Advantage of Hardy's nonlocal correlation in reverse zero-error channel coding

Mir Alimuddin [,1] Ananya Chakraborty,[1] Govind Lal Sidhardh,[1] Ram Krishna Patra [,1] Samrat Sen,[1]
Snehasish Roy Chowdhury,[2] Sahil Gopalkrishna Naik,[1] and Manik Banik [1]

[1]*Department of Physics of Complex Systems, S. N. Bose National Center for Basic Sciences,
Block JD, Sector III, Salt Lake, Kolkata 700106, India*
[2]*Physics and Applied Mathematics Unit, Indian Statistical Institute, 203 BT Road, Kolkata, India*

Hardy's argument constitutes an elegant proof of quantum nonlocality. In this work, we report an exotic application of Hardy's nonlocal correlations in two-party communication setup. We come up with a task, wherein a positive payoff can be through 1 bit of communication from the sender to the receiver if and only if the communication channel is assisted with a no-signaling correlation exhibiting Hardy's nonlocality. This further prompts us to establish a counterintuitive result in a correlation-assisted reverse zero-error channel coding scenario, where the aim is to simulate a higher input-output noisy classical channel by a lower input-output noiseless one in assistance with preshared correlations. We show that there exist such reverse zero-error channel simulation tasks where nonmaximally entangled states are preferable over the assistance with a maximally entangled state, even when the former states carry an arbitrarily small amount of entanglement. Our work thus establishes that within the operational paradigm of local operations and *limited* classical communication the structure of entangled resources is even more complex to characterize.

I. INTRODUCTION

The pioneering work of J. S. Bell establishes one of the most striking departures of quantum theory from the profound classical worldview of *local realism* [1] (see also Refs. [2,3]). Violation of a Bell-type inequality, as demonstrated in several milestone experiments [4–9], endorses the nonlocal nature of the quantum world. Apart from its foundational implications, Bell nonlocality has also been identified as a useful resource for several practical tasks, such as device-independent cryptography [10–12], reduction of communication complexity [13], and device-independent randomness certification and amplification [14–17].

A quite popular technique, known by the name "nonlocality without inequality" proof, is often used to establish the nonlocal behavior of quantum theory. Unlike the Bell-type inequalities, where statistics of many events are collected, these proofs focus on a single event whose occurrence shows the incompatibility of quantum theory with the notion of local realism. While the first proof of this kind for tripartite quantum systems is due to Greenberger-Horne-Zeilinger [18] (see also Ref. [19]), for bipartite systems, such proof was first proposed by Hardy [20], which is considered to be "simpler and more compelling than the arguments that underlie the derivation of Bell-CH inequality" [21,22]. More recently, Hardy-type nonlocality proofs have also been shown to be useful in several practical tasks [23–28].

In this work, we report an application of Hardy's nonlocal correlation in the simplest communication scenario. We show that Hardy's nonlocal correlation shared between two distant parties can empower the communication utility of a perfect classical channel. This is quite striking, as such a correlation by itself cannot be used for information transfer, which

otherwise will imply violation of the no-signaling (NS) principle. We also argue that the advantage reported in this work is different than the advantage of nonlocal correlations known in communication complexity tasks [13].

We consider a guessing game played between two distant players—a sender and a receiver. First, we show that the expected collaborative payoff of this game cannot be positive whenever only 1-bit classical communication is allowed from the sender to the receiver, who otherwise can share an unlimited amount of classical correlation between them. Interestingly, assistance of Hardy's nonlocal correlation to the same classical channel can ensure a strictly positive payoff, establishing a nontrivial advantage of Hardy's correlation in communication tasks. The advantage can be better understood in the framework of a correlation-assisted reverse zero-error coding scenario [29], where the aim is to simulate a higher input-output noisy classical channel by a lower input-output identity channel in the presence of NS correlations. In this scenario, the aforesaid game makes Hardy's nonlocal correlations special as we show that among all the 2-input-2-output NS correlations only those exhibiting Hardy's nonlocality can ensure a positive payoff. This further motivates us to show that pure entangled states which are not maximally entangled are preferable over the maximally entangled one for simulating certain noisy classical channels. At the end we show that a similar sort of results can also be obtained by considering a generalization of Hardy's nonlocality argument as proposed by Cabello [30]. The present work, therefore, establishes that the comparison of entanglement in quantum states even for bipartite systems is quite complex when classical communication among distant parties are treated as costly.

The manuscript is organized as follows. In Sec. II we introduce a two-party guessing game which we call the distributed
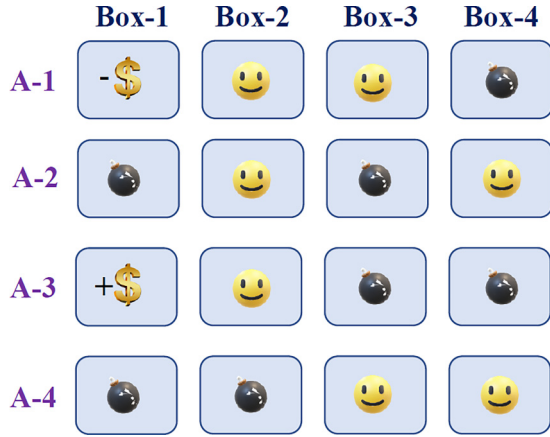
052430-1

FIG. 1. Distributed mine-hunting (DMH) game. Opening a box with $+$ \$ assures dollar bill gain for Bob, whereas a box with $-$ \$ demands him to pay a dollar bill to Charlie. A box with a smile neither offers nor demands any dollar bill, but a box with a bomb turns out to be fatal to Bob. Alice knows which of the arrangements {A-1, ..., A-4} Charlie chooses in a particular run and tries with 1-bit classical channel to help Bob to optimize his expected dollar gain. NS correlation can be used as assistance to the channel

mine-hunting game. Here we also discuss the reverse zero-error channel coding setup. In Sec. III we present our main results, and in Sec. IV we discuss implications of our results along with future outlooks.

II. A TWO-PARTY GUESSING GAME

The game involves two distant players, Alice and Bob, and a referee named Charlie. In each run of the game, Charlie provides four closed boxes numbered 1–4 to Bob, who has to open one of these boxes. Some of the boxes contain a bomb that will explode upon opening it. Among the boxes that do not contain the bomb, some are empty, some contain a dollar bill, and others may even prompt Bob to pay a dollar bill to Charlie. In each run of the game, Charlie uniformly and randomly picks one of four different arrangements of these boxes, as shown in Fig. 1. Charlie then informs Alice about the arrangement of the boxes in that particular run, and Alice tries to help Bob in choosing a box. However, only 1 bit of classical communication is allowed from Alice to Bob, which may be further assisted with NS correlations shared between them *a priori*. From now on, we will refer to this as the distributed mine-hunting (DMH) game.

General scenario

The aforesaid game can be formally studied within a more generic setup as considered in the reverse zero-error channel coding scenario [29]. In the zero-error communication scenario the core goal is to characterize the ability of noisy classical channels to transmit classical information with zero probability of error [31] (see also Ref. [32]). A channel from Alice to Bob, with input $m \in \mathcal{M}$ at Alice's end and output $z \in \mathcal{Z}$ at Bob's end, can be represented as a matrix $S \equiv (s_{mz})$, where s_{mz} denotes the probability of producing the output $z \in \mathcal{Z}$ by Bob given that Alice receives the message $m \in \mathcal{M}$.

The cardinality of $\mathcal{M}$ and $\mathcal{Z}$ is referred to as the input and output dimensions of the channel, respectively. The direct or forward zero-error coding theorem tries to find the maximum number of distinct input alphabets that can be sent perfectly from Alice to Bob through a noisy channel S. In other words, the aim is to find the largest dimensional identity channel (i.e., a noiseless channel) that can be simulated by the given channel S. The reverse problem, on the other hand, aims to simulate a higher-dimensional noisy channel with the help of a lower-dimensional identity channel. While both the direct and reverse problems can be studied in the single-shot setup as well as in the asymptotic limit, here we will restrict our study to the reverse problem in the single-shot setup. Interestingly, nonlocal correlations arising from entangled quantum states can provide a nontrivial advantage in the reverse zero-error coding scenario [29]. The present work, however, reports quite an exotic behavior of entanglement by establishing the precedence of nonmaximally entangled states over the maximal one in some instances of the reverse zero-error channel coding problem.

Within the aforesaid notation, a game (such as DMH) is entirely specified by the payoff matrix $\mathcal{G} \equiv (g_{mz})$, where $g_{mz} \in \overline{\mathbb{R}}$ is the reward or payoff given when Bob produced the index z provided Alice received the message m. For instance, the DMH game is specified by the following payoff matrix:

$$\mathcal{G}_{\text{DMH}} \equiv \begin{array}{c|c|c|c|c|} \mathcal{M}\backslash\mathcal{Z} & 1 & 2 & 3 & 4 \\ \hline\hline 1 & -1 & 0 & 0 & -\infty \\ \hline 2 & -\infty & 0 & -\infty & 0 \\ \hline 3 & +1 & 0 & -\infty & -\infty \\ \hline 4 & -\infty & -\infty & 0 & 0 \\ \hline \end{array} \qquad (1)$$

Payoffs in (1) quantitatively capture the scenario of the DMH game. A reward of $-\infty$ for the box containing the bomb captures the notion that choosing such a box must be avoided at all costs [33]. The reward 0 corresponds to the event where the players survive but do not receive any reward. Events with reward $+1 \, (-1)$ correspond to the scenario where the players receive (pay) some dollar bills from (to) Charlie. The game matrix and the sampling distribution of Alice's inputs are common knowledge to the players.

Alice and Bob are cooperative in nature and aim to maximize the payoff. Their collaborative strategy depends on the available resources, which can be broadly categorized into two types: (i) correlation shared between them before the game starts and (ii) communication from Alice to Bob. Any strategy employed by the players can be represented as an input-output channel $S \equiv (s_{mz})$. Given such a strategy matrix S, the average payoff can be obtained as

$$\mathcal{P}(S) = \sum_{z,m} p(m) \, g_{mz} \, s_{mz} . \qquad (2)$$

As it is evident, there will always be a perfect strategy for such a game if $\log_2 |\mathcal{M}|$ bits of communication are allowed from Alice to Bob. Interesting situations arise when communication is limited, which can further be aided by preshared correlations of different kinds.

III. RESULTS

Let $\Omega_{n_c+SR}(|\mathcal{M}|, |\mathcal{Z}|)$ denote the set of strategy matrices obtained when n bits of classical communication and an unlimited amount of shared randomness are available. The set $\Omega_{n_c+SR}(|\mathcal{M}|, |\mathcal{Z}|)$ forms a polytope with extreme points S^e representing strategy matrices obtained through deterministic encoding $\mathbb{E}: \mathcal{M} \to \{0, 1\}^n$ at Alice's end and deterministic decoding $\mathbb{D}: \{0, 1\}^n \to \mathcal{Z}$ at Bob's end [34] (see also Ref. [35]). Our first technical result is to limit the optimal success probability of the game in Eq. (1) for classical strategies with limited communication. .

Theorem 1. The average payoff of the DMH game is upper bounded by zero while following a strategy from the set $\Omega_{1_c+SR}(4, 4)$, *i.e.*, $\mathcal{P}(S) \leqslant 0, \ \forall S \in \Omega_{1_c+SR}(4, 4)$.

Proof. Since average payoff depends linearly on the strategy [see Eq. (8)], the polytope structure of $\Omega_{1_c+SR}(4, 4)$ ensures that maximum payoff will be attained at one of its vertices. For $\Omega_{n_c+SR}(|\mathcal{M}|, |\mathcal{Z}|)$, the number of vertices N can be calculated using the formula [35]

$$N = \sum_{k=1}^{2^n} k! \binom{|\mathcal{Z}|}{k} \left\{ \begin{matrix} |\mathcal{M}| \\ k \end{matrix} \right\},$$

$$\binom{|\mathcal{Z}|}{k} := \frac{|\mathcal{Z}|!}{k!(|\mathcal{Z}| - k)!}, \qquad (3)$$

$$\left\{ \begin{matrix} |\mathcal{M}| \\ k \end{matrix} \right\} := \sum_{j=0}^{k} \frac{1}{k!} (-1)^{k-j} \binom{k}{j} j^{|\mathcal{M}|}.$$

Note that the players' first priority is to avoid the bomb at any cost, i.e., the $-\infty$ reward in Eq. (1). This will exclude some vertices. For instance, consider the extreme strategy matrix,

$$S_{ns}^e := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad (4)$$

Comparing with Eq. (1), it is evident that in this strategy the bomb will be triggered with a nonzero probability (for $m = 2$). Among all possible vertices, only in the following five cases,

$$\left\{ S_{s1}^e := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ S_{s2}^e := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \right.$$

$$S_{s3}^e := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ S_{s4}^e := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\left. S_{s5}^e := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \right\} \qquad (5)$$

the bomb will never be triggered. However, we have $\mathcal{P}(S_{sK}^e) = 0, \ \forall K \in \{1, \ldots, 5\}$. Since all the deterministic strategies result in either zero or $-\infty$ payoff, the optimal payoff with 1-bit
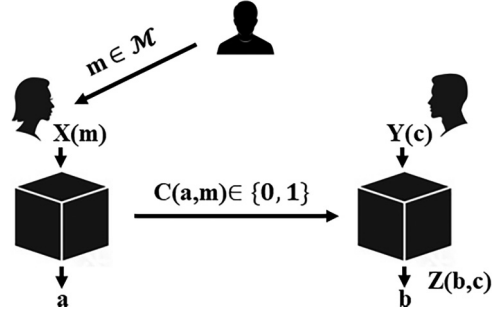

FIG. 2. General strategy to play a game $\mathcal{G}$ when the 1-cbit communication channel is assisted with NS correlation. Alice computes the input $x = X(m)$ to her part of the nonlocal box based on the message $m \in \mathcal{M}$ received from Referee. The output a of the nonlocal box at her end and the message m determine the classical bit $c = C(a, m)$ sent to Bob. Bob then inputs $y = Y(c)$ into his end of the NS box, obtaining the output b. Finally, he generates his guess as $z = Z(b, c)$.

classical communication and shared randomness is simply zero. This completes the claim. ∎

We now consider the scenario where the communication line from Alice to Bob is aided with a generic NS correlation $\mathbb{P} \equiv \{p(a, b|x, y)\}$, where $p(a, b|x, y) \geqslant 0, \ \forall a, b, x, y \ \& \ \sum_{a,b} p(a, b|x, y) = 1, \ \forall x, y$. Here, $p(a, b|x, y)$ denotes the probability of obtaining outcome $a \in \mathcal{A}$ at Alice's end and $b \in \mathcal{B}$ at Bob's end for their respective inputs $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Classical correlations that allow a local-realistic description, $p(a, b|x, y) = \int_\Lambda \mu(\lambda) p(a|x, \lambda) p(b|y, \lambda) d\lambda$, form a strict subset (a subpolytope) of the NS polytope; here, $\lambda \in \Lambda$ is some classical variable shared between Alice and Bob and $\mu(\lambda)$ is a probability distribution on Λ [3]. Quite surprisingly, entangled quantum states can lead to correlations that are not local realistic and nonlocality of those correlations can be certified through violation of some Bell-type inequalities [1]. Given such an NS correlation (possibly nonlocal) as an assistance to 1-bit classical communication from Alice to Bob, the general strategy to play a game $\mathcal{G}$ is described in Fig. 2.

For the binary input-output case, Hardy proposed an elegant argument according to which any NS correlation $\mathbb{H} \equiv \{h(a, b|x, y)\}$ satisfying the constraints

$$h(0, 0|0, 0) := h_0 > 0, \quad h(0, 1|0, 1) := h_5 = 0,$$

$$h(1, 0|1, 0) := h_{10} = 0, \quad h(0, 0|1, 1) := h_3 = 0,$$

$$\text{with } h(a, b|x, y) := h_{a \times 2^3 + b \times 2^2 + x \times 2^1 + y \times 2^0} \qquad (6)$$

must be nonlocal in nature [20] (for the sake of completeness we analyze the argument in Appendix A). A correlation exhibiting Hardy's nonlocality is termed as Hardy's nonlocal correlation. Our next result proves a nontrivial advantage of Hardy correlation while playing the DMH game.

Theorem 2. A strictly positive average payoff in the DMH game is achievable with 1-bit perfect classical channel from Alice to Bob when the channel is assisted with a 2-input-2-output Hardy's nonlocal correlation.

Proof. Consider the following strategy by Alice and Bob.
Alice's action:

(i) Depending on $m \in \mathcal{M}$ Alice computes her input in the NS box. For $m \in \{1, 3\}$ she chooses $x = 0$, otherwise she chooses $x = 1$.

(ii) Based on the tuple $(m, a) \in \mathcal{M} \times \mathcal{A}$ she communicates to Bob. She sends $c = 0$ to Bob when $(m, a) \in \{(1, 1), (2, 1), (3, 0), (3, 1)\}$, else she sends $c = 1$.

Bob's action:

(i) The communicated bit from Alice is used as input in Bob's part of the NS box.

(ii) Depending on the tuple $(c, b) \in C \times \mathcal{B}$ he chooses the box as follows: $(0, 0) \mapsto 1$, $(0, 1) \mapsto 2$, $(1, 0) \mapsto 3$, $(1, 1) \mapsto 4$.

The above strategy with 1-bit communication and 2-input-2-output Hardy's correlation $\mathbb{H} \equiv \{h(a, b|x, y)\}$ leads to the strategy matrix

$$
S_{\mathbb{H}} \equiv
\begin{array}{c|c|c|c|c}
\mathcal{M} \backslash \mathcal{Z} & 1 & 2 & 3 & 4 \\
\hline
1 & h_8 & h_{12} & h_1 & 0 \\
2 & 0 & h_{14} & 0 & h_7 \\
3 & h_0 + h_8 & h_4 + h_{12} & 0 & 0 \\
4 & 0 & 0 & h_{11} & h_7 + h_{15}
\end{array}
\tag{7}
$$

As evident from the payoff matrix (1) and the strategy (7), a box containing a bomb will never be opened. Furthermore, assuming Charlie's choice to be completely random, we have the average payoff

$$
\mathcal{P}(S_{\mathbb{H}}) = \tfrac{1}{4} \operatorname{Tr}\left[\mathcal{G}_{\text{DMH}}^T S_{\mathbb{H}} \right] = \tfrac{1}{4} h_0 > 0.
\tag{8}
$$

This completes the proof. ∎

This in turn establishes that the 4-input-4-output noisy channel $S_{\mathbb{H}}$ can be perfectly simulated by the two-dimensional identity channel (i.e., a 1-bit perfect channel from Alice to Bob) when assisted with Hardy's nonlocal correlation. However, as follows from Theorem 1, assistance of an arbitrary amount of shared randomness fails to achieve the goal. A correlation with Hardy success h_0 yields Clauser-Horne-Shimony-Holt (CHSH) [4] value $2 + 4 h_0$ [36]. Accordingly, the CHSH value corresponding to the optimal quantum Hardy correlation is strictly less than the Cirel'son bound [37,38]. Therefore, a natural question is whether other nonlocal quantum correlations can lead to better success in the DMH game. Our next result answers this question in negation.

Theorem 3. Any 2-input-2-output NS correlation providing a strictly positive payoff in the DMH game as an assistance to the 1-bit of perfect classical channel must exhibit Hardy's nonlocality.

Proof. The set of strategy matrices simulable by 1-cbit communication with the assistance of a given NS correlation $\mathbb{P} \equiv \{p(a, b|x, y) \,|\, a, b, x, y \in \{0, 1\}\}$ and unlimited shared randomness (SR) forms a polytope. We denote this set by $\Omega_{1_c + SR + P}$. Vertices of this polytope correspond to strategies where the players follow an encoding and decoding scheme characterized by deterministic functions of the form $x = X(m)$, $c = C(m, a)$, $y = Y(c)$, $z = Z(c, b)$. The linearity of the payoff function again implies that the maximum payoff occurs at one of the vertices of $\Omega_{1_c + SR + P}$.

Elementary counting shows that there are $2^4 \times 2^8 \times 2^2 \times 4^4$ such deterministic strategies. Furthermore, elements of the strategy matrix are related linearly to the NS correlation $P = \{p(a, b|x, y)\}$,

$$
s_{mz} = \sum_{x, a, c, b, y \in \{0, 1\}} \delta_{x, X(m)} \times \delta_{c, C(m, a)} \times \delta_{y, Y(c)}
$$
$$
\times \ \delta_{z, Z(c, b)} \times p(a, b|x, y).
\tag{9}
$$

For a nonnegative payoff, the game matrix enforces some of the entries of $S \equiv (s_{mz})$ to be zero. Since Eq. (9) is linear, one can solve these equality constraints to see what restrictions are imposed on the NS correlation $P(a, b|x, y)$. By brute-forcing through all the deterministic strategies and by symbolic programming, we were able to verify that for each vertex of $\Omega_{1_c + SR + P}$, the positivity of average payoff imposes the conditions in Eq. (3) in the main manuscript (or its local reversible relabeling) to the NS correlation $\{p(a, b|x, y)\}$. This proves the claim that among all 2-input-2-output correlations, only Hardy's nonlocal correlations can provide a positive payoff in the DMH game. ∎

Although it is known that a two-qubit maximally entangled state does not exhibit Hardy's nonlocality [39,40], still Theorem 3 is not sufficient to make a claim that such a state shared between Alice and Bob cannot lead to a strategy yielding strictly positive payoff in the DMH game. Increasing the cardinality of the input-output sets $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{A}$, $\mathcal{B}$, Alice and Bob can generate a more general NS correlation and then try to utilize this correlation to assist the 1-cbit channel to obtain a nonzero payoff in the DMH game. However, our next result proves a no-go to this aim.

Theorem 4. Two-qubit maximally entangled state together with 1-bit perfect classical channel from Alice to Bob does not result in a strategy ensuring a strictly positive average payoff in the DMH game.

Proof. Let Alice and Bob share the two-qubit maximally entangled state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Note that the existence of a strategy involving SR that yields an advantage in winning the game necessarily implies the existence of a strategy without involving any SR. We now prove that such a strategy does not exist. Without loss of generality, we can assume that Alice does a two-outcome measurement depending on the classical message m she receives, i.e., she performs $\{E_0^{(m)}, E_1^{(m)} \,|\, E_0^{(m)} + E_1^{(m)} = \mathbb{I}\}$ for $m \in \{1, \dots, 4\}$. Alice then communicates $c \in \{0, 1\}$ if the outcome $E_c^{(m)}$ clicks. Based on the communicated bit c, Bob performs a four-outcome measurement, and based on the measurement outcome, he chooses a box. Bob's measurement is denoted by $\{N_z^{(c)}\}_{z=1}^4$ with $\sum_{z=1}^4 N_z^{(c)} = \mathbb{I}$ when the communication $c \in \{0, 1\}$ is received. This leads to the strategy matrix with elements

$$
s_{mz} = \sum_{c=0}^1 s_{mzc} := \sum_{c=0}^1 \operatorname{Tr}\left[|\phi^+\rangle\langle\phi^+| \left(E_c^{(m)} \otimes N_z^{(c)} \right) \right],
$$
$$
= \frac{1}{2} \sum_{c=0}^1 \operatorname{Tr}\left[E_c^{(m)} N_z^{*(c)} \right].
\tag{10}
$$

The aim is to find positive operator-valued measure (POVM) elements $E_c^{(m)}$ and $N_z^{*(c)}$ such that the resulting strategy yields a positive payoff in the DMH game. For simplicity, we drop the notation "$*$" in $N_z^{*(c)}$, keeping in mind that if we do indeed find a solution for Eq. (10), we need to complex conjugate the matrices $N_z^{(c)}$. We also note that if $N_z^{(c)}$ forms a measurement,

so does $N_z^{*(c)}$. Therefore, a strictly positive payoff in the DMH game demands the following conditions to be satisfied:

$$\text{Tr}\left[E_0^{(1)}N_4^{(0)}\right] = 0 = \text{Tr}\left[E_1^{(1)}N_4^{(1)}\right], \tag{11a}$$

$$\text{Tr}\left[E_0^{(2)}N_1^{(0)}\right] = 0 = \text{Tr}\left[E_1^{(2)}N_1^{(1)}\right], \tag{11b}$$

$$\text{Tr}\left[E_0^{(2)}N_3^{(0)}\right] = 0 = \text{Tr}\left[E_1^{(2)}N_3^{(1)}\right], \tag{11c}$$

$$\text{Tr}\left[E_0^{(3)}N_3^{(0)}\right] = 0 = \text{Tr}\left[E_1^{(3)}N_3^{(1)}\right], \tag{11d}$$

$$\text{Tr}\left[E_0^{(3)}N_4^{(0)}\right] = 0 = \text{Tr}\left[E_1^{(3)}N_4^{(1)}\right], \tag{11e}$$

$$\text{Tr}\left[E_0^{(4)}N_1^{(0)}\right] = 0 = \text{Tr}\left[E_1^{(4)}N_1^{(1)}\right], \tag{11f}$$

$$\text{Tr}\left[E_0^{(4)}N_2^{(0)}\right] = 0 = \text{Tr}\left[E_1^{(4)}N_2^{(1)}\right], \tag{11g}$$

$$\sum_{c=0}^{1}\text{Tr}\left[E_c^{(3)}N_1^{(c)}\right] > \sum_{c=0}^{1}\text{Tr}\left[E_c^{(1)}N_1^{(c)}\right]. \tag{12}$$

The inequality (12) can further be written as

$$\text{Tr}\left[E_0^{(3)}\left(N_1^{(0)} - N_1^{(1)}\right)\right] > \text{Tr}\left[E_0^{(1)}\left(N_1^{(0)} - N_1^{(1)}\right)\right]. \tag{13}$$

Note that if a solution of POVMs exists satisfying (11) and (13), then there also exists a solution with $\{E_0^{(m)}, E_1^{(m)}\}$ being projective measurements $\forall m \in \{1, \ldots, 4\}$. This can be argued easily by expanding all the operators $\{E_0^{(m)}, E_1^{(m)}\}$ in the spectral form. For example, in inequality (13) on the left-hand side, we can choose the projector formed by the eigenvector of $E_0^{(3)}$ giving the maximum value of the trace. Similarly, on the right-hand side, we can choose the projector formed by the eigenvector of $E_0^{(1)}$ giving the lowest value of trace. Moreover, notice than for Eqs. (11) all the projectors corresponding to different eigenvalues of $\{E_0^{(m)}, E_1^{(m)}\}$ must satisfy the Eqs. (11) individually. Thus, we start by assuming all measurements $\{E_0^{(m)}, E_1^{(m)}\}$ are projective. In particular, let

$$E_0^{(3)} = |\psi\rangle\langle\psi|, \quad E_1^{(3)} = |\psi^\perp\rangle\langle\psi^\perp|,$$
$$E_0^{(4)} = |\phi\rangle\langle\phi|, \quad E_1^{(4)} = |\phi^\perp\rangle\langle\phi^\perp|.$$

Thus, looking into the left-hand side of Eqs. (11d) and (11g), we have

$$N_1^{(0)} = p_1|\phi^\perp\rangle\langle\phi^\perp|, \quad N_2^{(0)} = p_2|\phi^\perp\rangle\langle\phi^\perp|,$$
$$N_3^{(0)} = p_3|\psi^\perp\rangle\langle\psi^\perp|, \quad N_4^{(0)} = p_4|\psi^\perp\rangle\langle\psi^\perp|,$$

with $0 \leqslant p_1, \ldots, p_4 \leqslant 1$. Since $\{N_1^{(0)}, \ldots, N_4^{(0)}\}$ form a measurement, we therefore have

$$(p_1 + p_2)|\phi^\perp\rangle\langle\phi^\perp| + (p_3 + p_4)|\psi^\perp\rangle\langle\psi^\perp| = \mathbb{I},$$

which will be satisfied *if and only if*

$$p_1 + p_2 = p_3 + p_4 = 1, \ \& \ |\phi\rangle = |\psi^\perp\rangle.$$

A similar argument can be made for the other measurement of Bob, and we get the following:

$$N_1^{(0)} = p_1|\psi\rangle\langle\psi|, \qquad N_1^{(1)} = q_1|\psi^\perp\rangle\langle\psi^\perp|, \tag{14a}$$

$$N_2^{(0)} = p_2|\psi\rangle\langle\psi|, \qquad N_2^{(1)} = q_2|\psi^\perp\rangle\langle\psi^\perp|, \tag{14b}$$

$$N_3^{(0)} = p_3|\psi^\perp\rangle\langle\psi^\perp|, \qquad N_3^{(1)} = q_3|\psi\rangle\langle\psi|, \tag{14c}$$

$$N_4^{(0)} = p_4|\psi^\perp\rangle\langle\psi^\perp|, \qquad N_4^{(1)} = q_4|\psi\rangle\langle\psi|, \tag{14d}$$

where $0 \leqslant q_1, \ldots, q_4 \leqslant 1$ and $q_1 + q_2 = q_3 + q_4 = 1$. Now, let $E_0^{(1)} = |\chi\rangle\langle\chi|$ and $E_1^{(1)} = |\chi^\perp\rangle\langle\chi^\perp|$. From the left-hand side of Eqs. (11a) and (14d) we must have $p_4|\psi^\perp\rangle\langle\psi^\perp| = s_1|\chi^\perp\rangle\langle\chi^\perp|$, with $0 \leqslant s_1 \leqslant 1$. A solution is $p_4 = s_1 > 0$ and $|\psi\rangle = |\chi\rangle$, which implies $E_0^{(1)} = E_0^{(3)}$ and $E_1^{(1)} = E_1^{(3)}$. This further tells us that Alice uses the same strategy for $m = 1$ and $m = 3$, and this will never yield a positive payoff. The only other solution is

$$p_4 = s_1 = 0 \Rightarrow N_4^{(0)} = 0 \Rightarrow N_3^{(0)} = |\psi^\perp\rangle\langle\psi^\perp|.$$

Similarly, we can argue that we must have $N_3^{(1)} = |\psi\rangle\langle\psi|$. Now, from Eq. (11c) we have $E_0^{(2)} \propto |\psi\rangle\langle\psi|$ and $E_1^{(2)} \propto |\psi^\perp\rangle\langle\psi^\perp| \Rightarrow E_0^{(2)} = |\psi\rangle\langle\psi|$ and $E_1^{(2)} = |\psi^\perp\rangle\langle\psi^\perp|$. From Eqs. (11b) and (14a), the only solution is $N_1^{(0)} = N_1^{(1)} = 0$, which leads to violation of the inequality in Eq. (12). Thus, a consistent solution cannot be found satisfying the conditions (11) and (12). This completes the proof. ∎

This theorem has an interesting implication. It shows that there exists a communication task wherein a nonmaximally pure entangled state can be preferable over the maximally entangled one even when the entanglement of the former is vanishingly zero. More formally, we can deduce the following corollary.

Corollary 1. For every nonmaximally entangled state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ there exists a strategy matrix S_ψ such that $S_\psi \in \Omega_{1_c+SR+|\psi\rangle}(4, 4)$ but $S_\psi \notin \Omega_{1_c+SR+|\phi^+\rangle}(4, 4)$.

Here, $|\phi^+\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$ and $\Omega_{1_c+SR+|\chi\rangle}(4, 4)$ denotes the convex set of strategy matrices simulable with 1-cbit communication from Alice to Bob when the communication channel is further assisted with preshared quantum state $|\chi\rangle$ and unlimited shared randomness. Corollary 1 follows when results of Theorems 2 and 4 are combined with the fact that all nonmaximally pure entangled states exhibit Hardy's nonlocality [39]. More precisely, for every nonmaximally entangled state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ there exists a noisy channel of the form $S_\mathbb{H}$ that can be perfectly simulated with 1-bit perfect classical channel when assisted with the state $|\psi\rangle$, but not with $|\phi^+\rangle$. A more detailed discussion on the implications of theses results is presented in Appendix B.

IV. DISCUSSION AND OUTLOOK

The seminal quantum superdense coding protocol is worth mentioning as it shows that quantum entanglement, preshared between a sender and a receiver, can increase the classical communication capacity of a quantum system [41] (see also Ref. [42]). While in quantum superdense coding protocol a quantum channel is considered, here we show that quantum entanglement can even empower the communication utility of a perfect classical channel. As already mentioned, such an advantage can be better understood in zero-error and reverse zero-error communication setup. While such an advantage of quantum entanglement is already known (see Proposition 21 in Ref. [29] and see also Refs. [43–45]), the full picture of entanglement assistance is not well understood. Our Theorem 4 proves a nontrivial result to this direction. It shows that in the correlation-assisted reverse zero-error coding scenario,

there exist noisy channel simulation tasks wherein nonmaximally entangled states, with arbitrarily less entanglement, are preferable over a maximally entangled state. In the resource theory of quantum entanglement [46], where local operations and classical communication (LOCC) are considered to be free, a maximally entangled state is more useful than a nonmaximally entangled one, and a deterministic LOCC transformation is always possible from the former to the latter [47]. Our work, however, establishes that within the operational paradigm of local operations and *limited* classical communication the structures of entangled resources are quite complex to characterize.

A nonlocal advantage is also known in a variant of the communication scenario known as the communication complexity problem [13]. In such a scenario, Bob's goal is not to determine Alice's data $\mathcal{M}$ but to determine some information that is a function of $\mathcal{M}$ in a way that may depend on the other data $\mathcal{N}$ that resides with Bob while $\mathcal{N}$ is unknown to Alice. In that sense, our scenario is closer to the standard framework of Shannon [48] (and considered by Holevo in quantum setup [49]), where at Bob's end, no further data set $\mathcal{N}$ is considered, albeit in a single-shot setup.

In conclusion, the present work establishes exotic uses of quantum entanglement in zero-error information theory [31] (see also Ref. [32]) whose motivation arises from the fact that in many real-world critical applications, no errors can be tolerated, and in practice, the communication channel can only be available for a finite number of times. In particular, we show that quantum correlations exhibiting Hardy's nonlocality can empower the communication *utility* of a perfect classical communication channel. In Appendix C we show that similar results can be obtained by considering the generalization of Hardy's nonlocality argument as proposed by Cabello [30]. Our work also motivates many questions for future study. For instance, it would be interesting to see whether any nonlocal correlation can be made useful as a communication resource in the sense discussed here. It will also be interesting to see whether maximally entangled states of higher dimensions provide some advantage in the DMH game. More generally, characterizing the set $\Omega_{n_c + SR + \chi}(|\mathcal{M}|, |\mathcal{Z}|)$ for an arbitrary quantum state $|\chi\rangle$ would be very interesting.

APPENDIX A: HARDY'S NONLOCALITY ARGUMENT

Hardy's argument is a popular method to check the Bell nonlocality of a given NS correlation [20]. A 2-input-2-output NS correlation $\mathbb{H} \equiv \{h(a, b|x, y)\}$ with $a, b, x, y \in \{0, 1\}$ will exhibit Bell nonlocality if they satisfy the constraints

$$h(0, 0|0, 0) > 0, \quad \text{(A1a)}$$

$$h(0, 1|0, 1) = 0, \quad \text{(A1b)}$$

$$h(1, 0|1, 0) = 0, \quad \text{(A1c)}$$

$$h(0, 0|1, 1) = 0. \quad \text{(A1d)}$$

Recall that the "Bell locality" condition demands that such a correlation $\{h(a, b|x, y)\}$ must be factorizable in the form

$$h(a, b|x, y) = \int_{\lambda \in \Lambda} d\lambda \mu(\lambda) p_A(a|x, \lambda) p_B(b|y, \lambda),$$
$$\times \ \forall a, b, x, y, \quad \text{(A2)}$$

where $\lambda \in \Lambda$ is the local hidden variable; $\mu(\lambda)$ is the distribution of the hidden variable over Λ, which according to freedom of choice is assumed to be independent of Alice's and Bob's choices of inputs, i.e., $\mu(\lambda|x, y) = \mu(\lambda)$; and $p_x(i|j, \lambda)$ is the probability that party X observes outcome i given that they have performed measurement j when the hidden state is λ. Applying Eq. (A2) to Eq. (A1a), we can say that there exists a set $\tilde{\Lambda} \subseteq \Lambda$ of nonzero measures with respect to Λ which can be defined as

$$\tilde{\Lambda} \equiv \{\lambda \in \lambda | \mu(\lambda) > 0, p_A(0|0, \lambda) > 0, p_B(0|0, \lambda) > 0\}.$$

Now, Eq. (A2) can be written as

$$h(a, b|x, y) = \int_{\lambda \in \tilde{\Lambda}} d\lambda \mu(\lambda) p_A(a|x, \lambda) p_B(b|y, \lambda)$$
$$+ \int_{\lambda \in \tilde{\Lambda}^c} d\lambda \mu(\lambda) p_A(a|x, \lambda) p_B(b|y, \lambda), \quad \text{(A3)}$$

where $\tilde{\Lambda}^c$ is defined as the complement of $\tilde{\Lambda}$ with respect to Λ. It can be noted that since both the terms appearing on the right-hand side of Eq. (A3) are nonnegative, if the left-hand side of Eq. (A3) is 0 then both the individual terms on the right-hand side must be 0. Thus, from Eq. (A1b) we must have

$$\int_{\lambda \in \tilde{\Lambda}} d\lambda \mu(\lambda) p_A(0|0, \lambda) p_B(1|1, \lambda) = 0,$$

$$\Rightarrow p_B(1|1, \lambda) = 0 \forall \lambda \in \tilde{\Lambda}.$$

Since $p_A(0|0, \lambda) > 0 \ \& \ \mu(\lambda) > 0 \forall \lambda \in \tilde{\Lambda}$,

$$\Rightarrow p_B(0|1, \lambda) = 1 \forall \lambda \in \tilde{\Lambda}. \quad \text{(A4)}$$

Similarly, from Eq. (A1c) we must have

$$p_A(0|1, \lambda) = 1 \forall \lambda \in \tilde{\Lambda}. \quad \text{(A5)}$$

Thus, from Eq. (A1d) we have

$$\left(\int_{\lambda \in \tilde{\Lambda}^c} + \int_{\lambda \in \tilde{\Lambda}} \right) d\lambda \mu(\lambda) p_A(0|1, \lambda) p_B(0|1, \lambda) = 0,$$

$$\Rightarrow \int_{\lambda \in \tilde{\Lambda}^c} d\lambda \mu(\lambda) p_A(0|1, \lambda) p_B(0|1, \lambda)$$

$$+ \int_{\lambda \in \tilde{\Lambda}} d\lambda \mu(\lambda) = 0. \quad \text{(A6)}$$

This is impossible due to the fact that $\mu(\lambda) > 0 \, \forall \lambda \in \tilde{\Lambda}$, which guarantees a strictly positive contribution from the second term on the left-hand side in Eq. (A6). Thus, a Bell local NS correlation does not satisfy all the constraints ((A1a)–(A1d)). Accordingly, a NS correlation satisfying all the constraints ((A1a)–(A1d)) must be Bell nonlocal, and such a correlation is called Hardy's nonlocal correlation.

APPENDIX B: ANALYSIS AND IMPLICATIONS OF THEOREMS 3 AND 4 AND COROLLARY 1

In the framework of resource theory, quantum entanglement is considered to be a useful resource under the free operation of LOCC. In this operational paradigm, Nielson's result [47] proves that a state $|\psi\rangle$ can be transferred to another state $|\phi\rangle$ under LOCC *if and only if* λ_ψ is majorized by λ_ϕ, where λ'_is are Schmidt coefficient of the respective states. As it turns out, for the $\mathbb{C}^2 \otimes \mathbb{C}^2$ system any nonmaximally entangled state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ can be deterministically prepared from the maximally entangled state $|\phi^+\rangle$ using 1-bit classical communication from Alice to Bob, whereas the reverse transformation is not possible. This implies that maximally entangled states $|\phi^+\rangle$ will surpass any nonmaximally entangled state $|\psi\rangle$ in all possible tasks if classical communication is considered as a free resource.

However, the situation is quite different if classical communication is treated as a costly resource. This becomes quite evident in the reverse zero-error channel simulation task considered in our work. For instance, as discussed in our Corollary 1, there exists a 4-input-4-output noisy channel S_ψ of the form $S_\mathbb{H}$ of Eq. (4) of the main manuscript, which can be perfectly simulated by 1-bit classical communication from Alice to Bob with the assistance of the nonmaximally entangled state $|\psi\rangle$. By performing suitable local measurements on the state $|\psi\rangle$ one first obtain a 2-input-2-output NS correlation that exhibits Hardy's nonlocality, which according to Theorem 3 is necessary to simulate a channel of the form $S_\mathbb{H}$ when communication from Alice to Bob is limited to 1-cbit. At this point it should be noted that 2-input-2-output correlation obtained from the state $|\phi^+\rangle$ cannot exhibit Hardy's nonlocality [39,40]. However, this itself does not discard the possibility of simulating a noisy channel of the form $S_\mathbb{H}$ with 1-cbit channel from Alice to Bob in assistance with $|\phi^+\rangle$. From the state $|\phi^+\rangle$ one can try to come up with higher input-output NS correlation by performing local POVM on $|\phi^+\rangle$, which can further be used for the targeted simulation task. Our Theorem 4, however, proves that this, in fact, is not possible. Given the state $|\phi^+\rangle$ one might aim to convert it to the state $|\psi\rangle$ following Nielson's protocol and then try to simulate the channel S_ψ. But, this also is not possible since the available 1-cbit channel is consumed at the state transformation step and hence makes the simulation impossible. Therefore, our results establish a nontrivial advantage of a nonmaximally entangled state over the maximally entangled state in the paradigm of limited classical communication scenario even when the nonmaximally entangled state contains an arbitrarily small amount of entanglement. Of course, there might be the possibility of a different 4-input-4-output noisy channel which can be simulated by a 1-cbit channel in assistance with $|\phi^+\rangle$,
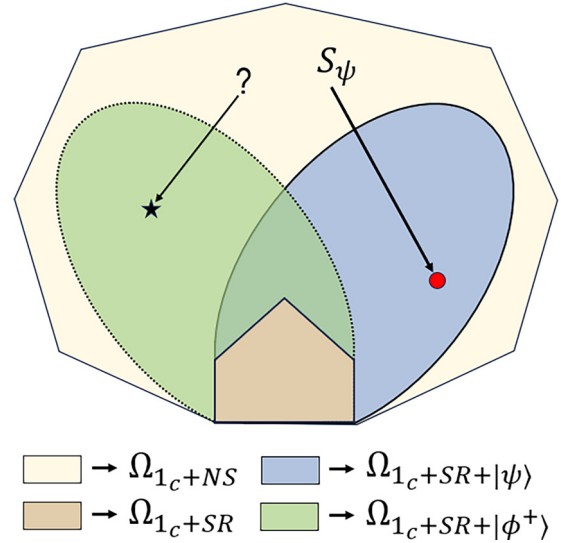

FIG. 3. Set of 4-input-4-output channels simulable by 1 bit of classical communication in assistance with different kinds of preshared resources. $\Omega_{1_c+SR}(4, 4)$ denotes the set of channels simulable by 1-bit classical communication when an unlimited amount of shared randomness is available as assistance. $\Omega_{1_c+NS}(4, 4)$ corresponds to the set when the communication line is assisted with an arbitrary amount of NS correlation. $\Omega_{1_c+SR+|\chi\rangle}(4, 4)$ denotes the set of channels simulable when an arbitrary amount of SR along with the state $|\chi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is available as assistance. Two convex sets are depicted for maximally and nonmaximally entangled states. The red dot denotes the channel S_ψ such that $S_\psi \in \Omega_{1_c+SR+|\psi\rangle}$ but $S_\psi \notin \Omega_{1_c+SR+|\phi^+\rangle}$. The star denotes a possible channel that lies within $\Omega_{1_c+SR+|\phi^+\rangle}$, but does not belong to the set $\Omega_{1_c+SR+|\psi\rangle}$.

but not in assistance with $|\psi\rangle$. Although we believe that such channels should exist, we could not come up with explicit examples, and leave the question for future research. The aforesaid discussion is depicted in Fig. 3.

APPENDIX C: ADVANTAGE OF CABELLO'S NONLOCAL CORRELATION

In the main manuscript, we have shown that Hardy's nonlocal correlation provides a communication advantage in the DMH game. Here, we extend this result for nonlocal correlations exhibiting Cabello's nonlocality [30], which can be thought of as a generalization of Hardy's nonlocality argument. As shown by Cabello, a binary input-output NS correlation $\{c(a, b|x, y)\}$ satisfying the constraints

$$c(0, 0|0, 0) := c_0 > c_3, \quad c(0, 1|0, 1) := c_5 = 0,$$
$$c(1, 0|1, 0) := c_{10} = 0, \quad c(0, 0|1, 1) := c_3,$$
$$\text{with } c(a, b|x, y) := c_{a \times 2^3 + b \times 2^2 + x \times 2^1 + y \times 2^0} \tag{C1}$$

must be nonlocal in nature. To establish the communication advantage of such a correlation, we consider a variant of the DMH game which will be denoted as DMH′ and specified by

the payoff matrix

$$
\mathcal{G}_{\text{DMH}'} \equiv
\begin{array}{c|cccc}
\mathcal{M}\backslash\mathcal{Z} & 1 & 2 & 3 & 4 \\
\hline
1 & -1 & 0 & 0 & -\infty \\
2 & -\infty & 0 & -1 & 0 \\
3 & +1 & 0 & -\infty & -\infty \\
4 & -\infty & -\infty & 0 & 0
\end{array}
\tag{C2}
$$

Our next result limits the success probability of this game when 1-cbit communication from Alice to Bob is allowed along with an unlimited amount of shared randomness.

Theorem 5. The average payoff of the DMH$'$ game is upper bounded by zero while following a strategy from the set $\Omega_{1_c+SR}(4,4)$, *i.e.*, $\mathcal{P}(S) \leqslant 0$, $\forall S \in \Omega_{1_c+SR}(4,4)$.

Proof. The proof follows similar reasoning to Theorem 1 in the main manuscript. The only extreme strategies which will not trigger the bomb are

$$
\left\{
S_{s1}^e := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\;
S_{s2}^e := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\right.
$$

$$
S_{s3}^e := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\;
S_{s4}^e := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},
$$

$$
S_{s5}^e := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\;
S_{s6}^e := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},
$$

$$
S_{s7}^e := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}\;
S_{s8}^e := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
\left.
S_{s9}^e := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\right\}
\tag{C3}
$$

As all the extreme strategies result in either zero or negative payoff, the optimal payoff with 1-bit classical communication and shared randomness is simply zero. ∎

Next, we proceed to establish the communication advantage of nonlocal correlation exhibiting Cabello's nonlocality.

Theorem 6. A strictly positive average payoff in the DMH$'$ game is achievable with a 1-bit perfect classical channel from Alice to Bob when the channel is assisted with a 2-input-2-output Cabello's nonlocal correlation.

Proof. Once again the proof structure is the same as Theorem 2 of the main manuscript. **Alice's action:**

(i) Depending on $m \in \mathcal{M}$, Alice computes her input in the NS box. For $m \in \{1, 3\}$ she chooses $x = 0$, otherwise she chooses $x = 1$.

(ii) Based on the tuple $(m, a) \in \mathcal{M} \times \mathcal{A}$, she communicates to Bob. She sends $c = 0$ to Bob when $(m, a) \in \{(1, 1), (2, 1), (3, 0), (3, 1)\}$, else she sends $c = 1$.

Bob's action:

(i) The communicated bit from Alice is used as input in Bob's part of the NS box.

(ii) Depending on the tuple $(c, b) \in C \times \mathcal{B}$, he chooses the box as follows: $(0, 0) \mapsto 1$, $(0, 1) \mapsto 2$, $(1, 0) \mapsto 3$, $(1, 1) \mapsto 4$.

The above strategy with 1-bit communication and 2-input-2-output Cabello's correlation $\{c(ab|xy)\}$ leads to the strategy matrix

$$
S_H \equiv
\begin{array}{c|cccc}
\mathcal{M}\backslash\mathcal{Z} & 1 & 2 & 3 & 4 \\
\hline
1 & c_8 & c_{12} & c_1 & 0 \\
2 & 0 & c_{14} & c_3 & c_7 \\
3 & c_0 + c_8 & c_4 + c_{12} & 0 & 0 \\
4 & 0 & 0 & c_3 + c_{11} & c_7 + c_{15}
\end{array}
\tag{C4}
$$

As evident from the payoff matrix (C2) and the strategy (C4), a box containing a bomb will never be opened. Furthermore, assuming Charlie's choice to be completely random, we have the average payoff

$$
\mathcal{P}(S_H) = \tfrac{1}{4}\,\text{Tr}\left[\mathcal{G}_{\text{DMH}'}^T\, S_H\right] = \tfrac{1}{4}(c_0 - c_3) > 0.
\tag{C5}
$$

This completes the proof. ∎

[1] J. S. Bell, On the Einstein Podolsky Rosen paradox, Phys. Phys. Fiz. **1**, 195 (1964); On the problem of hidden variables in quantum mechanics, Rev. Mod. Phys. **38**, 447 (1966).

[2] N. D. Mermin, Hidden variables and the two theorems of John Bell, Rev. Mod. Phys. **65**, 803 (1993).

[3] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Rev. Mod. Phys. **86**, 419 (2014).

[4] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed experiment to test local hidden-variable theories, Phys. Rev. Lett. **23**, 880 (1969).

[5] S. J. Freedman and J. F. Clauser, Experimental test of local hidden-variable theories, Phys. Rev. Lett. **28**, 938 (1972).

[6] A. Aspect, P. Grangier, and G. Roger, Experimental tests of realistic local theories via Bell's theorem, Phys. Rev. Lett. **47**, 460 (1981).

[7] A. Aspect, P. Grangier, and G. Roger, Experimental realization of Einstein-Podolsky-Rosen-Bohm *Gedankenexperiment*: A new violation of Bell's inequalities, Phys. Rev. Lett. **49**, 91 (1982).

[8] M. Żukowski, A. Zeilinger, M. A. Horne, and A. K. Ekert, "Event-ready-detectors" Bell experiment via entanglement swapping, Phys. Rev. Lett. **71**, 4287 (1993).

[9] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Violation of Bell's inequality under strict Einstein locality conditions, Phys. Rev. Lett. **81**, 5039 (1998).

[10] A. K. Ekert, Quantum cryptography based on Bell's theorem, Phys. Rev. Lett. **67**, 661 (1991).

[11] J. Barrett, L. Hardy, and A. Kent, No signaling and quantum key distribution, Phys. Rev. Lett. **95**, 010503 (2005).

[12] U. Vazirani and T. Vidick, Fully device-independent quantum key distribution, Phys. Rev. Lett. **113**, 140501 (2014).

[13] H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Nonlocality and communication complexity, Rev. Mod. Phys. **82**, 665 (2010).

[14] S. Pironio, A. Acín, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Random numbers certified by Bell's theorem, Nature (London) **464**, 1021 (2010).

[15] E. G. Cavalcanti and H. M. Wiseman, Bell nonlocality, signal locality and unpredictability (or what Bohr could have told Einstein at Solvay had he known about Bell experiments), Found Phys. **42**, 1329 (2012).

[16] A. Chaturvedi and M. Banik, Measurement-device-independent randomness from local entangled states, Europhys. Lett. **112**, 30003 (2015).

[17] R. Colbeck and R. Renner, Free randomness can be amplified, Nat. Phys. **8**, 450 (2012).

[18] D. M. Greenberger, M. A. Horne, and A. Zeilinger, Going beyond Bell's theorem, in *Bell's Theorem, Quantum Theory and Conceptions of the Universe*, edited by M. Kafatos, Fundamental Theories of Physics (Springer, Dordrecht, 1989), Vol. 37.

[19] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, Bell's theorem without inequalities, Am. J. Phys. **58**, 1131 (1990).

[20] L. Hardy, Quantum mechanics, local realistic theories, and Lorentz-invariant realistic theories, Phys. Rev. Lett. **68**, 2981 (1992).

[21] N. D. Mermin, Quantum mysteries refined, Am. J. Phys. **62**, 880 (1994).

[22] Here the CH inequality refers to the inequality derived in: J. F. Clauser and M. A. Horne, Experimental consequences of objective local theories, Phys. Rev. D **10**, 526 (1974).

[23] S. Das, M. Banik, A. Rai, MD R. Gazi, and S. Kunkri, Hardy's nonlocality argument as a witness for postquantum correlations, Phys. Rev. A **87**, 012112 (2013).

[24] A. Mukherjee, A. Roy, S. S. Bhattacharya, S. Das, Md. R. Gazi, and M. Banik, Hardy's test as a device-independent dimension witness, Phys. Rev. A **92**, 022302 (2015).

[25] H.-W. Li, M. Pawłowski, R. Rahaman, G.-C. Guo, and Z.-F. Han, Device- and semi-device-independent random numbers based on noninequality paradox, Phys. Rev. A **92**, 022327 (2015).

[26] R. Ramanathan, M. Horodecki, H. Anwer, S. Pironio, K. Horodecki, M. Grünfeld, S. Muhammad, M. Bourennane, and P. Horodecki, Practical no-signalling proof randomness amplification using Hardy paradoxes and its experimental implementation, arXiv:1810.11648.

[27] A. Rai, M. Pivoluska, M. Plesch, S. Sasmal, M. Banik, and S. Ghosh, Device-independent bounds from Cabello's nonlocality argument, Phys. Rev. A **103**, 062219 (2021).

[28] A. Rai, M. Pivoluska, S. Sasmal, M. Banik, S. Ghosh, and M. Plesch, Self-testing quantum states via nonmaximal violation in Hardy's test of nonlocality, Phys. Rev. A **105**, 052227 (2022).

[29] T. S. Cubitt, D. Leung, W. Matthews, and A. Winter, Zero-error channel capacity and simulation assisted by non-local correlations, IEEE Trans. Info. Theory **57**, 5509 (2011).

[30] A. Cabello, Bell's theorem with and without inequalities for the three-qubit Greenberger-Horne-Zeilinger and W states, Phys. Rev. A **65**, 032108 (2002).

[31] C. Shannon, The zero error capacity of a noisy channel, IEEE Trans. Inform. Theory **2**, 8 (1956).

[32] J. Korner and A. Orlitsky, Zero-error information theory, IEEE Trans. Inform. Theory **44**, 2207 (1998).

[33] Although uses of finite payoffs are common in game theory, the framework is quite flexible to incorporate a wide range of situations including those with $\pm\infty$ payoffs. Such extreme payoffs generally appear in financial trading games, gambling games, and decision-making games. A classic example is the "Stag Hunt" game. If the hunters are part of a community that relies on hunting for survival then the failure to catch a stag leads to starvation and the payoff for each hunter could be considered $-\infty$.

[34] P. E. Frenkel and M. Weiner, Classical information storage in an n-level quantum system, Commun. Math. Phys. **340**, 563 (2015).

[35] M. Dall'Arno, S. Brandsen, A. Tosini, F. Buscemi, and V. Vedral, No-hypersignaling principle, Phys. Rev. Lett. **119**, 020401 (2017).

[36] J. L. Cereceda, Quantum mechanical probabilities and general probabilistic constraints for Einstein-Podolsky-Rosen-Bohm experiments, Found. Phys. Lett. **13**, 427 (2000).

[37] R. Rabelo, L. Y. Zhi, and V. Scarani, Device-independent bounds for Hardy's experiment, Phys. Rev. Lett. **109**, 180401 (2012).

[38] B. S. Cirel'son, Quantum generalizations of Bell's inequality, Lett. Math. Phys. **4**, 93 (1980).

[39] S. Goldstein, Nonlocality without inequalities for almost all entangled states for two particles, Phys. Rev. Lett. **72**, 1951 (1994).

[40] T. F. Jordan, Testing Einstein-Podolsky-Rosen assumptions without inequalities with two photons or particles with spin 1/2, Phys. Rev. A **50**, 62 (1994).

[41] C. H. Bennett and S. J. Wiesner, Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states, Phys. Rev. Lett. **69**, 2881 (1992).

[42] C. H. Bennett, P. W. Shor, J. A. Smolin, and A. V. Thapliyal, Entanglement-assisted classical capacity of noisy quantum channels, Phys. Rev. Lett. **83**, 3081 (1999).

[43] T. S. Cubitt, D. Leung, W. Matthews, and A. Winter, Improving zero-error classical communication with entanglement, Phys. Rev. Lett. **104**, 230503 (2010).

[44] P. E. Frenkel and M. Weiner, On entanglement assistance to a noiseless classical channel, Quantum **6**, 662 (2022).

[45] R. K. Patra, S. G. Naik, E. P. Lobo, S. Sen, T. Guha, S. S. Bhattacharya, M. Alimuddin, and M. Banik, Classical analogue of quantum superdense coding and communication advantage of a single quantum, arXiv:2202.06796.

[46] M. B. Plenio and S. Virmani, An introduction to entanglement measures, Quant. Inf. Comput. **7**, 1 (2007).

[47] M. A. Nielsen, Conditions for a class of entanglement transformations, Phys. Rev. Lett. **83**, 436 (1999).

[48] C. E. Shannon, A mathematical theory of communication, Bell Syst. Tech. J. **27**, 379 (1948).

[49] A. S. Holevo, Bounds for the quantity of information transmitted by a quantum communication channel, Problems Inform. Transmission **9**, 177 (1973).