Imaginarity of Gaussian states

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It is a long-standing debate that why quantum mechanics uses complex numbers but not real numbers only. To address this topic, in recent years, the imaginarity theory has been developed in the way of quantum resource theory. However, the existing imaginarity theory mainly focuses on the quantum systems with finite dimensions. Gaussian states are widely used in many fields of quantum physics, but they are in the quantum systems with infinite dimensions. In this paper we establish a resource theory of imaginarity for bosonic Gaussian states. To do so, under the Fock basis, we determine the real Gaussian states and real Gaussian channels in terms of the means and covariance matrices of Gaussian states. Also, we provide two imaginary measures for Gaussian states based on the fidelity.

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I. INTRODUCTION

Complex numbers are widely used in both physics and mathematics. It has been a long-standing debate since the inception of quantum mechanics that why quantum mechanics uses complex numbers but not real numbers only. To improve this topic, recently, the imaginarity theory has been developed [1–11]. We consider a quantum system associated with the complex Hilbert space H and choose the orthonormal basis $\{|j\rangle\}_{j=1}^{d}$ of H, with d being the dimension of H. Imaginarity theory is basis dependent, and when we talk about imaginarity theory, we always preset an orthonormal basis. A quantum state represented by a density operator ρ is called real with respect to $\{|j\rangle\}_{j=1}^{d}$ if $\rho_{jk} = \langle j|\rho|k\rangle \in R$ for all j and k; here R denotes the set of all real numbers. A quantum operation [12] ϕ on H is often represented by a set of Kraus operators $\phi = \{K_\mu\}_\mu$ satisfying $\sum_\mu K_\mu^\dagger K_\mu \preceq I$, where $K_\mu^\dagger$ is the adjoint of K_μ, I is the identity operator, and $\sum_\mu K_\mu^\dagger K_\mu \preceq I$ means $I - \sum_\mu K_\mu^\dagger K_\mu \succeq 0$, i.e., $I - \sum_\mu K_\mu^\dagger K_\mu$ is positive semidefinite. A quantum operation $\phi = \{K_\mu\}_\mu$ is called a quantum channel if $\sum_\mu K_\mu^\dagger K_\mu = I$. In imaginarity theory, an operation ϕ is called real if ϕ can be expressed by a set of Kraus operators $\phi = \{K_\mu\}_\mu$ and $K_\mu \rho K_\mu^\dagger$ is real for any μ and any real state ρ.

Imaginarity theory can be viewed as a quantum resource theory. Quantum resource theories provide a powerful way to characterize certain quantum properties of a quantum system [13,14]. The well-known quantum resource theories are entanglement theory [15,16] and coherence theory [17–20]. Besides, other quantum resources have been developed, such as quantum thermodynamics [21,22], purity [23–25], nonlocality [26], quantum phase [27], and continuous-variable quantum resource theories [28–30]. A quantum resource theory for quantum states has two basic ingredients, free states and free operations. Resource measure and state transformation are two main topics in a quantum resource theory for quantum states. Imaginarity theory characterizes the property that a quantum state may be complex but not real. In imaginarity theory, the free states are real states and free operations are real operations. State transformations under real operations have been extensively studied [8]. Several imaginarity measures have been proposed [1–3,7,8]. Some results of imaginary theory have been experimentally tested [2,4–6,9].

The imaginarity theory discussed above mainly focuses on finite-dimensional quantum states. When we attempt to apply the concepts and results of imaginarity theory to infinite-dimensional quantum states, two problems occur. First, for the quantum states and quantum operations on infinite-dimensional systems, there may be some "divergence" difficulties, such as the energy of a quantum state, then some definitions for finite-dimensional states can no longer be well defined for infinite-dimensional states. Second, even if a definition or result is still well defined for infinite-dimensional states in a sense, there still may be a problem that this definition or result is hard to evaluate. These problems are similar to the cases of coherence theory. In coherence theory, the l_1 norm of coherence $C_{l_1}(\rho) = \sum_{j\neq k} |\langle j|\rho|k\rangle|$ is a valid coherence measure [17] and can be easily calculated for finite-dimensional states. But $C_{l_1}(\rho)$ may diverge for some infinite-dimensional states [31]. In coherence theory, the relative entropy of coherence $C_{\mathrm{r}}(\rho) = S(\rho_{\mathrm{diag}}) - S(\rho)$ is a valid coherence measure [17] which can be easily calculated for finite-dimensional states, but $C_{\mathrm{r}}(\rho)$ is hard to calculate for some infinite-dimensional states [31–33], where $S(\rho) = -\mathrm{tr}(\rho \log_2 \rho)$ is the Von Neumann entropy and ρ_{diag} is the diagonal part of ρ.

Bosonic Gaussian states are a class of infinite-dimensional states, which are widely used in quantum optics and quantum information theory [34–40]. A Gaussian state ρ is completely and conventionally described by its mean $\overline{X}$ and covariance matrix V; then we write ρ as $\rho(\overline{X}, V)$. The Fock basis is the orthonormal basis spanning the complex Hilbert space the Gaussian states are in; then it is natural to choose the Fock basis as the fixed basis for imaginarity theory of Gaussian states. So far, several imaginarity measures for finite-dimensional

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states have been proposed, such as $I_{tr}(\rho)$ based on the trace norm [1,2], $I_r(\rho)$ based on the Von Neumann entropy [7], and $I_f(\rho)$ based on the fidelity [3,8]; they are defined as

$$I_{tr}(\rho) = ||\rho - \rho^*||_{tr}, \tag{1}$$

$$I_r(\rho) = S(\text{Re}\rho) - S(\rho), \tag{2}$$

$$I_f(\rho) = 1 - F(\rho, \rho^*), \tag{3}$$

where ρ^* is the conjugate of ρ, $||\cdot||_{tr}$ denotes the trace norm, Reρ is the real part of ρ, and $F(\rho,\sigma) = \text{tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}$ is the fidelity of states ρ and σ [41,42]. We consider whether Eqs. (1), (2), and (3) are applicable to Gaussian states. Till now, to calculate ρ^*, Reρ, and $||\rho - \rho^*||_{tr}$ for general Gaussian states is very hard since it is hard to express general Gaussian states in the Fock basis [31–33,43–45]. $F(\rho,\sigma)$ has a closed expression for Gaussian states ρ and σ in terms of their means and covariance matrices [46], but we do not know whether ρ^* is a Gaussian state. Moreover, we do not even know which Gaussian states are real in terms of means and covariances.

In this paper we study the imaginarity of bosonic Gaussian states. We establish a resource theory of imaginarity for bosonic Gaussian states. This paper is structured as follows. In Sec. II, we determine the conditions for real Gaussian states and the conjugate of a Gaussian state under the Fock basis in terms of means and covariances. In Sec. III, we characterize the structure of real Gaussian channels. In Sec. IV, we provide two imaginary measures for Gaussian states based on the fidelity, and they all have explicit expressions. Section V is a brief summary and outlook. For structural clarity, we focus on stating the theoretical framework and results in the main text, and we put most of the proofs in the appendices.

II. REAL GAUSSIAN STATES AND THE CONJUGATE OF A GAUSSIAN STATE

In this section we determine the real Gaussian states and the conjugate of a Gaussian state. We first recall some basics and give the notation we use for Gaussian states. We denote the one-mode Fock basis by $\{|j\rangle\}_{j=0}^{\infty}$, with $j \in \{0, 1, 2, 3, \ldots\}$; $\{|j\rangle\}_{j=0}^{\infty}$ is an orthonormal basis spanning the complex Hilbert space $\overline{H}$. $\overline{H}$ is a countable but infinite-dimensional complex Hilbert space. The N-mode Fock basis is $\{|j\rangle\}_j^{\otimes N}$, the N-fold tensor product of $\{|j\rangle\}_{j=0}^{\infty}$, and $\{|j\rangle\}_j^{\otimes N}$ spans the complex Hilbert space $\overline{H}^{\otimes N} = \otimes_{l=1}^{N}\overline{H}_l$ with each $\overline{H}_l = \overline{H}$. On each $\overline{H}_l$, the bosonic field operators, annihilation operator $\widehat{a}_l$ and creation operator $\widehat{a}_l^{\dagger}$, are defined as

$$\widehat{a}_l|0\rangle = 0,\ \ \widehat{a}_l|j\rangle = \sqrt{j}|j-1\rangle \text{ for } j \geqslant 1; \tag{4}$$

$$\widehat{a}_l^{\dagger}|j\rangle = \sqrt{j+1}|j+1\rangle \text{ for } j \geqslant 0. \tag{5}$$

We arrange $\{\widehat{a}_l, \widehat{a}_l^{\dagger}\}_{l=1}^N$ as a vector as

$$\begin{aligned}\widehat{A} &= (\widehat{a}_1, \widehat{a}_1^{\dagger}, \widehat{a}_2, \widehat{a}_2^{\dagger}, \ldots, \widehat{a}_N, \widehat{a}_N^{\dagger})^T \\ &= (\widehat{A}_1, \widehat{A}_2, \widehat{A}_3, \widehat{A}_4, \ldots, \widehat{A}_{2N-1}, \widehat{A}_{2N})^T,\end{aligned} \tag{6}$$

with T standing for the transposition.

From the bosonic field operators $\{\widehat{a}_l, \widehat{a}_l^{\dagger}\}_{l=1}^N$, we can define the quadrature field operators $\{\widehat{q}_l, \widehat{p}_l^{\dagger}\}_{l=1}^N$ as

$$\widehat{q}_l = \widehat{a}_l + \widehat{a}_l^{\dagger},\ \ \widehat{p}_l = -i(\widehat{a}_l - \widehat{a}_l^{\dagger}), \tag{7}$$

where $i = \sqrt{-1}$. We arrange $\{\widehat{q}_l, \widehat{p}_l^{\dagger}\}_{l=1}^N$ as a vector as

$$\begin{aligned}\widehat{X} &= (\widehat{q}_1, \widehat{p}_1, \widehat{q}_2, \widehat{p}_2, \ldots, \widehat{q}_N, \widehat{p}_N)^T \\ &= (\widehat{X}_1, \widehat{X}_2, \widehat{X}_3, \widehat{X}_4, \ldots, \widehat{X}_{2N-1}, \widehat{X}_{2N})^T.\end{aligned} \tag{8}$$

Under these definitions, we obtain the canonical commutation relations

$$[\widehat{A}_l, \widehat{A}_m] = \Omega_{lm}, \tag{9}$$

$$[\widehat{X}_l, \widehat{X}_m] = 2i\Omega_{lm}, \tag{10}$$

where $[\widehat{A}_l, \widehat{A}_m] = \widehat{A}_l\widehat{A}_m - \widehat{A}_m\widehat{A}_l$ is the commutator of $\widehat{A}_l$ and $\widehat{A}_m$, and Ω_{lm} is the element of the $2N \times 2N$ matrix Ω with

$$\Omega = \oplus_{n=1}^N \omega,\ \ \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{11}$$

A quantum state ρ in $\overline{H}^{\otimes N}$ can be characterized by its characteristic function

$$\chi(\rho, \xi) = \text{tr}[\rho D(\xi)], \tag{12}$$

where $D(\xi)$ is the displacement operator,

$$D(\xi) = \exp(i\widehat{X}^T\Omega\xi), \tag{13}$$

$$\xi = (\xi_1, \xi_2, \ldots, \xi_{2N})^T \in R^{2N}. \tag{14}$$

For state ρ in $\overline{H}^{\otimes N}$, the mean of ρ is

$$\overline{X} = \text{tr}(\rho\widehat{X}) = (\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_{2N})^T; \tag{15}$$

the covariance matrix V is defined by its elements

$$V_{lm} = \tfrac{1}{2}\text{tr}(\rho\{\Delta\widehat{X}_l, \Delta\widehat{X}_m\}), \tag{16}$$

with $\Delta\widehat{X}_l = \widehat{X}_l - \overline{X}_l$, and $\{\Delta\widehat{X}_l, \Delta\widehat{X}_m\} = \Delta\widehat{X}_l\Delta\widehat{X}_m + \Delta\widehat{X}_m\Delta\widehat{X}_l$ is the anticommutator of $\Delta\widehat{X}_l$ and $\Delta\widehat{X}_m$. The covariance matrix $V = V^T$ is a $2N \times 2N$ real and symmetric matrix which must satisfy the uncertainty principle [47]

$$V + i\Omega \succeq 0. \tag{17}$$

Note that $V + i\Omega \succeq 0$ implies $V \succ 0$, meaning that V is positive definite.

With these preparations, we turn to the definition of Gaussian states. A quantum state ρ in $\overline{H}^{\otimes N}$ is called an N-mode Gaussian state if its characteristic function has the Gaussian form

$$\chi(\rho, \xi) = \exp\left[-\tfrac{1}{2}\xi^T(\Omega V\Omega^T)\xi - i(\Omega\overline{X})^T\xi\right], \tag{18}$$

where $\overline{X}$ is the mean of ρ and V is the covariance matrix of ρ. The Gaussian state ρ is determined by its characteristic function $\chi(\rho, \xi)$ via the inverse relation (see, for example, Chap. 4 in Ref. [40])

$$\rho = \int \frac{d^{2N}\xi}{\pi^N}\chi(\rho, \xi)D(-\xi), \tag{19}$$

where $\int = \int_{-\infty}^{\infty}$. $\overline{X}$ and V with Eq. (17) completely determine the Gaussian state ρ [47], and thus we write ρ as $\rho(\overline{X}, V)$.

Now we consider the question of under what conditions on $\overline{X}$ and V, ρ is a real Gaussian state, i.e., $\langle j_1|\langle j_2|\ldots|j_N|\rho|k_1\rangle|k_2\rangle\ldots|k_N\rangle\in R$ for any Fock basis vectors $\{|j_1\rangle,|j_2\rangle|,\ldots,|j_N\rangle;|k_1\rangle,|k_2\rangle,\ldots,|k_N\rangle\}$. For this question, we have Theorem 1 below; we provide a proof for Theorem 1 in Appendixes A and F.

Theorem 1. The N-mode Gaussian state $\rho(\overline{X},V)$ is real if and only if

$$\overline{X}_{2l}=0 \text{ for } l\in\{1,2,\ldots,N\}, \tag{20}$$

$$V_{2l-1,2m}=0 \text{ for } l,m\in\{1,2,\ldots,N\}. \tag{21}$$

If one of $\{\overline{X}_{2l},V_{2l-1,2m}\}_{l,m=1}^{N}$ is nonzero, then there exists $\langle j_1|\langle j_2|\ldots|j_N|\rho|k_1\rangle|k_2\rangle\ldots|k_N\rangle\notin R$ for $\rho(\overline{X},V)$; $\rho(\overline{X},V)$ is called not real. When $\rho(\overline{X},V)$ is not real, we further ask how about the conjugate ρ^* of $\rho(\overline{X},V)$. Is ρ^* still a Gaussian state? If ρ^* is still a Gaussian state, then how about the mean and covariance matrix of ρ^*? Theorem 2 below answers these questions; we provide a proof for Theorem 2 in Appendixes B and F.

Theorem 2. For the N-mode Gaussian state $\rho(\overline{X},V)$, the conjugate state of $\rho(\overline{X},V)$ is still a Gaussian state. We denote the conjugate state of $\rho(\overline{X},V)$ by $\rho^*(\overline{X}',V')$ with the mean $\overline{X}'$ and covariance matrix V', and then

$$\overline{X}_l'=(-1)^{l+1}\overline{X}_l,\ l\in\{1,2,\ldots,2N\}, \tag{22}$$

$$V_{lm}'=(-1)^{l+m}V_{lm},\ l,m\in\{1,2,\ldots,2N\}. \tag{23}$$

With Theorem 1, Theorem 2, and Eqs. (15) and (16), we see that, for the N-mode Gaussian state $\rho(\overline{X},V)$, the real part of ρ, $\mathrm{Re}\rho=\frac{\rho+\rho^*}{2}$, has the mean $\frac{\overline{X}+\overline{X}'}{2}$ and the covariance matrix $\frac{V+V'}{2}$, and $(\frac{\overline{X}+\overline{X}'}{2},\frac{V+V'}{2})$ determines a real Gaussian state since

$$\frac{V+V'}{2}+i\Omega=\frac{(V+i\Omega)+(V'+i\Omega)}{2}\succeq 0. \tag{24}$$

Then one may ask the question of whether $\mathrm{Re}\rho$ is a Gaussian state. The answer to this question is negative. That is, if ρ is a Gaussian state, then $\mathrm{Re}\rho$ is not a Gaussian state in general. We can check this fact using the Glauber coherent state in Example 2 below. With this observation, for the N-mode Gaussian state $\rho(\overline{X},V)$, we define a real Gaussian state $\overline{\rho}$ having the mean $\frac{\overline{X}+\overline{X}'}{2}$ and the covariance matrix $\frac{V+V'}{2}$; we write $\overline{\rho}$ as $\overline{\rho}(\frac{\overline{X}+\overline{X}'}{2},\frac{V+V'}{2})$. We call $\overline{\rho}(\frac{\overline{X}+\overline{X}'}{2},\frac{V+V'}{2})$ the real Gaussian state induced by the Gaussian state $\rho(\overline{X},V)$. Obviously, for the Gaussian state $\rho(\overline{X},V)$, we have

$$\overline{\rho}=(\overline{\rho})^*=\overline{\rho^*}, \tag{25}$$

and ρ is real (i.e., $\rho=\mathrm{Re}\rho$) if and only if $\rho=\overline{\rho}$. In general, $\mathrm{Re}\rho\neq\overline{\rho}$ for the Gaussian state $\rho(\overline{X},V)$.

III. REAL GAUSSIAN CHANNELS

A Gaussian channel ϕ on $\overline{H}^{\otimes N}$ can be represented by $\phi=(d,T,N)$, here $d=(d_1,d_2,\ldots,d_{2N})^T\in R^{2N}$, and T and $N=N^T$ are $2N\times 2N$ real matrices. $\phi=(d,T,N)$ maps the Gaussian state $\rho(\overline{X},V)$ to the Gaussian state with the mean and the covariance matrix given as

$$\overline{X}\to T\overline{X}+d,\quad V\to TVT^T+N, \tag{26}$$

and $\phi=(d,T,N)$ fulfills the complete positivity condition

$$N+i\Omega-iT\Omega T^T\succeq 0. \tag{27}$$

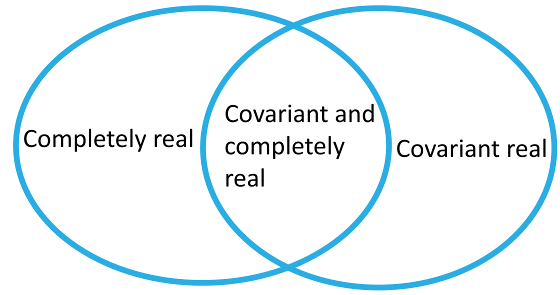

FIG. 1. Classification of real Gaussian channels.

We then define that a Gaussian channel is real if it maps any real Gaussian state to a real Gaussian state. For the structure of real Gaussian channels, we have Theorem 3 below; we provide a proof for Theorem 3 in Appendix C.

Theorem 3. The N-mode Gaussian channel $\phi=(d,T,N)$ is real if and only if

$$d_{2l}=0 \text{ for } l\in\{1,2,\ldots,N\}, \tag{28}$$

$$N_{2l-1,2m}=0 \text{ for } l,m\in\{1,2,\ldots,N\}, \tag{29}$$

and one of Eqs. (30) and (31) below:

$$T_{2l,2m-1}=T_{2l,2m}=0 \text{ for } l,m\in\{1,2,\ldots,N\}, \tag{30}$$

$$T_{2l-1,2m}=T_{2m,2l-1}=0 \text{ for } l,m\in\{1,2,\ldots,N\}. \tag{31}$$

We discuss the properties of real Gaussian channels. If a real Gaussian channel ϕ fulfills Eq. (30), we call it a completely real Gaussian channel. If a real Gaussian channel ϕ fulfills Eq. (31), we call it a covariant real Gaussian channel. The meanings of these definitions are explained in Theorem 4 below. We give a proof for Theorem 4 in Appendix D. In particular, if a real Gaussian channel ϕ fulfills both Eqs. (30) and (31), we call it a covariant and completely real Gaussian channel. Such a classification of real Gaussian channels is shown in Fig. 1.

Theorem 4. If ϕ is a completely real Gaussian channel, then $\phi(\rho)$ is real for any Gaussian state ρ. If ϕ is a covariant real Gaussian channel, then for any Gaussian state ρ, we have

$$[\phi(\rho)]^*=\phi(\rho^*), \tag{32}$$

$$\overline{\phi(\rho)}=\phi(\overline{\rho}). \tag{33}$$

IV. IMAGINARITY MEASURES OF GAUSSIAN STATES

An imaginarity measure $M(\rho)$ for N-mode Gaussian states is a real-valued functional on Gaussian states. In the spirit of quantum resource theory, we propose that any imaginarity measure $M(\rho)$ for N-mode Gaussian states should satisfy the following two conditions.

(M1) Faithfulness: $M(\rho)\geqslant 0$ for any state ρ and $M(\rho)=0$ if and only if ρ is real.

(M2) Monotonicity: $M(\phi(\rho)) \leqslant M(\rho)$ for any state ρ and any real Gaussian channel ϕ.

We provide two imaginarity measures based on the fidelity for Gaussian states in Theorem 5 below. We give a proof for Theorem 5 in Appendix E.

Theorem 5. For any N-mode Gaussian state $\rho(\overline{X}, V)$,

$$M(\rho) = 1 - F(\rho, \rho^*) \tag{34}$$

and

$$M'(\rho) = 1 - F(\rho, \overline{\rho}) \tag{35}$$

are all imaginarity measures; i.e., $M(\rho)$ and $M'(\rho)$ both satisfy (M1) and (M2).

From the definitions of $M(\rho)$ and $M'(\rho)$, we see that $M(\rho)$ and $M'(\rho)$ have the property of conjugation invariance:

$$M(\rho) = M(\rho^*), \quad M'(\rho) = M'(\rho^*). \tag{36}$$

It is shown that $1 - F(\rho, \rho^*)$ in Eq. (3) is a valid imaginarity measure for finite-dimensional states [3,8]. We have shown that if ρ is a Gaussian state, then ρ^* and $\overline{\rho}$ are all Gaussian states. Then the calculation of $M(\rho)$ and $M'(\rho)$ is about the calculation of the fidelity for two Gaussian states. The expression of the fidelity $F(\rho, \sigma)$ for two Gaussian states ρ and σ has been studied for many years [46,48–53], and in Ref. [46] an explicit expression of $F(\rho, \sigma)$ for any two N-mode Gaussian states was provided. Consequently, $M(\rho)$ and $M'(\rho)$ have explicit expressions via the explicit expression of $F(\rho, \sigma)$ for any two N-mode Gaussian states [46]. Below we discuss some special one-mode Gaussian states to demonstrate the calculation of $M(\rho)$ and $M'(\rho)$.

For any two one-mode Gaussian states $\rho(\overline{X}, V)$ and $\sigma(\overline{Y}, W)$, the fidelity $F(\rho, \sigma)$ has the following expression [46,53]:

$$F(\rho, \sigma) = \frac{\exp\left[-\frac{1}{4}(\overline{X} - \overline{Y})^T (V + W)^{-1} (\overline{X} - \overline{Y})\right]}{\sqrt{\sqrt{\det\left(\frac{V+W}{2}\right) + \Lambda} - \sqrt{\Lambda}}}, \tag{37}$$

$$\Lambda = 4 \det\left(\frac{V + i\Omega}{2}\right) \det\left(\frac{W + i\Omega}{2}\right). \tag{38}$$

With these expressions we can directly calculate $M(\rho)$ and $M'(\rho)$ for any one-mode Gaussian state ρ.

Corollary 1. For the one-mode Gaussian state $\rho(\overline{X}, V)$, the imaginarity measures $M(\rho)$ in Eq. (34) and $M'(\rho)$ in Eq. (35) become

$$M(\rho) = 1 - \frac{\exp\left(-\frac{\overline{X}_2^2}{2V_{22}}\right)}{\sqrt{\sqrt{V_{11}V_{22} + \Lambda} - \sqrt{\Lambda}}}, \tag{39}$$

$$\Lambda = \frac{\left(V_{11}V_{22} - V_{12}^2 - 1\right)^2}{4}; \tag{40}$$

$$M'(\rho) = 1 - \frac{\exp\left[-\frac{V_{11}\overline{X}_2^2}{2(4V_{11}V_{22} - V_{12}^2)}\right]}{\sqrt{\sqrt{\left(V_{11}V_{22} - \frac{1}{4}V_{12}^2\right) + \Lambda'} - \sqrt{\Lambda'}}}, \tag{41}$$

$$\Lambda' = \frac{1}{4}\left(V_{11}V_{22} - V_{12}^2 - 1\right)\left(V_{11}V_{22} - 1\right). \tag{42}$$

We discuss some classes of special one-mode Gaussian states: the thermal states, the Glauber coherent states, and the squeezed states. These classes of Gaussian states are widely used in quantum optics and quantum information theory. For the one-mode case, we also write the Fock basis as $\{|j\rangle\}_{j=0}^{\infty} = \{|n\rangle\}_{n=0}^{\infty}$, and we write the creation and annihilation operators as $\widehat{a}_1 = \widehat{a}$ and $\widehat{a}_1^{\dagger} = \widehat{a}^{\dagger}$.

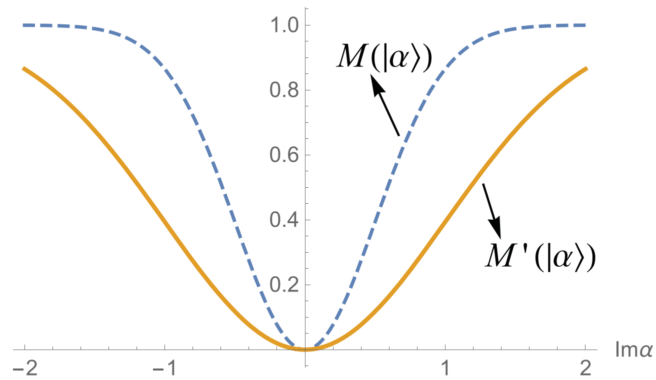

FIG. 2. $M(|\alpha\rangle)$ and $M'(|\alpha\rangle)$ versus Imα in Eqs. (45) and (46).

Example 1. Consider the one-mode thermal state

$$\rho_{\text{th}}(\overline{n}) = \sum_{n=0}^{\infty} \frac{\overline{n}^n}{(\overline{n} + 1)^{n+1}} |n\rangle\langle n|, \tag{43}$$

with $\overline{n} = \text{tr}[\widehat{a}^{\dagger}\widehat{a}\rho_{\text{th}}(\overline{n})]$ being the mean number of $\rho_{\text{th}}(\overline{n})$. The mean of $\rho_{\text{th}}(\overline{n})$ is $\overline{X} = (0, 0)^T$, and the covariance matrix of $\rho_{\text{th}}(\overline{n})$ is $V = (2\overline{n} + 1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then Eqs. (39)–(42) yield $M(\rho_{\text{th}}(\overline{n})) = M'(\rho_{\text{th}}(\overline{n})) = 0$. In fact, $M(\rho_{\text{th}}(\overline{n})) = M'(\rho_{\text{th}}(\overline{n})) = 0$ is an obvious result, since the matrix elements of $\rho_{\text{th}}(\overline{n})$ are all real in the Fock basis; i.e., $\rho_{\text{th}}(\overline{n})$ is a real Gaussian state.

Example 2. Consider the one-mode Glauber coherent state

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \tag{44}$$

with α being any complex number. The mean of $|\alpha\rangle\langle\alpha|$ is $\overline{X} = (2\text{Re}\alpha, 2\text{Im}\alpha)^T$, and the covariance matrix of $|\alpha\rangle\langle\alpha|$ is $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then Eqs. (39)–(42) yield

$$M(|\alpha\rangle) = 1 - e^{-2(\text{Im}\alpha)^2}, \tag{45}$$

$$M'(|\alpha\rangle) = 1 - e^{-\frac{(\text{Im}\alpha)^2}{2}}. \tag{46}$$

We see that $M(|\alpha\rangle) > M'(|\alpha\rangle) > 0$ when $\alpha \notin R$; $M(|\alpha\rangle) = M'(|\alpha\rangle) = 0$ if and only if $\alpha \in R$; $M(|\alpha\rangle)$ and $M(|\alpha\rangle)$ increase as $|\text{Im}\alpha|$ increases; and $M(|\alpha\rangle)$ and $M'(|\alpha\rangle)$ are independent of Reα. We depict Eqs. (45) and (46) in Fig. 2.

Example 3. Consider the one-mode squeezed state

$$|\zeta\rangle = \exp\left[\frac{1}{2}(\zeta^*\widehat{a}^2 - \zeta\widehat{a}^{\dagger 2})\right]|0\rangle \tag{47}$$

$$= \frac{1}{\sqrt{\cosh|\zeta|}} \sum_{n=0}^{\infty} (-e^{i\theta} \tanh|\zeta|)^n \frac{\sqrt{(2n)!}}{2^n n!} |2n\rangle, \tag{48}$$

with ζ being any complex number and $\zeta = |\zeta|e^{i\theta}$ its polar form. $\exp[\frac{1}{2}(\zeta^*\widehat{a}^2 - \zeta\widehat{a}^{\dagger 2})]$ is the squeezing operator. The mean of $|\zeta\rangle\langle\zeta|$ is $\overline{X} = (0, 0)^T$, and the covariance matrix V

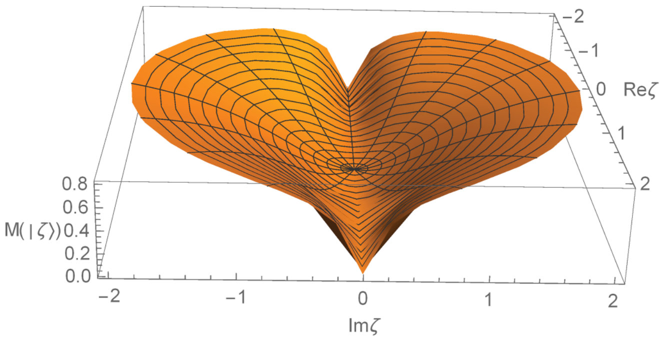

FIG. 3. $M(|\zeta\rangle)$ in Eq. (50).

of $|\zeta\rangle\langle\zeta|$ is

$$V_{11} = \cosh(2|\zeta|) + \cos\theta \sinh(2|\zeta|),$$
$$V_{12} = V_{21} = \sin\theta \sinh(2|\zeta|), \quad (49)$$
$$V_{22} = \cosh(2|\zeta|) - \cos\theta \sinh(2|\zeta|).$$

Then Eqs. (39)–(42) yield

$$M(|\zeta\rangle) = 1 - \frac{1}{\sqrt[4]{1 + \sin^2\theta \sinh^2(2|\zeta|)}}, \quad (50)$$

$$M'(|\zeta\rangle) = 1 - \frac{1}{\sqrt[4]{1 + \frac{3}{4}\sin^2\theta \sinh^2(2|\zeta|)}}. \quad (51)$$

We see that if $\zeta \notin R$ then $M(|\zeta\rangle) > M'(|\zeta\rangle) > 0$; $M(|\zeta\rangle)$ and $M'(|\zeta\rangle)$ increase as $|\zeta|$ increases; and $M(|\zeta\rangle) = M'(|\zeta\rangle) = 0$ if and only if $\zeta \in R$. We depict Eq. (50) in Fig. 3, and compare Eqs. (50) and (51) in Fig. 4.

V. SUMMARY AND OUTLOOK

We established a resource theory of imaginarity for Gaussian states. To this aim, under the Fock basis, we determined the real Gaussian states and real Gaussian channels via the means and covariances of Gaussian states. We provided two imaginarity measures based on the fidelity which all have closed expressions. As a by-product, we proved that the conjugate of a Gaussian state is still a Gaussian state. We also discussed the imaginarity of some one-mode Gaussian states.

There remained many open questions for future explorations. First, for the two imaginarity measures $M(\rho)$ and $M'(\rho)$ provided in this work, are there some physically operational interpretations linked to them? Second, does $M(\rho) \geqslant M'(\rho)$ hold for all Gaussian states? Third, are there some other imaginarity measures for Gaussian states satisfying the conditions (M1) and (M2) in this work? Last, the properties of state conversions under real Gaussian channels are worthy of further investigations.

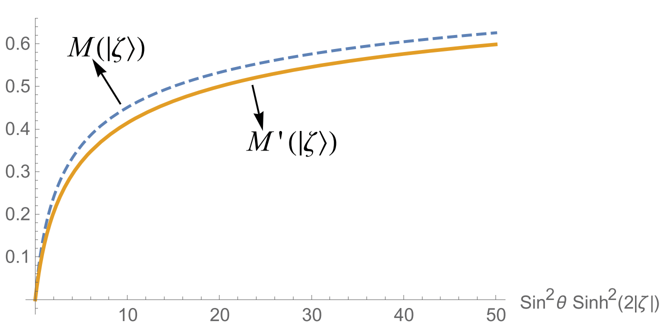

FIG. 4. $M(|\zeta\rangle)$ and $M'(|\zeta\rangle)$ versus $\sin^2\theta \sinh^2(2|\zeta|)$ in Eqs. (50) and (51).

ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF THEOREM 1

We set three steps to prove Theorem 1.

Step (A.i). We first prove that if the Gaussian state $\rho(\overline{X}, V)$ is real then $\{X_{2l} = 0\}_{l=1}^N$ and $\{V_{2l-1,2m} = 0\}_{l,m=1}^N$; this step is comparatively straightforward.

Expand the Gaussian state $\rho(\overline{X}, V)$ in the Fock basis $\{|j\rangle\}_j^{\otimes N}$ as

$$\rho(\overline{X}, V) = \sum_{j_1,k_1,\ldots,j_N,k_N=0}^{\infty} \rho_{j_1k_1,j_2k_2,\ldots,j_Nk_N} \times |j_1\rangle\langle k_1| \otimes |j_2\rangle\langle k_2| \otimes \ldots \otimes |j_N\rangle\langle k_N|, \quad (A1)$$

where $\rho_{j_1k_1,j_2k_2,\ldots,j_Nk_N} = \langle j_1|\langle j_2|\ldots\langle j_N|\rho|k_1\rangle|k_2\rangle\ldots|k_N\rangle \in R$ for any $\{j_1, k_1, j_2, k_2, \ldots, j_N, k_N\} \subset \{0, 1, 2, \ldots\}$. We also use the symbols $\rho_{j_1k_1} = \langle j_1|\rho^{(1)}|k_1\rangle$, with $\rho^{(1)}$ being the reduced state of ρ to the first mode, and $\rho_{j_1k_1;j_2k_2} = \langle j_1|\langle j_2|\rho^{(12)}|k_1\rangle|k_2\rangle$, with $\rho^{(12)}$ being the two-mode reduced state of ρ to the first and second modes.

Without loss of generality, we only need to prove that if $\rho(\overline{X}, V)$ is real then $\overline{X}_2 = 0$ and $V_{12} = V_{14} = V_{23} = 0$. Note that $\overline{X}_1$ is not on an equal footing with $\overline{X}_2$ by the definition of Eqs. (7), (8), and (15); any $\overline{X}_{2l-1}$ ($\overline{X}_{2l}$) is on an equal footing with $\overline{X}_1$ ($\overline{X}_2$). Similarly, V_{12}, V_{14}, and V_{23} have distinct meanings by the definition of Eqs. (7), (8), and (16); any $V_{2l-1,2m}$ has the similar situation with one of $\{V_{12}, V_{14}, V_{23}\}$.

From Eqs. (A1), (4), (5), (7), (8), and (15), direct calculations show that

$$\mathrm{tr}(\rho\widehat{a}_1) = \sum_{j_1=0}^{\infty} \rho^*_{j_1,j_1+1}\sqrt{j_1+1}, \quad (A2)$$

$$\mathrm{tr}(\rho\widehat{a}_1^\dagger) = \sum_{j_1=0}^{\infty} \rho_{j_1,j_1+1}\sqrt{j_1+1}, \quad (A3)$$

$$\overline{X}_1 = \mathrm{tr}(\rho\widehat{a}_1) + \mathrm{tr}(\rho\widehat{a}_1^\dagger), \quad (A4)$$

$$\overline{X}_2 = -i[\mathrm{tr}(\rho\widehat{a}_1) - \mathrm{tr}(\rho\widehat{a}_1^\dagger)]. \quad (A5)$$

We see that if $\rho(\overline{X}, V)$ is real, then $\overline{X}_2 = 0$.

To express V_{12}, V_{14}, and V_{23}, from Eqs. (A1), (4), (5), (7), (8), and (16) we derive that

$$\mathrm{tr}(\rho\widehat{a}_1^2) = \sum_{j_1=0}^{\infty}\rho^*_{j_1,j_1+2}\sqrt{(j_1+1)(j_1+2)}, \quad (A6)$$

$$\mathrm{tr}(\rho\widehat{a}_1^{\dagger 2}) = \sum_{j=0}^{\infty}\rho_{j_1,j_1+2}\sqrt{(j+1)(j+2)}, \quad (A7)$$

$$V_{12} = -i[\mathrm{tr}(\rho\widehat{a}_1^2) - \mathrm{tr}(\rho\widehat{a}_1^{\dagger 2})] - \overline{X}_1\overline{X}_2; \quad (A8)$$

$$\mathrm{tr}(\rho\widehat{a}_1\widehat{a}_2) = \sum_{j_1=0,j_2=0}^{\infty}\rho^*_{j_1,j_1+1;j_2,j_2+1}\sqrt{(j_1+1)(j_2+1)}, \quad (A9)$$

$$\mathrm{tr}(\rho\widehat{a}_1^\dagger\widehat{a}_2^\dagger) = \sum_{j_1=0,j_2=0}^{\infty}\rho_{j_1,j_1+1;j_2,j_2+1}\sqrt{(j_1+1)(j_2+1)}, \quad (A10)$$

$$\mathrm{tr}(\rho\widehat{a}_1\widehat{a}_2^\dagger) = \sum_{j_1=0,j_2=1}^{\infty}\rho^*_{j_1,j_1+1;j_2,j_2-1}\sqrt{(j_1+1)j_2}, \quad (A11)$$

$$\mathrm{tr}(\rho\widehat{a}_1^\dagger\widehat{a}_2) = \sum_{j_1=0,j_2=1}^{\infty}\rho_{j_1,j_1+1;j_2,j_2-1}\sqrt{(j_1+1)j_2}, \quad (A12)$$

$$V_{14} = -i\mathrm{tr}[\rho(\widehat{a}_1\widehat{a}_2 - \widehat{a}_1^\dagger\widehat{a}_2^\dagger - \widehat{a}_1\widehat{a}_2^\dagger + \widehat{a}_1^\dagger\widehat{a}_2)] - \overline{X}_1\overline{X}_4, \quad (A13)$$

$$V_{23} = -i\mathrm{tr}[\rho(\widehat{a}_1\widehat{a}_2 - \widehat{a}_1^\dagger\widehat{a}_2^\dagger + \widehat{a}_1\widehat{a}_2^\dagger - \widehat{a}_1^\dagger\widehat{a}_2)] - \overline{X}_2\overline{X}_3. \quad (A14)$$

It follows that if $\rho(\overline{X}, V)$ is real, then $V_{12} = V_{14} = V_{23} = 0$.

Step (A.ii). Next, we prove that, for the Gaussian state $\rho(\overline{X}, V)$, if $\{X_{2l} = 0\}_{l=1}^N$ and $\{V_{2l-1,2m} = 0\}_{l,m=1}^N$, then $\rho(\overline{X}, V)$ must be real. This step is comparatively difficult. Our proof is inspired by Ref. [32] (see Appendix A therein).

For a quantum state ρ in $\overline{H}^{\otimes N}$, its characteristic function $\chi(\rho, \xi)$ determines ρ via the relation [40]

$$\rho = \int \frac{d^{2N}\xi}{\pi^N}\chi(\rho,\xi)D(-\xi). \quad (A15)$$

Then

$$\langle j_1|\cdots\langle j_N|\rho|k_1\rangle\cdots|k_N\rangle = \int \frac{d^{2N}\lambda}{\pi^N}\chi(\rho,\xi)\langle j_1|D(-\lambda_1)|k_1\rangle\cdots\langle j_N|D(-\lambda_N)|k_N\rangle, \quad (A16)$$

where $\lambda_1 = \xi_1 + i\xi_2, \lambda_2 = \xi_3 + i\xi_4, \dots, \lambda_N = \xi_{2N-1} + i\xi_{2N}$,

$$D(\lambda_1) = D[(\xi_1,\xi_2)^T] = \exp(\lambda_1\widehat{a}_1^\dagger - \lambda_1^*\widehat{a}_1), \quad (A17)$$

$$D(\xi) = D(\lambda_1)D(\lambda_2)\cdots D(\lambda_N), \quad (A18)$$

$D(\lambda_2) = D[(\xi_3,\xi_4)^T], \dots, D(\lambda_N) = D[(\xi_{2N-1},\xi_{2N})^T]$. We further let $(\xi_1,\xi_2,\xi_3,\xi_4,\dots,\xi_{2N-1},\xi_{2N}) = (x_{\lambda_1}, y_{\lambda_1}, x_{\lambda_2}, y_{\lambda_2}, \dots, x_{\lambda_N}, y_{\lambda_N})$, $d^2\lambda_1 = dx_{\lambda_1}dy_{\lambda_1}$, and $d^{2N}\lambda = d^{2N}\xi = dx_{\lambda_1}dy_{\lambda_1}dx_{\lambda_2}dy_{\lambda_2}\cdots dx_{\lambda_N}dy_{\lambda_N}$. In Eq. (A16),

$$\begin{aligned}\langle j_1|D(-\lambda_1)|k_1\rangle &= \int \frac{d^2\alpha_1}{\pi}\frac{d^2\beta_1}{\pi}\langle j_1|\alpha_1\rangle\langle\alpha_1|D(-\lambda_1)|\beta_1\rangle\langle\beta_1|k_1\rangle \quad (A19)\\ &= \int \frac{d^2\alpha_1}{\pi}\frac{d^2\beta_1}{\pi}\frac{\alpha_1^{j_1}\beta_1^{*k_1}}{\sqrt{j_1!k_1!}}\exp b_1, \quad (A20)\end{aligned}$$

$$b_1 = -\frac{1}{2}(x_{\alpha_1}, y_{\alpha_1}, x_{\beta_1}, y_{\beta_1}, x_{\lambda_1}, y_{\lambda_1})Q_1 \times (x_{\alpha_1}, y_{\alpha_1}, x_{\beta_1}, y_{\beta_1}, x_{\lambda_1}, y_{\lambda_1})^T, \quad (A21)$$

$$Q_1 = \begin{pmatrix} 2 & 0 & -1 & -i & 1 & i\\ 0 & 2 & i & -1 & -i & 1\\ -1 & i & 2 & 0 & -1 & i\\ -i & -1 & 0 & 2 & -i & -1\\ 1 & -i & -1 & -i & 1 & 0\\ i & 1 & i & -1 & 0 & 1\end{pmatrix}. \quad (A22)$$

In Eq. (A19), $\alpha_1 = x_{\alpha_1} + iy_{\alpha_1}$, $\beta_1 = x_{\beta_1} + iy_{\beta_1}$, $\{x_{\alpha_1}, y_{\alpha_1}, x_{\beta_1}, y_{\beta_1}\} \subset R$, $d^2\alpha_1 = dx_{\alpha_1}dy_{\alpha_1}$, and $d^2\beta_1 = dx_{\beta_1}dy_{\beta_1}$. Below we use $\alpha_2, \beta_2, \dots, \alpha_N, \beta_N$ and $d^{2N}\alpha = dx_{\alpha_1}dy_{\alpha_1}\cdots dx_{\alpha_N}dy_{\alpha_N}$ and $d^{2N}\beta = dx_{\beta_1}dy_{\beta_1}\cdots dx_{\beta_N}dy_{\beta_N}$ similarly. In Eq. (A19),

$$|\alpha_1\rangle = D(\alpha_1)|0\rangle = e^{-\frac{|\alpha_1|^2}{2}}\sum_{j_1=0}^{\infty}\frac{\alpha_1^{j_1}}{\sqrt{j_1!}}|j_1\rangle \quad (A23)$$

is the Glauber coherent state, $|\beta_1\rangle$ similarly, and we have used the relations $\int\frac{d^2\alpha_1}{\pi}|\alpha_1\rangle\langle\alpha_1| = \int\frac{d^2\beta_1}{\pi}|\beta_1\rangle\langle\beta_1| = I$. In Eq. (A20), we have used $\langle\alpha_1|D(-\lambda_1)|\beta_1\rangle = \langle 0|D(-\alpha_1)D(-\lambda_1)D(\beta_1)|0\rangle$ and the relation

$$D(\alpha_1)D(\beta_1) = D(\alpha_1+\beta_1)\exp\frac{\alpha_1\beta_1^* - \alpha_1^*\beta_1}{2}. \quad (A24)$$

Now we consider the case where ρ is a Gaussian state $\rho = \rho(\overline{X}, V)$. Taking Eq. (A20) and the characteristic function $\chi(\rho,\xi)$ in Eq. (18) into Eq. (A16), we find

$$\langle j_1|\cdots\langle j_N|\rho|k_1\rangle\cdots|k_N\rangle = \int \frac{d^{2N}\lambda}{\pi^N}\frac{d^{2N}\alpha}{\pi^N}\frac{d^{2N}\beta}{\pi^N}\frac{\alpha_1^{j_1}\beta_1^{*k_1}\cdots\alpha_N^{j_N}\beta_N^{*k_N}}{\sqrt{j_1!k_1!\cdots j_N!k_N!}}\exp b_2, \quad (A25)$$

$$b_2 = -\frac{1}{2}\Gamma^T Q\Gamma + B'^T\Gamma, \quad (A26)$$

$$\Gamma = (x_{\alpha_1}, y_{\alpha_1}, x_{\beta_1}, y_{\beta_1}, x_{\alpha_2}, y_{\alpha_2}, x_{\beta_2}, y_{\beta_2}, \dots, x_{\alpha_N}, y_{\alpha_N}, x_{\beta_N}, y_{\beta_N}, x_{\lambda_1}, y_{\lambda_1}, x_{\lambda_2}, y_{\lambda_2}, \dots, x_{\lambda_N}, y_{\lambda_N})^T, \quad (A27)$$

$$B' = (0, 0, \dots, 0, 0, -i\overline{X}_2, i\overline{X}_1, -i\overline{X}_4, i\overline{X}_3, \dots, -i\overline{X}_{2N}, i\overline{X}_{2N-1})^T, \quad (A28)$$

$$Q=\begin{pmatrix} 2 & 0 & -1 & -i & 0 & 0 & 0 & 0 & \dots & 1 & i & 0 & 0 & \dots \\ 0 & 2 & i & -1 & 0 & 0 & 0 & 0 & \dots & -i & 1 & 0 & 0 & \dots \\ -1 & i & 2 & 0 & 0 & 0 & 0 & 0 & \dots & -1 & i & 0 & 0 & \dots \\ -i & -1 & 0 & 2 & 0 & 0 & 0 & 0 & \dots & -i & -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 2 & 0 & -1 & -i & \dots & 0 & 0 & 1 & i & \dots \\ 0 & 0 & 0 & 0 & 0 & 2 & i & -1 & \dots & 0 & 0 & -i & 1 & \dots \\ 0 & 0 & 0 & 0 & -1 & i & 2 & 0 & \dots & 0 & 0 & -1 & i & \dots \\ 0 & 0 & 0 & 0 & -i & -1 & 0 & 2 & \dots & 0 & 0 & -i & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & -i & -1 & -i & 0 & 0 & 0 & 0 & \dots & 1+V_{22} & -V_{21} & V_{24} & -V_{23} & \dots \\ i & 1 & i & -1 & 0 & 0 & 0 & 0 & \dots & -V_{12} & 1+V_{11} & -V_{14} & V_{13} & \dots \\ 0 & 0 & 0 & 0 & 1 & -i & -1 & -i & \dots & V_{42} & -V_{41} & 1+V_{44} & -V_{43} & \dots \\ 0 & 0 & 0 & 0 & i & 1 & i & -1 & \dots & -V_{32} & V_{31} & -V_{34} & 1+V_{33} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \quad \text{(A29)}$$

We introduce the Gaussian integral

$$J=\int d^{2N}\lambda d^{2N}\alpha d^{2N}\beta \exp\left(b_2+\sum_{l=1}^{N}(u_l\alpha_l+v_l\beta_l^*)\right), \quad \text{(A30)}$$

where $\{u_l, v_l\}_{l=1}^N \subset R$, and thus

$$\langle j_1|\cdots\langle j_N|\rho|k_1\rangle\cdots|k_N\rangle = \frac{\left(\frac{\partial^{j_1}}{\partial u_1^{j_1}}\frac{\partial^{k_1}}{\partial v_1^{k_1}}\cdots\frac{\partial^{j_N}}{\partial u_N^{j_N}}\frac{\partial^{k_N}}{\partial v_N^{k_N}}J\right)\Big|_{\{u_l=v_l=0\}_{l=1}^N}}{\pi^{3N}\sqrt{j_1!k_1!\cdots j_N!k_N!}}. \quad \text{(A31)}$$

To calculate J, we write

$$b_2+\sum_{l=1}^{N}(u_l\alpha_l+v_l\beta_l^*)=-\frac{1}{2}\Gamma^TQ\Gamma+B^T\Gamma, \quad \text{(A32)}$$

with

$$B=(u_1, iu_1, v_1, -iv_1, \dots, u_N, iu_N, v_N, -iv_N, -i\overline{X}_2, i\overline{X}_1, -i\overline{X}_4, i\overline{X}_3, \dots, -i\overline{X}_{2N}, i\overline{X}_{2N-1})^T. \quad \text{(A33)}$$

Employing the Gaussian integral formula one gets

$$J=\frac{(2\pi)^{3N}}{\sqrt{\det Q}}\exp\left(\frac{1}{2}B^TQ^{-1}B\right). \quad \text{(A34)}$$

Now we prove that if a Gaussian state $\rho(\overline{X},V)$ satisfies $\{X_{2l}=0\}_{l=1}^N$ and $\{V_{2l-1,2m}=0\}_{l,m=1}^N$, then $J\in R$, and hence, Eq. (A31) implies that $\rho(\overline{X},V)$ must be real. Observe that if $\{\overline{X}_{2l}=0\}_{l=1}^N$ and $\{V_{2l-1,2m}=0\}_{l,m=1}^N$, then in Eq. (A29)

$$\{Q_{2l-1,2m-1}, Q_{2l,2m}\}_{l,m=1}^N \subset R, \quad \text{(A35)}$$

$$\{Q_{2l-1,2m}, Q_{2l,2m-1}\}_{l,m=1}^N \subset iR, \quad \text{(A36)}$$

$$\{\overline{X}_{2l-1}\}_{l=1}^N \subset R, \quad \{\overline{X}_{2l}\}_{l=1}^N \subset iR, \quad \text{(A37)}$$

$$iR=\{ix|x\in R\}. \quad \text{(A38)}$$

Let

$$Q=E^*Q'E, \quad \text{(A39)}$$

with

$$E=\mathrm{diag}\{i,1,i,1,\dots,i,1\}, \quad \text{(A40)}$$

and hence,

$$E^*=E^{-1}, \quad \text{(A41)}$$

$$Q'_{2l-1,2m-1}=Q_{2l-1,2m-1}, \quad \text{(A42)}$$

$$Q'_{2l,2m}=Q_{2l,2m}, \quad \text{(A43)}$$

$$Q'_{2l-1,2m}=iQ_{2l-1,2m}, \quad \text{(A44)}$$

$$Q'_{2l,2m-1}=-iQ_{2l,2m-1}. \quad \text{(A45)}$$

$$B^TQ^{-1}B=B^TE^*Q'^{-1}EB. \quad \text{(A46)}$$

We see that Q' is a real matrix, $\{(EB)_l\}_{l=1}^N\subset iR$, and $\{(B^TE^*)_l\}_{l=1}^N\subset iR$. It follows that $B^TQ^{-1}B$ in Eq. (A46) is real and

$$\exp\left(\tfrac{1}{2}B^TQ^{-1}B\right)>0. \quad \text{(A47)}$$

Let $\{j_1, k_1, j_2, k_2, \dots j_N, k_N\}$ all be zero, then

$$[0,1]\ni\langle 0|\langle 0|\cdots\langle 0|\rho|0\rangle|0\rangle\cdots|0\rangle = \frac{1}{\pi^{3N}}J_{\{u_l=v_l=0\}_{l=1}^N} = \frac{2^{3N}}{\sqrt{\det Q}}\exp\left(\frac{1}{2}B^TQ^{-1}B\right)_{\{u_l=v_l=0\}_{l=1}^N}. \quad \text{(A48)}$$

As a result,

$$\det Q>0. \quad \text{(A49)}$$

Equations (A34) and (A31) imply that $J\in R$ and $\rho(\overline{X},V)$ must be real.

Step (A.iii). From the viewpoint of mathematical rigor, we need to prove that the Gaussian integral in Eqs. (A30), (A32), (A33), (A29), and (A27) is convergent, then Eq. (A34) is valid. To this aim, we prove ReQ is positive definite (the convergence condition of the Gaussian integral, see, for example, Ref. [54]).

Decompose Q as

$$Q = Q_2 + Q_3, \tag{A50}$$

where Q_2 is obtained by deleting all $\{V_{lm}\}_{l,m=1}^{2N}$ in Q. Since $V \succ 0$,

$$Q_3 = \Omega V \Omega^T \succ 0. \tag{A51}$$

There exists a permutation matrix P_0 such that

$$Q_2 = P_0\big(\oplus_{l=1}^N Q_1\big)P_0^T, \tag{A52}$$

where Q_1 is defined in Eq. (A22), P_0 permutes the rows of $(\oplus_{l=1}^N Q_1)$ while P_0^T permutes the columns of $(\oplus_{l=1}^N Q_1)$ in the same way. Taking the real part of Eqs. (A50) and (A52) gives

$$\text{Re}Q = \text{Re}Q_2 + Q_3, \tag{A53}$$

$$\text{Re}Q_2 = P_0\big(\oplus_{l=1}^N \text{Re}Q_1\big)P_0^T. \tag{A54}$$

ReQ_1 is symmetric and direct calculation shows that ReQ_1 has the eigenvalues $\{2+\sqrt{3}, 2+\sqrt{3}, 2-\sqrt{3}, 2-\sqrt{3}, 1, 1\}$; thus, Re$Q_1 \succ 0$ and Re$Q_2 \succ 0$. Together with $Q_3 = \Omega V \Omega^T \succ 0$, we get Re$Q \succ 0$.

This completes the proof of Theorem 1.

APPENDIX B: PROOF OF THEOREM 2

We set three steps to prove Theorem 2.

Step (B.i). For the N-mode Gaussian state $\rho(\overline{X}, V)$, we define the real column vector $\overline{X}' = (\overline{X}'_1, \overline{X}'_2, \ldots, \overline{X}'_{2N})^T$ and the $2N \times 2N$ real, symmetric matrix $V' = (V'_{lm})_{l,m=1}^{2N}$ as

$$\overline{X}'_l = (-1)^{l+1}\overline{X}_l,\ l \in \{1, 2, \ldots, 2N\}, \tag{B1}$$

$$V'_{lm} = (-1)^{l+m}V_{lm},\ l, m \in \{1, 2, \ldots, 2N\}. \tag{B2}$$

We first show that $(\overline{X}', V')$ determines a Gaussian state ρ' with $\overline{X}'$ being the mean and V' the covariance matrix. To do so, we need to prove that V' satisfies the uncertainty relation

$$V' + i\Omega \succeq 0. \tag{B3}$$

Since V is the covariance matrix of Gaussian state $\rho(\overline{X}, V)$, the uncertainty relation

$$V + i\Omega \succeq 0 \tag{B4}$$

holds. Taking the conjugate of the left-hand side of Eq. (B4) gives

$$V - i\Omega \succeq 0. \tag{B5}$$

That is, $V + i\Omega \succeq 0$ and $V - i\Omega \succeq 0$ are essentially equivalent.

Now we introduce the matrix

$$O = \oplus_{l=1}^N \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B6}$$

O is a real orthogonal matrix and $O^\dagger = O$. We can check that $OVO^\dagger = V'$ and $O\Omega O^\dagger = -\Omega$. Hence, $O(V - i\Omega)O^\dagger \succeq 0$ and Eq. (B3) follows.

Step (B.ii). We prove that the conjugate of $\rho(\overline{X}, V)$, ρ^*, has the mean $\overline{X}'$ and the covariance matrix V'.

Similar to Eqs. (A6)–(A14), we have

$$\text{tr}(\rho\widehat{a}_1^\dagger\widehat{a}_1) = \sum_{j=0}^{\infty} j_1\rho_{j_1 j_1}, \tag{B7}$$

$$V_{11} = \text{tr}\big[\rho\big(\widehat{a}_1^2 + \widehat{a}_1^{\dagger 2} + 2\widehat{a}_1^\dagger\widehat{a}_1 + 1\big)\big] - \overline{X}_1^2, \tag{B8}$$

$$V_{22} = -\text{tr}\big[\rho\big(\widehat{a}_1^2 + \widehat{a}_1^{\dagger 2} - 2\widehat{a}_1^\dagger\widehat{a}_1 - 1\big)\big] - \overline{X}_2^2, \tag{B9}$$

$$V_{13} = \text{tr}[\rho(\widehat{a}_1\widehat{a}_2 + \widehat{a}_1^\dagger\widehat{a}_2^\dagger + \widehat{a}_1\widehat{a}_2^\dagger + \widehat{a}_1^\dagger\widehat{a}_2)] - \overline{X}_1\overline{X}_3, \tag{B10}$$

$$V_{24} = -\text{tr}[\rho(\widehat{a}_1\widehat{a}_2 + \widehat{a}_1^\dagger\widehat{a}_2^\dagger - \widehat{a}_1\widehat{a}_2^\dagger - \widehat{a}_1^\dagger\widehat{a}_2)] - \overline{X}_2\overline{X}_4. \tag{B11}$$

We see that if replace ρ by its conjugate ρ^*, then $(\overline{X}, V)$ become $(\overline{X}', V')$. This says that the mean and the covariance matrix of ρ^* are $(\overline{X}', V')$.

Step (B.iii). Last, we prove that the Gaussian state $\rho'(\overline{X}', V')$ is just ρ^*. To this aim, we calculate the matrix elements $\langle j_1|\cdots\langle j_N|\rho'|k_1\rangle\cdots|k_N\rangle$ of $\rho'(\overline{X}', V')$ in the Fock basis $\{|j\rangle\}_j^{\otimes N}$. Since

$$\langle j_1|\cdots\langle j_N|\rho^*|k_1\rangle\cdots|k_N\rangle = \langle k_1|\cdots\langle k_N|\rho|j_1\rangle\cdots|j_N\rangle, \tag{B12}$$

we need to show

$$\langle j_1|\cdots\langle j_N|\rho'|k_1\rangle\cdots|k_N\rangle = \langle k_1|\cdots\langle k_N|\rho|j_1\rangle\cdots|j_N\rangle \tag{B13}$$

for any $\{j_1, k_1, \ldots, j_N, k_N\} \subset \{0, 1, 2, \ldots\}$.

We now show that when replacing $(\overline{X}, V)$ and $(u_1, v_1; \ldots; u_N, v_N)$ by $(\overline{X}', V')$ and $(v_1, u_1; \ldots; v_N, u_N)$, respectively, in Eq. (A34), the integral J in Eq. (A34) remains invariant. Together with Eq. (A31), we obtain Eq. (B13). With such replacements, in Eq. (A34), we consider what det Q and $B^TQ^{-1}B$ become.

Q becomes

$$\begin{aligned} Q_2 + \Omega V'\Omega^T &= Q_2 + \Omega OVO\Omega^T && \text{(B14)}\\ &= Q_2 + O\Omega V\Omega^T O && \text{(B15)}\\ &= O_1(Q_2^* + \Omega V\Omega^T)O_1 && \text{(B16)}\\ &= O_1Q^*O_1, && \text{(B17)}\end{aligned}$$

det Q then becomes det Q^*. In Eq. (B14) we use Eqs. (A50) and (A51). In Eq. (B14) we use the matrix O defined in Eq. (B6), and the fact that $V' = OVO$. In Eq. (B15) we use the fact that $O\Omega = -\Omega O$. In Eq. (B16) we extend the $2N \times 2N$ matrix O defined in Eq. (B6) to a larger $6N \times 6N$ real symmetric orthogonal matrix O_1 as

$$O_1 = \oplus_{m=1}^3 O = \oplus_{l=1}^{3N}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{B18}$$

and we use the following facts:

$$O_1 Q_2^* O_1 = Q_2, \tag{B19}$$

$$O\Omega V \Omega^T O = O_1 \Omega V \Omega^T O_1. \tag{B20}$$

Equations (B19) and (B20) can be directly checked. In Eq. (B17) we again used Eqs. (A50) and (A51) and the fact that $\Omega V \Omega^T$ is real.

B becomes $O_2 B$ with O_2 being a $6N \times 6N$ real symmetric orthogonal matrix as

$$O_2 = \left[\oplus_{l=1}^{N} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\right] \oplus \left[\oplus_{l=1}^{N} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right]. \tag{B21}$$

$B^T Q^{-1} B$ becomes

$$\begin{aligned} &B^T O_2 O_1 (Q^*)^{-1} O_1 O_2 B \\ &= B^T [(O_2 O_1 Q O_1 O_2)^*]^{-1} B = B^T Q^{-1} B, \end{aligned} \tag{B22}$$

where we have used the fact that

$$O_2 O_1 Q O_1 O_2 = Q^* \tag{B23}$$

which can be directly checked. Further, Eq. (B23) implies

$$\det Q = \det Q^*. \tag{B24}$$

With Eqs. (B22), (B24), and (A34), we obtain Eq. (B13), and $\rho'(\overline{X}', V') = \rho^*$ then follows.

APPENDIX C: PROOF OF THEOREM 3

Step (C.i): *One-mode case*.

Consider the real Gaussian state $\rho(\overline{X}, V)$ with

$$V = \begin{pmatrix} V_{11} & 0 \\ 0 & V_{22} \end{pmatrix}, \ \overline{X} = \begin{pmatrix} \overline{X}_1 \\ 0 \end{pmatrix}. \tag{C1}$$

Suppose $\phi = (d, T, N)$ is a real Gaussian channel,

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}, \ N = \begin{pmatrix} N_{11} & N_{12} \\ N_{12} & N_{22} \end{pmatrix}, \ d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}. \tag{C2}$$

Equation (26) yields

$$T\overline{X} + d = \begin{pmatrix} T_{11}\overline{X}_1 + d_1 \\ T_{21}\overline{X}_1 + d_2 \end{pmatrix}. \tag{C3}$$

Varying $\overline{X}_1 \in R$, $T_{21}\overline{X}_1 + d_2 = 0$ implies $T_{21} = d_2 = 0$. Further,

$$\begin{aligned} &TVT^T + N \\ &= \begin{pmatrix} T_{11}^2 V_{11} + T_{12}^2 V_{22} + N_{11} & T_{12} T_{22} V_{22} + N_{12} \\ T_{12} T_{22} V_{22} + N_{12} & T_{22}^2 V_{22} + N_{22} \end{pmatrix}. \end{aligned} \tag{C4}$$

Varying $V_{22} \in R$, $T_{12} T_{22} V_{22} + N_{12} = 0$ yields $T_{12} T_{22} = N_{12} = 0$. Then Theorem 3 holds for the one-mode case.

Step (C.ii): *N-mode case*.

Define the $2N \times 2N$ permutation matrix P as

$$P_{l,2l-1} = P_{N+l,2l} = 1 \text{ for } l \in \{1, 2, \dots, N\}, \tag{C5}$$

and other elements are all zero. P reorders the indices $(1, 2, 3, \dots, 2N)^T$ to $(1, 3, 5, \dots, 2N-1, 2, 4, \dots, 2N)^T$. Consider the real Gaussian state $\rho(\overline{X}, V)$, we find

$$P\overline{X} = (\overline{X}_1, \overline{X}_3, \dots, \overline{X}_{2N-1}, 0, \dots, 0)^T, \tag{C6}$$

$$\begin{aligned} PVP^T &= \begin{pmatrix} V_{11} & V_{13} & \dots & 0 & 0 & \dots \\ V_{31} & V_{33} & \dots & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & V_{22} & V_{24} & \dots \\ 0 & 0 & \dots & V_{42} & V_{44} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \\ &= \begin{pmatrix} V_1 & 0 \\ 0 & V_4 \end{pmatrix}, \end{aligned} \tag{C7}$$

with $V_1 = V_o$ and $V_4 = V_e$ all being $N \times N$ real matrices.

Suppose $\phi = (d, T, N)$ is a real Gaussian channel. PTP^T and PNP^T have reordered structures similar to those of PVP^T. We write

$$PTP^T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}, \ PNP^T = \begin{pmatrix} N_1 & N_2 \\ N_2^T & N_4 \end{pmatrix}, \tag{C8}$$

$$Pd = (d_1, d_3, \dots, d_{2N-1}, d_2, d_4, \dots, d_{2N})^T, \tag{C9}$$

with T_1, T_2, T_3, T_4, N_1, N_2, and N_4 all being $N \times N$ real matrices. Equation (26) yields

$$T\overline{X} + d = P^T[(PTP^T)(P\overline{X}) + (Pd)]. \tag{C10}$$

Varying $\{\overline{X}_{2l-1}\}_{l=1}^N \subset R$, we get $T_3 = 0$ and $d_{2l} = 0$ for $l \in \{1, 2, \dots, N\}$. Equation (26) further yields

$$\begin{aligned} &TVT^T + N \\ &= P^T[(PTP^T)(PVP^T)(PT^TP^T) + (PNP^T)]P \\ &= P^T \begin{pmatrix} T_1 V_1 T_1^T + T_2 V_4 T_2^T + N_1 & T_2 V_4 T_4^T + N_2 \\ T_4 V_4 T_2^T + N_2^T & T_4 V_4 T_4^T + N_4 \end{pmatrix} P, \\ &\times T_2 V_4 T_4^T + N_2 = 0. \end{aligned} \tag{C11}$$

Varying $\{V_{2l,2m}\}_{l,m=1}^N \subset R$ in Eq. (C11), we get $N_2 = 0$, $T_2 = 0$, or $T_4 = 0$. Then Theorem 3 follows for the N-mode case.

APPENDIX D: PROOF OF THEOREM 4

Suppose $\phi = (d, T, N)$ is an N-mode real Gaussian channel and $\rho(\overline{X}, V)$ is any N-mode Gaussian state. P, $P\overline{X}$, PTP^T, PNP^T, and Pd are defined similarly to Eqs. (C5), (C6), (C8), and (C9). We also denote

$$\overline{X}_o = (\overline{X}_1, \overline{X}_3, \overline{X}_5, \dots, \overline{X}_{2N-1})^T, \tag{D1}$$

$$\overline{X}_e = (\overline{X}_2, \overline{X}_4, \overline{X}_6, \dots, \overline{X}_{2N})^T, \tag{D2}$$

$$PVP^T = \begin{pmatrix} V_1 & V_2 \\ V_2^T & V_4 \end{pmatrix}, \tag{D3}$$

$$d_o = (d_1, d_3, d_5, \dots, d_{2N-1})^T, \tag{D4}$$

where $\{V_2, V_1 = V_o, V_4 = V_e\}$ are all $N \times N$ matrices.

Step (D.i). If $\phi = (d, T, N)$ is a completely real Gaussian channel, then

$$PTP^T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix},\ PNP^T = \begin{pmatrix} N_1 & 0 \\ 0 & N_4 \end{pmatrix}. \tag{D5}$$

We calculate the mean and the covariance of $\phi[\rho(\overline{X}, V)]$. Equation (26) yields

$$\begin{aligned} T\overline{X} + d &= P^T[(PTP^T)(P\overline{X}) + (Pd)] \\ &= P^T\begin{pmatrix} T_1\overline{X}_o + T_2\overline{X}_e + d_o \\ 0 \end{pmatrix}, \end{aligned} \tag{D6}$$

$$\begin{aligned} TVT^T + N &= P^T[(PTP^T)(PVP^T)(PT^TP^T) + (PNP^T)]P \\ &= P^T\begin{pmatrix} V_1' & 0 \\ 0 & N_4 \end{pmatrix}P, \\ V_1' &= T_1V_1T_1^T + T_1V_2T_2^T + T_2V_2^TT_1^T + T_2V_4T_2^T + N_1. \end{aligned} \tag{D7}$$

Consequently, $\phi[\rho(\overline{X}, V)]$ is a real Gaussian state.

Step (D.ii). If $\phi = (d, T, N)$ is a covariant real Gaussian channel, then

$$PTP^T = \begin{pmatrix} T_1 & 0 \\ 0 & T_4 \end{pmatrix},\ PNP^T = \begin{pmatrix} N_1 & 0 \\ 0 & N_4 \end{pmatrix}. \tag{D8}$$

We calculate the mean and the covariance of $\phi[\rho(\overline{X}, V)]$. Equation (26) yields

$$\begin{aligned} T\overline{X} + d &= P^T[(PTP^T)(P\overline{X}) + (Pd)] \\ &= P^T\begin{pmatrix} T_1\overline{X}_o + d_o \\ T_4\overline{X}_e \end{pmatrix}, \end{aligned} \tag{D9}$$

$$\begin{aligned} TVT^T + N &= P^T[(PTP^T)(PVP^T)(PT^TP^T) + (PNP^T)]P \\ &= P^T\begin{pmatrix} T_1V_1T_1^T + N_1 & T_1V_2T_4^T \\ T_4V_2^TT_1^T & T_4V_4T_4^T + N_4 \end{pmatrix}P. \end{aligned} \tag{D10}$$

Applying Theorem 2, $\{\phi[\rho(\overline{X}, V)]\}^*$ is still a Gaussian state with the mean and the covariance matrix

$$P^T\begin{pmatrix} T_1\overline{X}_o + d_o \\ -T_4\overline{X}_e \end{pmatrix}, \tag{D11}$$

$$P^T\begin{pmatrix} T_1V_1T_1^T + N_1 & -T_1V_2T_4^T \\ -T_4V_2^TT_1^T & T_4V_4T_4^T + N_4 \end{pmatrix}P. \tag{D12}$$

We see that Eqs. (D11) and (D12) are just the mean and the covariance matrix of $\phi[\rho^*(V', \overline{X}')]$, that is, $\overline{X}_e \to -\overline{X}_e$ and $V_2 \to -V_2$. Thus, Eq. (32) holds.

The proof of Eq. (33) is similar to the proof of Eq. (32).

APPENDIX E: PROOF OF THEOREM 5

We prove that Eq. (34) fulfills conditions (M1) and (M2). Equation (34) fulfilling condition (M1) is apparent since for any two quantum states ρ and σ, the fidelity $F(\rho, \sigma) \geqslant 0$ and $F(\rho, \sigma) = 0$ if and only if $\rho = \sigma$ [12]. Now we prove Eq. (34) fulfills condition (M2).

For a real Gaussian channel ϕ, if ϕ is a completely real Gaussian channel, then $\phi(\rho)$ is a real Gaussian state and $M(\phi(\rho)) = 0 \leqslant M(\rho)$ for any Gaussian state ρ.

For a real Gaussian channel ϕ, if ϕ is a covariant real Gaussian channel, then from Theorem 4 we have $[\phi(\rho)]^* = \phi(\rho^*)$ and

$$\begin{aligned} M(\phi(\rho)) &= 1 - F[\phi(\rho), (\phi(\rho))^*] \\ &= 1 - F[\phi(\rho), \phi(\rho^*)] \\ &\leqslant 1 - F(\rho, \rho^*) = M(\rho). \end{aligned} \tag{E1}$$

In the inequality we have used the monotonicity of the fidelity under a quantum channel ϕ, $F[\phi(\rho), \phi(\sigma)] \geqslant F(\rho, \sigma)$ for any two quantum states ρ and σ [12].

The proof of Eq. (35) fulfilling conditions (M1) and (M2) is similar to Eq. (34). Theorem 5 then follows.

APPENDIX F: ALTERNATIVE PROOFS FOR STEPS (A.ii) AND (B.iii)

In this section we provide alternative proofs for Steps (A.ii) and (B.iii) based on the Husimi function. For the N-mode Gaussian state $\rho(\overline{X}, V)$, the Husimi function of ρ is defined as [55]

$$Q(\alpha) = \frac{1}{\pi^N}\langle\alpha_1|\cdots\langle\alpha_N|\rho|\alpha_1\rangle\cdots|\alpha_N\rangle, \tag{F1}$$

where $\{|\alpha_j\rangle\}_{j=1}^N$ are Glauber coherent states defined in Eq. (A23). Note that $Q(\alpha) \geqslant 0$ and $\int Q(\alpha)d^{2N}\alpha = 1$. Inserting Eq. (A23) into Eq. (F1), we have (see, for example, chapter 3 in Ref. [56])

$$Q(\alpha) = \frac{1}{\pi^N}e^{-|\alpha|^2}\sum_{j_1,k_1,\ldots,j_N,k_N=0}^{\infty}\frac{(\alpha_1^*)^{j_1}\alpha_1^{k_1}\cdots(\alpha_N^*)^{j_N}\alpha_N^{k_N}}{\sqrt{j_1!\cdots j_N!k_1!\cdots k_N!}}\langle j_1|\cdots\langle j_N|\rho|k_1\rangle\cdots|k_N\rangle, \tag{F2}$$

where $|\alpha|^2 = \sum_{j=1}^N|\alpha_j|^2 = \sum_{j=1}^N\alpha_j^*\alpha_j$. We regard $\{\alpha_j, \alpha_j^*\}_{j=1}^N$ as formally independent variables, and we regard $Q(\alpha)$, $e^{-|\alpha|^2}$, and $e^{|\alpha|^2}$ as functions of $\{\alpha_j, \alpha_j^*\}_{j=1}^N$.

From Eq. (F2), we can calculate $\langle j_1|\cdots\langle j_N|\rho|k_1\rangle\cdots|k_N\rangle$ by

$$\langle j_1|\cdots\langle j_N|\rho|k_1\rangle\cdots|k_N\rangle = \frac{\pi^N\left(\frac{\partial}{\partial\alpha_1^*}\right)^{j_1}\left(\frac{\partial}{\partial\alpha_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial\alpha_N^*}\right)^{j_N}\left(\frac{\partial}{\partial\alpha_N}\right)^{k_N}}{\sqrt{j_1!\cdots j_N!k_1!\cdots k_N!}}[Q(\alpha)e^{|\alpha|^2}]\big|_{\alpha_1=\alpha_1^*=\cdots=\alpha_N=\alpha_N^*=0}. \tag{F3}$$

Expand $e^{-|\alpha|^2}$ and $e^{|\alpha|^2}$ as power series (of $\{\alpha_j, \alpha_j^*\}_{j=1}^N$)

$$e^{-|\alpha|^2} = \sum_{j_1=0}^{\infty} \frac{(-\alpha_1^*\alpha_1)^{j_1}}{j_1!} \cdots \sum_{j_N=0}^{\infty} \frac{(-\alpha_N^*\alpha_N)^{j_N}}{j_N!}, \tag{F4}$$

$$e^{|\alpha|^2} = \sum_{j_1=0}^{\infty} \frac{(\alpha_1^*\alpha_1)^{j_1}}{j_1!} \cdots \sum_{j_N=0}^{\infty} \frac{(\alpha_N^*\alpha_N)^{j_N}}{j_N!}. \tag{F5}$$

Evidently, all coefficients of the power series of $e^{-|\alpha|^2}$ ($e^{|\alpha|^2}$) are real. Consequently, all coefficients of the power series of $Q(\alpha)$ are real if and only if all coefficients of the power series of $Q(\alpha)e^{|\alpha|^2}$ ($Q(\alpha)e^{-|\alpha|^2}$) are real.

From Eq. (F2) we see that if ρ is real, then all coefficients of the power series of $Q(\alpha)e^{|\alpha|^2}$ are real. While Eq. (F3) implies that if all coefficients of the power series of $Q(\alpha)e^{|\alpha|^2}$ are real, then ρ is real. As a result, ρ is real if and only if all coefficients of the power series of $Q(\alpha)$ are real.

Taking Eqs. (18) and (19) into Eq. (F1), we calculate $Q(\alpha)$ as

$$\begin{aligned} Q(\alpha) &= \int \frac{d^{2N}\xi}{\pi^{2N}} \chi(\rho,\xi) \langle\alpha_1|\cdots\langle\alpha_N|D(-\xi)|\alpha_1\rangle\cdots|\alpha_N\rangle \\ &= \int \frac{d^{2N}\xi}{\pi^{2N}} \chi(\rho,\xi) \langle\alpha_1|D(-\lambda_1)|\alpha_1\rangle\cdots\langle\alpha_N|D(-\lambda_N)|\alpha_N\rangle & \text{(F6)} \\ &= \int \frac{d^{2N}\xi}{\pi^{2N}} \chi(\rho,\xi) \exp\left(-\frac{|\lambda_1|^2}{2} + \alpha_1\lambda_1^* - \alpha_1^*\lambda_1\right)\cdots\exp\left(-\frac{|\lambda_N|^2}{2} + \alpha_N\lambda_N^* - \alpha_N^*\lambda_N\right) & \text{(F7)} \\ &= \int \frac{d^{2N}\xi}{\pi^{2N}} \exp\left[-\frac{1}{2}\xi^T(\Omega V\Omega^T)\xi - i(\Omega\overline{X})^T\xi\right] \exp\left(-\frac{1}{2}\xi^T\xi - 2i\alpha^T\Omega\xi\right) & \text{(F8)} \\ &= \int \frac{d^{2N}\xi}{\pi^{2N}} \exp\left[-\frac{1}{2}\xi^T(\Omega V\Omega^T + I_{2N})\xi + i(\overline{X} - 2\alpha)^T\Omega\xi\right] & \text{(F9)} \\ &= \left(\frac{2}{\pi}\right)^N \frac{1}{\sqrt{\det(\Omega V\Omega^T + I_{2N})}} \exp\left[-\frac{1}{2}(\overline{X} - 2\alpha)^T\Omega(\Omega V\Omega^T + I_{2N})^{-1}\Omega^T(\overline{X} - 2\alpha)\right] & \text{(F10)} \\ &= \left(\frac{2}{\pi}\right)^N \frac{1}{\sqrt{\det(V + I_{2N})}} \exp\left[-\frac{1}{2}(2\alpha - \overline{X})^T(V + I_{2N})^{-1}(2\alpha - \overline{X})\right]. & \text{(F11)} \end{aligned}$$

In Eq. (F6), we use $\lambda_1 = \xi_1 + i\xi_2, \ldots, \lambda_N = \xi_{2N-1} + i\xi_{2N}$. In Eq. (F7), we have used

$$\begin{aligned} \langle\alpha_1|D(-\lambda_1)|\alpha_1\rangle &= \langle 0|D(-\alpha_1)D(-\lambda_1)D(\alpha_1)|0\rangle \\ &= \exp(\alpha_1\lambda_1^* - \alpha_1^*\lambda_1)\langle 0|D(-\lambda_1)|0\rangle \\ &= \exp\left(-\frac{|\lambda_1|^2}{2} + \alpha_1\lambda_1^* - \alpha_1^*\lambda_1\right). \end{aligned}$$

In Eq. (F8), we define the real vector $\alpha = (x_{\alpha_1}, y_{\alpha_1}, x_{\alpha_2}, y_{\alpha_2}, \ldots, x_{\alpha_N}, y_{\alpha_N})^T$, with $\alpha_1 = x_{\alpha_1} + iy_{\alpha_1}$, $\alpha_2 = x_{\alpha_2} + iy_{\alpha_2}$, ..., $\alpha_N = x_{\alpha_N} + iy_{\alpha_N}$. In Eq. (F9), I_{2N} is the identity matrix of size $2N$. In Eq. (F10), we use the Gaussian integral formula. In Eq. (F11), we use the facts $\Omega^T = -\Omega$ and $\Omega^2 = -I_{2N}$.

Using the matrix P defined in Eq. (C5), in the exponent of Eq. (F11), we define the function $q(\alpha)$ and calculate it as

$$\begin{aligned} q(\alpha) &= (2\alpha - \overline{X})^T(V + I_{2N})^{-1}(2\alpha - \overline{X}) \\ &= [(2\alpha - \overline{X})^T P^T][P(V + I_{2N})^{-1}P^T][P(2\alpha - \overline{X})] \\ &= [P(2\alpha - \overline{X})]^T[P(V + I_{2N})P^T]^{-1}[P(2\alpha - \overline{X})] \\ &= [P(2\alpha - \overline{X})]^T(PVP^T + I_{2N})^{-1}[P(2\alpha - \overline{X})]. \end{aligned} \tag{F12}$$

Now we prove the result of Step (A.ii) that if an N-mode Gaussian state $\rho(\overline{X}, V)$ satisfies $\{X_{2l} = 0\}_{l=1}^N$ and $\{V_{2l-1,2m} = 0\}_{l,m=1}^N$, then $\rho(\overline{X}, V)$ must be real. We only need to prove that if $\rho(\overline{X}, V)$ satisfies $\{X_{2l} = 0\}_{l=1}^N$ and $\{V_{2l-1,2m} = 0\}_{l,m=1}^N$, then all coefficients of the power series of $Q(\alpha)$ are real.

Suppose $\{X_{2l} = 0\}_{l=1}^N$ and $\{V_{2l-1,2m} = 0\}_{l,m=1}^N$. With Eqs. (D1)–(D3), then $q(\alpha)$ in Eq. (F12) reads

$$\begin{aligned} q(\alpha) &= [P(2\alpha - \overline{X})]^T[(V_o \oplus V_e) + I_{2N}]^{-1}[P(2\alpha - \overline{X})] \\ &= [P(2\alpha - \overline{X})]^T[(V_o + I_N)^{-1} \oplus (V_e + I_N)^{-1}] \\ &\quad \times [P(2\alpha - \overline{X})] \\ &= (2\alpha_o - \overline{X}_o)^T(V_o + I_N)^{-1}(2\alpha_o - \overline{X}_o) \\ &\quad + 2\alpha_e^T(V_e + I_N)^{-1}(2\alpha_e), \end{aligned} \tag{F13}$$

where I_N is the identity matrix of size N, and we define the real vectors $\alpha_o = (x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_N})^T$ and $\alpha_e = (y_{\alpha_1}, y_{\alpha_2}, \ldots, y_{\alpha_N})^T$. Notice that $2\alpha_o = (\alpha_1 + \alpha_1^*, \alpha_2 + \alpha_2^*, \ldots, \alpha_N + \alpha_N^*)^T$ and $2\alpha_e = -i(\alpha_1 - \alpha_1^*, \alpha_2 - \alpha_2^*, \ldots, \alpha_N - \alpha_N^*)^T$. Hence, Eqs. (F13) and (F11) imply that all coefficients of the power series of $q(\alpha)$ and $Q(\alpha)$ are real and the result of Step (A.ii) follows.

Next we prove the result of Step (B.iii) that, for the N-mode Gaussian state $\rho(\overline{X}, V)$, the Gaussian state

$\rho'(\overline{X}', V')$ with $(\overline{X}', V')$ defined in Eqs. (22) and (23) is just ρ^*. We use Eq. (F3) to show $\langle j_1|\ldots\langle j_N|\rho|k_1\rangle\ldots|k_N\rangle = \langle k_1|\ldots\langle k_N|\rho'|j_1\rangle\ldots|j_N\rangle$. In Eq. (F11), if we replace α_j, α_j^*, V, and $\overline{X}$ by α_j^*, α_j, V', and $\overline{X}'$, respectively, then α, V, and $\overline{X}$ become $O\alpha$, $V' = OVO$, and $\overline{X}' = O\overline{X}$, respectively, with O defined in Eq. (B6). Further, $\det(V + I_{2N})$ becomes

$$\det(V' + I_{2N}) = \det[O(V + I_{2N})O] = \det(V + I_{2N}); \quad \text{(F14)}$$

$(V + I_{2N})^{-1}$ becomes

$$(V' + I_{2N})^{-1} = [O(V + I_{2N})O]^{-1} = O(V + I_{2N})^{-1}O; \quad \text{(F15)}$$

and $(2\alpha - \overline{X})^T(V + I_{2N})^{-1}(2\alpha - \overline{X})$ becomes

$$\begin{aligned}&(2O\alpha - O\overline{X})^T O(V + I_{2N})^{-1}O(2O\alpha - O\overline{X})\\ &= (2\alpha - \overline{X})^T(V + I_{2N})^{-1}(2\alpha - \overline{X}).\end{aligned} \quad \text{(F16)}$$

Equations (F11), (F14), and (F16) show that if we replace α_j, α_j^*, V, and $\overline{X}$ by α_j^*, α_j, V', and $\overline{X}'$, respectively, then $Q(\alpha)$ remains invariant. $e^{|\alpha|^2}$ remains invariant obviously under such replacements. From Eq. (F3), we then conclude that the Gaussian state $\rho'(\overline{X}', V')$ is just ρ^*.

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