Quantum stochastic thermodynamics: A semiclassical theory in phase space

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A formalism for quantum many-body systems is proposed through a semiclassical treatment in phase space, allowing us to establish stochastic thermodynamics incorporating quantum statistics. Specifically, we utilize a stochastic Fokker-Planck equation as the dynamics at the mesoscopic level. Here, the noise term characterizing the fluctuation of the flux density accounts for the finite-N effects arising from random collisions between the system and the reservoir. Accordingly, the stationary solution is a quasiequilibrium state in a canonical system. We define stochastic thermodynamic quantities based on the trajectories of the phase-space distribution. The conservation law of energy, the H theorem, and fluctuation theorems are therefore obtained. Our work sets an alternative formalism of quantum stochastic thermodynamics that is independent of the two-point measurement scheme. The numerous projective measurements of quantum systems are replaced by the sampling of the phase-space distribution, offering hope for experimental verifications in the future.

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I. INTRODUCTION

Stochastic thermodynamics uses stochastic variables to better understand the nonequilibrium dynamics present in microscopic systems. In these systems, thermal fluctuations are significant and the laws of thermodynamics need to be understood from probabilistic perspectives. Typical results include the definition of stochastic thermodynamic quantities and fluctuation theorems [1–4], the latter of which quantify the statistical behavior of nonequilibrium systems and generalize the second law. In addition, exact thermodynamic statements beyond the realm of linear response are obtained. These exact results refer to distribution functions of thermodynamic quantities, such as exchanged heat, applied work, and entropy production for these systems [2].

Stochastic thermodynamics is also applied to quantum systems by usually introducing the two-point measurement scheme [5]. Here, work is not an observable [6], but is defined as the energy difference between the initial and final projective measurements. By applying the two-point measurement scheme to open quantum systems, stochastic thermodynamics is also established for Lindblad-type systems [7] and Caldeira-Leggett-type systems [8]. For a comprehensive overview of quantum stochastic thermodynamics, readers are referred to Ref. [9].

The two-point measurement scheme assumes numerous projective measurements of quantum systems. However, such an assumption is impractical for quantum many-body systems or the reservoir in realistic experiments, presenting a significant challenge to this scheme. Therefore, it is necessary to establish quantum stochastic thermodynamics within alternative formalisms.

In our view, the reason for the conundrum is that too much information is involved in projective measurements to establish quantum stochastic thermodynamics. We circumvent this issue by semiclassically treating quantum systems in phase space. When the largest energy-level spacing of the system is small compared to the thermal excitation energy, a phase-space description of the system under a proper error is possible through a distribution function $\rho(z), z = (x, p)$ (a one-dimensional system is considered here for simplicity) [10]. Meanwhile, the temperature is low enough that the system still presents quantum statistics, i.e., the thermal wavelength of a particle is comparable to the interparticle spacing [11].

In our formalism, the system is governed by a stochastic Fokker-Planck equation that is a nonlinear equation incorporating a noise term that characterizes the fluctuation of the flux density. The nonlinear equation determines an equilibrium state satisfying non-Boltzmann statistics, especially for quantum systems [12–14]. A simple derivation using heuristic arguments is provided in Ref. [15]. As an approximation of quantum Boltzmann equations, such a nonlinear equation has been used to study finite fermionic or bosonic systems [16,17], and the electron collisions in dense plasmas [18]. The noise term keeps track of the finite-N effect arising from random collisions between the system and the reservoir. Consequently, the system implies a quasiequilibrium state in a canonical system, which originates from the reverse form of the Boltzmann entropy, i.e., the exponential of the entropy denotes the number of the corresponding microscopic states according to Einstein's interpretation [19,20].

In this paper, we establish the stochastic thermodynamics based on the trajectories of the phase-space distribution. Accordingly, the stochastic thermodynamic quantities are defined and reproduce their counterparts in previous studies for systems consisting of distinguishable noninteracting particles

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[1–3]. The conservation law of energy, the H theorem, and fluctuation theorems are therefore obtained. A connection between stochastic thermodynamics and the quasiequilibrium state is also established.

For experimental verifications of our formalism, we would like to point out that the current experimental techniques do not support the simultaneous sampling of the phase-space distribution (but for the possible separate measurements of the density distribution with high-resolution optics [21] or the momentum distribution using the time-of-flight image [22]). Our formalism is hopefully verified in experiments once appropriate techniques become available. It is emphasized that, although formulated in phase space for the purpose of a straightforward connection to the Wigner function, our formalism can be extended to various other spaces, where the current experimental techniques may be sufficient.

II. NONLINEAR FOKKER-PLANCK EQUATION INCORPORATING QUANTUM STATISTICS

Let us first introduce a nonlinear Fokker-Planck equation [12–14],

$$\frac{\partial \rho_t}{\partial t} = L_{\mathrm{st}}\rho_t + \frac{\partial j_t}{\partial p}, \tag{1}$$

with a flux density in phase space j_t originating from collisions between the system and the reservoir. It reads

$$j_t = \gamma p \rho_t(1+\epsilon\rho_t) + \gamma m k_{\mathrm{B}} T \frac{\partial \rho_t}{\partial p}, \tag{2}$$

where $L_{\mathrm{st}} = -\frac{p}{m}\frac{\partial}{\partial x} + \frac{\partial U}{\partial x}\frac{\partial}{\partial p}$ denotes the streaming operator, m the mass of the particle, γ the damping coefficient, U the potential energy of the particle, k_{B} the Boltzmann constant, and T the temperature of the reservoir. Here, $\epsilon = 1, -1, 0$ for bosons, fermions, and distinguishable particles, respectively. In the derivation, we have used the Kramers-Moyal expansion of the master equation and truncated the expansion up to the second order [12,23].

For bosons, Eq. (1) describes the evolution of particles above the critical temperature of the Bose-Einstein condensation. The Bose-Einstein condensation is not considered in this paper. For fermions, due to the Pauli exclusion principle, we require $0 \leqslant \rho_t \leqslant 1$ [24].

Equation (1) is conservative in particle numbers, $N[\rho_t] = \int \mathrm{d}z \rho_t(z)$, where $\mathrm{d}z = \mathrm{d}x\mathrm{d}p/h$, and h denotes the Planck constant. Also, it determines a steady state (a semiclassical equilibrium state in phase space)

$$\rho_{\mathrm{eq}}(z) = \frac{1}{\mathrm{e}^{\beta[p^2/(2m)+U(x)-\mu]} - \epsilon}, \tag{3}$$

where $\beta = 1/(k_{\mathrm{B}}T)$ is the inverse temperature, and μ the chemical potential. Equation (3) can be found in Refs. [26–28] for bosons by ignoring the two-body interaction, and in Ref. [29] for fermions.

To highlight the connection between Eq. (1) and thermodynamics, we rewrite it in the following form (also see Ref. [14]),

$$\frac{\partial \rho_t}{\partial t} = L_{\mathrm{st}}\rho_t + L_{\mathrm{fp}}(\rho_t)\frac{\delta F[\rho_t]}{\delta \rho_t}, \tag{4}$$

where the operator $L_{\mathrm{fp}}(\rho) = m\gamma\frac{\partial}{\partial p}[\rho(1+\epsilon\rho)\frac{\partial}{\partial p}]$. Here, for the system, $F[\rho_t] = E[\rho_t] - TS[\rho_t]$ is the free energy, where the internal energy $E[\rho_t]$ and the entropy $S[\rho_t]$ are respectively given by

$$E[\rho_t] = \int \mathrm{d}z\rho_t\left(\frac{p^2}{2m} + U\right), \tag{5}$$

and

$$S[\rho_t] = k_{\mathrm{B}}\int \mathrm{d}z[-\rho_t \ln\rho_t + \epsilon^{-1}(1+\epsilon\rho_t)\ln(1+\epsilon\rho_t)]. \tag{6}$$

III. STOCHASTIC FOKKER-PLANCK EQUATION

It seems straightforward to construct a Langevin equation for a single particle corresponding to the nonlinear Fokker-Planck equation (1) (see examples in Sec. 6.5.4 of Ref. [13]), and establish stochastic thermodynamics on that. However, it is inconsistent to incorporate quantum statistics in a Langevin equation for a single particle. This is because that quantum statistics is an effect of the exchange symmetry of a many-body system. In contrast, we introduce fluctuations of the many-body system in another manner: adding a noise term in the flux density j_t [Eq. (2)].

Due to the discreteness of particle numbers and the randomness of collisions between the system and the reservoir, Eq. (1) only describes the evolution of ρ_t on average for a finite-size system. Hence, we add a noise term η_t into j_t to characterize the finite-N effects of the dynamics. The introduction of the noise term in the position-space flux density has been used to study many systems, such as many-body Brownian motion [30], and hydrodynamic fluctuations [31]. In particular, it is used to describe the turbulence in randomly stirred fluids [32], the onset of hydrodynamic stabilities [33], and charge transport in semiconducting devices [34]. Moreover, the connection between the stochastic Fokker-Planck equation [Eq. (7)] and the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy is discussed in Ref. [35].

According to the above discussions, we postulate a stochastic Fokker-Planck equation (similar results for density in position space are shown in Refs. [36,37])

$$\frac{\partial \rho_t^N}{\partial t} = L_{\mathrm{st}}\rho_t^N + L_{\mathrm{fp}}(\rho_t^N)\frac{\delta F[\rho_t^N]}{\delta \rho_t^N} + \frac{\partial \eta_t}{\partial p}, \tag{7}$$

where η_t is a Gaussian white noise satisfying $\langle \eta_t(z)\rangle = 0$, $\langle \eta_t(z)\eta_{t'}(z')\rangle = 2hm\gamma k_{\mathrm{B}} T \rho_t^N(z)[1 + \epsilon\rho_t^N(z)]\delta(z-z')\delta(t-t')$. Such an equation is still conservative in particle number. In the thermodynamic limit, the suppression of the noise η_t is shown in Refs. [36,37].

Since the phase-space distribution ρ_t^N now is a random variable, we define the probability distribution of observing a fixed distribution ϕ at time t as $P_t[\phi] = \langle \prod_z \delta[\phi(z) - \rho_t^N(z)]\rangle$. Then, following the similar procedure in Refs. [36–39], its evolution equation (the functional Fokker-Planck equation) is given by

$$\frac{\partial P_t[\phi]}{\partial t} = -\int \mathrm{d}z \frac{\delta}{\delta\phi(z)}\Bigg\{P_t[\phi]L_{\mathrm{st}}\phi(z) + L_{\mathrm{fp}}(\phi(z))\left[\frac{\delta F[\phi]}{\delta\phi(z)}P_t[\phi] + k_{\mathrm{B}}T\frac{\delta P_t[\phi]}{\delta\phi(z)}\right]\Bigg\}. \tag{8}$$

Its stationary solution is

$$P_{\rm s}[\phi] = \mathcal{Z}^{-1}{\rm e}^{-\beta F[\phi]}\delta(N - N[\phi])\delta(L_{\rm st}\phi), \tag{9}$$

where $\mathcal{Z} = \int \mathcal{D}\phi {\rm e}^{-\beta F[\phi]}\delta(N - N[\phi])\delta(L_{\rm st}\phi)$ is the generalized partition function. Here, the condition $L_{\rm st}\phi = 0$ means that ϕ is a constant along the equienergy surface in the phase space. Also, the integral $\int \mathcal{D}\phi$ is constrained by the condition: $\int {\rm d}z\phi(z) = 1, \phi \geqslant 0$ $(0 \leqslant \phi \leqslant 1$ for fermions). The stationary solution is actually a quasiequilibrium state in a canonical system according to the theory of equilibrium fluctuations [19], which precisely determines the form of the noise term [37]. In the thermodynamic limit, ϕ converges to $\rho_{\rm eq}$ in probability by using the method of steepest descent. In other words, according to the minimum free-energy principle in a canonical ensemble, $P_{\rm s}[\phi]$ now is replaced by its most probable distribution $\rho_{\rm eq}$.

The functional Fokker-Planck equation has an equivalent path-integral form. Let us define a trajectory of the stochastic phase-space distribution as $\phi_{[0,t]} := \{\phi_s | s \in [0, t]\}$. Then, the probability distribution of the trajectory conditioned with a fixed distribution ϕ_0 at initial time $t = 0$ reads $P[\phi_{[0,t]}|\phi_0] \propto {\rm e}^{-\beta\mathcal{S}[\phi_{[0,t]}]}$, where the action $\mathcal{S}$ is a generalized Onsager-Machlup functional,

$$\mathcal{S}[\phi_{[0,t]}] = -\frac{1}{4}\int_0^t {\rm d}s \int {\rm d}z\left[\frac{\partial\phi_s}{\partial s} - L_{\rm st}\phi_s - L_{\rm fp}(\phi_s)\frac{\delta F[\phi_s]}{\delta\phi_s}\right] \times L_{\rm fp}(\phi_s)^{-1}\left[\frac{\partial\phi_s}{\partial s} - L_{\rm st}\phi_s - L_{\rm fp}(\phi_s)\frac{\delta F[\phi_s]}{\delta\phi_s}\right]. \tag{10}$$

Similar results can be found in Refs. [36,37,40,41]. Here, the expression $L_{\rm fp}(\phi)^{-1}$ should be understood as the Green's function of $L_{\rm fp}(\phi)$. According to the principle of least action (the steepest descent approximation of $\mathcal{S}[\phi_{[0,t]}]$), the dynamics of ϕ_t converges to the nonlinear Fokker-Planck equation (4) in probability in the thermodynamic limit [42].

IV. STOCHASTIC THERMODYNAMICS

When the potential energy is tuned by a time-dependent parameter λ_t, we can establish the stochastic thermodynamics incorporating quantum statistics along a given trajectory of the phase-space distribution $\phi_{[0,t]}$.

The stochastic work $w[\phi_{[0,t]}]$ and the stochastic heat $q[\phi_{[0,t]}]$ are defined as

$$w[\phi_{[0,t]}] = \int_0^t {\rm d}s \int {\rm d}z\phi_s(z)\frac{{\rm d}\lambda_s}{{\rm d}s}\frac{\partial U(x, \lambda_s)}{\partial\lambda_s}, \tag{11}$$

and

$$q[\phi_{[0,t]}] = \int_0^t {\rm d}s \int {\rm d}z\frac{\partial\phi_s(z)}{\partial s}\left[\frac{p^2}{2m} + U(x, \lambda_s)\right]. \tag{12}$$

They satisfy the conservation law of energy

$$E[\phi_t; \lambda_t] - E[\phi_0; \lambda_0] = w[\phi_{[0,t]}] + q[\phi_{[0,t]}]. \tag{13}$$

It is worth mentioning that for a deterministic trajectory of ϕ_s, the definitions of work (11) and heat (12) are the semiclassical phase-space counterparts of the quantum work and the quantum heat given in Ref. [43].

The stochastic entropy $s[\phi_t]$, stochastic total entropy production $s_{\rm p}[\phi_{[0,t]}]$, and stochastic free energy $f[\phi_t; \lambda_t]$ are respectively given by

$$s[\phi_t] = S[\phi_t] - k_{\rm B}\ln P_t[\phi_t], \tag{14}$$

$$s_{\rm p}[\phi_{[0,t]}] = s[\phi_t] - s[\phi_0] + s_{\rm r}[\phi_{[0,t]}], \tag{15}$$

$$f[\phi_t; \lambda_t] = E[\phi_t; \lambda_t] - Ts[\phi_t] = F[\phi_t; \lambda_t] + k_{\rm B}T\ln P_t[\phi_t], \tag{16}$$

where $P_t[\phi]$ is the solution of the functional Fokker-Planck equation (8), and $s_{\rm r}[\phi_{[0,t]}] = -q[\phi_{[0,t]}]/T$ denotes the entropy change of the reservoir.

While the stochastic internal energy $E[\phi_t; \lambda_t]$, the stochastic work $w[\phi_{[0,t]}]$, and the stochastic heat $q[\phi_{[0,t]}]$ are solely determined by ϕ_t, Eq. (14) means that the stochastic entropy $s[\phi_t]$ also depends on the probability distribution of the stochastic phase-space distribution $P_t[\phi]$. That is to say, the explicit expression of $s[\phi_t]$ cannot be determined until we have obtained the solution of the functional Fokker-Planck equation. Moreover, if we take an average of $s[\phi_t]$ over $P_t[\phi]$, we have

$$\langle s[\phi_t]\rangle = \langle S[\phi_t]\rangle - k_{\rm B}\int \mathcal{D}\phi P_t[\phi]\ln P_t[\phi]. \tag{17}$$

The term $-k_{\rm B}\ln P_t[\phi_t]$ actually has a contribution to the average entropy due to the probability distribution $P_t[\phi]$ from the perspective of information theory [44,45]. Also, this term is essential to prove the H theorem $\frac{\rm d}{{\rm d}t}\langle s_{\rm p}[\phi_{[0,t]}]\rangle \geqslant 0$ (see Appendix A).

We prove the fluctuation theorems according to these stochastic thermodynamic quantities. For this purpose, let $P[\phi_{[0,t]}|\phi_0]$ $(P^\dagger[\phi^\dagger_{[0,t]}|\phi^\dagger_0])$ denote the conditional probability distribution of the forward (reverse) trajectory $\phi_{[0,t]}$ $[\phi^\dagger_{[0,t]} := \{\phi^\dagger_s | s \in [0, t]\}, \phi^\dagger_s(x, p) := \phi_{t-s}(x, -p)]$ with a fixed initial distribution ϕ_0 $(\phi^\dagger_0)$ and a forward (reverse) protocol λ_s $(\lambda^\dagger_s := \lambda_{t-s})$. Then, using the generalized Onsager-Machlup functional [Eq. (10)], we obtain the detailed fluctuation theorems (see Appendix C),

$$\ln\frac{P[\phi_{[0,t]}|\phi_0]}{P^\dagger[\phi^\dagger_{[0,t]}|\phi^\dagger_0]} = \ln\frac{P_t[\phi_t]}{P_0[\phi_0]} + \frac{s_{\rm p}[\phi_{[0,t]}]}{k_{\rm B}}. \tag{18}$$

By adding an arbitrary normalized distribution of ϕ at the initial time of the forward (reverse) process $P_0[\phi_0]$ $(P'_0[\phi^\dagger_0])$, we obtain the integral fluctuation theorems,

$$\left\langle\frac{P'_0[\phi^\dagger_0]}{P_t[\phi_t]}{\rm e}^{-s_{\rm p}/k_{\rm B}}\right\rangle = 1. \tag{19}$$

Such an equality is formally consistent with the integral fluctuation theorems in previous studies [2,3].

For a choice of $P'_0[\phi^\dagger_0] = P_t[\phi_t]$, we obtain the integral fluctuation theorem for total entropy production [2,3],

$$\langle{\rm e}^{-s_{\rm p}/k_{\rm B}}\rangle = 1. \tag{20}$$

Then, as a corollary, the second law $\langle s_{\rm p}\rangle \geqslant 0$ follows from the fluctuation theorem by using Jensen's inequality.

Moreover, when both $P_0[\phi_0]$, $P'_0[\phi_0^\dagger]$ are stationary solutions of the functional Fokker-Planck equation [Eq. (9)], i.e., $P_0[\phi_0] = P_{\rm s}[\phi_0;\lambda_0]$, $P'_0[\phi_0^\dagger] = P_{\rm s}[\phi_t;\lambda_t]$, we obtain the generalized Jarzynski equality,

$$\langle {\rm e}^{-\beta w}\rangle = \frac{\mathcal{Z}(\lambda_t)}{\mathcal{Z}(\lambda_0)}, \tag{21}$$

and the generalized principle of maximum work

$$\langle w\rangle \geqslant -k_{\rm B}T\ln\left[\frac{\mathcal{Z}(\lambda_t)}{\mathcal{Z}(\lambda_0)}\right], \tag{22}$$

by using Jensen's inequality. In the limit $N\to\infty$, $\mathcal{Z}(\lambda_t)/\mathcal{Z}(\lambda_0) = {\rm e}^{-\beta(F[\rho_{\rm eq}(\lambda_t)]-F[\rho_{\rm eq}(\lambda_0)])}$ by using the steepest descent approximation of $\mathcal{Z}$, which further results in the Jarzynski equality $\langle {\rm e}^{-\beta w}\rangle = {\rm e}^{-\beta(F[\rho_{\rm eq}(\lambda_t)]-F[\rho_{\rm eq}(\lambda_0)])}$, and the principle of maximum work $\langle w\rangle \geqslant F[\rho_{\rm eq}(\lambda_t)] - F[\rho_{\rm eq}(\lambda_0)]$ [1–3].

We would like to emphasize that for the stationary solution $P_{\rm s}[\phi;\lambda]$ (9), we have

$$\langle f[\phi;\lambda]\rangle = -k_{\rm B}T\ln\mathcal{Z}(\lambda), \tag{23}$$

$$\langle E[\phi;\lambda]\rangle = -\frac{\partial\ln\mathcal{Z}(\lambda)}{\partial\beta}, \tag{24}$$

$$\frac{\langle s[\phi;\lambda]\rangle}{k_{\rm B}} = \ln\mathcal{Z}(\lambda) - \frac{\partial\ln\mathcal{Z}(\lambda)}{\partial\beta}. \tag{25}$$

Hence, the generalized partition function $\mathcal{Z}(\lambda)$ as the characteristic state function of a canonical quasiequilibrium state plays the same role as the partition function of a canonical equilibrium state.

For distinguishable particles ($\epsilon = 0$), our formalism coincides with previous stochastic thermodynamics. It is demonstrated by following Dean's study of a system of N-particle Langevin equations [30],

$$\begin{aligned}\frac{{\rm d}x_i(t)}{{\rm d}t} &= \frac{p_i(t)}{m},\\ \frac{{\rm d}p_i(t)}{{\rm d}t} &= -\frac{\partial U(x_i(t))}{\partial x_i(t)} - \gamma p_i(t) + \xi_i(t),\end{aligned} \tag{26}$$

where the white noises are uncorrelated, $\langle\xi_i(t)\rangle = 0$, $\langle\xi_i(t)\xi_j(t')\rangle = 2m\gamma k_{\rm B}T\delta_{ij}\delta(t-t')$. Then, the empirical density $\tilde{\rho}_t^N(z) = \sum_{i=1}^N \delta(z - z_i(t))$ of particles satisfies the stochastic Fokker-Planck equation (7), where $\eta_t = \sum_{i=1}^N \xi_i(t)$ correspondingly. Substituting $\tilde{\rho}_t^N(z)$ into the definition of stochastic thermodynamic quantities $E[\phi_t;\lambda_t]$, $w[\phi_{[0,t]}]$, $q[\phi_{[0,t]}]$, and $s[\phi_t]$ [the second term in Eq. (14)], we reproduce their counterparts in previous studies of stochastic thermodynamics [1–3].

V. CONCLUSION

For a quantum many-body system semiclassically treated by a phase-space distribution at the mesoscopic level, we consider a stochastic Fokker-Planck equation as the dynamics of the system at the mesoscopic level. Here, the noise term finds its origin in the discreteness of the particle number and the randomness of collisions between the system and the reservoir, embodying Einstein's interpretation of the reverse form of the Boltzmann entropy.

Based on trajectories of the stochastic phase-space distribution, we propose a formalism of stochastic thermodynamics that accounts for quantum statistics. Consequently, a connection between stochastic thermodynamics and the quasiequilibrium state is established.

Independent of a previous formalism that relies on the two-point measurement scheme, our formalism is based on the sampling of the stochastic phase-space distribution, offering hope for experimental verifications. Moreover, by incorporating mean-field interactions and considering other types of non-Boltzmann entropy [12–14], our formalism can be readily extended to interacting systems and other non-Boltzmann systems. It is worth mentioning that an efficient treatment of fractional exclusion statistics is given by using the nonlinear Fokker-Planck equation in Ref. [46].

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APPENDIX A: PROOF OF THE *H* THEOREM

It follows from Eqs. (15) and (16) and the conservation of energy that

$$\frac{\rm d}{{\rm d}t}\langle s_{\rm p}[\phi_{[0,t]}]\rangle = \frac{1}{T}\frac{\rm d}{{\rm d}t}(\langle w[\phi_{[0,t]}]\rangle - \langle f[\phi_t;\lambda_t]\rangle) = \frac{1}{T}\frac{\rm d}{{\rm d}t}(\langle w[\phi_{[0,t]}]\rangle - \langle F[\phi_t;\lambda_t] + k_{\rm B}T\ln P_t[\phi_t]\rangle). \tag{A1}$$

Substituting Eq. (11) into Eq. (A1), and using Eq. (8) we have [noticing $\frac{\partial F[\phi;\lambda_t]}{\partial\lambda_t} = \int {\rm d}z\phi(z)\frac{{\rm d}\lambda_t}{{\rm d}t}\frac{\partial U(x,\lambda_t)}{\partial\lambda_t}$]

$$\begin{aligned}\frac{1}{T}\frac{\rm d}{{\rm d}t}(\langle w[\phi_{[0,t]}]\rangle - \langle F[\phi_t;\lambda_t] + k_{\rm B}T\ln P_t[\phi_t]\rangle) &= \frac{1}{T}\left(\int {\rm d}z\langle\phi_t(z)\rangle\frac{{\rm d}\lambda_t}{{\rm d}t}\frac{\partial U(x,\lambda_t)}{\partial\lambda_t} - \frac{\rm d}{{\rm d}t}\langle F[\phi_t;\lambda_t] + k_{\rm B}T\ln P_t[\phi_t]\rangle\right)\\ &= \frac{1}{T}\left(\int\mathcal{D}\phi P_t[\phi]\int {\rm d}z\phi(z)\frac{{\rm d}\lambda_t}{{\rm d}t}\frac{\partial U(x,\lambda_t)}{\partial\lambda_t} - \frac{\rm d}{{\rm d}t}\langle F[\phi_t;\lambda_t] + k_{\rm B}T\ln P_t[\phi_t]\rangle\right)\\ &= \frac{1}{T}\left(\int\mathcal{D}\phi P_t[\phi]\frac{{\rm d}\lambda_t}{{\rm d}t}\frac{\partial F[\phi;\lambda_t]}{\partial\lambda_t} - \frac{\rm d}{{\rm d}t}\int\mathcal{D}\phi P_t[\phi](F[\phi;\lambda_t] + k_{\rm B}T\ln P_t[\phi])\right)\end{aligned}$$

$$
\begin{aligned}
&= -\int \mathcal{D}\phi \frac{\partial P_t[\phi]}{\partial t}(T^{-1}F[\phi;\lambda_t] + k_\mathrm{B}\ln P_t[\phi] + k_\mathrm{B}) \\
&= \int \mathcal{D}\phi \int \mathrm{d}z \frac{\delta}{\delta\phi(z)}\left\{P_t[\phi]L_\mathrm{st}\phi(z) + L_\mathrm{fp}(\phi(z))\left[\frac{\delta F[\phi;\lambda_t]}{\delta\phi(z)}P_t[\phi] + k_\mathrm{B}T\frac{\delta P_t[\phi]}{\delta\phi(z)}\right]\right\} \\
&\quad \times (T^{-1}F[\phi;\lambda_t] + k_\mathrm{B}\ln P_t[\phi] + k_\mathrm{B}). \qquad \text{(A2)}
\end{aligned}
$$

Finally, using integral by parts, we obtain

$$
\begin{aligned}
&\int \mathcal{D}\phi \int \mathrm{d}z \frac{\delta}{\delta\phi(z)}\left\{P_t[\phi]L_\mathrm{st}\phi(z) + L_\mathrm{fp}(\phi(z))\left[\frac{\delta F[\phi;\lambda_t]}{\delta\phi(z)}P_t[\phi] + k_\mathrm{B}T\frac{\delta P_t[\phi]}{\delta\phi(z)}\right]\right\}(T^{-1}F[\phi;\lambda_t] + k_\mathrm{B}\ln P_t[\phi] + k_\mathrm{B}) \\
&= \int \mathcal{D}\phi P_t[\phi] \int \mathrm{d}z \left\{-\frac{1}{T}\frac{\delta F[\phi;\lambda_t]}{\delta\phi(z)} + k_\mathrm{B}\frac{\delta}{\delta\phi(z)}\right\}L_\mathrm{st}\phi(z) \\
&\quad + \int \mathcal{D}\phi \int \mathrm{d}z \frac{\gamma\phi(z)[1+\epsilon\phi(z)]}{TP_t[\phi]}\left\{\frac{\partial}{\partial p}\left[\frac{\delta F[\phi;\lambda_t]}{\delta\phi(z)}P_t[\phi] + k_\mathrm{B}T\frac{\delta P_t[\phi]}{\delta\phi(z)}\right]\right\}^2. \qquad \text{(A3)}
\end{aligned}
$$

Because the first term actually equals 0 by using the two identities in Appendix B and the second term is not less than 0, we obtain the H theorem $\frac{\mathrm{d}}{\mathrm{d}t}\langle s_\mathrm{p}[\phi_{[0,t]}]\rangle \geqslant 0$.

APPENDIX B: TWO IDENTITIES IN EQ. (A3)

For the first identity, using $F[\phi] = E[\phi] - TS[\phi]$ and integral by parts, we have

$$
\begin{aligned}
\int \mathrm{d}z \frac{\delta F[\phi]}{\delta\phi(z)}L_\mathrm{st}\phi(z) &= \int \mathrm{d}z\left[\frac{p^2}{2m} + U(x,\lambda)\right]L_\mathrm{st}\phi(z) - T\int \mathrm{d}z\frac{\delta S[\phi]}{\delta\phi(z)}L_\mathrm{st}\phi(z) \\
&= -\int \mathrm{d}z\phi L_\mathrm{st}\left[\frac{p^2}{2m} + U(x,\lambda)\right] - k_\mathrm{B}T\int \mathrm{d}z L_\mathrm{st}[-\phi\ln\phi + \epsilon^{-1}(1+\epsilon\phi)\ln(1+\epsilon\phi)] \\
&= 0. \qquad \text{(B1)}
\end{aligned}
$$

For the second identity, let us first define a set of real orthonormal complete square-integrable functions $\{u_i(z)\}$ in the phase space. Then, we decompose $\phi(z)$ as

$$
\phi(z) = \sum_i \phi_i u_i(z), \qquad \text{(B2)}
$$

where

$$
\phi_i = \int dz\phi(z)u_i(z). \qquad \text{(B3)}
$$

Hence, it follows from Eq. (B3) that

$$
\begin{aligned}
\int \mathrm{d}z\frac{\delta}{\delta\phi(z)}L_\mathrm{st}\phi(z) &= \sum_i \int \mathrm{d}z\frac{\delta\phi_i}{\delta\phi(z)}L_\mathrm{st}u_i(z) \\
&= \sum_i \int \mathrm{d}z u_i(z)L_\mathrm{st}u_i(z) \\
&= -\sum_i \int \mathrm{d}z u_i(z)L_\mathrm{st}u_i(z) \\
&= 0. \qquad \text{(B4)}
\end{aligned}
$$

The minus sign in the third equality results from integral by parts.

APPENDIX C: PROOF OF THE DETAILED FLUCTUATION THEOREM

We first prove that the normalization factor of the forward propagator $P[\phi_{[0,t]}|\phi_0]$ equals that of the reverse propagator $P^\dagger[\phi^\dagger_{[0,t]}|\phi^\dagger_0]$. Let us define another white noise $\zeta_t = \frac{\partial}{\partial p}\eta_t$, where $\langle\zeta_t(z)\rangle = 0$ and

$$\begin{aligned}\langle\zeta_t(z)\zeta_{t'}(z')\rangle &= \frac{\partial^2}{\partial p\partial p'}\langle\eta_t(z)\eta_{t'}(z')\rangle\\ &= 2hk_{\rm B}T\frac{\partial}{\partial p}\left\{\rho^N_t(z)[1+\epsilon\rho^N_t(z)]\frac{\partial}{\partial p'}\delta(z-z')\right\}\delta(t-t')\\ &= -2hk_{\rm B}T\frac{\partial}{\partial p}\left\{\rho^N_t(z)[1+\epsilon\rho^N_t(z)]\frac{\partial}{\partial p}\delta(z-z')\right\}\delta(t-t')\\ &= -2hk_{\rm B}TL_{\rm fp}(\phi_t(z))\delta(z-z')\delta(t-t').\end{aligned} \tag{C1}$$

Thus, the probability distribution of a trajectory $\zeta_{[0,t]}$ reads

$$P[\zeta_{[0,t]}] = \mathcal{N}[\phi_{[0,t]}]\mathrm{e}^{\frac{\beta}{4}\int_0^t \mathrm{d}s\int\mathrm{d}z\zeta_s(z)L_{\rm fp}(\phi_s)^{-1}\zeta_s(z)}, \tag{C2}$$

where

$$\mathcal{N}[\phi_{[0,t]}] = \int\mathcal{D}\zeta_{[0,t]}\mathrm{e}^{\frac{\beta}{4}\int_0^t \mathrm{d}s\int\mathrm{d}z\zeta_s(z)L_{\rm fp}(\phi_s)^{-1}\zeta_s(z)}. \tag{C3}$$

We obtain the propagator $P[\phi_{[0,t]}|\phi_0]$ through

$$\begin{aligned}P[\phi_{[0,t]}|\phi_0] &= \int\mathcal{D}\zeta_{[0,t]}P[\zeta_{[0,t]}]\prod_{(s,z)}\delta\left(\frac{\partial\phi_s(z)}{\partial s} - L_{\rm st}\phi_s(z) - L_{\rm fp}(\phi_s(z))\frac{\delta F[\phi_s]}{\delta\phi_s(z)} - \zeta_s(z)\right)\\ &= \mathcal{J}[\phi_{[0,t]}]\mathcal{N}[\phi_{[0,t]}]\mathrm{e}^{-\beta\mathcal{S}[\phi_{[0,t]}]},\end{aligned} \tag{C4}$$

where the Jacobian $\mathcal{J}[\phi_{[0,t]}]$ stemming from the variable transformation from ζ to ϕ is just a constant when the standard Ito forward discretization is applied [40].

According to Eq. (C3) and $\phi^\dagger_s(x,p) = \phi_{t-s}(x,-p)$, we have the following equality,

$$\begin{aligned}\mathcal{N}[\phi^\dagger_{[0,t]}] &= \int\mathcal{D}\zeta_{[0,t]}\mathrm{e}^{\frac{\beta}{4}\int_0^t \mathrm{d}s\int\mathrm{d}z\zeta_s(z)L_{\rm fp}(\phi^\dagger_s)^{-1}\zeta_s(z)}\\ &= \int\mathcal{D}\zeta_{[0,t]}\mathrm{e}^{\frac{\beta}{4}\int_0^t \mathrm{d}s\int\mathrm{d}z\zeta_s(z)L_{\rm fp}(\phi_s)^{-1}\zeta_s(z)}\\ &= \mathcal{N}[\phi_{[0,t]}],\end{aligned} \tag{C5}$$

where we have used the transformation $s\to t-s$, $p\to -p$, $\zeta_{t-s}(x,-p)\to\zeta_s(x,p)$.

Next, using the transformation $s\to t-s$, $p\to -p$, it is straightforward to verify that the action $\mathcal{S}[\phi^\dagger_{[0,t]}]$ [Eq. (10)] can be rewritten in terms of $\phi_{[0,t]}$:

$$\mathcal{S}[\phi^\dagger_{[0,t]}] = -\frac{1}{4}\int_0^t\mathrm{d}s\int\mathrm{d}z\left[\frac{\partial\phi_s}{\partial s} - L_{\rm st}\phi_s + L_{\rm fp}(\phi_s)\frac{\delta F[\phi_s]}{\delta\phi_s}\right]L_{\rm fp}(\phi_s)^{-1}\left[\frac{\partial\phi_s}{\partial s} - L_{\rm st}\phi_s + L_{\rm fp}(\phi_s)\frac{\delta F[\phi_s]}{\delta\phi_s}\right]. \tag{C6}$$

Finally, we derive the detailed fluctuation theorem,

$$\begin{aligned}\ln\frac{P[\phi_{[0,t]}|\phi_0]}{P^\dagger[\phi^\dagger_{[0,t]}|\phi^\dagger_0]} &= -\beta(\mathcal{S}[\phi_{[0,t]}] - \mathcal{S}[\phi^\dagger_{[0,t]}])\\ &= -\beta\int_0^t\mathrm{d}s\int\mathrm{d}z\frac{\partial\phi_s}{\partial s}\frac{\delta F[\phi_s;\lambda_s]}{\delta\phi_s}\\ &= -\beta(F[\phi_t;\lambda_t] - F[\phi_0;\lambda_0] - w[\phi_{[0,t]}])\\ &= \ln\frac{P_t[\phi_t]}{P_0[\phi_0]} + \frac{s_{\rm p}[\phi_{[0,t]}]}{k_{\rm B}},\end{aligned} \tag{C7}$$

where in the first line we have used Eq. (C5), in the second line we have used Eqs. (10) and (C6), in the third line we have used $\frac{\mathrm{d}\lambda_t}{\mathrm{d}t}\frac{\partial F[\phi_t;\lambda_t]}{\partial\lambda_t} = \frac{\mathrm{d}\lambda_t}{\mathrm{d}t}\frac{\partial E[\phi_t;\lambda_t]}{\partial\lambda_t} = \frac{\mathrm{d}}{\mathrm{d}t}w[\phi_{[0,t]}]$, and in the fourth line we have used the definitions of the stochastic thermodynamic quantities [Eqs. (11)–(16)].

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