Casimir and Casimir-Polder interactions for magneto-dielectric materials: Surface scattering expansion

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We develop a general multiple scattering expansion (MSE) for computing Casimir forces between magneto-dielectric bodies and Casimir-Polder forces between polarizable particles and magneto-dielectric bodies. The approach is based on fluctuating electric and magnetic surface currents and charges. The surface integral equations for these surface fields can be formulated in terms of surface scattering operators (SSOs). We show that there exists an entire family of such operators. One particular member of this family is only weakly divergent and allows for a MSE that appears to be convergent for general magneto-dielectric bodies. We prove a number of properties of this operator, and demonstrate explicitly convergence for sufficiently low and high frequencies, and for perfect conductors. General expressions are derived for the Casimir interaction between macroscopic bodies and for the Casimir-Polder interaction between particles and macroscopic bodies in terms of the SSO, both at zero and finite temperatures. An advantage of our approach over previous scattering methods is that it does not require the knowledge of the scattering amplitude (T operator) of the bodies. A number of simple examples are provided to demonstrate the use of the method. Some applications of our approach have appeared previously [T. Emig and G. Bimonte, Phys. Rev. Lett. **130**, 200401 (2023)]. Here we provide additional technical aspects and details of our approach.

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I. INTRODUCTION

It is a quite common situation in physics, biology, and chemistry to find surfaces of macroscopic objects and particles in close proximity to each other. Although these structures carry often no charge, they still experience a long-ranged interaction which results from modifications of the quantum and thermal fluctuations of the electromagnetic (EM) field by the objects. A well-known manifestation of this interaction is the Casimir force between two parallel perfectly conducting plates [1]. Microscopically, this interaction can be understood as a collective, nonadditive force between induced dipoles in the bodies. Indeed, the connection between an atomistic description and nonideal macroscopic dielectric materials was established by Lifshitz, who considered random currents within the interacting bodies to obtain the Casimir force between planar bodies [2]. This approach has been the core theory for interpreting most of the precision measurements of Casimir interactions between various materials and surface shapes which were enabled by an enormous progress in force sensing techniques and the fabrication of nanostructures [3–11]. Naturally, in practice macroscopic bodies have curved or structured surfaces. Hence, an approximation by planar surfaces is often not justified. Indeed, recent experiments [12–14] have demonstrated large deviations from common proximity approximations [15], making theoretical formulations for a precise force computation highly desirable.

An exact computation of Casimir forces in nonplanar geometries is extremely hard. To date, the only nonplanar configurations for which the force can be computed exactly are the sphere-plate and the sphere-sphere systems, for Drude conductors in the high-temperature limit [16,17]. In principle, there exist methods to compute Casimir forces in arbitrary geometries. However, they are often limited in their practical applicability. Indeed, enormous efforts have been put forward by many groups to develop theoretical and numerical methods that can cope with more general surface shapes [18–20]. Specifically, the scattering method [21–23], originally devised for mirrors [24,25], expresses the interaction between dielectric bodies in terms of their scattering amplitude, known as T operator. While this approach has enabled most of the recent theoretical progress, the T operator is known only for highly symmetric bodies, such as spheres and cylinders, or for a few perfectly conducting shapes [26], practically exhausting this method. The Casimir interaction of dielectric gratings has been computed using a generalization of the Rayleigh expansion in [27–29]. The scattering approach can be augmented by advanced numerical methods, for example for gratings [30], but they can be limited by computational power required for convergence. A more fundamental

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limitation is that interlocked geometries evade this method due to lack of convergence of the partial wave expansions [14]. If the surface is only gently curved, a gradient expansion can be used to obtain first-order corrections to the proximity approximation [31,32]. The theoretical treatment of nonideal materials with sharp surface features, such as used in atomic force microscopy or fabricated by lithographical techniques, is beyond the scope of existing methods.

Substantial progress has been made over the last decade with fully numerical methods to compute Casimir forces for general shapes and materials. An important example is an approach based on a boundary element method (SCUFF-EM) for computing the interaction of fluctuating surface currents [33,34]. It is believed that this approach can provide in principle the exact force for arbitrary shapes, with computational power the only but practically important limiting factor [14]. This method depends on a suitable refinement of the surface mesh for a broad band of relevant wavelengths. Therefore, the numerical effort for keeping discretization errors sufficiently small can be challenging. To the best of our knowledge, complementary, not fully numerical methods with comparably broad application range do not exist to date.

Here we develop an approach for computing Casimir forces for magneto-dielectric bodies of arbitrary shape. Conceptional, the Casimir force is related to fluctuating electric and magnetic surface currents and charges by the fluctuation-dissipation theorem [35]. This allows for a formulation of a general theory for Casimir forces that is based on scattering operators which are localized only on the surfaces of the interacting bodies. The important features of our method are the following.

(i) No knowledge of the scattering amplitude (the T matrix) of the bodies is required. Hence, an important practical problem of the existing scattering approaches is overcome.

(ii) No expansion of the EM field in partial waves, or expansion of currents in multipoles, is required. This eliminates the problems of convergence in geometries where surfaces interlock.

(iii) Explicit expressions for the surface scattering operators (SSOs) are given in terms of free Green functions.

(iv) Any basis for the tangential surface currents can be used, simplifying the computation of surface integrals appearing in the operator products.

(v) The Casimir interaction can be expanded in the number of surface scatterings, leading to a rapidly converging estimate for the interaction energy.

We underline that not all of the above five features are unique to our approach. For example, the surface approach developed in [33,34], which is implemented in the boundary element method (SCUFF-EM), possesses the features (i)–(iv) listed above. However, differently from our SSO, the surface operator $\mathbf{M}$ used in [33,34] [corresponding to case (C3) in Sec. IV below] is not of Fredholm form, and therefore it does not allow for a multiple scattering expansion (MSE). The approach based on the volume T operator (for a review see [36]) shares with our approach properties (i) and (ii) above. However, this approach is based on a three-dimensional integral kernel supported on the *volumes* occupied by the bodies, which involves the free Green tensor of the background and the polarizabilities of the bodies. When the resulting formula for the Casimir energy is expanded in powers of the polarizabilities, one obtains a Born expansion of the Casimir energy, expressed in terms of iterated volume integrals extended over the volumes occupied by the bodies [37,38]. This is of course different from our MSE, which instead is an expansion in terms of iterated *surface* integrals extended over the surfaces of the bodies.

In our approach, the general multiple scattering expansion is enabled by treating the back and forth scatterings of waves between different objects on an equal footing as the scatterings within an isolated object, eliminating the necessity to resort to the concept of a T operator. In this formulation, a wave propagates freely in a magneto-dielectric medium between successive scattering points on the surfaces, no matter if the points belong to different objects or the same object. For perfectly conducting objects, in a seminal work Balian and Duplantier had demonstrated the very existence and convergence of a multiple scattering expansion for Casimir forces [39,40]. Our approach shows that a conceptionally similar theory can be developed for arbitrary dissipative magneto-dielectric materials. We provide a number of simple examples which show rapid convergence in the number of scatterings even at short surface separations. Our paper represents a powerful approach to substantially extend accurate predictions of Casimir forces to materials and shapes for which only computationally intensive fully numerical methods were available.

A brief report of our findings has appeared previously [41]. Here we provide details of the derivation of the multiple scattering expansion and derive some important properties of the SSO. The paper is organized as follows. In Sec. II we derive the general expression of the SSO for a collection of N magneto-dielectric bodies of any shape, placed at arbitrary relative positions in space. In Sec. III we express the Casimir interaction of two bodies, and the Casimir-Polder (CP) interaction between a polarizable particle and general magneto-dielectric body in terms of the SSO. Several equivalent formulations of the SSO are discussed in Sec. IV. The limits of perfect conductors, and high and low frequencies, are analyzed in Sec. V. In Sec. VI we address the convergence properties of the MSE in general. A number of simple examples demonstrate the application of the MSE in Sec. VII. In Sec. VIII we present our conclusions and a discussion of future applications of the MSE. Finally, several appendices provide further technical details.

II. ELECTRIC AND MAGNETIC SURFACE CURRENTS FROM A MULTIPLE SCATTERING EXPANSION

Before considering Casimir interactions, we first develop in this section the concept of surface currents and show how they naturally lead to an expansion of the EM field in the number of surface scatterings. This shall enable us to formulate a scattering expansion for the scattering Green tensor $\mathbb{\Gamma}(\mathbf{r}, \mathbf{r}') = \mathbb{G}(\mathbf{r}, \mathbf{r}') - \mathbb{G}_0(\mathbf{r}, \mathbf{r}')$, where $\mathbb{G}$ is the N-body EM Green tensor and $\mathbb{G}_0$ is the empty space Green tensor for a homogenous medium with contrast ϵ_0, μ_0 (see Appendix E). Physically, $\mathbb{\Gamma}(\mathbf{r}, \mathbf{r}')$ describes the *modification* of the EM field at position $\mathbf{r}$, due to the presence of the bodies, when it is generated by a source at position $\mathbf{r}'$. This naturally implies to construct $\mathbb{\Gamma}$ from the surface fields which are induced by an external source at the bodies. However, the primary current induced directly by the source induces in turn a secondary current, which induces again higher-order currents, leading to an infinite sequence of induction processes. As we shall demonstrate subsequently, an exact mathematical description of these processes is provided by our MSE for $\mathbb{\Gamma}$. While Green functions have been constructed in terms of surface currents, the existence and convergence of a MSE between magneto-dielectric bodies is not obvious, particularly for Casimir interactions, and to the best of our knowledge had been demonstrated only for perfect electric conductors [39,40,42]. The MSE is based on surface integral equations that determine the tangential electric and magnetic fields at the surfaces S_σ which can be considered as magnetic surface currents $\mathbf{m}_\sigma$ and electric surface currents $\mathbf{j}_\sigma$, acting as equivalent sources for the scattered field [43]. This can be viewed as a mathematical reformulation of the Huygens principle. We note that in the static limit, it shall turn out that it is sufficient to consider the normal components of the EM field at the surfaces, corresponding to electric and magnetic surface charge densities. For finite frequencies, these charge densities are related to the surface currents by surface continuity equations.

In the following, we consider a configuration of N material bodies with dielectric and magnetic permittivities ϵ_σ and μ_σ ($\sigma = 1, \dots, N$). The bodies are bounded by closed surfaces S_σ which can be of arbitrary shape and separate their bulk from the surrounding homogeneous medium with dielectric and magnetic permittivities ϵ_0 and μ_0 [44]. From the uniqueness of an EM field in a region specified by sources within the region and the tangential components of the field over the boundary of the region, one can construct the total EM field $(\mathbf{E}, \mathbf{H})$ separately in the region external to the bodies, and inside the N interior regions of the bodies. When doing so, one can vary the field outside a given region at will as long as the surface currents are adjusted according to the jump conditions $\mathbf{j} = \mathbf{n} \times (\mathbf{H}_+ - \mathbf{H}_-)$, $\mathbf{m} = -\mathbf{n} \times (\mathbf{E}_+ - \mathbf{E}_-)$ where $\mathbf{n}$ is the surface normal pointing to the outside and the label $+(-)$ indicates the value when the surface is approached from the outside (inside). To proceed, we make the choice that the field outside a given region vanishes as this allows us to replace the magneto-dielectric media outside the region by the medium inside the region, so that the surface currents on the boundary of the region radiate in homogenous unbounded space. Hence the field can be expressed in the interior of the bodies as the surface integral

$$(\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})(\mathbf{r}) = \int_{S_\sigma} ds_{\mathbf{u}}\, \mathbb{G}_\sigma(\mathbf{r}, \mathbf{u})(\mathbf{j}_{\sigma-}, \mathbf{m}_{\sigma-})(\mathbf{u}), \tag{1}$$

where $\mathbb{G}_\sigma$ is the free Green tensor in a medium with permittivities ϵ_σ and μ_σ, and $\mathbf{j}_{\sigma-} = -\mathbf{n}_\sigma \times \mathbf{H}_-$ and $\mathbf{m}_{\sigma-} = \mathbf{n}_\sigma \times \mathbf{E}_-$ are the tangential fields when S_σ is approached from the inside of the bodies. Exterior to the bodies the field

$$(\mathbf{E}^{(0)}, \mathbf{H}^{(0)})(\mathbf{r}) = \int d\mathbf{r}'\, \mathbb{G}_0(\mathbf{r}, \mathbf{r}')(\mathbf{J}, \mathbf{M})(\mathbf{r}') + \sum_{\sigma=1}^{N} \int_{S_\sigma} ds_{\mathbf{u}}\, \mathbb{G}_0(\mathbf{r}, \mathbf{u})(\mathbf{j}_{\sigma+}, \mathbf{m}_{\sigma+})(\mathbf{u}), \tag{2}$$

where now $\mathbf{j}_{\sigma+} = \mathbf{n}_\sigma \times \mathbf{H}_+$ and $\mathbf{m}_{\sigma+} = -\mathbf{n}_\sigma \times \mathbf{E}_+$ are the tangential fields when S_σ is approached from the outside of the bodies and we assumed an external source of electric and magnetic currents $(\mathbf{J}, \mathbf{M})$ outside the bodies to generate the incident field $(\mathbf{E}_{\text{inc}}, \mathbf{H}_{\text{inc}})$. Surface integral equations for the surface fields follow by taking advantage of the property of the surface integrals that they are also defined when $\mathbf{r}$ is located *on* the surfaces and their corresponding value is the average of the limits taken from the inside and the outside [45], and that one of the two limits vanishes by construction, leading to

$$\begin{aligned}(\mathbf{m}_{\sigma-}, -\mathbf{j}_{\sigma-})(\mathbf{u}) &= 2\mathbf{n}_\sigma(\mathbf{u}) \times (\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})(\mathbf{u}),\\ (\mathbf{m}_{\sigma+}, -\mathbf{j}_{\sigma+})(\mathbf{u}) &= -2\mathbf{n}_\sigma(\mathbf{u}) \times (\mathbf{E}^{(0)}, \mathbf{H}^{(0)})(\mathbf{u})\end{aligned} \tag{3}$$

for $\mathbf{u}$ located on surface S_σ. Associated with the surface currents must be surface charges which are given by the (rescaled) surface charge densities, defined on both sides of the surfaces as

$$\begin{aligned}(\varrho_{j,\sigma-}, \varrho_{m,\sigma-})(\mathbf{u}) &= -2\mathbf{n}_\sigma(\mathbf{u}) \cdot (\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})(\mathbf{u}),\\ (\varrho_{j,\sigma+}, \varrho_{m,\sigma+})(\mathbf{u}) &= 2\mathbf{n}_\sigma(\mathbf{u}) \cdot (\mathbf{E}^{(0)}, \mathbf{H}^{(0)})(\mathbf{u}).\end{aligned} \tag{4}$$

Finally, to couple the interior and exterior solutions, we impose the usual continuity conditions on the tangential components of $(\mathbf{E}, \mathbf{H})$ at the interfaces between different media, leading to one unique set of surface currents $(\mathbf{j}_\sigma, \mathbf{m}_\sigma) \equiv (\mathbf{j}_{\sigma+}, \mathbf{m}_{\sigma+}) = -(\mathbf{j}_{\sigma-}, \mathbf{m}_{\sigma-})$. Similarly, imposing continuity on the normal components of $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ leads to the relation

$$\varrho_{j,\sigma-} = -\frac{\epsilon_0}{\epsilon_\sigma}\varrho_{j,\sigma+}, \quad \varrho_{m,\sigma-} = -\frac{\mu_0}{\mu_\sigma}\varrho_{m,\sigma+} \tag{5}$$

between the interior and exterior charge densities. Hence, it is sufficient to consider the unique set of surface charge densities $(\varrho_{j,\sigma}, \varrho_{m,\sigma}) \equiv (\varrho_{j,\sigma+}, \varrho_{m,\sigma+})$. Since the field $(\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})$ obeys the source free Maxwell equations in the interior region of the surface S_σ, the interior surface currents and charges are related by the continuity equations

$$\begin{aligned}\nabla\mathbf{j}_{\sigma-} &= -\kappa\epsilon_\sigma\varrho_{j,\sigma-},\\ \nabla\mathbf{m}_{\sigma-} &= -\kappa\mu_\sigma\varrho_{m,\sigma-},\end{aligned} \tag{6}$$

or, due to Eq. (5), equivalently by the continuity equations for the unique surface currents and charges:

$$\begin{aligned}\nabla\mathbf{j}_\sigma &= -\kappa\epsilon_0\varrho_{j,\sigma},\\ \nabla\mathbf{m}_\sigma &= -\kappa\mu_0\varrho_{m,\sigma}.\end{aligned} \tag{7}$$

Now we have expressed the surface currents $(\mathbf{j}_\sigma, \mathbf{m}_\sigma)$ and charges $(\varrho_{j,\sigma}, \varrho_{m,\sigma})$ in terms of both the interior field $(\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})$ and the exterior field $(\mathbf{E}^{(0)}, \mathbf{H}^{(0)})$. This yields the surface integral equations

$$(\mathbf{m}_\sigma, -\mathbf{j}_\sigma)(\mathbf{u}) = -2\mathbf{n}_\sigma(\mathbf{u}) \times (\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})(\mathbf{u}), \tag{8}$$

$$(\mathbf{m}_\sigma, -\mathbf{j}_\sigma)(\mathbf{u}) = -2\mathbf{n}_\sigma(\mathbf{u}) \times (\mathbf{E}^{(0)}, \mathbf{H}^{(0)})(\mathbf{u}), \tag{9}$$

$$(\varrho_{j,\sigma}, \varrho_{m,\sigma})(\mathbf{u}) = 2\mathbf{n}_\sigma(\mathbf{u}) \cdot \left(\frac{\epsilon_\sigma}{\epsilon_0}\mathbf{E}^{(\sigma)}, \frac{\mu_\sigma}{\mu_0}\mathbf{H}^{(\sigma)}\right)(\mathbf{u}), \tag{10}$$

$$(\varrho_{j,\sigma}, \varrho_{m,\sigma})(\mathbf{u}) = 2\mathbf{n}_\sigma(\mathbf{u}) \cdot (\mathbf{E}^{(0)}, \mathbf{H}^{(0)})(\mathbf{u}) \tag{11}$$

where the fields are given by the integrals in Eqs. (1) and (2) with $(\mathbf{j}_{\sigma+}, \mathbf{m}_{\sigma+}) = (\mathbf{j}_\sigma, \mathbf{m}_\sigma)$ and $(\mathbf{j}_{\sigma-}, \mathbf{m}_{\sigma-}) = -(\mathbf{j}_\sigma, \mathbf{m}_\sigma)$. These $8N$ surface integral equations constitute an overdetermined system for the $2N$ surface currents or tangential surface fields, and the $2N$ surface charge densities, which must be related to the surface currents by the continuity equations (7). Existence of a unique solution requires that only $4N$ equations are independent, agreeing with the number of constraints imposed by the continuity of the tangential and normal field components. The additional $4N$ constraints, implicitly fulfilled by construction of the fields, must account for the unique relation between the components of the electric and magnetic fields on both sides of the surfaces as specification of either tangential $\mathbf{E}$ or tangential $\mathbf{H}$ determines a unique solution to the exterior and interior problems. For this reason, a consistent set of $4N$ integral equations with a unique solution can be obtained by taking linear combinations of the set of $4N$ equations involving $(\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})$ *and* the corresponding set involving $(\mathbf{E}^{(0)}, \mathbf{H}^{(0)})$ but not by considering only one of the two sets as this would ignore the coupling of the interior and exterior fields.

We first consider the integral equations for the surface currents, Eqs. (8) and (9). In general, when taking linear combinations of the integral equations, one can choose $4N$ suitable coefficients which form $2N$ diagonal 2×2 matrices $\mathbb{C}^i_\sigma$ and $\mathbb{C}^e_\sigma$ acting on the two field components of the interior and exterior integral equations. To interpret the integral equations as successive scatterings, we introduce the SSOs $\mathbb{K}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}')$ which describe free propagation from $\mathbf{u}'$ on surface $S_{\sigma'}$ to $\mathbf{u}$ on surface S_σ and scattering at point $\mathbf{u}$

$$
\begin{aligned}
\mathbb{K}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}') &= 2\mathbb{P}\big(\mathbb{C}^i_\sigma + \mathbb{C}^e_\sigma\big)^{-1}\mathbf{n}_\sigma(\mathbf{u}) \\
&\quad\times\big[\delta_{\sigma\sigma'}\mathbb{C}^i_\sigma\mathbb{G}_\sigma(\mathbf{u}, \mathbf{u}') - \mathbb{C}^e_\sigma\mathbb{G}_0(\mathbf{u}, \mathbf{u}')\big],\\
\mathbb{P} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\end{aligned} \tag{12}
$$

acting on electric and magnetic tangential surface fields at $\mathbf{u}'$ ($\delta_{\sigma\sigma'}$ is the Kronecker delta). The action of $\mathbf{n}_\sigma(\mathbf{u})\times$ on the 3×3 matrices $\mathbb{G}^{(pq)}_\sigma$ and $\mathbb{G}^{(pq)}_0$ ($p, q \in \{E, H\}$) are respectively defined by $(\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}^{(pq)}_\sigma)\mathbf{v} \equiv \mathbf{n}_\sigma(\mathbf{u}) \times (\mathbb{G}^{(pq)}_\sigma \mathbf{v})$ and $(\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}^{(pq)}_0)\mathbf{v} \equiv \mathbf{n}_\sigma(\mathbf{u}) \times (\mathbb{G}^{(pq)}_0 \mathbf{v})$, for any vector $\mathbf{v}$.

With these SSOs the surface currents are determined in terms of the external source $(\mathbf{J}, \mathbf{M})$ by the Fredholm integral equations of the second kind

$$
\begin{aligned}
&\sum_{\sigma'=1}^{N}\int_{S_{\sigma'}} ds_{\mathbf{u}'}\,[\mathbb{1} - \mathbb{K}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}')]\begin{pmatrix}\mathbf{j}_{\sigma'}\\ \mathbf{m}_{\sigma'}\end{pmatrix}(\mathbf{u}')\\
&\quad= \int d\mathbf{r}\,\mathbb{M}_\sigma(\mathbf{u}, \mathbf{r})\begin{pmatrix}\mathbf{J}\\ \mathbf{M}\end{pmatrix}(\mathbf{r})
\end{aligned} \tag{13}
$$

with

$$
\mathbb{M}_\sigma(\mathbf{u}, \mathbf{r}) = -2\mathbb{P}\big(\mathbb{C}^i_\sigma + \mathbb{C}^e_\sigma\big)^{-1}\mathbb{C}^e_\sigma\,\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}_0(\mathbf{u}, \mathbf{r}). \tag{14}
$$

(For an alternative derivation of the SSO we refer to Appendix A.) More explicit expressions for the SSO for different choices of the coefficient matrices will be given below in Sec. IV. As we shall see, the choice of coefficients $\mathbb{C}^i_\sigma$ and $\mathbb{C}^e_\sigma$ provides a powerful tool to engineer convergence of the MSE. Uniqueness of the solution of the integral equation (13) is ensured if one can show that the operator $\mathbb{K}$ does not have an eigenvalue equal to 1. Such a proof for any (complex) frequency can be found in the book [45] for a particular choice of coefficients, denoted by choice 1 in Sec. IV below, and for a single body. A simple generalization of the proof allows us to show that the result remains true for any number of bodies. After an appropriate rescaling of the EM field, one can show that the result holds also for all values of the coefficients as long as $\mathbb{C}^e_\sigma + \mathbb{C}^i_\sigma$ is different from zero. Explicit computation of the SSO requires integration of the *free* space Green tensor in homogenous media over the bodies' surfaces which can be performed analytically in some cases. Contributions to the Casimir energy from scatterings between remote surface positions are exponentially damped with distance as we need to consider the Green tensor only for purely imaginary frequencies.

Next, we consider the integral equations which determine the surface charge densities. While the electromagnetic scattering problem is basically solved in terms of surface currents determined by Eq. (13), it turns out that the zero frequency limit $\kappa = 0$ requires a separate treatment due to a divergent term in the SSO for $\kappa \to 0$. The corresponding static problem is described in terms of surface charges only, as we shall see now. We take linear combinations of the integral equations for the surface currents, Eqs. (10) and (11), with scalar interior coefficients $\mathbb{c}^i_{j,\sigma}$ and $\mathbb{c}^i_{m,\sigma}$ and exterior coefficients $\mathbb{c}^e_{j,\sigma}$ and $\mathbb{c}^e_{m,\sigma}$. Using the surface divergence theorem and the continuity Eqs. (7), one gets two Fredholm integral equations of the second kind,

$$
\begin{aligned}
&\varrho_{j,\sigma}(\mathbf{u}) + \frac{2}{\mathbb{c}^e_{j,\sigma} + \mathbb{c}^i_{j,\sigma}}\sum_{\sigma'=1}^{N}\int_{S_{\sigma'}} ds_{\mathbf{u}'}\bigg[\kappa\,\mathbf{n}_\sigma(\mathbf{u})\cdot\mathbf{j}_{\sigma'}(\mathbf{u}')\Big(\mathbb{c}^e_{j,\sigma}\mu_0 g_0(\mathbf{u}-\mathbf{u}') - \frac{\epsilon_\sigma}{\epsilon_0}\mathbb{c}^i_{j,\sigma}\mu_\sigma g_\sigma(\mathbf{u}-\mathbf{u}')\delta_{\sigma\sigma'}\Big)\\
&\quad+\varrho_{j,\sigma'}(\mathbf{u})\big(\mathbb{c}^e_{j,\sigma}\partial_{\mathbf{n}_\sigma(\mathbf{u})}g_0(\mathbf{u}-\mathbf{u}') - \mathbb{c}^i_{j,\sigma}\partial_{\mathbf{n}_\sigma(\mathbf{u})}g_\sigma(\mathbf{u}-\mathbf{u}')\delta_{\sigma\sigma'}\big)\\
&\quad+\mathbf{n}_\sigma(\mathbf{u})\cdot\bigg(\Big(\mathbb{c}^e_{j,\sigma}\nabla_{\mathbf{u}}g_0(\mathbf{u}-\mathbf{u}') - \frac{\epsilon_\sigma}{\epsilon_0}\mathbb{c}^i_{j,\sigma}\nabla_{\mathbf{u}}g_\sigma(\mathbf{u}-\mathbf{u}')\delta_{\sigma\sigma'}\Big)\times\mathbf{m}_{\sigma'}(\mathbf{u}')\bigg)\bigg]\\
&= \frac{2\mathbb{c}^e_{j,\sigma}}{\mathbb{c}^e_{j,\sigma} + \mathbb{c}^i_{j,\sigma}}\mathbf{n}_\sigma(\mathbf{u})\cdot\mathbf{E}_{\rm inc}(\mathbf{u}),
\end{aligned} \tag{15}
$$

$$\begin{aligned}
&\varrho_{m,\sigma}(\mathbf{u}) + \frac{2}{\mathbb{c}^e_{m,\sigma}+\mathbb{c}^i_{m,\sigma}} \sum_{\sigma'=1}^{N} \int_{S_{\sigma'}} ds_{\mathbf{u}'} \bigg[\kappa\, \mathbf{n}_\sigma(\mathbf{u})\cdot \mathbf{m}_{\sigma'}(\mathbf{u}')\bigg(\mathbb{c}^e_{m,\sigma}\epsilon_0 g_0(\mathbf{u}-\mathbf{u}') - \frac{\mu_\sigma}{\mu_0}\mathbb{c}^i_{m,\sigma}\epsilon_\sigma g_\sigma(\mathbf{u}-\mathbf{u}')\delta_{\sigma\sigma'}\bigg)\\
&\quad + \varrho_{m,\sigma'}(\mathbf{u})\big(\mathbb{c}^e_{m,\sigma}\partial_{\mathbf{n}_\sigma(\mathbf{u})} g_0(\mathbf{u}-\mathbf{u}') - \mathbb{c}^i_{m,\sigma}\partial_{\mathbf{n}_\sigma(\mathbf{u})} g_\sigma(\mathbf{u}-\mathbf{u}')\delta_{\sigma\sigma'}\big)\\
&\quad - \mathbf{n}_\sigma(\mathbf{u})\cdot\bigg(\bigg(\mathbb{c}^e_{m,\sigma}\nabla_{\mathbf{u}} g_0(\mathbf{u}-\mathbf{u}') - \frac{\mu_\sigma}{\mu_0}\mathbb{c}^i_{m,\sigma}\nabla_{\mathbf{u}} g_\sigma(\mathbf{u}-\mathbf{u}')\delta_{\sigma\sigma'}\bigg)\times \mathbf{j}_{\sigma'}(\mathbf{u}')\bigg)\bigg]\\
&= \frac{2\mathbb{c}^e_{m,\sigma}}{\mathbb{c}^e_{m,\sigma}+\mathbb{c}^i_{m,\sigma}}\mathbf{n}_\sigma(\mathbf{u})\cdot\mathbf{H}_{\text{inc}}(\mathbf{u}),
\end{aligned} \tag{16}$$

where g_σ is the scalar free Green function (see Appendix E). Different choices for the coefficients $\mathbb{c}^{i/e}_{j/m,\sigma}$ will be discussed in Sec. IV. Remarkably, there exists a choice of the coefficients [see Eq. (31)] such that in the static limit, $\kappa \to 0$, the above integral equations can be expressed in terms of the surface charges only, as we shall show in Sec. V A below.

III. INTERACTIONS DUE TO FLUCTUATIONS OF THE ELECTROMAGNETIC FIELD AND SURFACE CURRENTS

A. Scattering Green tensor

The scattering Green tensor $\mathbb{\Gamma}(\mathbf{r},\mathbf{r}') = \mathbb{G}(\mathbf{r},\mathbf{r}') - \mathbb{G}_0(\mathbf{r},\mathbf{r}')$ is essential to compute the expectation value of the stress tensor, and hence Casimir forces. It is determined by the field generated by the surface currents, and hence

$$\mathbb{\Gamma}(\mathbf{r},\mathbf{r}') = \int_S ds_{\mathbf{u}} \int_S ds_{\mathbf{u}'}\, \mathbb{G}_0(\mathbf{r},\mathbf{u})(\mathbb{1}-\mathbb{K})^{-1}(\mathbf{u},\mathbf{u}')\mathbb{M}(\mathbf{u}',\mathbf{r}') \tag{17}$$

where the integration extends over all surfaces S_σ and a summation over all surface labels σ is understood. The operator $\mathbb{M}(\mathbf{u}',\mathbf{r}')$ is proportional to the free Green tensor,

$$\begin{aligned}
\mathbb{M}(\mathbf{u}',\mathbf{r}') &= \mathbb{V}(\mathbf{u}')\mathbb{G}_0(\mathbf{u}',\mathbf{r}'),\\
\mathbb{V}(\mathbf{u}') &= -2\mathbb{P}\big(\mathbb{C}^i_\sigma + \mathbb{C}^e_\sigma\big)^{-1}\mathbb{C}^e_\sigma\, \mathbf{n}_\sigma(\mathbf{u}')\times\cdot,
\end{aligned} \tag{18}$$

where $\cdot$ is a placeholder for the argument on which the operator acts. The existence of a MSE follows from the Fredholm type of the operator $(\mathbb{1}-\mathbb{K})^{-1}$ that permits an expansion in powers of $\mathbb{K}$ [46] and hence in the number of scatterings, as illustrated for one body in Fig. 1(a).

B. Casimir force between magneto-dielectric bodies

We now derive the Casimir interaction among the bodies. Following the method in [36], we first express the Casimir force $\mathbf{F}_\sigma$ on one of the bodies, labeled by σ, as the integral of the expectation value of the EM stress tensor at discrete Matsubara imaginary frequencies $\xi = i\omega$ with $\xi = \xi_n = 2\pi n k_B T/\hbar$ with $n = 0, 1, \ldots$, over the surface S_σ using the fluctuation-dissipation theorem. A divergence in the surface integral, originating from the empty space stress tensor and hence unrelated to the Casimir force, is readily removed by replacing the N-body EM Green tensor $\mathbb{G}$ by the scattering Green tensor $\mathbb{\Gamma}(\mathbf{r},\mathbf{r}')$.

The regularized stress tensor involves only $\mathbb{\Gamma}$, and it can be shown [36] that the Casimir force on body σ is determined by the operator $(\mathbb{1}-\mathbb{K})^{-1}(\mathbf{u},\mathbf{u}')\mathbb{V}(\mathbf{u}')$ which is sandwiched between the free Green tensors in the scattering Green tensor [see Eq. (17)]. Hence, the Casimir force is given by

$$\mathbf{F}_\sigma = k_B T {\sum_{n=0}^{\infty}}' \mathrm{Tr}\big[(\mathbb{1}-\mathbb{K})^{-1}\mathbb{V}\,\nabla_{\mathbf{r}_\sigma}\mathbb{G}_0\big]. \tag{19}$$

Due to the important general relation

$$\nabla_{\mathbf{r}_\sigma}\mathbb{K} = \mathbb{V}\,\nabla_{\mathbf{r}_\sigma}\mathbb{G}_0 \tag{20}$$

the force can be written solely in terms of the SSO, expressed as a sum over Matsubara frequencies ξ_n by

$$\mathbf{F}_\sigma = k_B T {\sum_{n=0}^{\infty}}' \mathrm{Tr}\big[(\mathbb{1}-\mathbb{K})^{-1}\nabla_{\mathbf{r}_\sigma}\mathbb{K}\big] \tag{21}$$

where $\nabla_{\mathbf{r}_\sigma}$ is the gradient with respect to the position of the body, and the bare Casimir energy assumes the simple

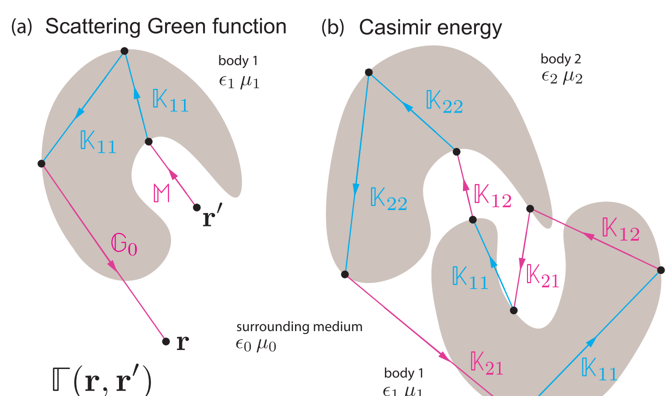

FIG. 1. .Multiple scattering expansion. Diagrammatic representation of contributions to the MSE, shown in panel (a) for the scattering Green function $\mathbb{\Gamma}(\mathbf{r},\mathbf{r}')$ of a single body with source point $\mathbf{r}'$ and observation point $\mathbf{r}$, and in panel (b) for the Casimir energy between two bodies. In the displayed examples, lines with arrows represent free propagation between surface points of the same body (blue lines or lines with equal indices on $\mathbb{K}$) and to external points or between surface points of different bodies (magenta lines or lines with unequal indices on $\mathbb{K}$, connecting points on different bodies or external points). Each free propagation between two surface points, followed by a scattering, is described by a surface operator $\mathbb{K}_{\sigma\sigma'}$. The bodies have dielectric and magnetic permittivities ϵ_1 and μ_1 and ϵ_2 and μ_2, respectively, and they are surrounded by a medium with permittivities ϵ_0 and μ_0. $\mathbb{G}_0$ is the free Green tensor of the surrounding medium, and $\mathbb{M}$ describes the tangential surface components of the incident field generated by a source at position $\mathbf{r}'$.

expression

$$\mathcal{E} = k_B T \sum_{n=0}^{\infty}{}' \operatorname{Tr} \ln(\mathbb{1} - \mathbb{K}) \tag{22}$$

(where the primed sum gives a weight of $1/2$ to the $n = 0$ term). Here the trace Tr involves a sum over vector indices of the electric and magnetic components and an integration over all surfaces. To gain insight into the structure of the MSE for the Casimir energy, we consider two bodies. After subtracting the self-energies, arising from isolated scatterings on a single body, the energy is expressed in terms of four SSOs as

$$\mathcal{E} = k_B T \sum_{n=0}^{\infty}{}' \operatorname{Tr} \ln[\mathbb{1} - (\mathbb{1} - \mathbb{K}_{11})^{-1}\mathbb{K}_{12}(\mathbb{1} - \mathbb{K}_{22})^{-1}\mathbb{K}_{21}]. \tag{23}$$

We note that this formula provides the exact representation of the Casimir energy for all allowed choices of the coefficients $\mathbb{C}_\sigma^i$ and $\mathbb{C}_\sigma^e$ (see also next section). After expanding both the logarithm and the inverse operators in powers of the SSOs we obtain the MSE which involves at least one scattering on each body with closed paths going from body 1 to body 2 and back ($\mathbb{K}_{12}$ and $\mathbb{K}_{21}$), possibly multiple times, and with an arbitrary number (including zero) of scatterings on each body ($\mathbb{K}_{11}$ and $\mathbb{K}_{22}$), as illustrated in Fig. 1(b). Comparison with scattering approaches relying on the knowledge of the bodies' T matrix shows that our MSE constructs the T matrix in the number of scatterings on individual bodies by expanding $(\mathbb{1} - \mathbb{K}_{\sigma\sigma})^{-1}$, treating scatterings inside individual bodies and between them on an equal footing. It is important to compare the MSE with the so-called Born series expansion of the Green tensor [37,38], which is an expansion in terms of iterated integrals over the *volumes* occupied by the bodies. Since our MSE is instead an expansion in terms of iterated integrals over the bodies' *surfaces*, it is clear that compared with the Born expansion, the MSE saves an enormous amount of computing time, especially when high orders are considered. We note also that while the Born series is an expansion in the dielectric contrast, our MSE is instead an expansion in the number of scatterings.

Previously, scatterings of EM waves at dielectric media have been described in terms of electric and magnetic surface currents for real frequencies, revealing sometimes poor convergence of expansions in the number of scatterings. However, since Casimir interactions can be formulated in terms of correlations of the EM field for purely imaginary frequencies, the exponential decay of Green tensors in separation can be expected to lead to rather fast convergence of the MSE for the scattering Green function and the Casimir energy. This had been demonstrated only for perfect electric conductors, based on a MSE that ignores the coupling between electric and magnetic surface currents [39]. One remarkable property of this previous approach, the cancellation of an odd overall number of scatterings, is explained in retrospect by our general MSE by the observation that ignorance of the coupling leads to SSOs with opposite signs for the electric and magnetic components.

C. Casimir-Polder force between a polarizable particle and a magneto-dielectric body

The Casimir-Polder interaction between a polarizable particle and a magneto-dielectric body can be obtained as a simple byproduct of our general approach. We assume that the particle is characterized by a frequency dependent electric polarizability tensor $\boldsymbol{\alpha}(\omega)$ and a magnetic polarizability tensor $\boldsymbol{\beta}(\omega)$. The classical energy of an induced dipole is then given by

$$\mathcal{E}_{\rm cl} = -\frac{1}{2} \sum_{i,j=1}^{3} [\alpha_{ij} E_i E_j + \beta_{ij} H_i H_j]. \tag{24}$$

Using the fluctuation-dissipation theorem, this expression is averaged over EM field fluctuations. After removing a divergent contribution from empty space, the Casimir-Polder energy is expressed in terms of the scattering Green tensor as

$$\mathcal{E}_{\rm CP} = -4\pi k_B T \sum_{n=0}^{\infty}{}' \kappa_n \sum_{i,j=1}^{3} \big[\alpha_{ij}(\mathrm{i}\,\xi_n) \mathbb{\Gamma}_{ij}^{(EE)}(\mathbf{r}_0, \mathbf{r}_0; \kappa_n) + \beta_{ij}(\mathrm{i}\,\xi_n) \mathbb{\Gamma}_{ij}^{(HH)}(\mathbf{r}_0, \mathbf{r}_0; \kappa_n)\big], \tag{25}$$

where we assumed that the particle is located at position $\mathbf{r}_0$. Substitution of $\mathbb{\Gamma}$ from Eq. (17) yields the interaction energy of the particle with a body in terms of the SSO. This energy can be computed by a MSE with respect to the number of scatterings at the surface of the body. It is instructive to write down explicitly the first terms of the scattering expansion of the Casimir-Polder energy, assuming for simplicity that the electric polarizability of the particle is isotropic, $\alpha_{ij} = \alpha\,\delta_{ij}$, and that its magnetic polarizability β is negligible:

$$\mathcal{E}_{\rm CP} = -4\pi k_B T \sum_{n=0}^{\infty}{}' \kappa_n \,\alpha(\mathrm{i}\,\xi_n) \Bigg\{ \sum_{p=E,H} \int_S ds_{\mathbf{u}} \operatorname{tr}\big[\mathbb{G}_0^{(Ep)}(\mathbf{r}_0, \mathbf{u}; \kappa_n) \mathbb{M}^{(pE)}(\mathbf{u}, \mathbf{r}_0; \kappa_n)\big] + \sum_{p,q=E,H} \int_S ds_{\mathbf{u}} \int_S ds_{\mathbf{u}'} \operatorname{tr}\big[\mathbb{G}_0^{(Ep)}(\mathbf{r}_0, \mathbf{u}; \kappa_n) \mathbb{K}^{(pq)}(\mathbf{u}, \mathbf{u}'; \kappa_n) \mathbb{M}^{(qE)}(\mathbf{u}', \mathbf{r}_0; \kappa_n)\big] \Bigg\} + \cdots \tag{26}$$

where tr denotes a trace over tensor spatial indices. Recalling that the kernels $\mathbb{K}(\mathbf{u}, \mathbf{u}')$ and $\mathbb{M}(\mathbf{u}, \mathbf{r})$ are combinations of free-space Green tensors $\mathbb{G}_0$ and $\mathbb{G}_\sigma$, and that the latter are elementary functions, we see from the above equation that the CP energy is expressed in terms of iterated integrals of elementary functions extended on the surface S of the body.

Since for imaginary frequencies the Green tensors decay exponentially with distance, Eq. (26) makes evident the intuitive fact that the points of the surface that are closest to the particle dominate the interaction.

IV. EQUIVALENT FORMULATIONS OF THE SSO

With different interior coefficient matrices $\mathbb{C}_\sigma^i$ and exterior coefficient matrices $\mathbb{C}_\sigma^e$ the SSOs form an equivalence class of operators in the sense that Eq. (13) yields the same surface currents for a given external source for all coefficients, as long as neither the interior nor the exterior matrices vanish for any σ, and the sum $\mathbb{C}_\sigma^i + \mathbb{C}_\sigma^e$ is invertible. Consequently, the scattering Green tensor and the Casimir energy must be also independent of the choice made for the coefficients. The surface currents and the Casimir energy at any *finite* order of the MSE, however, in general do depend on the chosen coefficients, and hence the rate of convergence of the MSE does as well. This remarkable property provides an effective method to optimize convergence for different permittivities and even frequencies by suitable adjustment of coefficients.

Physically, the required relation between the tangential surface fields $\mathbf{n}_\sigma \times \mathbf{E}$ and $\mathbf{n}_\sigma \times \mathbf{H}$ is in general obeyed only approximately at any finite order of the MSE, with the approximation converging to the exact relation with increasing MSE order. Indeed, at first order, $\mathbf{E}$ and $\mathbf{H}$ of the incident field are rescaled differently at each body by the chosen coefficients $\mathbb{C}_\sigma^i$ and $\mathbb{C}_\sigma^e$ [see Eq. (14)]. The coefficients hence set the initial field for the MSE iteration and they control how the exact tangential surface fields are build up successively by the MSE.

Among the infinitely many choices there are a few which we consider important to discuss explicitly and for which we shall provide detailed expressions of the SSOs.

(1) In general, the SSO has a leading singularity that diverges as $1/|\mathbf{u}-\mathbf{u}'|^\gamma$ with $\gamma = 3$ when the two surface positions $\mathbf{u}$ and $\mathbf{u}'$ approach each other. There exists a choice of coefficients [45], however, for which the singularity is reduced to a weaker divergence with exponent $\gamma = 1$, presumably accelerating convergence. The coefficient matrices are

$$\mathbb{C}_\sigma^i = \mathrm{diag}(\epsilon_\sigma, \mu_\sigma), \quad \mathbb{C}_\sigma^e = \mathrm{diag}(\epsilon_0, \mu_0). \tag{27}$$

The corresponding explicit expressions of the SSO $\mathbb{K}$ and of the operator $\mathbb{M}$ read

$$\begin{aligned}
\mathbb{K}_{\sigma\sigma'}^{(EE)}(\mathbf{u},\mathbf{u}') &= \frac{2}{\mu_0+\mu_\sigma}\mathbf{n}_\sigma(\mathbf{u}) \times \big[\mu_0\mathbb{G}_0^{(HE)}(\mathbf{u},\mathbf{u}') - \delta_{\sigma\sigma'}\mu_\sigma\mathbb{G}_\sigma^{(HE)}(\mathbf{u},\mathbf{u}')\big],\\
\mathbb{K}_{\sigma\sigma'}^{(HH)}(\mathbf{u},\mathbf{u}') &= \frac{2}{\epsilon_0+\epsilon_\sigma}\mathbf{n}_\sigma(\mathbf{u}) \times \big[-\epsilon_0\mathbb{G}_0^{(EH)}(\mathbf{u},\mathbf{u}') + \delta_{\sigma\sigma'}\epsilon_\sigma\mathbb{G}_\sigma^{(EH)}(\mathbf{u},\mathbf{u}')\big],\\
\mathbb{K}_{\sigma\sigma'}^{(EH)}(\mathbf{u},\mathbf{u}') &= \frac{2}{\mu_0+\mu_\sigma}\mathbf{n}_\sigma(\mathbf{u}) \times \big[\mu_0\mathbb{G}_0^{(HH)}(\mathbf{u},\mathbf{u}') - \delta_{\sigma\sigma'}\mu_\sigma\mathbb{G}_\sigma^{(HH)}(\mathbf{u},\mathbf{u}')\big],\\
\mathbb{K}_{\sigma\sigma'}^{(HE)}(\mathbf{u},\mathbf{u}') &= \frac{2}{\epsilon_0+\epsilon_\sigma}\mathbf{n}_\sigma(\mathbf{u}) \times \big[-\epsilon_0\mathbb{G}_0^{(EE)}(\mathbf{u},\mathbf{u}') + \delta_{\sigma\sigma'}\epsilon_\sigma\mathbb{G}_\sigma^{(EE)}(\mathbf{u},\mathbf{u}')\big]
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
\mathbb{M}_\sigma^{(EE)}(\mathbf{u},\mathbf{r}) &= \frac{2\mu_0}{\mu_0+\mu_\sigma}\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}_0^{(HE)}(\mathbf{u},\mathbf{r}),\\
\mathbb{M}_\sigma^{(EH)}(\mathbf{u},\mathbf{r}) &= \frac{2\mu_0}{\mu_0+\mu_\sigma}\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}_0^{(HH)}(\mathbf{u},\mathbf{r}),\\
\mathbb{M}_\sigma^{(HE)}(\mathbf{u},\mathbf{r}) &= -\frac{2\epsilon_0}{\epsilon_0+\epsilon_\sigma}\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}_0^{(EE)}(\mathbf{u},\mathbf{r}),\\
\mathbb{M}_\sigma^{(HH)}(\mathbf{u},\mathbf{r}) &= -\frac{2\epsilon_0}{\epsilon_0+\epsilon_\sigma}\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}_0^{(EH)}(\mathbf{u},\mathbf{r}),
\end{aligned} \tag{29}$$

with the free Green tensor $\mathbb{G}_\sigma$ which can be found in Appendix E. Substitution of this tensor yields the more explicit form in terms of the scalar Green functions $g_\sigma(\mathbf{u}-\mathbf{u}')$:

$$\begin{aligned}
\mathbb{K}_{\sigma\sigma'}^{(EE)}(\mathbf{u},\mathbf{u}') &= \frac{2}{\mu_0+\mu_\sigma}[\mathbf{n}(\mathbf{u}) \times (\cdot \times \nabla(-\mu_0 g_0(\mathbf{u}-\mathbf{u}') + \delta_{\sigma\sigma'}\mu_\sigma g_\sigma(\mathbf{u}-\mathbf{u}')))],\\
\mathbb{K}_{\sigma\sigma'}^{(HH)}(\mathbf{u},\mathbf{u}') &= \frac{2}{\epsilon_0+\epsilon_\sigma}[\mathbf{n}(\mathbf{u}) \times (\cdot \times \nabla(-\epsilon_0 g_0(\mathbf{u}-\mathbf{u}') + \delta_{\sigma\sigma'}\epsilon_\sigma g_\sigma(\mathbf{u}-\mathbf{u}')))],\\
\mathbb{K}_{\sigma\sigma'}^{(EH)}(\mathbf{u},\mathbf{u}') &= \frac{2}{\mu_0+\mu_\sigma}[\kappa(-\epsilon_0\mu_0 g_0(\mathbf{u}-\mathbf{u}') + \delta_{\sigma\sigma'}\epsilon_\sigma\mu_\sigma g_\sigma(\mathbf{u}-\mathbf{u}'))\mathbf{n}(\mathbf{u}) \times \cdot + \frac{1}{\kappa}\mathbf{n}(\mathbf{u})(\cdot\nabla)\nabla(g_0(\mathbf{u}-\mathbf{u}')) - g_\sigma(\mathbf{u}-\mathbf{u}')))],\\
\mathbb{K}_{\sigma\sigma'}^{(HE)}(\mathbf{u},\mathbf{u}') &= -\frac{\mu_0+\mu_\sigma}{\epsilon_0+\epsilon_\sigma}\mathbb{K}_{\sigma\sigma'}^{(EH)}(\mathbf{u},\mathbf{u}').
\end{aligned} \tag{30}$$

This surface operator $\mathbb{K}$ has unique mathematical properties which we shall discuss in detail in Sec. VI.

The corresponding choices for the coefficients of the integral equations for the surface charges [Eqs. (15) and (16)] are

$$\mathbb{c}^i_{j,\sigma} = \epsilon_0, \quad \mathbb{c}^i_{m,\sigma} = \mu_0, \quad \mathbb{c}^e_{j,\sigma} = \epsilon_\sigma, \quad \mathbb{c}^e_{m,\sigma} = \mu_\sigma. \tag{31}$$

(2) An asymmetric, material independent choice of coefficient matrices is

$$\mathbb{C}^i_\sigma = \text{diag}(1, 0), \quad \mathbb{C}^e_\sigma = \text{diag}(0, 1). \tag{32}$$

For good conductors, we have observed fast convergence of the MSE with this choice, while for materials with a moderately high permittivity, like Si, convergence is slow, which made us prefer the choice 1 in the numerical computations in [41]. The corresponding expressions of the SSO $\mathbb{K}$ and of the operator $\mathbb{M}$ are

$$\begin{aligned}
\mathbb{K}^{(EE)}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}') &= 2\,\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}^{(HE)}_0(\mathbf{u}, \mathbf{u}'),\\
\mathbb{K}^{(HH)}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}') &= 2\,\delta_{\sigma\sigma'}\,\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}^{(EH)}_\sigma(\mathbf{u}, \mathbf{u}'),\\
\mathbb{K}^{(EH)}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}') &= 2\,\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}^{(HH)}_0(\mathbf{u}, \mathbf{u}'),\\
\mathbb{K}^{(HE)}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}') &= 2\,\delta_{\sigma\sigma'}\,\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}^{(EE)}_\sigma(\mathbf{u}, \mathbf{u}')
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
\mathbb{M}^{(EE)}_\sigma(\mathbf{u}, \mathbf{r}) &= 2\,\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}^{(HE)}_0(\mathbf{u}, \mathbf{r}),\\
\mathbb{M}^{(EH)}_\sigma(\mathbf{u}, \mathbf{r}) &= 2\,\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}^{(HH)}_0(\mathbf{u}, \mathbf{r}),\\
\mathbb{M}^{(HE)}_\sigma(\mathbf{u}, \mathbf{r}) &= 0,\\
\mathbb{M}^{(HH)}_\sigma(\mathbf{u}, \mathbf{r}) &= 0.
\end{aligned} \tag{34}$$

(3) Finally, we note that the singular choice with $\mathbb{C}^i_\sigma + \mathbb{C}^e_\sigma = \mathbb{0}$, which we excluded, does not yield a Fredholm integral equation and hence does not permit a MSE. A corresponding popular choice [47] is

$$\mathbb{C}^i_\sigma = \text{diag}(-1, -1), \quad \mathbb{C}^e_\sigma = \text{diag}(1, 1). \tag{35}$$

The resulting integral equations for the surface currents are

$$\sum_{\sigma'=1}^{N} \int_{S_{\sigma'}} ds_{\mathbf{u}'}\, \mathbb{B}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}') \begin{pmatrix} \mathbf{j}_{\sigma'} \\ \mathbf{m}_{\sigma'} \end{pmatrix}(\mathbf{u}') = \int d\mathbf{r}\, \mathbb{M}_\sigma(\mathbf{u}, \mathbf{r}) \begin{pmatrix} \mathbf{J} \\ \mathbf{M} \end{pmatrix}(\mathbf{r}), \tag{36}$$

with

$$\mathbb{B}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}') = [\mathbb{G}_0(\mathbf{u}, \mathbf{u}') + \delta_{\sigma\sigma'}\mathbb{G}_\sigma(\mathbf{u}, \mathbf{u}')]_t, \tag{37}$$

$$\mathbb{M}_\sigma(\mathbf{u}, \mathbf{r}) = -[\mathbb{G}_0(\mathbf{u}, \mathbf{r})]_t, \tag{38}$$

where the subscript t means that when the argument of the tensor belongs to the surface S_σ, the tangential projection of the corresponding index of the tensor onto S_σ at that position is taken. In [47] it is shown that Eq. (36) determines uniquely the surface current at all frequencies. These integral equations (36) have been employed in a computationally intensive boundary element method [33], implemented in the open-source software SCUFF-EM [48].

V. LIMITING CASES

A. Zero frequency

The surface integral equations for the currents become singular in the limit of zero frequency. This singularity does not constitute a problem for evaluation of forces and energies at zero temperature since both involve integration over all imaginary frequencies. However, it impedes evaluation of the $n = 0$ term of the Matsubara sum at finite temperatures. Independent of this, one feels that solving the EM scattering problem at zero frequency in terms of surface currents is somewhat unnatural, and that a simpler approach based solely on surface charges should be possible in the static limit. We show below that this expectation is indeed correct.

Let us consider the electrostatic problem first. At points $\mathbf{r}$ away from the bodies' surfaces, the electrostatic potential ϕ satisfies the equation

$$\nabla \cdot [\epsilon_0 \nabla\phi(\mathbf{r})] = -\rho_j(\mathbf{r}), \quad \mathbf{r} \in V_0, \tag{39}$$

$$\nabla \cdot [\epsilon_\sigma \nabla\phi(\mathbf{r})] = 0, \quad \mathbf{r} \in V_\sigma, \tag{40}$$

where ρ_j are the external sources of the incident electrostatic field. The potential is continuous across the surfaces of the bodies, while its normal derivative satisfies the boundary condition

$$\epsilon_\sigma\, \hat{\mathbf{n}}_\sigma \cdot \nabla_-\phi = \epsilon_0\, \hat{\mathbf{n}}_\sigma \cdot \nabla_+\phi, \tag{41}$$

i.e., the normal component of the induction vector $\mathbf{D}(\mathbf{r}) = \epsilon(\mathbf{r})\,\mathbf{E}(\mathbf{r})$ is continuous across the surfaces. It is known from potential theory that the scalar potential ϕ is determined, within each of the regions $V_0, V_1, \dots, V_N$ by knowledge of the external source ρ_j and of the normal derivative of ϕ, or what is the same by knowledge of the normal component $\mathbf{D}_n$ of the induction vector, on the surfaces $S_1, \dots, S_N$. This means that the scattering problem is solved, if we can set up an equation to compute $\mathbf{D}_n$. To achieve this, we can use a variant of the equivalence principle. One notes that it is immaterial to replace ϵ_σ by ϵ_0 in Eq. (40). This means that away from the surfaces S_σ the potential ϕ also satisfies the Poisson equation for a *homogeneous* medium with permittivity ϵ_0:

$$\triangle\phi(\mathbf{r}) = -\frac{\rho_j(\mathbf{r})}{\epsilon_0}. \tag{42}$$

When considered in such a homogeneous medium, the normal component of the corresponding induction vector $\mathbf{D}_0 = \epsilon_0\,\mathbf{E}$ has a jump across the surfaces of the bodies. This discontinuity of $\mathbf{D}_0$ can be interpreted as arising from an *unphysical* surface distribution of charge $\bar{\varrho}_\sigma$ such that

$$\bar{\varrho}_{j,\sigma} = -\epsilon_0[\hat{\mathbf{n}}_\sigma \cdot \nabla_+\phi - \hat{\mathbf{n}}_\sigma \cdot \nabla_-\phi]. \tag{43}$$

In view of Eqs. (42) and (43), the potential can be then expressed *everywhere* as

$$\phi(\mathbf{r}) = \phi_{\text{inc}}(\mathbf{r}) + \bar{\phi}(\mathbf{r}), \tag{44}$$

where

$$\phi_{\text{inc}}(\mathbf{r}) = \frac{1}{\epsilon_0}\int_{V_0} d^3\mathbf{r}'\, g_0(|\mathbf{r} - \mathbf{r}'|)\, \rho_j(\mathbf{r}') \tag{45}$$

and

$$\bar{\phi}(\mathbf{r}) = \frac{1}{\epsilon_0}\sum_{\sigma=1}^{N}\int_{S_\sigma} ds_{\mathbf{u}}\, g_0(|\mathbf{r}-\mathbf{u}|)\,\bar{\varrho}_{j,\sigma}(\mathbf{u}), \tag{46}$$

with

$$g_0(\mathbf{r}) = \frac{1}{4\pi|\mathbf{r}|}. \tag{47}$$

We note that according to Eq. (44), the field $\bar{\phi}(\mathbf{r})$ can be identified with the scattered field, at points $\mathbf{r}$ outside the bodies:

$$\phi_{\text{scat}}(\mathbf{r}) = \bar{\phi}(\mathbf{r}), \quad \mathbf{r}\in V_0. \tag{48}$$

An integral equation for $\bar{\varrho}_\sigma$ can be derived as follows. By taking the gradient of Eq. (44), one derives the identity

$$\hat{\mathbf{n}}_\sigma\cdot\nabla\phi(\mathbf{u}) = \hat{\mathbf{n}}_\sigma\cdot\nabla\phi_{\text{inc}}(\mathbf{u}) + \hat{\mathbf{n}}_\sigma\cdot\nabla\bar{\phi}(\mathbf{u}). \tag{49}$$

Now, the normal derivative of ϕ satisfies the identity

$$\hat{\mathbf{n}}_\sigma\cdot\nabla\phi(\mathbf{u}) = \tfrac{1}{2}[\hat{\mathbf{n}}_\sigma\cdot\nabla_+\phi(\mathbf{u}) + \hat{\mathbf{n}}_\sigma\cdot\nabla_-\phi(\mathbf{u})]. \tag{50}$$

On the other hand, using Eqs. (41) and (43) one finds

$$\begin{aligned}\hat{\mathbf{n}}_\sigma\cdot\nabla_+\phi(\mathbf{u}) &= \frac{\epsilon_\sigma}{\epsilon_0}\frac{1}{\epsilon_0-\epsilon_\sigma}\bar{\varrho}_{j,\sigma}(\mathbf{u}),\\ \hat{\mathbf{n}}_\sigma\cdot\nabla_-\phi(\mathbf{u}) &= \frac{1}{\epsilon_0-\epsilon_\sigma}\bar{\varrho}_{j,\sigma}(\mathbf{u}).\end{aligned} \tag{51}$$

Plugging the right-hand side (rhs) of the above identities into the rhs of Eq. (50), we then find

$$\hat{\mathbf{n}}_\sigma\cdot\nabla\phi(\mathbf{u}) = \frac{\epsilon_0+\epsilon_\sigma}{2\epsilon_0(\epsilon_0-\epsilon_\sigma)}\,\bar{\varrho}_{j,\sigma}(\mathbf{u}). \tag{52}$$

Upon substituting the rhs of Eq. (52) into the left-hand side (lhs) of Eq. (49), and expressing the incident field in terms of the external charge ρ_j, after a little algebra one obtains a Fredholm integral equation for $\bar{\varrho}_\sigma$. The magnetostatic problem can be treated in exactly the same way by doing the substitutions $\bar{\varrho}_{j,\sigma}\to\bar{\varrho}_{m,\sigma}$, $\epsilon_0\to\mu_0$, and $\epsilon_\sigma\to\mu_\sigma$. The resulting integral equations for the surface charges are

$$\bar{\varrho}_{j,\sigma}(\mathbf{u}) - \sum_{\sigma'=1}^{N}\int_{S_{\sigma'}} ds_{\mathbf{u}'}\,\mathbb{k}^{(j)}_{\sigma\sigma'}(\mathbf{u},\mathbf{u}')\bar{\varrho}_{j,\sigma'}(\mathbf{u}') = \int d\mathbf{r}\,\mathbb{m}^{(j)}_{\sigma}(\mathbf{u},\mathbf{r})\begin{pmatrix}\rho_j\\ \rho_m\end{pmatrix}(\mathbf{r}), \tag{53}$$

$$\bar{\varrho}_{m,\sigma}(\mathbf{u}) - \sum_{\sigma'=1}^{N}\int_{S_{\sigma'}} ds_{\mathbf{u}'}\,\mathbb{k}^{(m)}_{\sigma\sigma'}(\mathbf{u},\mathbf{u}')\bar{\varrho}_{m,\sigma'}(\mathbf{u}') = \int d\mathbf{r}\,\mathbb{m}^{(m)}_{\sigma}(\mathbf{u},\mathbf{r})\begin{pmatrix}\rho_j\\ \rho_m\end{pmatrix}(\mathbf{r}). \tag{54}$$

Here the kernels are given by

$$\begin{aligned}\mathbb{k}^{(j)}_{\sigma\sigma'}(\mathbf{u},\mathbf{u}') &= 2\,\frac{\epsilon_0-\epsilon_\sigma}{\epsilon_0+\epsilon_\sigma}\,\partial_{\mathbf{n}_\sigma(\mathbf{u})}\,g_0(\mathbf{u}-\mathbf{u}'),\\ \mathbb{k}^{(m)}_{\sigma\sigma'}(\mathbf{u},\mathbf{u}') &= 2\,\frac{\mu_0-\mu_\sigma}{\mu_0+\mu_\sigma}\,\partial_{\mathbf{n}_\sigma(\mathbf{u})}\,g_0(\mathbf{u}-\mathbf{u}'),\end{aligned} \tag{55}$$

which turn out to be independent of σ', and

$$\begin{aligned}\mathbb{m}^{(j)}_{\sigma}(\mathbf{u},\mathbf{r}) &= 2\,\frac{\epsilon_0-\epsilon_\sigma}{\epsilon_0+\epsilon_\sigma}\,\partial_{\mathbf{n}_\sigma(\mathbf{u})}\,g_0(\mathbf{u}-\mathbf{r}),\\ \mathbb{m}^{(m)}_{\sigma}(\mathbf{u},\mathbf{r}) &= 2\,\frac{\mu_0-\mu_\sigma}{\mu_0+\mu_\sigma}\,\partial_{\mathbf{n}_\sigma(\mathbf{u})}\,g_0(\mathbf{u}-\mathbf{r}).\end{aligned} \tag{56}$$

We note that the above integral equations are of the same form as the ones for the surface currents, Eq. (13).

It is nice to verify that the integral equations (53) and (54) can be also derived by taking the static limit of Eqs. (15) and (16), respectively. Consider indeed the integral equations (15) and (16) for $\kappa=0$. The first term of the integrand $\sim\kappa$ obviously vanishes. In addition, $\lim_{\kappa\to0} g_\sigma(\mathbf{u}-\mathbf{u}') = \lim_{\kappa\to0} g_0(\mathbf{u}-\mathbf{u}') = 1/(4\pi|\mathbf{u}-\mathbf{u}'|)$. We make now the choice 1 for the coefficients [see Eq. (31)]. Then, in the static limit, the integral equations read

$$\begin{aligned}&\varrho_{j,\sigma}(\mathbf{u}) - \frac{2\epsilon_\sigma}{\epsilon_0+\epsilon_\sigma}\sum_{\sigma'=1}^{N}\int_{S_{\sigma'}} ds_{\mathbf{u}'}\Big[((1-\delta_{\sigma\sigma'})\nabla_{\mathbf{u}}g_0(\mathbf{u}-\mathbf{u}')\\ &\quad\times\mathbf{n}_\sigma(\mathbf{u}))\cdot\mathbf{m}_{\sigma'}(\mathbf{u}') - \Big(1-\frac{\epsilon_0}{\epsilon_\sigma}\delta_{\sigma\sigma'}\Big)\partial_{\mathbf{n}_\sigma(\mathbf{u})}\varrho_{j,\sigma'}(\mathbf{u}')\Big]\\ &= \frac{2\epsilon_\sigma}{\epsilon_0+\epsilon_\sigma}\mathbf{n}_\sigma(\mathbf{u})\cdot\mathbf{E}_{\text{inc}}(\mathbf{u}),\end{aligned} \tag{57}$$

$$\begin{aligned}&\varrho_{m,\sigma}(\mathbf{u}) - \frac{2\mu_\sigma}{\mu_0+\mu_\sigma}\sum_{\sigma'=1}^{N}\int_{S_{\sigma'}} ds_{\mathbf{u}'}\Big[((\delta_{\sigma\sigma'}-1)\nabla_{\mathbf{u}}g_0(\mathbf{u}-\mathbf{u}')\\ &\quad\times\mathbf{n}_\sigma(\mathbf{u}))\cdot\mathbf{j}_{\sigma'}(\mathbf{u}') - \Big(1-\frac{\mu_0}{\mu_\sigma}\delta_{\sigma\sigma'}\Big)\partial_{\mathbf{n}_\sigma(\mathbf{u})}\varrho_{m,\sigma'}(\mathbf{u}')\Big]\\ &= \frac{2\mu_\sigma}{\mu_0+\mu_\sigma}\mathbf{n}_\sigma(\mathbf{u})\cdot\mathbf{H}_{\text{inc}}(\mathbf{u}).\end{aligned} \tag{58}$$

The term of the sum with $\sigma'=\sigma$ is independent of the surface currents $\mathbf{j}_\sigma$ and $\mathbf{m}_\sigma$ due to the delta function. To simplify the terms with $\sigma'\neq\sigma$ we note that the EM field for $\mathbf{r}$ located in the interior region of the surface $S_{\sigma'}$ in the static limit can be written as

$$\begin{aligned}\mathbf{E}^{(\sigma')}(\mathbf{r}) = \int_{S_{\sigma'}} ds_{\mathbf{u}}\Big[&\frac{\epsilon_0}{\epsilon_{\sigma'}}\varrho_{j,\sigma'}(\mathbf{u})\nabla g_0(\mathbf{r}-\mathbf{u})\\ &+\nabla g_0(\mathbf{r}-\mathbf{u})\times\mathbf{m}_{\sigma'}(\mathbf{u})\Big],\end{aligned} \tag{59}$$

$$\begin{aligned}\mathbf{H}^{(\sigma')}(\mathbf{r}) = \int_{S_{\sigma'}} ds_{\mathbf{u}}\Big[&\frac{\mu_0}{\mu_{\sigma'}}\varrho_{m,\sigma'}(\mathbf{u})\nabla g_0(\mathbf{r}-\mathbf{u})\\ &-\nabla g_0(\mathbf{r}-\mathbf{u})\times\mathbf{j}_{\sigma'}(\mathbf{u})\Big].\end{aligned} \tag{60}$$

If $\mathbf{r}$ is located in the region exterior to the surface $S_{\sigma'}$, the above integrals vanish. Since $\mathbf{u}$ in Eqs. (57) and (58) is located outside of the surface $S_{\sigma'}$ for $\sigma'\neq\sigma$ we can use this relation to eliminate the surface currents. Upon expressing now $(\varrho_{j,\sigma},\varrho_{m,\sigma})$ in terms of $(\bar{\varrho}_{j,\sigma},\bar{\varrho}_{m,\sigma})$ via the relations

$$\varrho_{j,\sigma}(\mathbf{u}) = -\frac{\epsilon_\sigma}{\epsilon_0}\frac{1}{\epsilon_0-\epsilon_\sigma}\,\bar{\varrho}_{j,\sigma}(\mathbf{u}), \tag{61}$$

$$\varrho_{m,\sigma}(\mathbf{u}) = -\frac{\mu_\sigma}{\mu_0}\frac{1}{\mu_0-\mu_\sigma}\,\bar{\varrho}_{j,\sigma}(\mathbf{u}), \tag{62}$$

which follow from a comparison of the second of Eqs. (4) with Eq. (52) (and the analogous relation for the magnetic field), and taking as incident fields the electrostatic and magnetostatic fields generated by external charges ρ_j and ρ_m, respectively, one finds that Eqs. (57) and (58) actually coincide with Eqs. (53) and (54), respectively.

For the benefit of the reader, we write below the expressions of the classical $n = 0$ contributions to the Casimir and CP energies, in terms of the static SSO introduced above. They are

$$\mathcal{E}|_{n=0} = \frac{k_B T}{2} \sum_{p=j,m} \mathrm{Tr}\log\left[\mathbb{1} - \left(\mathbb{1} - \mathbb{k}_{11}^{(p)}\right)^{-1}\mathbb{k}_{12}^{(p)} \times \left(\mathbb{1} - \mathbb{k}_{22}^{(p)}\right)^{-1}\mathbb{k}_{21}^{(p)}\right], \quad (63)$$

$$\mathcal{E}_{\mathrm{CP}}|_{n=0} = -2\pi k_B T \sum_{i,j=1}^{3}\left[\alpha_{ij}(0)\tilde{\mathbb{\Gamma}}_{ij}^{(EE)}(\mathbf{r}_0, \mathbf{r}_0) + \beta_{ij}(0)\tilde{\mathbb{\Gamma}}_{ij}^{(HH)}(\mathbf{r}_0, \mathbf{r}_0)\right], \quad (64)$$

where

$$\tilde{\mathbb{\Gamma}}^{(EE)}(\mathbf{r}, \mathbf{r}') = \vec{\nabla}_{\mathbf{r}} \int_S ds_{\mathbf{u}} \int_S ds_{\mathbf{u}'}\, g_0(\mathbf{r}, \mathbf{u})(\mathbb{1} - \mathbb{k}^{(j)})^{-1}(\mathbf{u}, \mathbf{u}') \times \mathbb{m}^{(j)}(\mathbf{u}', \mathbf{r}')\, \overleftarrow{\nabla}_{\mathbf{r}'}, \quad (65)$$

$$\tilde{\mathbb{\Gamma}}^{(HH)}(\mathbf{r}, \mathbf{r}') = \vec{\nabla}_{\mathbf{r}} \int_S ds_{\mathbf{u}} \int_S ds_{\mathbf{u}'}\, g_0(\mathbf{r}, \mathbf{u})(\mathbb{1} - \mathbb{k}^{(m)})^{-1}(\mathbf{u}, \mathbf{u}') \times \mathbb{m}^{(m)}(\mathbf{u}', \mathbf{r}')\, \overleftarrow{\nabla}_{\mathbf{r}'}. \quad (66)$$

B. High frequencies

It is instructive to study the limit of asymptotically high frequencies. We do this here by assuming *fixed*, i.e., frequency independent, permittivities. In the high-frequency limit, the SSO becomes ultralocal, and hence the surface can be approximated by its tangent plane at each position. Then a simple computation yields the following limits:

$$\lim_{\kappa\to\infty} \mathbb{K}_{\sigma\sigma'}^{(EE)} = \lim_{\kappa\to\infty} \mathbb{K}_{\sigma\sigma'}^{(HH)} = 0,$$
$$\lim_{\kappa\to\infty} \mathbb{K}_{\sigma\sigma'}^{(EH)} = \begin{pmatrix} 0 & \frac{\sqrt{\epsilon_0\mu_0}-\sqrt{\epsilon_\sigma\mu_\sigma}}{\mu_0+\mu_\sigma} \\ \frac{-\sqrt{\epsilon_0\mu_0}+\sqrt{\epsilon_\sigma\mu_\sigma}}{\mu_0+\mu_\sigma} & 0 \end{pmatrix}\delta_{\sigma\sigma'},$$
$$\lim_{\kappa\to\infty} \mathbb{K}_{\sigma\sigma'}^{(HE)} = \begin{pmatrix} 0 & \frac{-\sqrt{\epsilon_0\mu_0}+\sqrt{\epsilon_\sigma\mu_\sigma}}{\epsilon_0+\epsilon_\sigma} \\ \frac{\sqrt{\epsilon_0\mu_0}-\sqrt{\epsilon_\sigma\mu_\sigma}}{\epsilon_0+\epsilon_\sigma} & 0 \end{pmatrix}\delta_{\sigma\sigma'}. \quad (67)$$

Here the matrix elements are expressed in an orthogonal basis of tangential unit vectors. This shows that for $\kappa \to \infty$, the N-body $\mathbb{K}$ operator splits into N independent off-diagonal single-body multiplicative operators $\mathbb{K}_{\sigma\sigma}|_{\kappa=\infty}$. Using the above limits, it is straightforward to verify that the eigenvalues of $\mathbb{K}_{\sigma\sigma}|_{\kappa=\infty}$ are

$$\lambda_{\sigma;\kappa=\infty} = \pm\frac{\sqrt{\epsilon_\sigma\mu_\sigma} - \sqrt{\epsilon_0\mu_0}}{\sqrt{(\mu_\sigma + \mu_0)(\epsilon_\sigma + \epsilon_0)}}. \quad (68)$$

It can be easily verified that for all constant values of the permittivities $|\lambda_{\sigma;\kappa=\infty}| < 1$, which shows that the MSE converges in the $\kappa \to \infty$ limit.

C. Perfect conductors

In the limit of perfect conductors, the boundary conditions reduce to the requirement that the tangential component of the electric field vanishes. Hence, it is sufficient to consider only electric surface currents. Those currents are determined by a Fredholm integral equation of the second kind with the operators

$$\mathbb{K}_{\sigma\sigma'}^{(\mathrm{PC})}(\mathbf{u}, \mathbf{u}') = 2\,\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}_0^{(HE)}(\mathbf{u}, \mathbf{u}') \quad (69)$$

and

$$\mathbb{M}_{\sigma\sigma'}^{(\mathrm{PC})}(\mathbf{u}, \mathbf{r}) = 2\,\mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}_0^{(HE)}(\mathbf{u}, \mathbf{r}) \quad (70)$$

acting only on the electric surface currents $\mathbf{j}_\sigma(\mathbf{u}) = \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{H}_+(\mathbf{u})$. This result was derived in the study of the Casimir effect for perfectly conducting bodies in [39]. Details of the derivation of this result are provided in Appendix C.

VI. CONVERGENCE PROPERTIES OF THE MSE

Now we turn to the important problem of the convergence of the Neumann series with the choice 1 for the coefficients for the SSO $\mathbb{K}$. It has been shown that the equation $\mathbb{K}\mathbf{v} = \mathbf{v}$ does not have any solutions, apart from the trivial one $\mathbf{v} = \mathbf{0}$ [45]. Since $\mathbb{K}$ is compact, general theorems on compact operators then ensure that the operator $(\mathbb{I} - \mathbb{K})^{-1}$ exists and is a bounded operator [49]. Inversion of the Fredholm integral equation then gives

$$\begin{pmatrix}\mathbf{j}\\ \mathbf{m}\end{pmatrix} = (\mathbb{I} - \mathbb{K})^{-1}\,\mathbb{M}\begin{pmatrix}\mathbf{J}\\ \mathbf{M}\end{pmatrix}. \quad (71)$$

If the Neumann series converges, $\binom{\mathbf{j}}{\mathbf{m}}$ can be computed by means of the MSE:

$$\begin{pmatrix}\mathbf{j}\\ \mathbf{m}\end{pmatrix} = \left(\sum_{k=0}^{\infty}\mathbb{K}^k\right)\mathbb{M}\begin{pmatrix}\mathbf{J}\\ \mathbf{M}\end{pmatrix}. \quad (72)$$

An important question is whether this series converges. Convergence is ensured if all eigenvalues of $\mathbb{K}$ are smaller than 1 in modulus. Unfortunately, a general proof of convergence does not seem possible. However, we can provide several arguments supporting the conjecture that the Neumann series indeed converges at all frequencies, and for all passive materials. The first argument comes from [39] where it was shown that for an isolated compact perfect conductor with a smooth surface the eigenvalues of $\mathbb{K}$ are smaller than 1 in modulus. Below we show that this conclusion remains true also for any number of perfect conductors. In addition, our results below will show explicitly that the Neumann series converges in three distinct limits, namely at all frequencies for perfect conductors, and for magneto-dielectric bodies in the limits of asymptotically large frequencies and vanishing frequencies.

A. Zero frequency

We begin by considering the static limit $\kappa \to 0$. Since in the computation of the Casimir energy of two bodies one only needs consider the separate Neumann series of the single-body operators $\mathbb{k}_{11}$ and $\mathbb{k}_{22}$, we here only consider the static limit for a single isolated body. In the static limit, the SSO operator of an isolated body is given by the electric and magnetic kernels $\mathbb{k}_{\sigma\sigma}^{(j)}$ and $\mathbb{k}_{\sigma\sigma}^{(m)}$ in Eq. (55). We now provide a proof of convergence of the Neumann series $(\mathbb{1} - \mathbb{k}_{\sigma\sigma}^{(j)})^{-1}$ and $(\mathbb{1} - \mathbb{k}_{\sigma\sigma}^{(m)})^{-1}$. Convergence is demonstrated by proving that the moduli of the eigenvalues λ of $\mathbb{k}_{\sigma\sigma}$ are smaller than

1. Here and in the following we drop the indices (j) and (m). Let us consider the eigenvalue equation for $\mathbb{k}_{\sigma\sigma}$:

$$\mathbb{k}_{\sigma\sigma}\,\bar{\varrho}_\sigma = \lambda\,\bar{\varrho}_\sigma. \tag{73}$$

Note that the eigenvalues λ may be complex, *a priori*, since $\mathbb{k}_{\sigma\sigma}$ is not Hermitian. Let $\bar{\phi}_\sigma$ denote the field generated by the surface charge distribution $\bar{\varrho}_\sigma$:

$$\bar{\phi}_\sigma(\mathbf{r}) = \int_{S_\sigma} ds_{\mathbf{u}}\, g_0(\mathbf{r}-\mathbf{u})\,\bar{\varrho}_\sigma(\mathbf{u}). \tag{74}$$

The eigenvalue equation is then equivalent to the integral equation

$$2\,\frac{\epsilon_0-\epsilon_\sigma}{\epsilon_0+\epsilon_\sigma}\;\hat{\mathbf{n}}_\sigma\cdot\nabla\bar{\phi}_\sigma(\mathbf{u}) = \lambda\,\bar{\varrho}_\sigma(\mathbf{u}). \tag{75}$$

By construction, $\bar{\phi}_\sigma$ satisfies the Laplace equation at all points away from the surface S_σ:

$$\triangle\bar{\phi}_\sigma = 0. \tag{76}$$

Moreover, at points on the surface S_σ the normal derivative of $\bar{\phi}_\sigma$ satisfies the identities

$$\bar{\varrho}_\sigma = \hat{\mathbf{n}}_\sigma\cdot\nabla_-\bar{\phi}_\sigma - \hat{\mathbf{n}}_\sigma\cdot\nabla_+\bar{\phi}_\sigma, \tag{77}$$

$$\hat{\mathbf{n}}_\sigma\cdot\nabla\bar{\phi}_\sigma = \tfrac{1}{2}[\hat{\mathbf{n}}_\sigma\cdot\nabla_-\bar{\phi}_\sigma + \hat{\mathbf{n}}_\sigma\cdot\nabla_+\bar{\phi}_\sigma]. \tag{78}$$

From the above identities, we obtain

$$\hat{\mathbf{n}}_\sigma\cdot\nabla\bar{\phi}_\sigma = \hat{\mathbf{n}}_\sigma\cdot\nabla_\pm\bar{\phi}_\sigma \pm \tfrac{1}{2}\,\bar{\varrho}_\sigma. \tag{79}$$

Substitution of the rhs of this identity into the lhs of Eq. (75) gives

$$\hat{\mathbf{n}}_\sigma\cdot\nabla_\pm\bar{\phi}_\sigma = \frac{1}{2}\left(\frac{\epsilon_0+\epsilon_\sigma}{\epsilon_0-\epsilon_\sigma}\,\lambda\mp 1\right)\varrho_\sigma. \tag{80}$$

Now, consider the positive-definite integrals $\mathcal{I}_0$ and $\mathcal{I}_\sigma$ defined by

$$\mathcal{I}_0 = \int_{\mathbb{R}^3-V_\sigma} d^3\mathbf{r}\,\nabla\bar{\phi}^*_\sigma(\mathbf{r})\cdot\nabla\bar{\phi}_\sigma(\mathbf{r}),$$
$$\mathcal{I}_\sigma = \int_{V_\sigma} d^3\mathbf{r}\,\nabla\bar{\phi}^*_\sigma(\mathbf{r})\cdot\nabla\bar{\phi}_\sigma(\mathbf{r}). \tag{81}$$

By using Green's theorem, and then considering the identities in Eq. (80), one finds that the above integrals become

$$\mathcal{I}_0 = -\int_{S_\sigma} ds_{\mathbf{u}}\,\bar{\phi}^*_\sigma(\mathbf{u})\,\hat{\mathbf{n}}_\sigma\cdot\nabla_+\bar{\phi}_\sigma(\mathbf{u}) = \frac{1}{2}\left(1-\lambda\frac{\epsilon_0+\epsilon_\sigma}{\epsilon_0-\epsilon_\sigma}\right)J_\sigma,$$
$$\mathcal{I}_\sigma = \int_{S_\sigma} ds_{\mathbf{u}}\,\bar{\phi}^*_\sigma(\mathbf{u})\,\hat{\mathbf{n}}_\sigma\cdot\nabla_-\bar{\phi}_\sigma(\mathbf{u}) = \frac{1}{2}\left(1+\lambda\,\frac{\epsilon_0+\epsilon_\sigma}{\epsilon_0-\epsilon_\sigma}\right)J_\sigma, \tag{82}$$

where

$$J_\sigma = \int_{S_\sigma} ds_{\mathbf{u}}\,\bar{\phi}^*_\sigma(\mathbf{u})\varrho_\sigma(\mathbf{u}). \tag{83}$$

Since $\mathcal{I}_\sigma$ are obviously positive, the integrals J_σ cannot be zero. Upon multiplying the first of Eqs. (82) by the conjugate of the second, we obtain

$$\mathcal{I}_0\mathcal{I}_\sigma = \frac{1}{4}\left[1-|\lambda|^2\left(\frac{\epsilon_0+\epsilon_\sigma}{\epsilon_0-\epsilon_\sigma}\right)^2 - 2\,\mathrm{i}\,\frac{\epsilon_0+\epsilon_\sigma}{\epsilon_0-\epsilon_\sigma}\,\mathrm{Im}\,\lambda\right]|J_\sigma|^2. \tag{84}$$

This identity implies that $\mathrm{Im}\,\lambda = 0$, and one obtains the inequality

$$1-|\lambda|^2\left(\frac{\epsilon_0+\epsilon_\sigma}{\epsilon_0-\epsilon_\sigma}\right)^2 > 0, \tag{85}$$

which directly implies

$$|\lambda|^2 < \left(\frac{\epsilon_0-\epsilon_\sigma}{\epsilon_0+\epsilon_\sigma}\right)^2 < 1, \tag{86}$$

since ϵ_0 and ϵ_σ are both positive numbers. An analogous proof shows that $|\lambda| < 1$ for the magnetostatic problem.

B. Perfect conductors

In this subsection we prove that the Neumann series for a collection of perfectly conducting bodies converges for all imaginary frequencies. The proof applies to compact bodies, with smooth surfaces. Convergence is demonstrated by proving that the absolute values of the eigenvalues λ of the operator $\mathbb{K}^{(\mathrm{PC})}$ in Eq. (69) are less than 1. We note that convergence of the Neumann series was proved in [39] for a single body, in a larger domain of complex frequencies ω, that includes the imaginary axis. Since for purely imaginary frequencies the proof becomes considerably simpler, we find it useful to present it here for a general system of N conductors. Let us consider the eigenvalue equation for $\mathbb{K}^{(\mathrm{PC})}$:

$$\mathbb{K}^{(\mathrm{PC})}\,\mathbf{j} = \lambda\,\mathbf{j}. \tag{87}$$

Note that the eigenvalues λ may be complex, *a priori*, since $\mathbb{K}$ is not Hermitian. Let $(\mathbf{E},\mathbf{H})$ denote the EM field generated by the surface current $\mathbf{j}$:

$$\mathbf{E}(\mathbf{r}) = \sum_{\sigma=1}^{N}\int_{S_\sigma} ds_{\mathbf{u}}\,\mathbb{G}_0^{(EE)}(\mathbf{r}-\mathbf{u})\mathbf{j}_\sigma(\mathbf{u}),$$
$$\mathbf{H}(\mathbf{r}) = \sum_{\sigma=1}^{N}\int_{S_\sigma} ds_{\mathbf{u}}\,\mathbb{G}_0^{(HE)}(\mathbf{r}-\mathbf{u})\mathbf{j}_\sigma(\mathbf{u}). \tag{88}$$

In view of Eq. (69), we see that the eigenvalue equation is equivalent to the relation

$$2\,\hat{\mathbf{n}}_\sigma(\mathbf{u})\times\mathbf{H}(\mathbf{u}) = \lambda\,\mathbf{j}_\sigma(\mathbf{u}). \tag{89}$$

By construction, the EM field $(\mathbf{E},\mathbf{H})$ satisfies Maxwell equations at points $\mathbf{r}$ not lying on any of the surfaces S_σ:

$$-\nabla\times\mathbf{E}(\mathbf{r}) = \kappa\,\mu_0\,\mathbf{H}(\mathbf{r}), \tag{90}$$

$$\nabla\times\mathbf{H}(\mathbf{r}) = \kappa\,\epsilon_0\,\mathbf{E}(\mathbf{r}). \tag{91}$$

At points $\mathbf{u}$ on S_σ the field $(\mathbf{E},\mathbf{H})$ satisfies the jump conditions

$$\hat{\mathbf{n}}_\sigma(\mathbf{u})\times[\mathbf{E}_+(\mathbf{u})-\mathbf{E}_-(\mathbf{u})] = 0,$$
$$\hat{\mathbf{n}}_\sigma(\mathbf{u})\times[\mathbf{H}_+(\mathbf{u})-\mathbf{H}_-(\mathbf{u})] = \mathbf{j}_\sigma. \tag{92}$$

Moreover, it holds that

$$\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{E}(\mathbf{u}) = \tfrac{1}{2}\,\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times [\mathbf{E}_+(\mathbf{u}) + \mathbf{E}_-(\mathbf{u})],$$
$$\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{H}(\mathbf{u}) = \tfrac{1}{2}\,\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times [\mathbf{H}_+(\mathbf{u}) + \mathbf{H}_-(\mathbf{u})]. \quad (93)$$

Combining Eqs. (92) and (93) we obtain

$$\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{E}(\mathbf{u}) = \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{E}_\pm(\mathbf{u}),$$
$$\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{H}(\mathbf{u}) = \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{H}_\pm(\mathbf{u}) \mp \tfrac{1}{2}\,\mathbf{j}_\sigma. \quad (94)$$

Upon substituting the rhs of the second of the above equations into the lhs of the eigenvalue Eq. (89), we obtain the relation

$$2\,\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{H}_\pm(\mathbf{u}) = (\lambda \pm 1)\,\mathbf{j}_\sigma. \quad (95)$$

Now, consider the energy fluxes across the inner and the outer sides of the surface S_σ, given by surface integrals of the Poynting vector:

$$\mathcal{J}_{\sigma\pm} = 2\int_{S_\sigma} ds_{\mathbf{u}}\,\hat{\mathbf{n}}_\sigma(\mathbf{u}) \cdot (\mathbf{E}^*_\pm \times \mathbf{H}_\pm)(\mathbf{u}). \quad (96)$$

By using the divergence theorem, one obtains the identities

$$\mathcal{J}_{\sigma-} = \mathcal{I}_{\sigma-},$$
$$\sum_{\sigma=1}^{N} \mathcal{J}_{\sigma+} = -\mathcal{I}_+, \quad (97)$$

where $\mathcal{I}_{\sigma-}$ and $\mathcal{I}_+$ denote the following positive-definite integrals:

$$\mathcal{I}_+ = 2\kappa \int_{V_0} d^3\mathbf{r}\,(\epsilon_0\,\mathbf{E}^* \cdot \mathbf{E} + \mu_0\,\mathbf{H}^* \cdot \mathbf{H}),$$
$$\mathcal{I}_{\sigma-} = 2\kappa \int_{V_\sigma} d^3\mathbf{r}\,(\epsilon_0\,\mathbf{E}^* \cdot \mathbf{E} + \mu_0\,\mathbf{H}^* \cdot \mathbf{H}). \quad (98)$$

Upon substituting Eqs. (95) into the rhs of Eq. (96), and recalling the first of Eqs. (94), we find that the identities in Eq. (97) can be recast as

$$\mathcal{I}_+ = (1+\lambda)\sum_{\sigma=1}^{N}\int_{S_\sigma} ds_{\mathbf{u}}\,\mathbf{E}^* \cdot \mathbf{j}_\sigma,$$
$$\mathcal{I}_{\sigma-} = (1-\lambda)\int_{S_\sigma} ds_{\mathbf{u}}\,\mathbf{E}^* \cdot \mathbf{j}_\sigma. \quad (99)$$

Since $\mathcal{I}_+$ and $\mathcal{I}_{\sigma-}$ are positive, neither of the surface integrals on the rhs of the above equations can be zero. Upon adding the identities in the second line of the above equation, and then dividing the sum by the identity in the first line, we find

$$\frac{1-\lambda}{1+\lambda} = \frac{\mathcal{I}_-}{\mathcal{I}_+}, \quad (100)$$

where we set $\mathcal{I}_- = \sum_{\sigma=1}^{N}\mathcal{I}_{\sigma-}$. By solving for λ, we get

$$\lambda = \frac{\mathcal{I}_+ - \mathcal{I}_-}{\mathcal{I}_+ + \mathcal{I}_-}. \quad (101)$$

This relation shows that the eigenvalues are real, and that $|\lambda| < 1$ since $\mathcal{I}_+, \mathcal{I}_- > 0$. This establishes convergence of the Neumann series.

C. Some general properties of the SSO in the formulation of choice 1

In this section, we derive the main properties of the SSO $\mathbb{K}$ with the coefficient choice 1 for magneto-dielectric bodies. We assume throughout that the frequency ω is imaginary, $\omega = \mathrm{i}\xi$, with $\xi > 0$. We underline though that most of the properties discussed below are in fact valid for arbitrary frequencies ω belonging to the upper complex plane $\mathcal{C}^+ = \{\omega : \mathrm{Im}(\omega) \geqslant 0\}$, as the reader may easily verify in each case. We recall that along the positive imaginary frequency axis the permittivities of dissipative and dispersive media are positive numbers, and therefore we assume below $\epsilon_\sigma > 0$ and $\mu_\sigma > 0$.

The unique feature of the formulation of choice 1, which distinguishes it from all other formulations, is its weak short-distance singularity, since $\mathbb{K}$ behaves as $|\mathbf{u} - \mathbf{u}'|^{-1}$ when $\mathbf{u} \to \mathbf{u}'$. We note that an analogous weak singularity is also displayed by the SSO $\mathbb{K}^{(\mathrm{PC})}$ for perfect conductors in Eq. (69). This has to be contrasted with the $|\mathbf{u} - \mathbf{u}'|^{-3}$ singularity displayed by $\mathbb{K}$, for all other choices of the coefficients. As a result of its weak singularity, the SSO $\mathbb{K}$ is a compact operator [45]. As it is well known [49], the spectrum $\sigma(\mathbb{A})$ of a compact operator $\mathbb{A}$ consists only of discrete eigenvalues, and the set of its nonvanishing eigenvalues (each counted as many times as its multiplicity) is either empty or finite or it is a sequence converging to zero. The latter property implies that the number of eigenvalues whose modulus exceeds any positive constant is necessarily finite. An important consequence of this general property of compact operators is that the number of eigenvalues of $\mathbb{K}$ that exceed 1 in modulus is finite, which implies that the MSE of $(\mathbb{I} - \mathbb{K})^{-1}$ converges in general, except possibly in a finite-dimensional subspace.

Before we study some mathematical properties of the operator $\mathbb{K}$, it is instructive to consider its general structure and behavior of low and high imaginary frequencies κ. Consider the expression for $\mathbb{K}$ given in Eq. (30). For small κ the EH and HE components vanish, as can be seen by expanding $g_\sigma(\mathbf{u} - \mathbf{u}')$ for small κ. In the opposite limit of large κ, the EE and HH components of $\mathbb{K}$ vanish, as we had seen explicitly already in Eq. (67). We have already shown before that in both limits the eigenvalues of $\mathbb{K}$ are smaller than 1. This implies that the MSE must converge for sufficiently small and for sufficiently large κ. However, this does not guarantee convergence for all values of κ since the eigenvalues are not monotonous functions of κ, as we shall see in the examples given in the next section.

We proceed with some mathematical properties of $\mathbb{K}$. On the space of surface currents $\binom{\mathbf{j}}{\mathbf{m}}$ we define the scalar product

$$\left\langle \begin{pmatrix}\mathbf{j}'\\ \mathbf{m}'\end{pmatrix} \middle| \begin{pmatrix}\mathbf{j}\\ \mathbf{m}\end{pmatrix} \right\rangle = \sum_{\sigma=1}^{N}\int_{S_\sigma} ds_{\mathbf{u}}[\,\mathbf{j}'^*_\sigma(\mathbf{u}) \cdot \mathbf{j}_\sigma(\mathbf{u}) + \mathbf{m}'^*_\sigma(\mathbf{u}) \cdot \mathbf{m}_\sigma(\mathbf{u})]. \quad (102)$$

It is a simple matter to verify that the $\mathbb{K}$ operator in Eq. (28) can be factorized as

$$\mathbb{K} = \mathbb{R}\,\mathbb{U}, \quad (103)$$

where $\mathbb{R}$ is the local multiplicative operator

$$\mathbb{R}\begin{pmatrix}\mathbf{j}_\sigma\\ \mathbf{m}_\sigma\end{pmatrix}(\mathbf{u}) = \begin{pmatrix}\hat{\mathbf{n}}_\sigma(\mathbf{u})\times\mathbf{j}_\sigma(\mathbf{u})\\ -\hat{\mathbf{n}}_\sigma(\mathbf{u})\times\mathbf{m}_\sigma(\mathbf{u})\end{pmatrix}, \tag{104}$$

and $\mathbb{U}$ is the surface operator

$$\begin{aligned}
\mathbb{U}^{(EE)}_{\sigma\sigma'}(\mathbf{u},\mathbf{u}') &= \frac{2}{\mu_0+\mu_\sigma}\big[\mu_0\mathbb{G}_0^{(HE)}(\mathbf{u},\mathbf{u}') - \delta_{\sigma\sigma'}\mu_\sigma\mathbb{G}_\sigma^{(HE)}(\mathbf{u},\mathbf{u}')\big]_t,\\
\mathbb{U}^{(HH)}_{\sigma\sigma'}(\mathbf{u},\mathbf{u}') &= \frac{2}{\epsilon_0+\epsilon_\sigma}\big[\epsilon_0\mathbb{G}_0^{(EH)}(\mathbf{u},\mathbf{u}') - \delta_{\sigma\sigma'}\epsilon_\sigma\mathbb{G}_\sigma^{(EH)}(\mathbf{u},\mathbf{u}')\big]_t,\\
\mathbb{U}^{(EH)}_{\sigma\sigma'}(\mathbf{u},\mathbf{u}') &= \frac{2}{\mu_0+\mu_\sigma}\big[\mu_0\mathbb{G}_0^{(HH)}(\mathbf{u},\mathbf{u}') - \delta_{\sigma\sigma'}\mu_\sigma\mathbb{G}_\sigma^{(HH)}(\mathbf{u},\mathbf{u}')\big]_t,\\
\mathbb{U}^{(HE)}_{\sigma\sigma'}(\mathbf{u},\mathbf{u}') &= \frac{2}{\epsilon_0+\epsilon_\sigma}\big[\epsilon_0\mathbb{G}_0^{(EE)}(\mathbf{u},\mathbf{u}') - \delta_{\sigma\sigma'}\epsilon_\sigma\mathbb{G}_\sigma^{(EE)}(\mathbf{u},\mathbf{u}')\big]_t,
\end{aligned} \tag{105}$$

where the subscript t denotes projection of tensors onto the tangent plane at S_σ. We note that both $\mathbb{R}$ and $\mathbb{U}$ are real operators. Let us define the transpose $\mathbb{A}^{\mathrm{T}}$ of an operator $\mathbb{A}$:

$$(\mathbb{A}^{\mathrm{T}})^{(\alpha\beta)}_{ij;\sigma\sigma'}(\mathbf{u},\mathbf{u}') = \mathbb{A}^{(\beta\alpha)}_{ji;\sigma'\sigma}(\mathbf{u}',\mathbf{u}). \tag{106}$$

The operator $\mathbb{R}$ is orthogonal:

$$\mathbb{R}\,\mathbb{R}^{\mathrm{T}} = -\mathbb{R}^2 = \mathbb{1}. \tag{107}$$

It can be verified that $\mathbb{U}$ satisfies the relation

$$\mathbb{g}\,\mathbb{U} = \mathbb{U}^T\,\mathbb{g}, \tag{108}$$

where $\mathbb{g}$ is the local positive and symmetric operator

$$\mathbb{g}\begin{pmatrix}\mathbf{j}_\sigma\\ \mathbf{m}_\sigma\end{pmatrix}(\mathbf{u}) = \begin{pmatrix}(\mu_\sigma+\mu_0)\,\mathbf{j}_\sigma\\ (\epsilon_\sigma+\epsilon_0)\,\mathbf{m}_\sigma\end{pmatrix}(\mathbf{u}). \tag{109}$$

We note also that $\mathbb{R}$ and $\mathbb{g}$ commute:

$$[\,\mathbb{R},\mathbb{g}\,] = 0. \tag{110}$$

The symmetry property Eq. (108) implies that the operator $\mathbb{U}$ is self-adjoint with respect to the following material-dependent inner product $\langle\,|\,\rangle_g$:

$$\begin{aligned}
&\left\langle\begin{pmatrix}\mathbf{j}'\\ \mathbf{m}'\end{pmatrix}\middle|\begin{pmatrix}\mathbf{j}\\ \mathbf{m}\end{pmatrix}\right\rangle_g\\
&\equiv\left\langle\begin{pmatrix}\mathbf{j}'\\ \mathbf{m}'\end{pmatrix}\middle|\,\mathbb{g}\begin{pmatrix}\mathbf{j}\\ \mathbf{m}\end{pmatrix}\right\rangle\\
&=\sum_{\sigma=1}^{N}\int_{S_\sigma} ds_{\mathbf{u}}[(\mu_\sigma+\mu_0)\,\mathbf{j}'^*_\sigma(\mathbf{u})\cdot\mathbf{j}_\sigma(\mathbf{u})\\
&\quad+(\epsilon_\sigma+\epsilon_0)\,\mathbf{m}'^*_\sigma(\mathbf{u})\cdot\mathbf{m}_\sigma(\mathbf{u})].
\end{aligned} \tag{111}$$

Thus, Eq. (103) shows that for imaginary frequencies the operator $\mathbb{K}$ is the product of an orthogonal operator $\mathbb{R}$ times a self-adjoint real operator $\mathbb{U}$. This implies that if λ is an eigenvalue, then also $-\lambda$, λ^*, and $-\lambda^*$ are eigenvalues. We note first that reality of $\mathbb{K}$ implies that the set of its eigenvalues is formed by pairs (λ,λ^*) of complex conjugate eigenvalues. Consider now an eigenvalue λ of $\mathbb{K}$. Since the eigenvalues of an operator coincide with the eigenvalues of its transpose, there must exist a nonvanishing left eigenvector v of $\mathbb{K}$ such that

$$\mathbb{K}^{\mathrm{T}}v = -\mathbb{U}^{\mathrm{T}}\mathbb{R}\,v = \lambda\,v. \tag{112}$$

Now we define $w = \mathbb{g}^{-1}\mathbb{R}\,v$. The vector w is clearly different from zero, because $\mathbb{R}$ is orthogonal and $\mathbb{g}$ is a positive operator. Then, using the relation Eq. (108), we get

$$\begin{aligned}
\mathbb{K}\,w &= \mathbb{R}\,\mathbb{U}\,\mathbb{g}^{-1}\mathbb{R}\,v = \mathbb{R}\,\mathbb{g}^{-1}\,\mathbb{U}^{\mathrm{T}}\mathbb{R}\,v = -\lambda\mathbb{R}\,\mathbb{g}^{-1}v\\
&= -\lambda\,\mathbb{g}^{-1}\,\mathbb{R}\,v = -\lambda w,
\end{aligned} \tag{113}$$

which shows that w is an eigenvector of $\mathbb{K}$ with eigenvalue $-\lambda$. It is clear that all the above conclusions are true also for the SSO $\mathbb{K}_{\sigma\sigma}$ of the σth body in isolation.

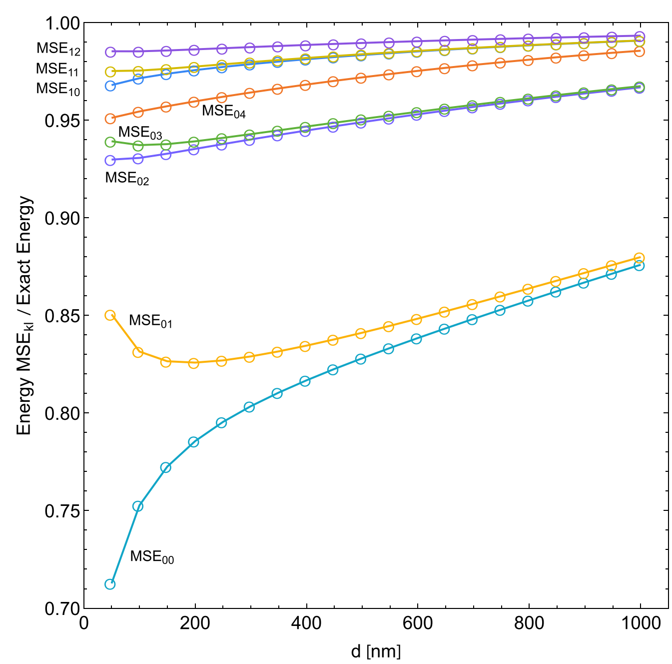

FIG. 2. Multiple scattering expansion of the Casimir energy between a silicon plate and a gold plate. Different orders of the MSE for the Casimir energy between a plate made of doped silicon and a plate made of gold, normalized to the known exact energy. Indices of MSE$_{kl}$ label the number of scatterings between the plates $[2(k+1)]$ and within the silicon plate (l) (see text for details).

VII. EXAMPLES

In order to strengthen the case for convergence of the Neumann series for the formulation of choice 1, we consider in this section explicitly the operator $\mathbb{K}$ for a magneto-dielectric plate, sphere, and cylinder. The eigenvalues can be computed exactly in these cases, using plane waves or the partial-wave representations of the free Green tensors. We considered several distinct values of the electric and magnetic permittivities, and always found that the moduli of the eigenvalues are less than 1, at all frequencies.

A. Example 1: Magneto-dielectric parallel plates

The Casimir interaction between two planar and parallel surfaces is determined by their Fresnel coefficients according to the Lifshitz formula [2]. In our formulation, the Casimir interaction is determined by the SSO $\mathbb{K}$. For an infinite, planar surface of a material with permittivities ϵ_σ and μ_σ in an external medium with permittivities ϵ_0 and μ_0, the SSO $\mathbb{K}_{\sigma\sigma}$ can be expressed easily in a plane-wave basis:

$$\begin{aligned}
\mathbb{K}^{(EE)}_{\sigma\sigma} &= 0,\\
\mathbb{K}^{(HH)}_{\sigma\sigma} &= 0,\\
\mathbb{K}^{(EH)}_{\sigma\sigma} &= (-1)^\sigma (2\pi)^2 \delta(\mathbf{k}_\parallel - \mathbf{k}'_\parallel)\frac{1}{\kappa}\frac{1}{\mu_0+\mu_\sigma}\left[\frac{1}{\sqrt{\epsilon_0\mu_0\kappa^2+\mathbf{k}_\parallel^2}}\begin{pmatrix} -k_1k_2 & -\epsilon_0\mu_0\kappa^2 - k_1^2 \\ \epsilon_0\mu_0\kappa^2 + k_2^2 & k_1k_2\end{pmatrix}\right.\\
&\quad\left. - \frac{1}{\sqrt{\epsilon_\sigma\mu_\sigma\kappa^2+\mathbf{k}_\parallel^2}}\begin{pmatrix} -k_1k_2 & -\epsilon_\sigma\mu_\sigma\kappa^2 - k_1^2 \\ \epsilon_\sigma\mu_\sigma\kappa^2 + k_2^2 & k_1k_2\end{pmatrix}\right],\\
\mathbb{K}^{(HE)}_{\sigma\sigma} &= -\frac{\mu_0+\mu_\sigma}{\epsilon_0+\epsilon_\sigma}\mathbb{K}^{(EH)}_{\sigma\sigma},
\end{aligned} \tag{114}$$

where $\mathbf{k}_\parallel = (k_1, k_2)$ is the k vector parallel to the surface. The factor $(-1)^\sigma$ accounts for the different orientation of the surface normal vector on the two plates. The eigenvalues of $\mathbb{K}_{\sigma\sigma}$ are

$$\lambda_\pm(\mathbf{k}_\parallel) = \pm\left[\frac{(s_1-s_0)(\epsilon_1\mu_1 s_0 - \epsilon_0\mu_0 s_1)}{s_0 s_1(\epsilon_0+\epsilon_1)(\mu_0+\mu_1)}\right]^{1/2} \tag{115}$$

with $s_\sigma = \sqrt{\epsilon_\sigma\mu_\sigma\kappa^2+\mathbf{k}_\parallel^2}$. Each eigenvalue has an algebraic multiplicity of 2. The eigenvalues are real valued, and $|\lambda_\pm(\mathbf{k}_\parallel)| < 1$ as can be easily checked.

The components of the operators $\mathbb{K}_{\sigma\sigma'}$ with $\sigma \neq \sigma'$ which couple surface currents on different surfaces are also easily expressed in plane waves, leading to

$$\begin{aligned}
\mathbb{K}^{(EE)}_{12} &= (2\pi)^2\delta(\mathbf{k}_\parallel - \mathbf{k}'_\parallel)\frac{\mu_0}{\mu_0+\mu_1}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}e^{-s_0 d},\\
\mathbb{K}^{(HH)}_{12} &= (2\pi)^2\delta(\mathbf{k}_\parallel - \mathbf{k}'_\parallel)\frac{\epsilon_0}{\epsilon_0+\epsilon_1}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}e^{-s_0 d},\\
\mathbb{K}^{(EH)}_{12} &= (2\pi)^2\delta(\mathbf{k}_\parallel - \mathbf{k}'_\parallel)\frac{1}{(\mu_0+\mu_1)\kappa s_0}\begin{pmatrix}-k_1k_2 & -k_1^2 - \epsilon_0\mu_0\kappa^2\\ k_2^2+\epsilon_0\mu_0\kappa^2 & k_1k_2\end{pmatrix}e^{-s_0 d},\\
\mathbb{K}^{(HE)}_{12} &= (2\pi)^2\delta(\mathbf{k}_\parallel - \mathbf{k}'_\parallel)\frac{1}{(\epsilon_0+\epsilon_1)\kappa s_0}\begin{pmatrix}k_1k_2 & k_1^2 + \epsilon_0\mu_0\kappa^2\\ -k_2^2-\epsilon_0\mu_0\kappa^2 & -k_1k_2\end{pmatrix}e^{-s_0 d}.
\end{aligned} \tag{116}$$

The elements of $\mathbb{K}_{21}$ are obtained from those of $\mathbb{K}_{12}$ by replacing ϵ_1 and μ_1 by ϵ_2 and μ_2 and changing the sign of the EH and HE components. When these operator components are substituted into Eq. (23), the Lifshitz formula [2] is recovered. We note that the inverse of $\mathbb{1} - \mathbb{K}_{\sigma\sigma}$ can be computed easily as the operator is diagonal. However, to examine the convergence rate of the MSE, we expand $(\mathbb{1} - \mathbb{K}_{\sigma\sigma})^{-1}$ into a Neumann series in $\mathbb{K}_{\sigma\sigma}$ and compute the Casimir energy at different orders of the MSE. Since $|\lambda_\pm(\mathbf{k}_\parallel)| < 1$, the MSE must converge. Indeed, when the SSO $\mathbb{K}_{11}$ describes the scatterings on one plate, expansion of the energy in Eq. (14) in this SSO yields MSE approximants to the Casimir interaction. MSE orders are labeled by MSE_{kl} where $2(k+1)$ is the number of scatterings between the surfaces (total number of $\mathbb{K}_{12}$ and $\mathbb{K}_{21}$ operators) and l is the number of single-body scatterings on the Si surface (number of $\mathbb{K}_{11}$ operators).

The majority of experiments measure forces between gold (Au) and/or doped silicon (Si) surfaces [3,4,6,14], and hence we consider these materials in this example. Figure 2 shows the energy for eight different orders of MSE relative to the known exact energy at $T = 300$ K for surface separations between 100 nm and 1 μm. While the lowest-order MSE_{00}

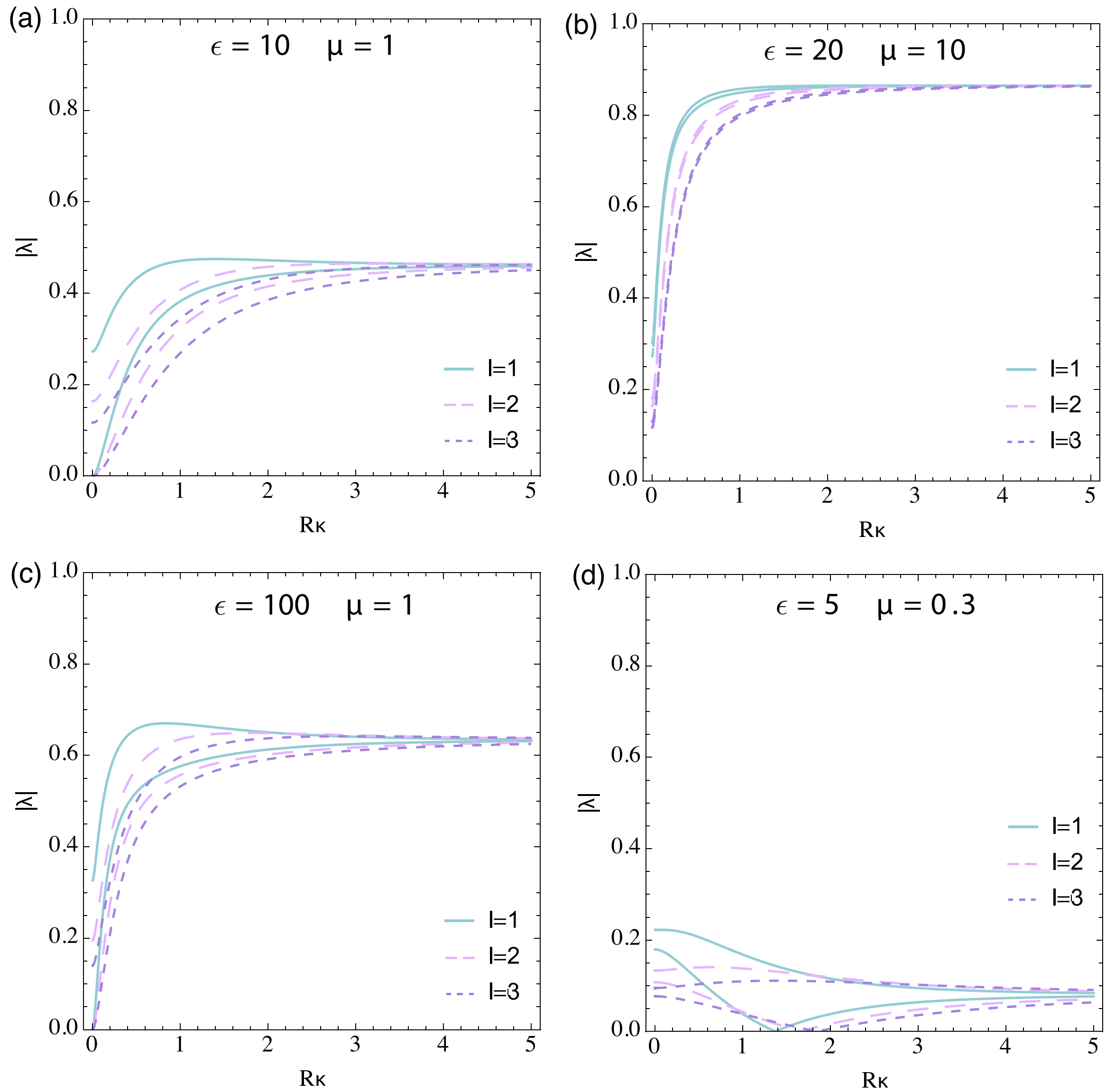

FIG. 3. Eigenvalues of $\mathbb{K}_{11}$ for a magneto-dielectric sphere of radius R. Shown are the absolute values of the eigenvalues for electric and magnetic polarization, as a function of the dimensionless quantity $R\kappa$ for different partial wave indices $l = 1, 2,$ and 3. For each value of l only two of the four eigenvalues are shown, as eigenvalues of $\mathbb{K}_{11}$ always appear in pairs $(\lambda, -\lambda)$. The permittivities are indicated in the plots.

with *no* single-body scattering on the Si surface yields already between 70 and 87% of the exact interaction, only four scatterings between the surfaces ($k = 1$) and two single-body scatterings on the Si surface ($l = 2$) are required for an accuracy of about 1%. This validation example demonstrates fast convergence of our MSE, with good homogeneity in separation.

B. Example 2: Magneto-dielectric sphere

For a magneto-dielectric sphere of radius R the SSO operator $\mathbb{K}_{11}$ can be computed easily in terms of vector spherical harmonics. A similar computation has been carried out in [40] for a perfectly conducting sphere. The elements of the infinite matrix representing $\mathbb{K}_{11}$ can be expressed in terms of Bessel functions $I_{l+1/2}(z)$ and $K_{l+1/2}(z)$ with half-integer index. The full expressions are not particularly illuminating and hence are not shown here. Figure 3 shows the eigenvalues of $\mathbb{K}_{11}$ for the first three partial waves as a function of the rescaled frequency κR. For all considered permittivities and frequencies, the moduli of the eigenvalues were found to be less than 1. For large $R\kappa$ the eigenvalues become independent of the partial wave index l as they approach the high-energy limit given by Eq. (68).

C. Example 3: Magneto-dielectric cylinder

A third validation example involves the eigenvalues of $\mathbb{K}_{11}$ and the scattering Green function $\mathbb{\Gamma}$ for a dielectric cylinder. The latter is fully specified by the scattering T operator $\mathbb{T}$ of the cylinder. It is known exactly and constitutes the only exact result for a curved dielectric body which couples electric and magnetic polarizations upon scattering [50]. When $\mathbb{T}$ is known, one can use the relation [36]

$$\Gamma(\mathbf{r}, \mathbf{r}') = \int d\tilde{\mathbf{r}} \int d\tilde{\mathbf{r}}' \, \mathbb{G}_0(\mathbf{r}, \tilde{\mathbf{r}})\mathbb{T}(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}')\mathbb{G}_0(\tilde{\mathbf{r}}', \mathbf{r}') \quad (117)$$

to compute the components of Γ in a partial wave expansion of $\mathbb{G}_0$ where the integrations now extend over the volume of the cylinder. Specifically, vector cylindrical waves are a convenient choice to obtain the SSO $\mathbb{K}_{11}$ of the cylinder and to extract from the MSE for Γ the T-operator elements $\mathbb{T}^{\alpha\alpha'}(m, \kappa, k_z)$ for $\alpha, \alpha' \in \{E, H\}$, the imaginary wave number $\kappa = \xi/c$, the wave vector k_z along the cylinder axis, and the angular quantum number m (see Appendix D for details). The elements of $\mathbb{K}_{11}$ can be expressed in terms of Bessel functions $K_m(z)$ and $I_m(z)$ but the expressions are too lengthy to be shown here. For each value κR and $k_z R$ and integer partial wave index $m \geqslant 0$ there are four eigenvalues of $\mathbb{K}_{11}$. They can be also expressed in terms of Bessel functions. For all considered permittivities, frequencies, and wave vectors, the moduli of the eigenvalues were found to be less than 1. Figures 4–6 show the absolute values of the eigenvalues for different permittivities. For large $R\kappa$ the eigenvalues become independent of Rk_z and m as they approach the high-energy limit given by Eq. (68).

Next, we study the scattering Green function. The panels in Fig. 7 display interesting aspects of the convergence of the approximant for $\mathbb{T}^{\alpha\alpha'}$ with the MSE $(1 - \mathbb{K}_{11})^{-1} = \sum_{n=0}^{p} \mathbb{K}_{11}^{p}$ for order $p = 3$. The contour plots show the ratio of the approximant and the exact T-operator elements for $m = 0$ and 1 as a function of the dimensionless wave numbers κR and $k_z R$ for a cylinder of radius R and permittivities $\epsilon_1 = 30$ and $\mu_1 = 1$. While at this low order overall convergence has reached already agreement of better than 85% with the exact result, the plots reveal a complex dependence of the convergence rate on wave numbers. Typically convergence accelerates with decreasing frequency scale κ and increasing wave number k_z, with the exception of lowest $m = 0$ elements which show slow convergence around the static, long-wavelength limits $\kappa = k_z = 0$. This slowdown can be understood from the presence of a logarithmic divergence in $\mathbb{T}$ for $m = 0$ which is a consequence of the infinite length of the cylinder [50]. The observation of fast convergence of the MSE for Γ is important as it determines directly the Casimir-Polder interaction between a surface and a polarizable particle [51].

VIII. CONCLUSION AND DISCUSSION

After decades of efforts by many researchers, the power of integral equations methods [52,53] in computational electromagnetism is by now an established fact. Only recently, however, these methods have been applied to Casimir physics [33,34]. The findings of [33,34] undoubtedly represent a significant progress in the field, because they make it possible to compute, at least in principle, Casimir interactions for arbitrary arrangements of any number of (homogeneous) magneto-dielectric bodies of any shape. This is very important in view of applications to micro- and nanomechanical devices of complex shapes, where the Casimir force may play an important role. While it is a huge step forward, the approach of

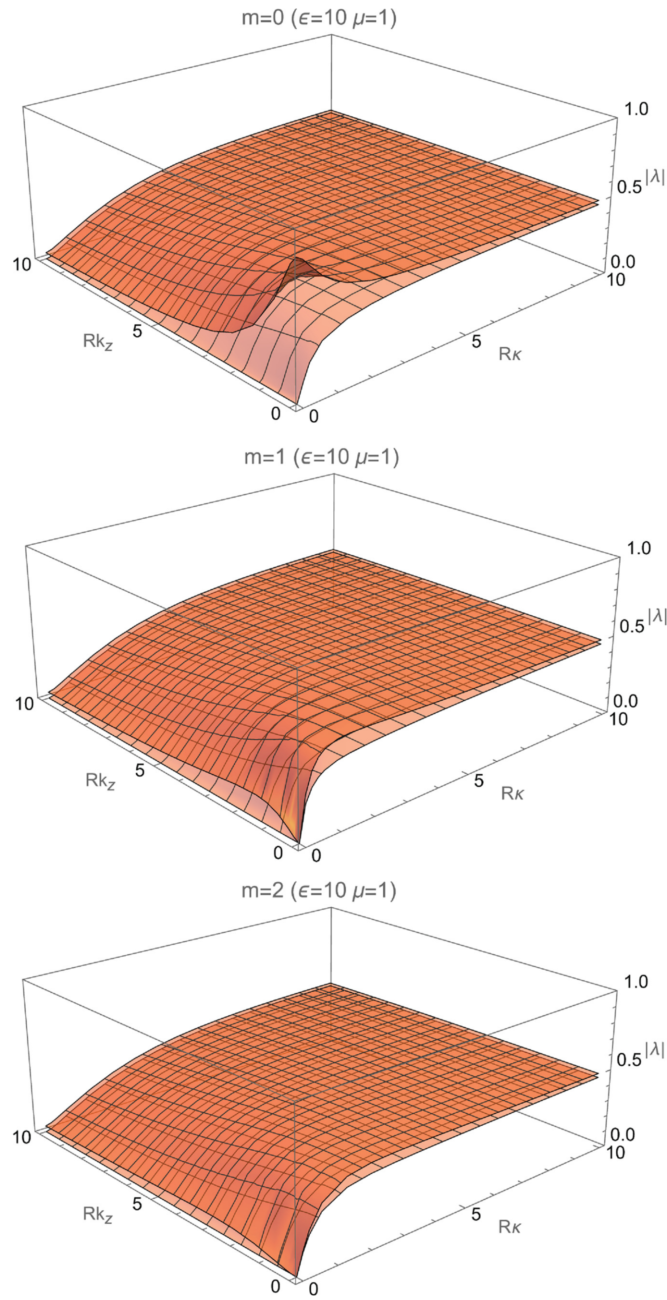

FIG. 4. Eigenvalues of $\mathbb{K}_{11}$ for a magneto-dielectric cylinder of radius R with $\epsilon = 10$ and $\mu = 1$. Shown are the absolute values of the eigenvalues for electric and magnetic polarization, as a function of the rescaled frequency $R\kappa$ and the rescaled dimensionless wave vector Rk_z, for different partial wave indices $m = 0, 1,$ and 2. Only two of the four eigenvalues are shown, as eigenvalues of $\mathbb{K}_{11}$ always appear in pairs $(\lambda, -\lambda)$.

[33,34] suffers from the drawback that its implementation is extremely costly in terms of the required computer resources, which may not be generally available to the interested researchers.

In [41] we introduced a class of exact integral-equation representations of the Casimir and Casimir-Polder interaction between bodies of arbitrary shape and material composition. The present paper offers a detailed and pedagogical presenta-

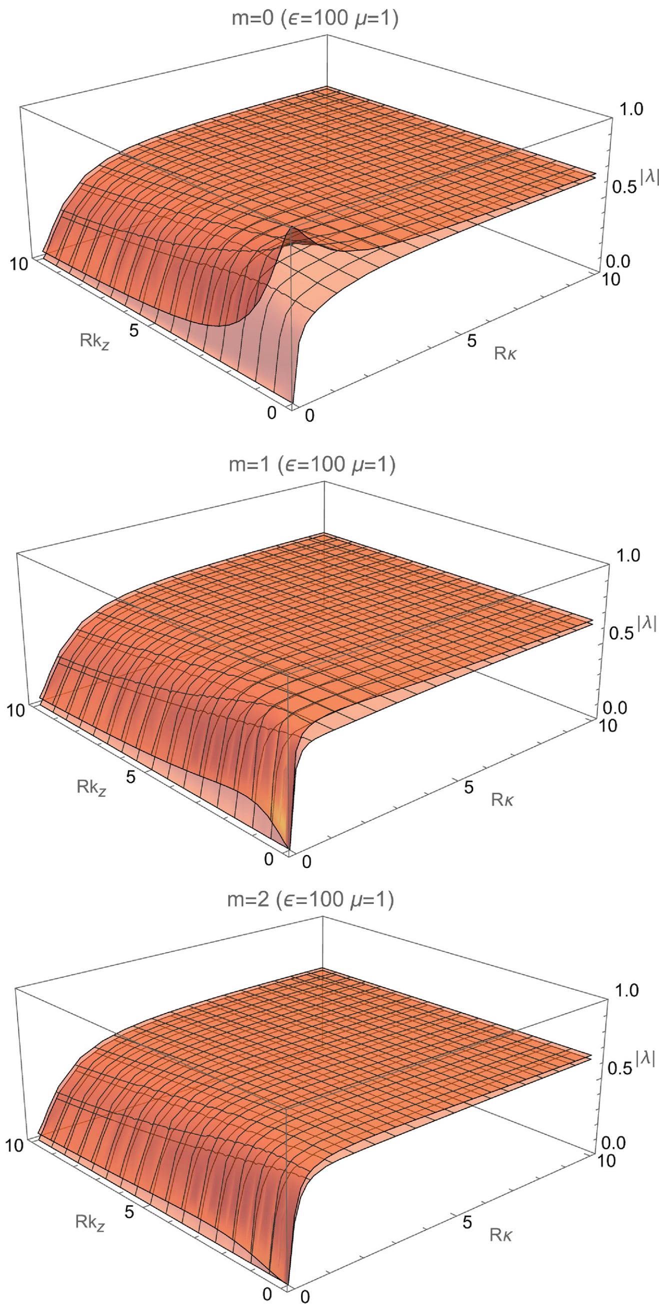

FIG. 5. Eigenvalues of $\mathbb{K}_{11}$ for a magneto-dielectric cylinder of radius R with $\epsilon = 100$ and $\mu = 1$.

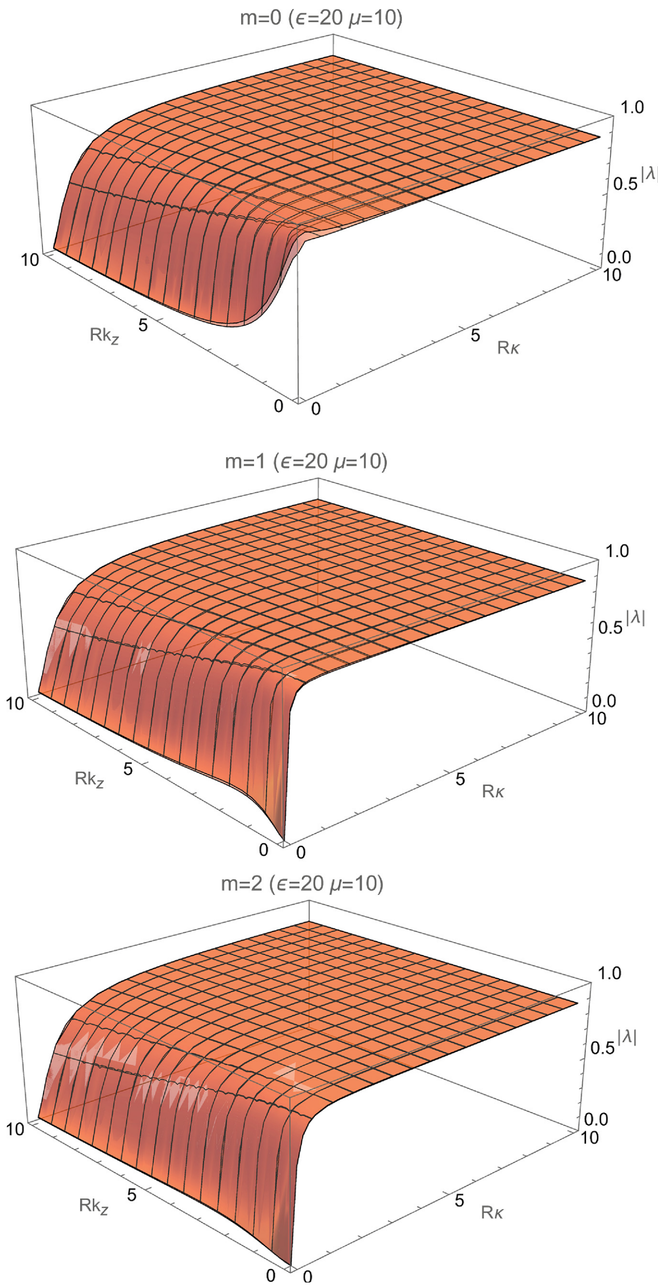

FIG. 6. Eigenvalues of $\mathbb{K}_{11}$ for a magneto-dielectric cylinder of radius R with $\epsilon = 20$ and $\mu = 10$.

tion of our methods, which may not be familiar to the majority of researchers in Casimir physics. A major difference with respect to [33,34] is that in our approach the Casimir and Casimir-Polder interactions are expressed in terms of surface integral equations of the second Fredholm type, which are amenable to a MSE in terms of elementary free-space propagators. We underline that our representation does not depend on the scattering amplitude of the bodies. Moreover, our semianalytical MSE does not involve a discrete mesh representation of the geometry and requires no numerical computation and inversion of large matrices over boundary elements, a computationally expensive task. In our approach, the interaction is in fact expressed in terms of iterated integrals of elementary functions extended on the surfaces of the bodies. For soft material bodies like those usually considered in biological systems, the MSE converges quickly, and then already the first terms of the expansion may provide a fairly accurate estimate of the interaction energy. The Si-Au wedge-plate system studied in [41] shows that even in the case of condensed bodies convergence of the MSE is rather fast. We believe that the possibility of getting, via the MSE, an estimate of the Casimir energy in a complex geometry, by just performing simple surface integrals, adds a useful tool to the toolbox of researchers in the field.

We envisage several possible future directions for our paper. One important advantage of our approach is that our representations of the Casimir and Casimir-Polder interac-

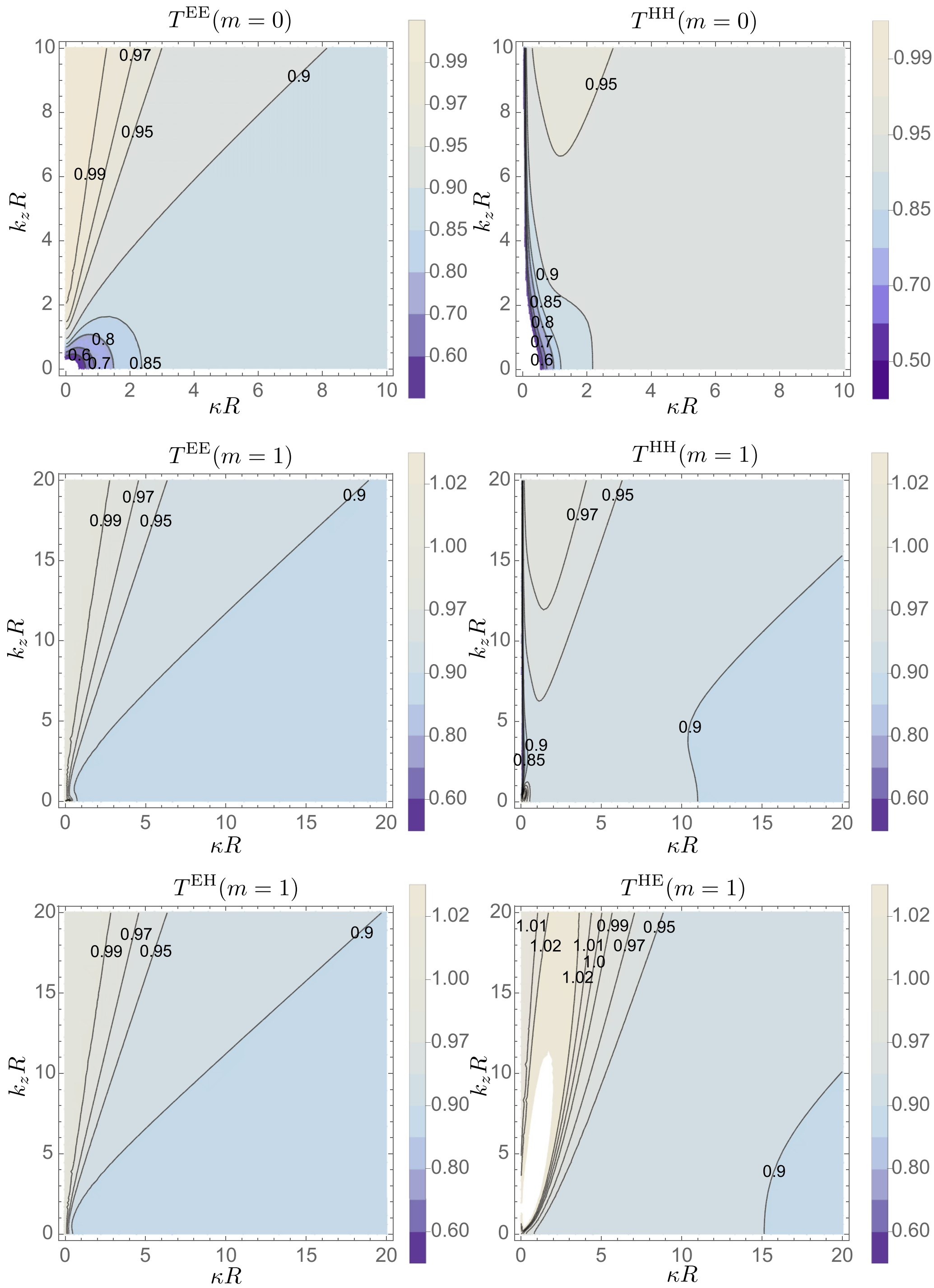

FIG. 7. Multiple scattering expansion of the scattering Green function of a dielectric cylinder. Contour plots of the ratio of the T-matrix elements of a dielectric cylinder of radius R computed with the MSE to order $p = 3$ and the exact results, as a function of rescaled imaginary frequency κR and the rescaled wave vector k_zR along the cylinder axis. The dielectric permittivities of the cylinder are $\epsilon = 30$ and $\mu = 1$. Shown are the lowest-order T-matrix elements with angular quantum numbers $m = 0$ and 1 for all four combinations of polarizations E and H. (For $m = 0$ the polarization couplings T^{EH} and T^{HE} vanish.)

tions involve several free parameters, that may be in principle adjusted to the dielectric properties of the bodies, in order to speed convergence of the expansion. The problem of determining the optimal choice of these coefficients is a very interesting topic, that we plan to investigate in future publications. Another clear direction is to apply the MSE in the real frequency domain, to the technologically important problem of radiative transfer at the micro- and nanoscales, a subject of intense study in recent years [54,55]. Our approach can be easily adapted to this problem, by following steps similar to those of [56]. The nontrivial issue that requires a systematic investigation is the domain of convergence of the MSE for real frequencies, for the materials and the frequency ranges that are relevant to the problem.

In conclusion, our rapidly convergent MSE can provide a powerful tool to delve deeper into Casimir and thermal phenomena in submicrometer structures composed of various materials which cannot be understood by simple additive power laws and planar or spherical surface interactions.

ACKNOWLEDGMENT

Early discussions with B. Duplantier are acknowledged.

APPENDIX A: SURFACE-INTEGRAL FORMULATION OF ELECTROMAGNETIC SCATTERING

In this Appendix, we briefly review surface-integral formulation of EM scattering by dielectric objects. This provides a convenient basis for the derivation of the SSO, which is the subject of the next Appendix.

We start from the formulation of our scattering problem. Let us consider a collection of N dielectric bodies, characterized by the respective (frequency dependent) permittivities ϵ_σ and μ_σ, embedded in a dielectric medium with permittivities ϵ_0 and μ_0. We let V_σ denote the volume occupied by the σth body, and S_σ denote its surface, with $\hat{\mathbf{n}}_\sigma(\mathbf{u})$ the unit outward normal to S_σ. We finally denote by V_0 the region of space, outside the collection of N bodies. We imagine a distribution of electric and magnetic sources $(\mathbf{J}, \mathbf{M})$ in V_0, and we let $(\mathbf{E}_{\rm inc}, \mathbf{H}_{\rm inc})$ denote the incident EM field radiated (in the absence of the N bodies) by $(\mathbf{J}, \mathbf{M})$:

$$(\mathbf{E}_{\rm inc}, \mathbf{H}_{\rm inc})(\mathbf{r}) = \int_{V_0} d\mathbf{r}' \, \mathbb{G}_0(\mathbf{r}-\mathbf{r}') \cdot (\mathbf{J}, \mathbf{M})(\mathbf{r}'), \tag{A1}$$

where $\mathbb{G}_0$ denote the Green tensors for a homogeneous and isotropic medium with permittivities ϵ_0 and μ_0, respectively (the explicit expressions of the Green tensors are provided in Appendix E). Solution of the N-body scattering problem requires solving Maxwell equations in the regions $V_0, V_1, \ldots, V_N$, with sources $(\mathbf{J}, \mathbf{M})$ in V_0, subjected to the boundary conditions that the tangential components of the EM field $\mathbf{E}$ and $\mathbf{H}$ are continuous across the N surfaces S_σ:

$$\begin{aligned} \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{E}_+(\mathbf{u}) &= \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{E}_-(\mathbf{u}), \\ \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{H}_+(\mathbf{u}) &= \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{H}_-(\mathbf{u}), \end{aligned} \tag{A2}$$

where $\mathbf{E}_+$ and $\mathbf{E}_-$ ($\mathbf{H}_+$ and $\mathbf{H}_-$) denote, respectively, the values of the electric (magnetic) field at points just outside and inside the surface S_σ. It is convenient to define the electric and magnetic "surface currents" $\mathbf{j}_\sigma(\mathbf{u})$ and $\mathbf{m}_\sigma(\mathbf{u})$, with $\mathbf{u} \in S_\sigma$, by the relations

$$\begin{aligned} \mathbf{j}_\sigma(\mathbf{u}) &\equiv \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{H}(\mathbf{u}), \\ \mathbf{m}_\sigma(\mathbf{u}) &\equiv -\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{E}(\mathbf{u}). \end{aligned} \tag{A3}$$

By using Green's theorem [43,57,58], one can prove the following four sets of integral identities, which relate the EM field $\mathbf{E}$ and $\mathbf{H}$ to the incident field $(\mathbf{E}_{\rm inc}, \mathbf{H}_{\rm inc})$ and to the boundary fields $\mathbf{j}_1, \ldots, \mathbf{m}_N$:

$$(\mathbf{E}, \mathbf{H})(\mathbf{r}) = (\mathbf{E}_{\rm inc}, \mathbf{H}_{\rm inc})(\mathbf{r}) + (\tilde{\mathbf{E}}^{(0)}, \tilde{\mathbf{H}}^{(0)})(\mathbf{r};\ \mathbf{j}_1, \ldots, \mathbf{m}_N), \quad \mathbf{r} \in V_0, \tag{A4}$$

$$0 = (\mathbf{E}_{\rm inc}, \mathbf{H}_{\rm inc})(\mathbf{r}) + (\tilde{\mathbf{E}}^{(0)}, \tilde{\mathbf{H}}^{(0)})(\mathbf{r};\ \mathbf{j}_1, \ldots, \mathbf{m}_N), \quad \mathbf{r} \notin V_0, \tag{A5}$$

$$(\mathbf{E}, \mathbf{H})(\mathbf{r}) = -(\tilde{\mathbf{E}}^{(\sigma)}, \tilde{\mathbf{H}}^{(\sigma)})(\mathbf{r};\ \mathbf{j}_\sigma, \mathbf{m}_\sigma), \quad \mathbf{r} \in V_\sigma, \tag{A6}$$

$$0 = (\tilde{\mathbf{E}}^{(\sigma)}, \tilde{\mathbf{H}}^{(\sigma)})(\mathbf{r};\ \mathbf{j}_\sigma, \mathbf{m}_\sigma), \quad \mathbf{r} \notin V_\sigma. \tag{A7}$$

In the above relations, $\tilde{\mathbf{E}}^{(\rho)}$ and $\tilde{\mathbf{H}}^{(\rho)}$ ($\rho = 0, 1, \ldots, N$) denote the following surface integrals:

$$\begin{aligned} &(\tilde{\mathbf{E}}^{(0)}, \tilde{\mathbf{H}}^{(0)})(\mathbf{r};\ \mathbf{j}_1, \ldots, \mathbf{m}_N) \\ &\quad \equiv \sum_{\sigma=1}^{N} \int_{S_\sigma} ds_{\mathbf{u}} \, \mathbb{G}_0(\mathbf{r}-\mathbf{u}) \cdot (\mathbf{j}_\sigma, \mathbf{m}_\sigma)(\mathbf{u}), \\ &(\tilde{\mathbf{E}}^{(\sigma)}, \tilde{\mathbf{H}}^{(\sigma)})(\mathbf{r};\ \mathbf{j}_\sigma, \mathbf{m}_\sigma) \\ &\quad \equiv \int_{S_\sigma} ds_{\mathbf{u}} \, \mathbb{G}_\sigma(\mathbf{r}-\mathbf{u}) \cdot (\mathbf{j}_\sigma, \mathbf{m}_\sigma)(\mathbf{u}), \end{aligned} \tag{A8}$$

where $ds_{\mathbf{u}}$ is the area element on S_σ, while $\mathbb{G}_\sigma^{(\alpha\beta)}$, $\sigma = 0, 1, \ldots, N$ denote the Green tensors for a homogeneous and isotropic medium with frequency dependent electric and magnetic permittivities $\epsilon_\sigma(\omega)$ and $\mu_\sigma(\omega)$, respectively.

Independent of the Green's theorem, validity of the identities Eqs. (A4)–(A7) can be easily understood by using a nice mathematical trick, that goes by the name of the "equivalence principle" [43]. The trick consists in introducing the following $N+1$ EM fields $(\mathbf{E}^{(0)}, \mathbf{H}^{(0)}), \ldots, (\mathbf{E}^{(N)}, \mathbf{H}^{(N)})$:

$$(\mathbf{E}^{(0)}, \mathbf{H}^{(0)})(\mathbf{r}) = \begin{cases} (\mathbf{E}, \mathbf{H})(\mathbf{r}), & \mathbf{r} \in V_0 \\ (\mathbf{0}, \mathbf{0}) & \mathbf{r} \notin V_0 \end{cases}, \tag{A9}$$

$$(\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})(\mathbf{r}) = \begin{cases} (\mathbf{E}, \mathbf{H})(\mathbf{r}), & \mathbf{r} \in V_\sigma \\ (\mathbf{0}, \mathbf{0}) & \mathbf{r} \notin V_\sigma \end{cases}. \tag{A10}$$

As we see, the field $(\mathbf{E}^{(0)}, \mathbf{H}^{(0)})$ coincides with the actual EM field $(\mathbf{E}, \mathbf{H})$ at points in the medium surrounding the bodies, and it *vanishes* at all points *inside* bodies. Vice versa, each of the fields $(\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})$ coincides with the total field $(\mathbf{E}, \mathbf{H})$ at points *inside* the respective body, and *vanishes* at all other points of space. All these fields are clearly *unphysical*, since they do not fulfill the boundary conditions Eq. (A2) on at least one among the surfaces S_σ. While unphysical, these fields have by construction the nice property of being solutions of Maxwell equations in infinite *homogeneous* space, with *constant* dielectric properties. More precisely, the field $(\mathbf{E}^{(0)}, \mathbf{H}^{(0)})$ satisfies [except on the surfaces $(S_1, \ldots, S_N)$, where it is discontinuous] Maxwell equations in a medium having *everywhere* the permittivities (ϵ_0, μ_0) of the medium surrounding the bodies, while each of the fields $(\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})$ satisfies (except on the surface of the σth body, where it is discontinuous) Maxwell equations in a medium having *everywhere* the permittivities $(\epsilon_\sigma, \mu_\sigma)$ of the material filling the σth body. Now comes the main observation. Since the media in which all these fields live are spatially homogeneous, one concludes that these fields are in fact free fields, and therefore they can be expressed as convolutions of free-space Green tensors with the appropriate sources. By construction, the sources of $(\mathbf{E}^{(0)}, \mathbf{H}^{(0)})$ are the original external sources $(\mathbf{J}, \mathbf{M})$ of our scattering problem, together with the $2N$ surface currents $(\mathbf{j}_1, \ldots, \mathbf{m}_N)$ arising from the discontinuity of $(\mathbf{E}^{(0)}, \mathbf{H}^{(0)})$ across the bodies surfaces. The identities in Eqs. (A4) and (A5) become obvious, if one realizes that they represent the expression of $(\mathbf{E}^{(0)}, \mathbf{H}^{(0)})$ as a convolution of $\mathbb{G}_0$ with its sources $(\mathbf{J}, \mathbf{M})$ and $(\mathbf{j}_1, \ldots, \mathbf{m}_N)$. An analogous argument applies to the fields $(\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})$. From the discontinuity of $(\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})$ across S_σ, one sees that $(\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})$ is sourced by the surface currents $(-\mathbf{j}_\sigma, -\mathbf{m}_\sigma)$. Upon expressing $(\mathbf{E}^{(\sigma)}, \mathbf{H}^{(\sigma)})$ as a convolution of $\mathbb{G}_\sigma$ with $(-\mathbf{j}_\sigma, -\mathbf{m}_\sigma)$, one recovers at once the identities in Eqs. (A6) and (A7).

Let us go back now to Eq. (A4): this integral relation shows that at points $\mathbf{r}$ outside the bodies, the scattered field $(\mathbf{E}_{\text{scat}}, \mathbf{H}_{\text{scat}})$ coincides with the surface integral $(\tilde{\mathbf{E}}^{(0)}, \tilde{\mathbf{H}}^{(0)})$:

$$(\mathbf{E}_{\text{scat}}, \mathbf{H}_{\text{scat}})(\mathbf{r}) = (\tilde{\mathbf{E}}^{(0)}, \tilde{\mathbf{H}}^{(0)})(\mathbf{r};\, \mathbf{j}_1, \ldots, \mathbf{m}_N), \quad \mathbf{r} \in V_0. \tag{A11}$$

This relation shows that the scattering problem is solved, provided that the $2N$ surface currents $(\mathbf{j}_1, \ldots, \mathbf{m}_N)$ can be actually computed. In the next Appendix, we show how this goal can be achieved, using the SSO.

APPENDIX B: ALTERNATIVE DERIVATION OF THE SSO

In this Appendix, we construct the SSO that allows us to compute the surface currents providing the solution of the EM scattering problem. The starting point is provided by the identities in Eqs. (A5) and (A7). Upon taking the limits of Eqs. (A5) and (A7) as the point $\mathbf{r}$ approaches the point $\mathbf{u}$ on the surface S_σ, and then taking a vector product with the unit normal to S_σ, one obtains the following identities:

$$\begin{aligned}
&\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \tilde{\mathbf{E}}^{(0)}_-(\mathbf{u};\, \mathbf{j}_1, \ldots, \mathbf{m}_N) + \hat{\mathbf{n}}_\sigma \times \mathbf{E}_{\text{inc}}(\mathbf{u}) = 0,\\
&\hat{\mathbf{n}}_\sigma(\mathbf{r}) \times \tilde{\mathbf{H}}^{(0)}_-(\mathbf{u};\, \mathbf{j}_1, \ldots, \mathbf{m}_N) + \hat{\mathbf{n}}_\sigma \times \mathbf{H}_{\text{inc}}(\mathbf{u}) = 0,\\
&\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \tilde{\mathbf{E}}^{(\sigma)}_+(\mathbf{u};\, \mathbf{j}_\sigma, \mathbf{m}_\sigma) = 0,\\
&\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \tilde{\mathbf{H}}^{(\sigma)}_+(\mathbf{u};\, \mathbf{j}_\sigma, \mathbf{m}_\sigma) = 0.
\end{aligned} \tag{B1}$$

The above relations constitute an overdetermined set of $4N$ integral equations in the $2N$ unknown boundary fields $(\mathbf{j}_1, \ldots, \mathbf{m}_N)$. A consistent set of equations can be obtained by taking $2N$ distinct linear combinations of the $4N$ Eqs. (B1):

$$\begin{aligned}
&C^{(e|E)}_\sigma \hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{E}}^{(0)}_-(\mathbf{j}_1, \ldots, \mathbf{m}_N) - C^{(i|E)}_\sigma \hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{E}}^{(\sigma)}_+(\mathbf{j}_\sigma, \mathbf{m}_\sigma)\\
&\quad = -C^{(e|E)}_\sigma \hat{\mathbf{n}}_\sigma \times \mathbf{E}_{\text{inc}},\\
&C^{(e|H)}_\sigma \hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{H}}^{(0)}_-(\mathbf{j}_1, \ldots, \mathbf{m}_N) - C^{(i|H)}_\sigma \hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{H}}^{(\sigma)}_+(\mathbf{j}_\sigma, \mathbf{m}_\sigma)\\
&\quad = -C^{(e|H)}_\sigma \hat{\mathbf{n}}_\sigma \times \mathbf{H}_{\text{inc}},
\end{aligned} \tag{B2}$$

where for brevity we do not display the explicit dependence of the boundary fields on the point $\mathbf{u}$. We remark that the coefficients in Eq. (B2) are defined up to rescalings by arbitrary nonvanishing factors $\lambda^{(\alpha)}_\sigma$:

$$\left(C^{(i|\alpha)}_\sigma, C^{(e|\alpha)}_\sigma\right) \to \lambda^{(\alpha)}_\sigma \left(C^{(i|\alpha)}_\sigma, C^{(e|\alpha)}_\sigma\right). \tag{B3}$$

It is convenient to reexpress Eqs. (B2) in terms of the values of the surface integrals $(\tilde{\mathbf{E}}^{(\rho)}, \tilde{\mathbf{H}}^{(\rho)})$ computed directly on the surfaces S_σ. This can be done by observing that, for an arbitrary choice of the surface currents, the surface integrals $(\tilde{\mathbf{E}}^{(0)}, \tilde{\mathbf{H}}^{(0)})$ and $(\tilde{\mathbf{E}}^{(\sigma)}, \tilde{\mathbf{H}}^{(\sigma)})$ satisfy the jump conditions

$$\begin{aligned}
&\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times [\tilde{\mathbf{E}}^{(0)}_+(\mathbf{u}) - \tilde{\mathbf{E}}^{(0)}_-(\mathbf{u})] = -\mathbf{m}_\sigma(\mathbf{u}),\\
&\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times [\tilde{\mathbf{E}}^{(\sigma)}_+(\mathbf{u}) - \tilde{\mathbf{E}}^{(\sigma)}_-(\mathbf{u})] = -\mathbf{m}_\sigma(\mathbf{u}),\\
&\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times [\tilde{\mathbf{H}}^{(0)}_+(\mathbf{u}) - \tilde{\mathbf{H}}^{(0)}_-(\mathbf{u})] = \mathbf{j}_\sigma(\mathbf{u}),\\
&\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times [\tilde{\mathbf{H}}^{(\sigma)}_+(\mathbf{u}) - \tilde{\mathbf{H}}^{(\sigma)}_-(\mathbf{u})] = \mathbf{j}_\sigma(\mathbf{u}),
\end{aligned} \tag{B4}$$

On the other hand, we know [45,58] that the fields $(\tilde{\mathbf{E}}^{(\rho)}, \tilde{\mathbf{H}}^{(\rho)})(\mathbf{u})$ are the averages of the corresponding values just inside and outside S_σ:

$$\begin{aligned}
&\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times [\tilde{\mathbf{E}}^{(0)}_+(\mathbf{u}) + \tilde{\mathbf{E}}^{(0)}_-(\mathbf{u})] = 2\, \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \tilde{\mathbf{E}}^{(0)}(\mathbf{u}),\\
&\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times [\tilde{\mathbf{E}}^{(\sigma)}_+(\mathbf{u}) + \tilde{\mathbf{E}}^{(\sigma)}_-(\mathbf{u})] = 2\, \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \tilde{\mathbf{E}}^{(\sigma)}(\mathbf{u}),\\
&\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times [\tilde{\mathbf{H}}^{(0)}_+(\mathbf{u}) + \tilde{\mathbf{H}}^{(0)}_-(\mathbf{u})] = 2\, \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \tilde{\mathbf{H}}^{(0)}(\mathbf{u}),\\
&\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times [\tilde{\mathbf{H}}^{(\sigma)}_+(\mathbf{u}) + \tilde{\mathbf{H}}^{(\sigma)}_-(\mathbf{u})] = 2\, \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \tilde{\mathbf{H}}^{(\sigma)}(\mathbf{u}).
\end{aligned} \tag{B5}$$

The above equations can be used to eliminate $\tilde{\mathbf{E}}^{(0)}_-, \tilde{\mathbf{H}}^{(0)}_-, \tilde{\mathbf{E}}^{(\sigma)}_+$, and $\tilde{\mathbf{H}}^{(\sigma)}_+$ from Eqs. (B2). By doing so, one arrives at the following set of integral equations for the surface currents:

$$\begin{aligned}
&\left(C^{(e|H)}_\sigma + C^{(i|H)}_\sigma\right) \mathbf{j}_\sigma - 2\, C^{(e|H)}_\sigma \hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{H}}^{(0)}(\mathbf{j}_1, \ldots, \mathbf{m}_N)\\
&\quad + 2\, C^{(i|H)}_\sigma \hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{H}}^{(\sigma)}(\mathbf{j}_\sigma, \mathbf{m}_\sigma) = 2\, C^{(e|H)}_\sigma \hat{\mathbf{n}}_\sigma \times \mathbf{H}_{\text{inc}},\\
&\left(C^{(e|E)}_\sigma + C^{(i|E)}_\sigma\right) \mathbf{m}_\sigma + 2\, C^{(e|E)}_\sigma \hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{E}}^{(0)}(\mathbf{j}_1, \ldots, \mathbf{m}_N)\\
&\quad - 2\, C^{(i|E)}_\sigma \hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{E}}^{(\sigma)}(\mathbf{j}_\sigma, \mathbf{m}_\sigma) = -2\, C^{(e|E)}_\sigma \hat{\mathbf{n}}_\sigma \times \mathbf{E}_{\text{inc}}.
\end{aligned} \tag{B6}$$

For generic values of the coefficients, both $C^{(e|H)}_\sigma + C^{(i|H)}_\sigma$ and $C^{(e|E)}_\sigma + C^{(i|E)}_\sigma$ are different from zero, and then the integral equations (B6) can be recast in the form of Eq. (13). The proof that Eqs. (B6) actually determine uniquely the surface currents $(\mathbf{j}_1, \ldots, \mathbf{m}_N)$ at all complex frequencies, and for any choice of the $4N$ coefficients $(C^{(e|E)}_\sigma, C^{(e|H)}_\sigma, C^{(i|E)}_\sigma, C^{(i|H)}_\sigma)$, such that both $C^{(e|H)}_\sigma + C^{(i|H)}_\sigma$ and $C^{(e|E)}_\sigma + C^{(i|E)}_\sigma$ are different from zero, can indeed be obtained by a simple adaptation of the proof given in [45] for a single body and for the particular choice of coefficients, denoted by choice 1 in Sec. IV.

APPENDIX C: PERFECT CONDUCTORS

In this Appendix we work out the SSO for a collection of perfect conductors. The scattering problem now involves a system of N perfectly conducting bodies placed in a medium characterized by electric and magnetic permittivities ϵ_0 and μ_0, respectively. Like before, we imagine a distribution of electric and magnetic sources $(\mathbf{J}, \mathbf{M})$ in the region V_0 outside the conductors. Solution of the N-body scattering problem now requires solving Maxwell equations in the region V_0, with sources $(\mathbf{J}, \mathbf{M})$ in V_0, subjected to the boundary conditions that the tangential component of the electric field $\mathbf{E}$ vanishes on the boundaries of the conductors:

$$\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{E}_+(\mathbf{u}) = 0. \tag{C1}$$

In view of this simple condition, we now have only one set of surface currents, namely the electric currents

$$\mathbf{j}_\sigma(\mathbf{u}) = \hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \mathbf{H}_+(\mathbf{u}). \tag{C2}$$

By Green's theorem [43,57,58], one finds the following two sets of integral identities, which relate the EM field $\mathbf{E}$ and $\mathbf{H}$ to the external field $(\mathbf{E}_{\rm inc}, \mathbf{H}_{\rm inc})$ and to the boundary fields $\mathbf{j}_1, \dots, \mathbf{j}_N$:

$$(\mathbf{E}, \mathbf{H})(\mathbf{r}) = (\mathbf{E}_{\rm inc}, \mathbf{H}_{\rm inc})(\mathbf{r}) + (\tilde{\mathbf{E}}^{(0)}, \tilde{\mathbf{H}}^{(0)})(\mathbf{r};\ \mathbf{j}_1, \dots, \mathbf{j}_N), \quad \mathbf{r} \in V_0, \tag{C3}$$

$$0 = (\mathbf{E}_{\rm inc}, \mathbf{H}_{\rm inc})(\mathbf{r}) + (\tilde{\mathbf{E}}^{(0)}, \tilde{\mathbf{H}}^{(0)})(\mathbf{r};\ \mathbf{j}_1, \dots, \mathbf{j}_N), \quad \mathbf{r} \notin V_0. \tag{C4}$$

The above equations show that the PM scattering problem is solved if one can determine the N surface currents $\mathbf{j}_1, \dots, \mathbf{j}_N$. Proceeding as in the case of dielectric bodies, we consider the limits of Eq. (C4) as $\mathbf{r}$ tends to a point $\mathbf{u}$ on the surfaces of the conductors. This gives us

$$\begin{aligned}\hat{\mathbf{n}}_\sigma(\mathbf{u}) \times \tilde{\mathbf{E}}^{(0)}_-(\mathbf{u};\ \mathbf{j}_1, \dots, \mathbf{j}_N) + \hat{\mathbf{n}}_\sigma \times \mathbf{E}_{\rm inc}(\mathbf{u}) = 0,\\ \hat{\mathbf{n}}_\sigma(\mathbf{r}) \times \tilde{\mathbf{H}}^{(0)}_-(\mathbf{u};\ \mathbf{j}_1, \dots, \mathbf{j}_N) + \hat{\mathbf{n}}_\sigma \times \mathbf{H}_{\rm inc}(\mathbf{u}) = 0.\end{aligned} \tag{C5}$$

The above relations constitute an overdetermined set of $2N$ integral equations in the N unknown boundary fields $(\mathbf{j}_1, \dots, \mathbf{j}_N)$. A consistent set of equations can be obtained by taking N distinct linear combinations of the $2N$ Eqs. (C5):

$$\begin{aligned}&C^{(e|E)}_\sigma \hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{E}}^{(0)}_-(\mathbf{j}_1, \dots, \mathbf{j}_N) + C^{(e|H)}_\sigma \hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{H}}^{(0)}_-(\mathbf{j}_1, \dots, \mathbf{j}_N)\\ &\quad = -C^{(e|E)}_\sigma \hat{\mathbf{n}}_\sigma \times \mathbf{E}_{\rm inc} - C^{(e|H)}_\sigma \hat{\mathbf{n}}_\sigma \times \mathbf{H}_{\rm inc}.\end{aligned} \tag{C6}$$

Similar to what we did earlier, we can take advantage of the identities in the last two lines of Eqs. (B4) and (B5) to reexpress the above integral equation in terms of the values of $\tilde{\mathbf{E}}^{(0)}$ and $\tilde{\mathbf{H}}^{(0)}$ on the surfaces S_σ:

$$\begin{aligned}&C^{(e|H)}_\sigma \mathbf{j}_\sigma - 2\, C^{(e|H)}_\sigma \hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{H}}^{(0)}(\mathbf{j}_1, \dots, \mathbf{j}_N)\\ &\quad + 2\, C^{(e|E)}_\sigma \hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{E}}^{(0)}(\mathbf{j}_1, \dots, \mathbf{j}_N)\\ &\quad = 2\, C^{(e|H)}_\sigma \hat{\mathbf{n}}_\sigma \times \mathbf{H}_{\rm inc} - 2\, C^{(e|E)}_\sigma \hat{\mathbf{n}}_\sigma \times \mathbf{E}_{\rm inc}.\end{aligned} \tag{C7}$$

As in the general case for magneto-dielectric bodies, several different formulations exist for the perfectly conducting limit, depending on the choice of the coefficients in Eq. (C7). A possible choice is

$$C^{(e|H)}_\sigma = 0, \quad C^{(e|E)}_\sigma = 1. \tag{C8}$$

The resulting integral equation reads

$$\hat{\mathbf{n}}_\sigma \times \tilde{\mathbf{E}}^{(0)}(\mathbf{j}_1, \dots, \mathbf{j}_N) = -\hat{\mathbf{n}}_\sigma \times \mathbf{E}_{\rm inc}. \tag{C9}$$

Upon taking the vector product with $\hat{\mathbf{n}}_\sigma$ of both members of the above equation, we obtain the following integral equation for perfect conductors:

$$\sum_{\sigma'=1}^N \int_{S_{\sigma'}} ds_{\mathbf{u}'}\, \mathbb{B}^{(\rm PC)}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}')\, \mathbf{j}_{\sigma'}(\mathbf{u}') = \int d\mathbf{r}\, \tilde{\mathbb{M}}^{(\rm PC)}_\sigma(\mathbf{u}, \mathbf{r})\mathbf{J}(\mathbf{r}), \tag{C10}$$

where

$$\mathbb{B}^{(\rm PC)}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}') = \left[\mathbb{G}^{(EE)}_0(\mathbf{u}, \mathbf{u}')\right]_t, \tag{C11}$$

$$\tilde{\mathbb{M}}^{(\rm PC)}_\sigma(\mathbf{u}, \mathbf{r}) = -\left[\mathbb{G}^{(EE)}_0(\mathbf{u}, \mathbf{r})\right]_t. \tag{C12}$$

The integral equation is not of Fredholm form, and therefore it does not allow for a MSE. We note that this formulation was used in a numerical investigation of the Casimir effect in [33]. We now consider the alternative choice

$$C^{(e|H)}_\sigma = 1, \quad C^{(e|E)}_\sigma = 0, \tag{C13}$$

which leads to the following integral equation of second Fredholm type:

$$\sum_{\sigma'=1}^N \int_{S_{\sigma'}} ds_{\mathbf{u}'} \left[\mathbb{1} - \mathbb{K}^{(\rm PC)}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}')\right]\mathbf{j}_{\sigma'}(\mathbf{u}') = \int d\mathbf{r}\, \mathbb{M}^{(\rm PC)}_\sigma(\mathbf{u}, \mathbf{r})\mathbf{J}(\mathbf{r}) \tag{C14}$$

with

$$\mathbb{K}^{(\rm PC)}_{\sigma\sigma'}(\mathbf{u}, \mathbf{u}') = 2\, \mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}^{(HE)}_0(\mathbf{u}, \mathbf{u}') \tag{C15}$$

and

$$\mathbb{M}^{(\rm PC)}_\sigma(\mathbf{u}, \mathbf{r}) = 2\, \mathbf{n}_\sigma(\mathbf{u}) \times \mathbb{G}^{(HE)}_0(\mathbf{u}, \mathbf{r}). \tag{C16}$$

This is the integral equation for CP used in [39,40].

APPENDIX D: T MATRIX OF A MAGNETO-DIELECTRIC CYLINDER

The T operator of a dielectric cylinder of radius R assumes a 2×2 block-diagonal form in vector cylindrical waves labeled by the angular quantum number m and the wave vector k_z along the cylinder axis [23]. It is assumed that the cylinder has electric and magnetic permittivities ϵ and μ, and the surrounding medium is vacuum ($\epsilon_0 = \mu_0 = 1$). On the imaginary frequency axis, and with $p_0 = \sqrt{\kappa^2 + k_z^2}$ and $p_1 = \sqrt{\epsilon\mu\kappa^2 + k_z^2}$, the diagonal elements are given by [50]

$$\begin{aligned}\mathbb{T}^{HH}(m, \kappa, k_z) &= -\frac{I_m(p_0R)}{K_m(p_0R)} \frac{\Delta_1\Delta_4 + K^2}{\Delta_1\Delta_2 + K^2},\\ \mathbb{T}^{EE}(m, \kappa, k_z) &= -\frac{I_m(p_0R)}{K_m(p_0R)} \frac{\Delta_2\Delta_3 + K^2}{\Delta_1\Delta_2 + K^2},\\ \mathbb{T}^{HE}(m, \kappa, k_z) &= -\mathbb{T}^{EH}(m, \kappa, k_z)\\ &= \frac{K}{\sqrt{\epsilon\mu}(p_0R)^2 K_m(p_0R)^2} \frac{1}{\Delta_1\Delta_2 + K^2},\end{aligned} \tag{D1}$$

with

$$K = \frac{mk_z}{\sqrt{\epsilon\mu}\kappa R^2}\left(\frac{1}{p_1^2} - \frac{1}{p_0^2}\right) \tag{D2}$$

and

$$\begin{aligned}\Delta_1 &= \frac{I'_m(p_1R)}{p_1RI_m(p_1R)} - \frac{1}{\epsilon}\frac{K'_m(p_0R)}{p_0RK_m(p_0R)},\\ \Delta_2 &= \frac{I'_m(p_1R)}{p_1RI_m(p_1R)} - \frac{1}{\mu}\frac{K'_m(p_0R)}{p_0RK_m(p_0R)},\\ \Delta_3 &= \frac{I'_m(p_1R)}{p_1RI_m(p_1R)} - \frac{1}{\epsilon}\frac{I'_m(p_0R)}{p_0RI_m(p_0R)},\\ \Delta_4 &= \frac{I'_m(p_1R)}{p_1RI_m(p_1R)} - \frac{1}{\mu}\frac{I'_m(p_0R)}{p_0RI_m(p_0R)},\end{aligned} \tag{D3}$$

where I_m and K_m are Bessel functions, and I'_m and K'_m are their derivatives. We note that the polarization is not conserved under scattering, i.e., $\mathbb{T}^{EH}, \mathbb{T}^{HE} \neq 0$. The scattering Green tensor Γ of the cylinder can be expressed in terms of these matrix elements, following the conventional scattering method [23]. The comparison to the MSE can be performed by suitable projection. For instance, from the projection $\hat{\mathbf{r}}\Gamma^{EE}\hat{\mathbf{r}}'$ on the radial directions $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}'$ of $\Gamma^{EE}(\mathbf{r},\mathbf{r}')$, all four elements $\mathbb{T}^{EE}$, $\mathbb{T}^{HH}$, $\mathbb{T}^{HE}$, and $\mathbb{T}^{EH}$ can be extracted as they are multiplied by different combinations of $K_m(p_0r)$, $K'_m(p_0r)$, $K_m(p_0r')$, and $K'_m(p_0r')$. Therefore, all components of the analytically computed MSE for Γ^{EE} can be compared to the above T-matrix elements.

APPENDIX E: FREE GREEN TENSORS

For completeness, we provide the explicit expressions of the Green tensors, for a homogeneous and isotropic magneto-dielectric medium with frequency dependent electric and magnetic permittivities $\epsilon_\sigma(\omega)$ and $\mu_\sigma(\omega)$, respectively. The external sources $(\mathbf{J}, \mathbf{M})$ are normalized such that Maxwell equations for imaginary frequencies $\omega = i\xi$ take the form

$$-\nabla \times \mathbf{E} = \kappa\, \mu\, \mathbf{H} + \mathbf{M}, \tag{E1}$$

$$\nabla \times \mathbf{H} = \kappa\, \epsilon\, \mathbf{E} + \mathbf{J}, \tag{E2}$$

where κ is wave number $\kappa = \xi/c$. The components of the 6×6 dimensional Green tensor then are

$$\begin{aligned}\mathbb{G}^{(EE)}_{\sigma,ij}(\mathbf{r},\mathbf{r}') &= -\frac{1}{\kappa}\left(\frac{1}{\epsilon}\frac{\partial^2}{\partial x_i \partial x'_j} + \mu\,\kappa^2\,\delta_{ij}\right) g_\sigma(\mathbf{r}-\mathbf{r}'),\\ \mathbb{G}^{(HH)}_{\sigma,ij}(\mathbf{r},\mathbf{r}') &= -\frac{1}{\kappa}\left(\frac{1}{\mu}\frac{\partial^2}{\partial x_i \partial x'_j} + \epsilon\,\kappa^2\,\delta_{ij}\right) g_\sigma(\mathbf{r}-\mathbf{r}'),\\ \mathbb{G}^{(HE)}_{\sigma,ij}(\mathbf{r},\mathbf{r}') &= -\,\epsilon_{ijk}\frac{\partial}{\partial x_k}\, g_\sigma(\mathbf{r}-\mathbf{r}'),\\ \mathbb{G}^{(EH)}_{\sigma,ij}(\mathbf{r},\mathbf{r}') &= -\,\epsilon_{ijk}\frac{\partial}{\partial x'_k}\, g_\sigma(\mathbf{r}-\mathbf{r}'),\end{aligned} \tag{E3}$$

where $i, j \in \{x, y, z\}$ denote the spatial components, ϵ_{ijk} is the Levi-Civita symbol, and the scalar Green function is

$$g_\sigma(\mathbf{r}-\mathbf{r}') = \frac{e^{-\kappa\sqrt{\epsilon_\sigma \mu_\sigma}|\mathbf{r}-\mathbf{r}'|}}{4\pi\,|\mathbf{r}-\mathbf{r}'|}. \tag{E4}$$

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