3A_2B_2C_1 + 4A_2B_1C_2 - A_2B_2C_2 - A_1B_1 - A_2B_1 + 2A_2B_2 - A_1C_1 - A_2C_2 - A_1 + A_2 + B_1C_1 - B_1C_2 + 2B_2C_2 + C_1 + C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B\leftrightarrow C, A_1\to -A_1} \mathbb{B}_3^{(--)}$
14	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_2\to -A_2} \mathbb{B}_3^{(+-)}, \mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1} \mathbb{B}_3^{(-+)}, \mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1, A_2\to -A_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 1.63$

TABLE XLIV. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{43}$.

Śliwa$_{43}$	$\frac{1}{8}(2A_1 + 2B_1 - A_1B_1 + A_2B_1 + A_1B_2 - A_2B_2 + A_1C_1 + A_2C_1 + B_1C_1 - 2A_1B_1C_1 - 3A_2B_1C_1 - B_2C_1 + A_1B_2C_1 + 2A_2B_2C_1 + A_1C_2 - A_2C_2 + B_1C_2 - 3A_1B_1C_2 + B_2C_2 - 4A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = \sqrt{2}$
Number	New tight inequalities	Remarks
4	$\frac{1}{16}(-3A_1B_1C_1D_1 - A_1B_2C_1D_1 + A_1B_1C_2D_1 - 7A_1B_2C_2D_1 - A_1B_1C_1D_2 + 3A_1B_2C_1D_2 - 7A_1B_1C_2D_2 - A_1B_2C_2D_2 - 5A_2B_1C_1D_1 - A_2B_2C_1D_1 - A_2B_1C_2D_1 + A_2B_2C_2D_1 - A_2B_1C_1D_2 + 5A_2B_2C_1D_2 + A_2B_1C_2D_2 + A_2B_2C_2D_2 - 2A_1B_1D_1 + 2A_1B_2D_2 + 2A_2B_1D_1 - 2A_2B_2D_2 + 2A_1C_1 + 2A_1C_2 + 2A_2C_1 - 2A_2C_2 + 4A_1 + 2B_1C_1D_1 + 2B_2C_2D_1 - 2B_2C_1D_2 + 2B_1C_2D_2 + 2B_1D_1 + 2B_2D_1 + 2B_1D_2 - 2B_2D_2) \leqslant 1$	$\mathcal{Q} = \frac{25+\sqrt{577}}{24}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{8}(-A_1B_1C_1 - 2A_1B_2C_1 + 4A_1B_1C_2 - 3A_1B_2C_2 - 2A_2B_1C_1 - 3A_2B_2C_1 - A_2B_1C_2 - A_1B_1 - A_1B_2 + A_2B_1 + A_2B_2 + A_1C_1 + A_1C_2 + A_2C_1 - A_2C_2 + 2A_1 + B_1C_1 + B_2C_1 - B_1C_2 + B_2C_2 + 2B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\leftrightarrow 2, B_1\to -B_1} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{8}(A_1B_1C_1 + 2A_1B_2C_1 - 4A_1B_1C_2 + 3A_1B_2C_2 + 2A_2B_1C_1 + 3A_2B_2C_1 + A_2B_1C_2 + A_1B_1 + A_1B_2 - A_2B_1 - A_2B_2 + A_1C_1 + A_1C_2 + A_2C_1 - A_2C_2 + 2A_1 - B_1C_1 - B_2C_1 + B_1C_2 - B_2C_2 - 2B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\leftrightarrow 2, B_2\to -B_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{8}(2A_1B_1C_1 - A_1B_2C_1 + 3A_1B_1C_2 + 4A_1B_2C_2 + 3A_2B_1C_1 - 2A_2B_2C_1 - A_2B_2C_2 + A_1B_1 - A_1B_2 - A_2B_1 + A_2B_2 + A_1C_1 + A_1C_2 + A_2C_1 - A_2C_2 + 2A_1 - B_1C_1 + B_2C_1 - B_1C_2 - B_2C_2 - 2B_1) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, B_2\to -B_2} \mathbb{B}_3^{(--)}$
107	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_2\to -B_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, B_2\to -B_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \sqrt{2}$

TABLE XLV. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{44}$.

Śliwa$_{44}$	$\frac{1}{8}(2A_1 + 2A_2 + 2A_1B_1 - 2A_2B_1 + A_1C_1 - A_2C_1 - 2B_1C_1 + 2A_1B_1C_1 + 2A_2B_1C_1 + 2B_2C_1 - A_1B_2C_1 - 3A_2B_2C_1 + A_1C_2 - A_2C_2 - 2B_1C_2 + 2A_1B_1C_2 + 2A_2B_1C_2 - 2B_2C_2 + 3A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = \frac{3\sqrt{2}-1}{2}$
Number	New tight inequalities	Remarks
11	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\leftrightarrow 2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\leftrightarrow 2, B_1\to -B_1} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{3\sqrt{2}-1}{2}$
17	$\frac{1}{8}(2A_1B_1C_1D_1 - 2A_1B_2C_1D_1 - A_2B_2C_1D_1 + 2A_2B_1C_1D_2 + A_1B_2C_1D_2 - 2A_2B_2C_1D_2 + 2A_1B_1C_2D_1 + 2A_1B_2C_2D_1 - A_2B_2C_2D_1 + 2A_2B_1C_2D_2 + A_1B_2C_2D_2 + 2A_2B_2C_2D_2 - 2A_2B_1D_1 + 2A_1B_1D_2 - A_2C_1D_1 + A_1C_1D_2 - A_2C_2D_1 + A_1C_2D_2 + 2A_1D_1 + 2A_2D_2 - 2B_1C_1 + 2B_2C_1 - 2B_1C_2 - 2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.95$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{8}(2A_1B_1C_1 - 3A_1B_2C_1 + 2A_1B_1C_2 + A_1B_2C_2 - 2A_2B_1C_1 + A_2B_2C_1 - 2A_2B_1C_2 - 3A_2B_2C_2 - 2A_1B_1 - 2A_2B_1 - A_1C_1 - A_1C_2 - A_2C_1 - A_2C_2 + 2A_1 - 2A_2 - 2B_1C_1 + 2B_2C_1 - 2B_1C_2 - 2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\leftrightarrow 2, A_2\to -A_2} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{8}(-2A_1B_1C_1 + 3A_1B_2C_1 - 2A_1B_1C_2 - A_1B_2C_2 + 2A_2B_1C_1 - A_2B_2C_1 + 2A_2B_1C_2 + 3A_2B_2C_2 + 2A_1B_1 + 2A_2B_1 + A_1C_1 + A_1C_2 + A_2C_1 + A_2C_2 - 2A_1 + 2A_2 - 2B_1C_1 + 2B_2C_1 - 2B_1C_2 - 2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\leftrightarrow 2, A_1\to -A_1} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{8}(-2A_1B_1C_1 + A_1B_2C_1 - 2A_1B_1C_2 - 3A_1B_2C_2 - 2A_2B_1C_1 + 3A_2B_2C_1 - 2A_2B_1C_2 - A_2B_2C_2 - 2A_1B_1 + 2A_2B_1 - A_1C_1 - A_1C_2 + A_2C_1 + A_2C_2 - 2A_1 - 2A_2 - 2B_1C_1 + 2B_2C_1 - 2B_1C_2 - 2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{A_1\to -A_1, A_2\to -A_2} \mathbb{B}_3^{(--)}$

TABLE XLVI. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{45}$.

Śliwa$_{45}$	$\frac{1}{8}(3A_1 + A_2 + 2A_1B_1 - 2A_2B_1 + A_1B_2 - A_2B_2 + 2A_1C_1 - 2A_2C_1 - 2B_1C_1 + 2A_1B_1C_1 + 2A_2B_1C_1 - 2B_2C_1 + 2A_1B_2C_1 + 2A_2B_2C_1 + A_1C_2 - A_2C_2 - 2B_1C_2 + 2A_1B_1C_2 + 2A_2B_1C_2 + 2B_2C_2 - 3A_1B_2C_2 - A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = \frac{3\sqrt{2}-1}{2}$
Number	New tight inequalities	Remarks
6	$\frac{1}{16}(4A_1B_1C_1D_1 - A_1B_2C_1D_1 - 5A_1B_2C_2D_1 + 5A_1B_2C_1D_2 + 4A_1B_1C_2D_2 - A_1B_2C_2D_2 + 4A_2B_1C_1D_1 + A_2B_2C_1D_1 - 3A_2B_2C_2D_1 + 3A_2B_2C_1D_2 + 4A_2B_1C_2D_2 + A_2B_2C_2D_2 + 4A_1B_1 + 2A_1B_2 - 4A_2B_1 - 2A_2B_2 + 3A_1C_1D_1 - A_1C_2D_1 + A_1C_1D_2 + 3A_1C_2D_2 - 3A_2C_1D_1 + A_2C_2D_1 - A_2C_1D_2 - 3A_2C_2D_2 + 6A_1 + 2A_2 - 4B_1C_1D_1 + 4B_2C_2D_1 - 4B_2C_1D_2 - 4B_1C_2D_2) \leqslant 1$	$\mathcal{Q} = \frac{5}{2}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{8}(2A_1B_1C_1 - 3A_1B_2C_1 - 2A_1B_1C_2 - 2A_1B_2C_2 + 2A_2B_1C_1 - A_2B_2C_1 - 2A_2B_1C_2 - 2A_2B_2C_2 + 2A_1B_1 + A_1B_2 - 2A_2B_1 - A_2B_2 + A_1C_1 - 2A_1C_2 - A_2C_1 + 2A_2C_2 + 3A_1 + A_2 - 2B_1C_1 + 2B_2C_1 + 2B_1C_2 + 2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\leftrightarrow 2, C_2\to -C_2} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{8}(-2A_1B_1C_1 + 3A_1B_2C_1 + 2A_1B_1C_2 + 2A_1B_2C_2 - 2A_2B_1C_1 + A_2B_2C_1 + 2A_2B_1C_2 + 2A_2B_2C_2 + 2A_1B_1 + A_1B_2 - 2A_2B_1 - A_2B_2 - A_1C_1 + 2A_1C_2 + A_2C_1 - 2A_2C_2 + 3A_1 + A_2 + 2B_1C_1 - 2B_2C_1 - 2B_1C_2 - 2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\leftrightarrow 2, C_1\to -C_1} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{8}(-2A_1B_1C_1 - 2A_1B_2C_1 - 2A_1B_1C_2 + 3A_1B_2C_2 - 2A_2B_1C_1 - 2A_2B_2C_1 - 2A_2B_1C_2 + A_2B_2C_2 + 2A_1B_1 + A_1B_2 - 2A_2B_1 - A_2B_2 - 2A_1C_1 - A_1C_2 + 2A_2C_1 + A_2C_2 + 3A_1 + A_2 + 2B_1C_1 + 2B_2C_1 + 2B_1C_2 - 2B_2C_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1, C_2\to -C_2} \mathbb{B}_3^{(--)}$
40	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_2\to -B_2} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, B_2\to -B_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = \frac{3\sqrt{2}-1}{2}$

TABLE XLVII. Two cases of the (4,2,2) tight inequalities generated from Śliwa$_{46}$.

Śliwa$_{46}$	$\frac{1}{10}(3A_1 + A_2 + 3B_1 - 2A_1B_1 - A_2B_1 + B_2 - A_1B_2 - 2A_2B_2 + 2A_1C_1 - 2A_2C_1 + B_1C_1 - 3A_1B_1C_1 + 4A_2B_1C_1 + B_2C_1 - A_1B_2C_1 + 2A_2B_2C_1 + A_1C_2 + A_2C_2 + 2B_1C_2 - 3A_1B_1C_2 - A_2B_1C_2 - 2B_2C_2 + 4A_1B_2C_2 + 2A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 1.3$
Number	New tight inequalities	Remarks
1	$\mathbb{B}_3^{(++)} \xleftrightarrow{C_2\to -C_2} \mathbb{B}_3^{(+-)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1} \mathbb{B}_3^{(-+)}$, $\mathbb{B}_3^{(++)} \xleftrightarrow{C_1\to -C_1, C_2\to -C_2} \mathbb{B}_3^{(--)}$	$\mathcal{Q} = 1.3$
3	$\frac{1}{20}(-2A_1B_1C_1D_1 - 4A_1B_2C_1D_1 - 7A_1B_1C_2D_1 + A_1B_2C_2D_1 - 4A_1B_1C_1D_2 + 2A_1B_2C_1D_2 + A_1B_1C_2D_2 + 7A_1B_2C_2D_2 + 2A_2B_1C_1D_1 + 6A_2B_2C_1D_1 - 3A_2B_1C_2D_1 + A_2B_2C_2D_1 + 6A_2B_1C_1D_2 - 2A_2B_2C_1D_2 + A_2B_1C_2D_2 + 3A_2B_2C_2D_2 - A_1B_1D_1 - 3A_1B_2D_1 - 3A_1B_1D_2 + A_1B_2D_2 + A_2B_1D_1 - 3A_2B_2D_1 - 3A_2B_1D_2 - A_2B_2D_2 + 4A_1C_1 + 2A_1C_2 - 4A_2C_1 + 2A_2C_2 + 6A_1 + 2A_2 + 2B_2C_1D_1 + 4B_1C_2D_1 + 2B_1C_1D_2 - 4B_2C_2D_2 + 2B_1D_1 + 4B_2D_1 + 4B_1D_2 - 2B_2D_2) \leqslant 1$	$\mathcal{Q} = \frac{11+\sqrt{65}}{10}$
$\mathbb{B}_3^{(+-)}$	$\frac{1}{10}(A_1B_1C_1 - 3A_1B_2C_1 - 4A_1B_1C_2 - 3A_1B_2C_2 - 2A_2B_1C_1 + 4A_2B_2C_1 - 2A_2B_1C_2 - A_2B_2C_2 + A_1B_1 - 2A_1B_2 + 2A_2B_1 - A_2B_2 + 2A_1C_1 + A_1C_2 - 2A_2C_1 + A_2C_2 + 3A_1 + A_2 - B_1C_1 + B_2C_1 + 2B_1C_2 + 2B_2C_2 - B_1 + 3B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\leftrightarrow 2, B_1\to -B_1} \mathbb{B}_3^{(+-)}$
$\mathbb{B}_3^{(-+)}$	$\frac{1}{10}(-A_1B_1C_1 + 3A_1B_2C_1 + 4A_1B_1C_2 + 3A_1B_2C_2 + 2A_2B_1C_1 - 4A_2B_2C_1 + 2A_2B_1C_2 + A_2B_2C_2 - A_1B_1 + 2A_1B_2 - 2A_2B_1 + A_2B_2 + 2A_1C_1 + A_1C_2 - 2A_2C_1 + A_2C_2 + 3A_1 + A_2 + B_1C_1 - B_2C_1 - 2B_1C_2 - 2B_2C_2 + B_1 - 3B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\leftrightarrow 2, B_2\to -B_2} \mathbb{B}_3^{(-+)}$
$\mathbb{B}_3^{(--)}$	$\frac{1}{10}(3A_1B_1C_1 + A_1B_2C_1 + 3A_1B_1C_2 - 4A_1B_2C_2 - 4A_2B_1C_1 - 2A_2B_2C_1 + A_2B_1C_2 - 2A_2B_2C_2 + 2A_1B_1 + A_1B_2 + A_2B_1 + 2A_2B_2 + 2A_1C_1 + A_1C_2 - 2A_2C_1 + A_2C_2 + 3A_1 + A_2 - B_1C_1 - B_2C_1 - 2B_1C_2 + 2B_2C_2 - 3B_1 - B_2) \leqslant 1$	$\mathbb{B}_3^{(++)} \xleftrightarrow{B_1\to -B_1, B_2\to -B_2} \mathbb{B}_3^{(--)}$

TABLE XLVIII. Two cases of the (5,2,2) tight inequalities generated from the third (4,2,2) inequality in Table IV.

The (4,2,2) Inequality	$\frac{1}{4}(A_1B_1C_1D_1 + A_2B_1C_1D_1 + A_1B_2C_1D_1 + A_2B_2C_1D_1 + A_2B_1C_1D_2 - A_1B_2C_1D_2 - A_1B_1C_2D_1 + A_2B_1C_2D_1 + A_1B_2C_2D_1 - A_2B_2C_2D_1 - A_2B_1C_2D_2 + A_1B_2C_2D_2 + A_1B_1C_1 - A_2B_2C_1 + A_1B_1C_2 - A_2B_2C_2) \leqslant 1$	$\mathcal{Q} = 2$
Number	New tight inequalities	Remarks
1	$\frac{1}{8}(2A_1B_1C_1D_1E_1 + 2A_1B_2C_1D_1E_1 - 2A_1B_2C_1D_2E_1 + 2A_2B_2C_1D_1E_2 + A_2B_1C_1D_2E_2 + A_2B_2C_1D_2E_2 + 2A_2B_1C_2D_1E_1 - 2A_2B_2C_2D_1E_1 - 2A_2B_1C_2D_2E_1 + 2A_1B_2C_2D_1E_2 + A_1B_1C_2D_2E_2 + A_1B_2C_2D_2E_2 + 2A_2B_1C_1D_1 + A_2B_1C_1D_2 - A_2B_2C_1D_2 - 2A_1B_1C_2D_1 - A_1B_1C_2D_2 + A_1B_2C_2D_2 + 2A_1B_1C_1E_1 + A_2B_1C_1E_2 - A_2B_2C_1E_2 - 2A_2B_2C_2E_1 + A_1B_1C_2E_2 - A_1B_2C_2E_2 - A_2B_1C_1 - A_2B_2C_1 + A_1B_1C_2 + A_1B_2C_2) \leqslant 1$	$\mathcal{Q} = 2.01$
$\mathbb{B}_4^{(+-)}$	$\frac{1}{4}(A_1B_1C_1D_1 + A_2B_1C_1D_1 + A_1B_2C_1D_1 - A_2B_2C_1D_1 - A_1B_2C_1D_2 - A_2B_2C_1D_2 - A_1B_1C_2D_1 + A_2B_1C_2D_1 - A_1B_2C_2D_1 - A_2B_2C_2D_1 - A_1B_1C_2D_2 - A_2B_1C_2D_2 + A_1B_1C_1 - A_2B_1C_1 + A_1B_2C_2 - A_2B_2C_2) \leqslant 1$	$\mathbb{B}_4^{(++)} \xleftrightarrow{A\leftrightarrow C, A_1\to -A_1, C_1\to -C_1} \mathbb{B}_4^{(+-)}$
$\mathbb{B}_4^{(-+)}$	$\frac{1}{4}(-A_1B_1C_1D_1 + A_2B_1C_1D_1 - A_1B_2C_1D_1 + A_2B_2C_1D_1 + A_2B_1C_1D_2 + A_1B_2C_1D_2 - A_1B_1C_2D_1 - A_2B_1C_2D_1 + A_1B_2C_2D_1 + A_2B_2C_2D_1 + A_2B_1C_2D_2 + A_1B_2C_2D_2 - A_1B_1C_1 - A_2B_2C_1 + A_1B_1C_2 + A_2B_2C_2) \leqslant 1$	$\mathbb{B}_4^{(++)} \xleftrightarrow{A_1\to -A_1, C_2\to -C_2} \mathbb{B}_4^{(-+)}$
$\mathbb{B}_4^{(--)}$	$\frac{1}{4}(-A_1B_1C_1D_1 + A_2B_1C_1D_1 - A_1B_2C_1D_1 - A_2B_2C_1D_1 + A_1B_2C_1D_2 - A_2B_2C_1D_2 - A_1B_1C_2D_1 - A_2B_1C_2D_1 - A_1B_2C_2D_1 + A_2B_2C_2D_1 - A_1B_1C_2D_2 + A_2B_1C_2D_2 - A_1B_1C_1 - A_2B_1C_1 + A_1B_2C_2 + A_2B_2C_2) \leqslant 1$	$\mathbb{B}_4^{(++)} \xleftrightarrow{A\leftrightarrow C, A_1\to -A_1, A_2\to -A_2} \mathbb{B}_4^{(--)}$
117	$\mathbb{B}_4^{(++)} \xleftrightarrow{C_1\leftrightarrow 2, D_1\to -D_1} \mathbb{B}_4^{(+-)}$, $\mathbb{B}_4^{(++)} \xleftrightarrow{A\leftrightarrow B, C_1\leftrightarrow 2} \mathbb{B}_4^{(-+)}$, $\mathbb{B}_4^{(++)} \xleftrightarrow{A\leftrightarrow B, D_1\to -D_1} \mathbb{B}_4^{(--)}$	$\mathcal{Q} = 2$

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