QED calculations of the nuclear recoil effect in muonic atoms

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The nuclear recoil effect, known also as the mass shift, is one of the theoretical contributions to the energy levels in muonic atoms. Accurate theoretical predictions are needed for extracting, e.g., the nuclear charge radii from experimental spectra. We report rigorous QED calculations of the nuclear recoil correction in muonic atoms, carried out to all orders in the nuclear binding strength parameter $Z\alpha$ (where Z is the nuclear charge number and α is the fine-structure constant). The calculations show differences with the previous approximate treatment of this effect, most pronounced for the lowest-lying bound states. The calculated recoil correction was found to be sensitive to the nuclear charge radius, which needs to be accounted for when extracting nuclear parameters from measured spectra.

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I. INTRODUCTION

Muonic atoms are a class of atomic systems where a negatively charged muon replaces an electron in the atomic cloud. Compared to electrons, muons penetrate about 200 times deeper into the atomic nucleus due to their larger mass. This phenomenon allows one to use muonic atoms as a powerful tool to investigate the inner structure and properties of atomic nuclei [1–3]. In combination with theoretical predictions, experimental studies of transition energies in muonic atoms provide accurate determinations of the nuclear charge radii, the magnetic dipole, and the electric quadrupole moments of the nuclei [4–8]. A new generation of these experiments is currently being implemented at the Paul Scherrer Institute [9,10], which requires complementary advances on the theory side.

The previous-generation theory did not always succeeded in adequately describing the experimental data. In particular, there are unexplained disagreements observed in muonic Pb, Zr, and Sn atoms [5,11–13]. At the time, these disagreements were ascribed to the insufficiently known nuclear polarization (NP) corrections, which were typically treated as free fitting parameters when interpreting the experimental spectra. However, recent studies of the NP correction [14,15] did not support this supposition and called for systematic *ab initio* QED recalculations of spectra of muonic atoms [16] and a search for alternative solutions [17].

In the present paper we report a rigorous QED calculation of the nuclear recoil effect in muonic atoms. Previously, the nuclear recoil calculations in muonic atoms were performed approximately [18], with a partial inclusion of relativistic effects. This is in contrast to the case of electronic atoms, where the nuclear recoil was studied rigorously within QED. In particular, formulas valid to all orders in the electron-nucleus coupling strength parameter $Z\alpha$ were derived by Shabaev [19,20] and calculated numerically in Refs. [21,22]. These formulas obtained for the electronic atoms can in principle be applied also to muonic systems, but the point-nucleus model assumed in the derivation was not an adequate approximation for muonic atoms. One therefore needed a generalization of the nuclear recoil theory for the extended nuclear size.

A partial inclusion of the finite nuclear size (FNS) into the nuclear recoil was reported in Refs. [23,24]. The complete treatment, however, required a derivation of the generalized photon propagator, which was accomplished only recently in Ref. [25]. In the present work we use the general expressions for the FNS nuclear recoil effect derived in these studies to perform rigorous numerical calculations for muonic atoms without any expansion in the parameter $Z\alpha$.

Relativistic units ($\hbar = c = 1$) and the Heaviside charge units $\alpha = e^2/(4\pi)$ are used throughout the paper.

II. THEORY

General formulas for the nuclear recoil correction to energies of hydrogenlike atoms were derived by Shabaev [19,20]. These formulas are valid to all orders in the nuclear binding strength parameter $Z\alpha$. They were derived for the point nuclear model; within this model they can be applied both for the electronic and the muonic atoms. The expression for the nuclear recoil correction to the energy of a bound lepton in a state a is

$$E_{\text{rec}} = \frac{i}{2\pi M}\int_{-\infty}^{\infty} d\omega \sum_n \frac{1}{\varepsilon_a + \omega - \varepsilon_n(1-i0)} \times \langle a|\vec{p} - \vec{D}(\omega)|n\rangle\langle n|\vec{p} - \vec{D}(\omega)|a\rangle, \tag{1}$$

where M is the nuclear mass, $\vec{p}$ is the momentum operator, ε_a is the Dirac energy of the state a, $\vec{D}(\omega)$ is obtained from the transverse part of the photon propagator in the Coulomb

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gauge D_C^{ij} by

$$D^j(\omega) = -4\pi Z\alpha\, \alpha^i\, D_C^{ij}(\omega,\vec{r}),$$

and α^i are the Dirac matrices. The summation over n in Eq. (1) is carried out over the complete Dirac spectrum. The transverse part of the photon propagator in the Coulomb gauge can be expressed in terms of the scalar function $\mathcal{D}(\omega, r)$ as

$$D_C^{ij}(\omega,\vec{r}) = \delta^{ij}\,\mathcal{D}(\omega,r) + \frac{\nabla^i\nabla^j}{\omega^2}[\mathcal{D}(\omega,r) - \mathcal{D}(0,r)]. \quad (2)$$

In the case of the standard photon propagator describing the interaction between the two pointlike particles, the function $\mathcal{D}(\omega, r)$ has a simple form [20]

$$\mathcal{D}(\omega,r) = -\frac{e^{i|\omega|r}}{4\pi r}. \quad (3)$$

It was recently demonstrated [25] that formula (1) can be generalized to describe the nuclear recoil effect for an extended-size nucleus if one replaces the standard photon propagator by the generalized photon propagator describing the interaction between a pointlike and an extended-size particle. The derivation of the generalized photon propagator presented in Ref. [25] yields the following result for the function $\mathcal{D}(\omega, r)$,

$$\mathcal{D}(\omega,r) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{r}} \frac{\rho(\vec{k}^2-\omega^2)}{\omega^2-\vec{k}^2}, \quad (4)$$

where $\rho(q^2)$ is the charge form factor of the nucleus in momentum space. An interesting feature of the generalized photon propagator is that it requires knowledge of the charge form factor $\rho(q^2)$ not only for the positive but also the negative arguments q^2. Generally, for calculating the nuclear recoil correction with the propagator (4), one needs the analytical continuation of the charge form factor into the whole complex plane of momenta.

Following Ref. [25], we use the exponential model of the nuclear charge distribution. Within this model, the nuclear charge density in coordinate and momentum space is given by

$$\rho(r) = \frac{\lambda^3}{8\pi}e^{-\lambda r}, \qquad \rho(\vec{k}^2) = \frac{\lambda^4}{(\lambda^2+\vec{k}^2)^2}, \quad (5)$$

respectively, where the parameter λ is expressed in terms of the root-mean-square nuclear charge radius r_C as $\lambda = 2\sqrt{3}/r_C$. Within this parametrization of the nuclear charge distribution, the extended-size photon propagator is written in coordinate space as [25]

$$\mathcal{D}(\omega,r) = -\frac{1}{4\pi}\left[\frac{e^{i|\omega|r}}{r} - \frac{e^{i\sqrt{\omega^2-\lambda^2}r}}{r} - \frac{i\lambda^2}{2}\frac{e^{i\sqrt{\omega^2-\lambda^2}r}}{\sqrt{\omega^2-\lambda^2}}\right]. \quad (6)$$

For numerical calculations, it is convenient to separate the nuclear recoil correction (1) into several parts, which are induced by the exchange of arbitrary number of Coulomb photons, by the Coulomb and one transverse photon ($E_{\rm tr1}$), and by the Coulomb and two transverse photons ($E_{\rm tr2}$). Furthermore, the Coulomb-photon part is separated into the leading-order part $E_{\rm L}$ and the higher-order Coulomb-photon part $E_{\rm C}$. We thus write

$$E_{\rm rec} = E_{\rm L} + E_{\rm C} + E_{\rm tr1} + E_{\rm tr2}. \quad (7)$$

The leading-order contribution is

$$E_{\rm L} = \frac{1}{2M}\langle a|\vec{p}^{\,2}|a\rangle. \quad (8)$$

We note that $E_{\rm L}$ has a form of an expectation value of the nonrelativistic reduced-mass operator $\vec{p}^{\,2}/(2M)$ with the Dirac wave functions. The remaining Coulomb-photon contribution is given by

$$E_{\rm C} = -\frac{1}{M}\sum_{\varepsilon_n<0}\langle a|\vec{p}|n\rangle\langle n|\vec{p}|a\rangle, \quad (9)$$

where the summation is performed over the *negative-continuum* part of the Dirac spectra. The correction $E_{\rm C}$ is suppressed by a factor of $(Z\alpha)^3$ as compared to the leading-order contribution $E_{\rm L}$.

The one-transverse-photon and the two-transverse-photon contributions are given by

$$E_{\rm tr1} = -\frac{i}{\pi M}\int_{-\infty}^{\infty} d\omega \sum_n \frac{1}{\varepsilon_a+\omega-\varepsilon_n(1-i0)} \times \langle a|\vec{p}|n\rangle\langle n|\vec{D}(\omega)|a\rangle, \quad (10)$$

$$E_{\rm tr2} = \frac{i}{2\pi M}\int_{-\infty}^{\infty} d\omega \sum_n \frac{1}{\varepsilon_a+\omega-\varepsilon_n(1-i0)} \times \langle a|\vec{D}(\omega)|n\rangle\langle n|\vec{D}(\omega)|a\rangle. \quad (11)$$

The one-transverse-photon and the two-transverse-photon contributions are suppressed with respect to the leading-order contribution $E_{\rm L}$ by factors $(Z\alpha)^2$ and $(Z\alpha)^3$, respectively.

III. NUMERICAL EVALUATION

We now bring the general formulas for the nuclear recoil correction to the form suitable for the numerical evaluation. The leading-order contribution $E_{\rm L}$ is calculated after transforming the matrix element as follows [26],

$$\langle a|\vec{p}^{\,2}|a\rangle = \langle a|(\vec{\alpha}\cdot\vec{p})^2|a\rangle = \langle a|(\varepsilon_a - \beta m - V_{\rm nucl})^2|a\rangle, \quad (12)$$

where β is the Dirac matrix, ε_a is the Dirac energy of the state a, and $V_{\rm nucl}(r)$ is the nuclear binding potential. Performing angular integration, we obtain

$$E_{\rm L} = \frac{1}{2M}\int_0^\infty dr\, r^2\big\{[(\varepsilon_a - V_{\rm nucl})^2 + m^2]\big[g_a^2(r)+f_a^2(r)\big] - 2m(\varepsilon_a - V_{\rm nucl})\big[g_a^2(r)-f_a^2(r)\big]\big\}, \quad (13)$$

where $g_a(r)$ and $f_a(r)$ are the upper and the lower radial components of the wave function of the state a, defined as in Ref. [27].

For calculating the remaining Coulomb-photon contribution $E_{\rm C}$ it is convenient to apply the identity [26]

$$\vec{p} = \tfrac{1}{2}\{\vec{\alpha}, h_D\} - \vec{\alpha}\,V_{\rm nucl}, \quad (14)$$

where $\{\vec{\alpha}, h_D\} = \vec{\alpha}\,h_D + h_D\vec{\alpha}$, and h_D is the Dirac Hamiltonian

$$h_D = \vec{\alpha}\cdot\vec{p} + \beta m + V_{\rm nucl}(r), \quad (15)$$

with $h_D|a\rangle = \varepsilon_a|a\rangle$. Therefore,

$$\langle a|\vec{p}|n\rangle = \langle a|\vec{\alpha}\,\phi(r)|n\rangle, \tag{16}$$

where $\phi(r) = (\varepsilon_a + \varepsilon_n)/2 - V_{\rm nucl}(r)$. The angular integration is performed analytically, yielding

$$E_{\rm C} = -\frac{1}{M}\sum_{\varepsilon_n<0}\frac{3}{2j_a+1}[R_C(an)]^2. \tag{17}$$

Here, j_a is the total angular momentum quantum number of the state a and the radial integral R_C is defined by

$$R_C(an) = \int_0^\infty dr\, r^2\,\phi(r)[g_a(r)f_n(r)S_{10}(\kappa_a,-\kappa_n) - f_a(r)g_n(r)S_{10}(-\kappa_a,\kappa_n)], \tag{18}$$

where κ_i is the Dirac angular-momentum quantum number of the state i and $S_{JL}(\kappa_1,\kappa_2)$ are the standard angular coefficients defined in the Appendix.

Calculations of the transverse-photon contribution are more complicated than that of $E_{\rm C}$ because of the integration over the photon energy ω. First, we make the Wick rotation of the ω integration contour. This rotation produces pole terms originating from the intermediate states more or equally deeply bound as the reference state. We obtain

$$E_{\rm tr1} = -\frac{2}{M}\sum_{0<\varepsilon_n\leqslant\varepsilon_a} a_n\langle a|\vec{p}|n\rangle\langle n|\vec{D}(\Delta_{an})|a\rangle + \frac{2}{\pi M}\int_0^\infty d\omega\sum_n\frac{\Delta_{an}}{\Delta_{an}^2+\omega^2}\langle a|\vec{p}|n\rangle\langle n|\vec{D}(i\omega)|a\rangle, \tag{19}$$

$$E_{\rm tr2} = \frac{1}{M}\sum_{0<\varepsilon_n\leqslant\varepsilon_a} a_n\langle a|\vec{D}(\Delta_{an})|n\rangle\langle n|\vec{D}(\Delta_{an})|a\rangle - \frac{1}{\pi M}\int_0^\infty d\omega\sum_n\frac{\Delta_{an}}{\Delta_{an}^2+\omega^2}\langle a|\vec{D}(i\omega)|n\rangle\langle n|\vec{D}(i\omega)|a\rangle, \tag{20}$$

where $a_n = 1$ for $\varepsilon_n \neq \varepsilon_a$ and $a_n = 1/2$ for $\varepsilon_n = \varepsilon_a$, and $\Delta_{an} = \varepsilon_a - \varepsilon_n$.

The angular integration for the one-transverse-photon contribution is carried out analytically using the standard Racah algebra. The result is

$$\sum_{\mu_n}\langle a|\vec{p}|n\rangle\langle n|\vec{D}(\omega)|a\rangle = \frac{3}{2j_a+1}R_C(an)\left[R_T^{(1)}(\omega,an) + \frac{1}{\sqrt{3}}C_1(\kappa_n,\,\kappa_a)R_T^{(2)}(\omega,an)\right], \tag{21}$$

where μ_n is the momentum projection of the state n, the angular coefficient $C_L(\kappa_1,\kappa_2)$ is defined in the Appendix, and the radial integrals are

$$R_T^{(1)}(\omega,an) = \int_0^\infty dr\,r^2\,\Phi_1(\omega,r)[g_a(r)f_n(r)S_{10}(\kappa_a,-\kappa_n) - f_a(r)g_n(r)S_{10}(-\kappa_a,\kappa_n)], \tag{22}$$

$$R_T^{(2)}(\omega,an) = \int_0^\infty dr\,r^2\,\Phi_2(\omega,r)[g_a(r)g_a(r)+f_a(r)f_n(r)]. \tag{23}$$

Furthermore,

$$\Phi_1(\omega,r) = -4\pi Z\alpha\,\mathcal{D}(\omega), \tag{24}$$

$$\Phi_2(\omega,r) = -4\pi Z\alpha\frac{\varepsilon_a-\varepsilon_n}{\omega^2}[\mathcal{D}'(\omega)-\mathcal{D}'(0)], \tag{25}$$

and $\mathcal{D}'(\omega) = d/(dr)\mathcal{D}(\omega,r)$.

Similarly, the integrand of the two-transverse-photon part is evaluated as

$$\sum_{\mu_n}\langle a|\vec{D}(\omega)|n\rangle\langle n|\vec{D}(\omega)|a\rangle = \frac{3}{2j_a+1}\left[R_T^{(1)}(\omega,an) + \frac{1}{\sqrt{3}}C_1(\kappa_n,\,\kappa_a)R_T^{(2)}(\omega,an)\right]^2. \tag{26}$$

IV. RESULTS

We performed numerical calculations of the nuclear recoil corrections by representing the Dirac muon spectrum with the finite basis set constructed with B splines, using the dual-kinetic balance method [28]. The numerical procedure is very similar to that developed for the electronic atoms in Ref. [29]. Numerical results are conveniently represented in terms of dimensionless function $P_{\rm rec}$ defined as

$$E_{\rm rec} = m_\mu c^2\frac{m_\mu}{M}\frac{(Z\alpha)^2}{2n^2}P_{\rm rec}(Z\alpha), \tag{27}$$

where $E_{\rm rec}$ is the recoil correction to the energy level. In the nonrelativistic limit $\alpha\to 0$, the nuclear recoil effect reduces to the multiplication of the binding energy by the reduced mass prefactor. Therefore, in the nonrelativistic limit $P_{\rm rec}(0) = 1$.

Table I presents our numerical results for the nuclear recoil correction obtained for the $n = 1$, $n = 2$, and $n = 3$ states of muonic atoms with the nuclear charge numbers $Z = 10 - 100$. For each Z, we present the leading-order contribution $E_{\rm L}$, the complete nuclear recoil correction $E_{\rm rec}$ calculated for the nuclear charge radius r_C specified in the table, and the derivative $E'_{\rm rec} = dE_{\rm rec}/dr_C$. It is interesting that the leading-order contribution $E_{\rm L}$ yields a very reasonable approximation for the complete recoil correction, even in the high-Z region. For example, for $Z = 90$ the difference between $E_{\rm L}$ and $E_{\rm rec}$ does not exceed 20%. This is in contrast to the electronic ions, where the corresponding difference for $Z = 90$ reaches 70% [21]. We conclude that for muonic atoms the QED corrections to the nuclear recoil effect are less prominent than for the electronic atoms. With the increase of the principal quantum number n and the orbital momentum l, we see a clear tendency that both $E_{\rm L}$ and $E_{\rm rec}$ rapidly approach the nonrelativistic limit of $P_{\rm rec}(0) = 1$.

In the point-nucleus limit the function $P_{\rm rec}$ is the same for electronic and muonic atoms. We checked that our numerical calculations with the point nuclear model agree with the results of Refs. [21,22] for the $n = 1$ and $n = 2$ states and with those of Ref. [30] for the $n = 3$ states. For the extended-size nuclei, the results for the muonic and electronic atoms are very much different. For the electronic atoms, the finite nuclear size effect is a small correction, whereas for the muonic atoms it changes the magnitude of the nuclear recoil effect drastically.

TABLE I. Nuclear recoil correction with inclusion of the finite nuclear size for muonic atoms, in terms of the function $P_{\rm rec}$. For each Z, the upper line presents results for the leading-order recoil correction $E_{\rm L}$, whereas the second and the third lines give the complete recoil correction $E_{\rm rec}$ and its derivative $E'_{\rm rec} = d/(dr_C)E_{\rm rec}$, in fm^{-1}.

Z	r_C (fm)		$1s$	$2s$	$2p_{1/2}$	$2p_{3/2}$	$3s$	$3p_{1/2}$	$3p_{3/2}$	$3d_{3/2}$	$3d_{5/2}$
10	3.0055	$E_{\rm L}$	0.96805	0.98776	1.00573	1.00174	0.99269	1.00470	1.00205	1.00160	1.00071
		$E_{\rm rec}$	0.95994	0.98114	1.00125	0.99994	0.98769	1.00112	1.00025	1.00029	1.00000
		$E'_{\rm rec}$	−0.02270	−0.01156	−0.00006	−0.00004	−0.00775	−0.00005	−0.00003	−0.00000	−0.00000
20	3.4776	$E_{\rm L}$	0.87756	0.95090	1.02196	1.00637	0.97030	1.01804	1.00773	1.00643	1.00285
		$E_{\rm rec}$	0.85493	0.92996	1.00399	0.99910	0.95400	1.00373	1.00052	1.00116	0.99997
		$E'_{\rm rec}$	−0.05638	−0.03025	−0.00103	−0.00076	−0.02062	−0.00081	−0.00060	−0.00000	−0.00000
30	3.9283	$E_{\rm L}$	0.75975	0.89972	1.04420	1.01123	0.93888	1.03650	1.01500	1.01455	1.00641
		$E_{\rm rec}$	0.72569	0.86303	1.00441	0.99506	0.90920	1.00489	0.99892	1.00258	0.99991
		$E'_{\rm rec}$	−0.07336	−0.04240	−0.00509	−0.00386	−0.02962	−0.00394	−0.00301	−0.00003	−0.00002
40	4.2694	$E_{\rm L}$	0.65040	0.84997	1.06504	1.01240	0.90861	1.05452	1.02086	1.02600	1.01137
		$E_{\rm rec}$	0.60926	0.79817	0.99720	0.98451	0.86500	1.00069	0.99292	1.00450	0.99977
		$E'_{\rm rec}$	−0.07614	−0.04787	−0.01363	−0.01051	−0.03440	−0.01030	−0.00809	−0.00017	−0.00012
50	4.6519	$E_{\rm L}$	0.54921	0.79955	1.07352	1.00411	0.87760	1.06440	1.02101	1.04057	1.01752
		$E_{\rm rec}$	0.50554	0.73483	0.97566	0.96297	0.82075	0.98653	0.97923	1.00663	0.99937
		$E'_{\rm rec}$	−0.07029	−0.04856	−0.02613	−0.02083	−0.03604	−0.01915	−0.01569	−0.00069	−0.00051
60	4.9123	$E_{\rm L}$	0.47194	0.75942	1.06937	0.98730	0.85370	1.06685	1.01636	1.05778	1.02451
		$E_{\rm rec}$	0.42713	0.68243	0.94210	0.93242	0.78339	0.96475	0.95957	1.00849	0.99841
		$E'_{\rm rec}$	−0.06341	−0.04813	−0.03905	−0.03214	−0.03689	−0.02783	−0.02369	−0.00192	−0.00141
70	5.3108	$E_{\rm L}$	0.39779	0.71411	1.03855	0.95315	0.82513	1.05278	1.00047	1.07573	1.03114
		$E_{\rm rec}$	0.35515	0.62843	0.88934	0.88633	0.74347	0.93068	0.92933	1.00852	0.99582
		$E'_{\rm rec}$	−0.05368	−0.04532	−0.05009	−0.04328	−0.03604	−0.03477	−0.03111	−0.00461	−0.00342
80	5.4648	$E_{\rm L}$	0.35068	0.68598	1.01045	0.92163	0.80935	1.04295	0.98798	1.09463	1.03764
		$E_{\rm rec}$	0.30833	0.58982	0.83977	0.84305	0.71454	0.89989	0.90175	1.00698	0.99194
		$E'_{\rm rec}$	−0.04800	−0.04448	−0.05784	−0.05159	−0.03650	−0.03972	−0.03665	−0.00835	−0.00621
82	5.5012	$E_{\rm L}$	0.34186	0.68027	1.00324	0.91434	0.80607	1.04013	0.98489	1.09824	1.03876
		$E_{\rm rec}$	0.29972	0.58225	0.82902	0.83366	0.70877	0.89324	0.89575	1.00628	0.99087
		$E'_{\rm rec}$	−0.04683	−0.04422	−0.05900	−0.05299	−0.03652	−0.04047	−0.03758	−0.00929	−0.00692
83	5.5211	$E_{\rm L}$	0.33747	0.67734	0.99935	0.91051	0.80436	1.03855	0.98323	1.10000	1.03929
		$E_{\rm rec}$	0.29546	0.57844	0.82350	0.82882	0.70584	0.88981	0.89266	1.00587	0.99028
		$E'_{\rm rec}$	−0.04623	−0.04408	−0.05953	−0.05366	−0.03652	−0.04082	−0.03801	−0.00979	−0.00729
90	5.7848	$E_{\rm L}$	0.30219	0.64980	0.95782	0.87374	0.78631	1.01813	0.96452	1.10922	1.04084
		$E_{\rm rec}$	0.26260	0.54731	0.77647	0.78709	0.68129	0.86019	0.86549	1.00007	0.98405
		$E'_{\rm rec}$	−0.04071	−0.04189	−0.06148	−0.05721	−0.03561	−0.04215	−0.04026	−0.01414	−0.01072
92	5.8571	$E_{\rm L}$	0.29309	0.64228	0.94545	0.86288	0.78133	1.01205	0.95897	1.11132	1.04090
		$E_{\rm rec}$	0.25417	0.53886	0.76302	0.77504	0.67454	0.85173	0.85765	0.99786	0.98186
		$E'_{\rm rec}$	−0.03929	−0.04131	−0.06172	−0.05792	−0.03537	−0.04236	−0.04072	−0.01550	−0.01181
100	5.8570	$E_{\rm L}$	0.27316	0.62987	0.92314	0.83949	0.77628	1.00674	0.95158	1.12470	1.04425
		$E_{\rm rec}$	0.23394	0.51805	0.72744	0.74374	0.65831	0.83026	0.83809	0.99208	0.97551
		$E'_{\rm rec}$	−0.03701	−0.04144	−0.06399	−0.06117	−0.03622	−0.04438	−0.04322	−0.02015	−0.01536

Figure 1 presents a plot of $P_{\rm rec}$ as a function of the nuclear radius r_C, for $Z = 90$ and the $1s$ state. It is seen that the finite nuclear size effect reduces the numerical value of the function $P_{\rm rec}$ by nearly an order of magnitude.

The dependence of $E_{\rm rec}$ on the nuclear charge radius r_C for muonic atoms is thus quite strong. For $Z = 90$ and the $1s$ state, a 10% change of the nuclear radius value given in Table I leads to a 9% change of $E_{\rm rec}$. In order to facilitate the analysis of the dependence of $E_{\rm rec}$ on the nuclear radius, Table I presents results for the first derivative of $E_{\rm rec}$ on r_C. The results for the derivative can be used to obtain $E_{\rm rec}$ for nuclear radii different from the ones listed in the table.

Table II presents a comparison of the nuclear recoil correction evaluated within the leading-order approximation, $E_{\rm L}$, the complete nuclear recoil correction evaluated within QED, $E_{\rm rec}$, and the results obtained previously within an approximate relativistic treatment [18]. We observe that the previous approach yields typically a slightly better approximation than the leading-order approximation $E_{\rm L}$. However, the approximate results are systematically lower than the full-QED values and the previous uncertainties underestimate the missing relativistic and QED effects.

In the present work we do not study the dependence of the nuclear recoil correction on the model of the nuclear charge distribution. The effect of this dependence is much smaller and completely overshadowed by the model dependence of the leading-order finite nuclear size contribution [31,32].

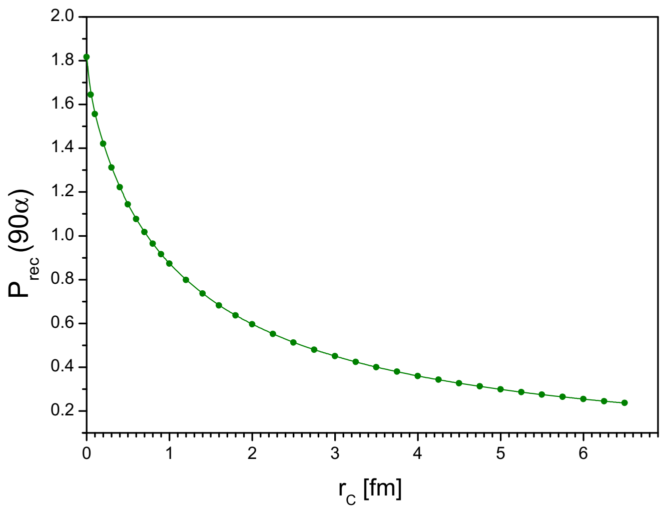

FIG. 1. Dependence of the nuclear recoil correction $P_{\rm rec}(Z\alpha)$ on the root-mean-square nuclear charge radius r_C, for $Z = 90$ and the $1s$ state.

V. CONCLUSION

We have calculated the nuclear recoil effect for muonic atoms rigorously within QED and to all orders in the nuclear binding strength parameter $Z\alpha$. This calculation significantly improves the accuracy of the nuclear recoil correction as compared to the previous treatments with partial inclusion of relativistic effects. The results of the full QED treatment differ from the nonrelativistic predictions for high values of Z and the deeply bound states, but rapidly converge to the nonrelativistic limit when Z decreases and (or) the highly excited states are considered. The nuclear recoil correction was shown to depend strongly on the nuclear charge radius. This dependence should be taken into account when nuclear parameters are extracted from experimental transition energies.

TABLE II. Nuclear recoil correction to the energies of muonic atoms, in keV. $E_{\rm L}$ is the leading-order contribution, and $E_{\rm rec}$ is the complete nuclear recoil correction.

Atom	r_C (fm)	State	$E_{\rm L}$	$E_{\rm rec}$	Previous [18]
$^{89}_{40}$Zr	4.2706	$1s_{1/2}$	3.735	3.499	3.21(15)
		$2s_{1/2}$	1.220	1.146	1.09(2)
		$2p_{1/2}$	1.529	1.432	1.42(1)
		$2p_{3/2}$	1.454	1.414	1.40(1)
		$3s_{1/2}$	0.580	0.552	0.53(1)
		$3p_{1/2}$	0.673	0.639	0.64
		$3p_{3/2}$	0.652	0.634	0.63
		$3d_{3/2}$	0.655	0.641	0.64
		$3d_{5/2}$	0.645	0.638	0.63
$^{147}_{62}$Sm	4.9892	$1s_{1/2}$	3.810	3.438	2.88(8)
		$2s_{1/2}$	1.566	1.402	1.26(5)
		$2p_{1/2}$	2.224	1.947	1.92(5)
		$2p_{3/2}$	2.050	1.930	1.92(4)
		$3s_{1/2}$	0.787	0.719	0.66(2)
		$3p_{1/2}$	0.988	0.890	0.88(1)
		$3p_{3/2}$	0.941	0.885	0.88(1)
		$3d_{3/2}$	0.985	0.936	0.92(1)
		$3d_{5/2}$	0.952	0.926	0.91(1)
$^{205}_{83}$Bi	5.5008	$1s_{1/2}$	3.633	3.179	2.41(6)
		$2s_{1/2}$	1.820	1.554	1.33(4)
		$2p_{1/2}$	2.685	2.212	2.12(3)
		$2p_{3/2}$	2.445	2.226	2.26(1)
		$3s_{1/2}$	0.960	0.842	0.75(3)
		$3p_{1/2}$	1.239	1.062	1.02(3)
		$3p_{3/2}$	1.173	1.065	1.03(3)
		$3d_{3/2}$	1.311	1.199	1.19(2)
		$3d_{5/2}$	1.239	1.180	1.17(2)

APPENDIX: ANGULAR COEFFICIENTS S_{JL} AND C_L

The coefficients $C_J(\kappa_b,\kappa_a)$ are given by

$$C_J(\kappa_b,\kappa_a) = (-1)^{j_b+1/2}\sqrt{(2j_a+1)(2j_b+1)} \times \begin{pmatrix} j_a & J & j_b \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \Pi(l_a,l_b,J), \tag{A1}$$

where the symbol $\Pi(l_a,l_b,J)$ is unity if l_a+l_b+J is even and zero otherwise. Furthermore, κ_i is the Dirac angular-momentum quantum number, $j_i = |\kappa_i| - 1/2$, and $l_i = |\kappa_i + 1/2| - 1/2$.

The angular coefficients $S_{JL}(\kappa_a,\kappa_b)$ are nonvanishing only for $L = J, J\pm 1$ and can be written for $J \neq 0$ as follows:

$$S_{J\,J+1}(\kappa_a,\kappa_b) = \sqrt{\frac{J+1}{2J+1}}\left(1+\frac{\kappa_a+\kappa_b}{J+1}\right)C_J(-\kappa_b,\kappa_a), \tag{A2}$$

$$S_{J\,J}(\kappa_a,\kappa_b) = \frac{\kappa_a-\kappa_b}{\sqrt{J(J+1)}}C_J(\kappa_b,\kappa_a), \tag{A3}$$

$$S_{J\,J-1}(\kappa_a,\kappa_b) = \sqrt{\frac{J}{2J+1}}\left(-1+\frac{\kappa_a+\kappa_b}{J}\right)C_J(-\kappa_b,\kappa_a). \tag{A4}$$

In the case $J = 0$ there is only one nonvanishing coefficient $S_{01}(\kappa_a,\kappa_b) = C_0(-\kappa_b,\kappa_a)$.

It can be immediate seen that $C_J(\kappa_b,\kappa_a) \propto \Delta(j_a,j_b,J)$, where Δ denotes the triangular condition. For the coefficients S_{JL} we have $S_{JL}(\kappa_a,\kappa_b) \propto \Delta(j_a,j_b,J)\Pi(l_a,l_b,L)$.

We note several useful symmetry relations of the angular coefficients:

$$C_J(\kappa_a,\kappa_b) = C_J(-\kappa_a,-\kappa_b) = (-1)^{j_a-j_b}C_J(\kappa_b,\kappa_a), \tag{A5}$$

$$S_{JL}(\kappa_a,\kappa_b) = (-1)^{J+L+1}(-1)^{j_b-j_a}S_{JL}(\kappa_b,\kappa_a). \tag{A6}$$

Several specific values of the angular coefficients relevant for this study are

$$S_{10}(-1,-1) = \sqrt{2}, \quad S_{10}(1,1) = -\frac{\sqrt{2}}{3}, \quad S_{10}(-1,2) = 0, \quad S_{10}(1,-2) = \frac{4}{3}. \tag{A7}$$

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