

Quantum limit to subdiffraction incoherent optical imaging. III. Numerical analysisXiao-Jie Tan¹ and Mankei Tsang^{1,2,*}¹*Department of Electrical and Computer Engineering, National University of Singapore, 4 Engineering Drive 3, Singapore 117583*²*Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117551*

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To investigate the fundamental limit to far-field incoherent imaging, the prequels to this work [M. Tsang, *Phys. Rev. A* **99**, 012305 (2019); **104**, 052411 (2021)], have studied a quantum lower bound on the error of estimating an object moment and proved a scaling law for the bound with respect to the object size. As the scaling law was proved only in the asymptotic limit of vanishing object size, this work performs a numerical analysis of the quantum bound to verify that the law works well for nonzero object sizes in reality. We also use the numerical bounds to study the optimality of a measurement called spatial-mode demultiplexing (SPADE), showing that SPADE not only follows the scaling but is also numerically close to being optimal, at least for low-order moments.

DOI: [10.1103/PhysRevA.108.052416](https://doi.org/10.1103/PhysRevA.108.052416)**I. INTRODUCTION**

The quantum nature of light imposes fundamental limits to the precision of optical measurements for information processing. To derive the most fundamental quantum limits, Helstrom invented a theory of quantum detection and estimation—also called quantum metrology in modern terminology—and first studied its implications for optics [1]. More recently, this line of research has yielded a pleasant surprise: the quantum limits to incoherent imaging turn out to be far less stringent than previously thought, but still achievable by physical measurements [2,3]. These results have a potential impact on both optical astronomy and fluorescence microscopy [3].

Beyond the assumption of two point sources in the earlier literature, much recent effort has been devoted to the case of multiple point sources. In particular, the prequels to this work [4,5] (henceforth called Paper I and Paper II, respectively) study a quantum Cramér-Rao lower bound on the error of estimating an object moment; see also Ref. [6]. As those works aim to derive the bound for an unknown distribution of many incoherent point sources, the dimension of the parameter space (or equivalently the number of scalar parameters) may be high or even infinite, making the analysis formidable and an exact expression for the bound elusive. In classical statistics, a problem or any concept associated with an infinite-dimensional parameter space is called semiparametric. Even in the classical case, semiparametric problems are difficult and require advanced mathematics [7].

Despite the daunting semiparametric setting of the moment estimation problem, progress has been made—Paper I proposes a scaling law for the quantum bound with respect to the object size, while Paper II proves the scaling law rigorously by using the quantum semiparametric theory developed in

Ref. [8]. The measurement method of spatial-mode demultiplexing (SPADE) can achieve the same scaling [5,9–11], but since the scaling law is proved only in the asymptotic limit of vanishing object size, it remains an open question how well the law holds for nonzero object sizes in reality. Another open question is the gap between the quantum limit and the SPADE error. This question is significant because a small gap would imply that SPADE is almost optimal, while a large gap would leave open the possibility that another measurement can significantly improve moment estimation but is yet to be discovered. The moment estimation problem also has significant implications for more general image reconstruction problems [12] and random displacement models [13], so it is worthwhile to devise a method that can compute quantum bounds more precisely for future studies along these lines.

To verify the scaling law and to investigate the gap between the quantum limit and the SPADE error, this work proposes a numerical method to compute a quantum Cramér-Rao bound for moment estimation. The numerical results demonstrate that the scaling law works well for nonzero object sizes, not just in asymptotics, and the SPADE errors are close to the quantum bounds, at least for low moment orders. For higher-order moments, the results are unfortunately less conclusive.

The proposed method is nontrivial because the bound needs to be valid for an infinite-dimensional parameter space, yet computable with finite resources. To achieve this goal, we use the concept of parametric submodels, which is established in classical semiparametric statistics [7] and generalized for the quantum case by Ref. [8]. Paper II has previously used the submodel concept to prove the scaling law analytically; here the approach is extended to create a numerical method.

Apart from the aforementioned papers, the closest prior work may be Ref. [14] by Bisketzi and coworkers, which numerically computes quantum bounds for the localization of multiple point sources. The most important difference between their method and ours is that, whereas their method works only for a finite number of point sources, our method

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can produce bounds for extended objects modeled by continuous intensity functions. Reference [14] also mentions moment estimation only in passing. Other outstanding prior works that deal with extended objects include Refs. [15–17], although those works assume objects with known shapes and low-dimensional parameter spaces (i.e., few scalar parameters) and have not explicitly shown that their bounds are also valid for semiparametric problems.

II. THEORY

A. Quantum optics

Following Papers I and II, we consider the far-field imaging of one-dimensional spatially incoherent optical sources. A weak-source approximation [2,3,18] results in the density operator

$$\rho = (1 - \epsilon)\tau_0 + \epsilon\tau \tag{2.1}$$

for each temporal optical mode, where τ_0 is the vacuum state, τ is the one-photon state on a Hilbert space \mathcal{H} that models the spatial degrees of freedom, and $\epsilon \ll 1$ is the average photon number per temporal mode. With M temporal modes, the density operator is assumed to be

$$\omega = \rho^{\otimes M}, \tag{2.2}$$

and the average photon number in all modes is then

$$N \equiv M\epsilon. \tag{2.3}$$

For one-dimensional spatially incoherent sources, τ can be expressed as

$$\tau = \int_{-\infty}^{\infty} F(x)e^{-i\hat{k}x}|\psi\rangle\langle\psi|e^{i\hat{k}x}dx, \tag{2.4}$$

where F is the non-negative density function for the sources with the normalization $\int_{-\infty}^{\infty} F(x)dx = 1$, $|\psi\rangle \in \mathcal{H}$ with $\langle\psi|\psi\rangle = 1$ models the point-spread function of the imaging system, and \hat{k} is a self-adjoint momentum operator on \mathcal{H} such that $\exp(-i\hat{k}x)$ models the photon displacement. To be specific, let $|k\rangle$ be the Dirac eigenket of \hat{k} such that $\hat{k}|k\rangle = k|k\rangle$ and $\langle k|k'\rangle = \delta(k - k')$. Then the optical transfer function is given by $\langle k|\psi\rangle$ and the point-spread function for the optical field is given by

$$\psi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \langle k|\psi\rangle dk. \tag{2.5}$$

To obtain tighter quantum bounds, we generalize Papers I and II slightly by assuming that both ϵ and F may depend on unknown parameters. Then it is convenient to use the theory of Poisson states recently proposed in Ref. [19], which simplifies quantum information calculations when the Poisson limit

$$\epsilon \rightarrow 0, \quad M \rightarrow \infty, \quad N \text{ fixed}, \tag{2.6}$$

is taken. A Poisson state is completely specified by the so-called intensity operator

$$\Gamma \equiv N\tau, \tag{2.7}$$

and quantum bounds in the Poisson limit can be computed by considering only Γ rather than ϵ and τ . For the imaging

problem, we can write

$$\Gamma(G) = \int_{-\infty}^{\infty} G(x)e^{-i\hat{k}x}|\psi\rangle\langle\psi|e^{i\hat{k}x}dx, \tag{2.8}$$

where G is an un-normalized object intensity function given by

$$G(x) \equiv NF(x). \tag{2.9}$$

B. Quantum Cramér-Rao bounds

To express the quantum bounds succinctly, we first establish some basic notations. Given an intensity operator Γ , define a weighted inner product between two Hermitian operators h and g as

$$\langle h, g \rangle_{\Gamma} \equiv \text{tr}(h \circ g)\Gamma, \tag{2.10}$$

where

$$h \circ g \equiv \frac{1}{2}(hg + gh) \tag{2.11}$$

denotes the Jordan product. The corresponding norm is defined as

$$\|h\|_{\Gamma} \equiv \sqrt{\langle h, h \rangle_{\Gamma}}. \tag{2.12}$$

Similarly, define the inner product

$$\langle h, g \rangle_G \equiv \int_{-\infty}^{\infty} G(x)h(x)g(x)dx \tag{2.13}$$

between real c-number functions h and g with respect to an object intensity G . The associated norm is then $\|h\|_G \equiv \sqrt{\langle h, h \rangle_G}$.

Now let the set of all possible object intensity functions in the estimation problem be

$$\mathbf{G} \equiv \left\{ G : G(x) \geq 0, \int_{-\infty}^{\infty} G(x)dx < \infty \right\}, \tag{2.14}$$

such that the set of intensity operators is

$$\Gamma(\mathbf{G}) \equiv \{\Gamma(G) : G \in \mathbf{G}\}. \tag{2.15}$$

Let $\beta : \mathbf{G} \rightarrow \mathbb{R}$ be a real scalar parameter of interest and $G_0 \in \mathbf{G}$ be the true object intensity. To define a quantum Cramér-Rao bound for this semiparametric estimation problem, consider a one-dimensional parametric submodel expressed as

$$\boldsymbol{\gamma} = \{\gamma_{\theta} = \Gamma(G_{\theta}) : \theta \in \Theta \subseteq \mathbb{R}\} \subset \Gamma(\mathbf{G}). \tag{2.16}$$

$\boldsymbol{\gamma}$ is required to contain the true $\Gamma(G_0)$, and a parametrization has been chosen to give $\gamma_0 = \Gamma(G_0)$ without loss of generality. The Hermitian score operator S of the submodel is defined in terms of the intensity operator γ_{θ} by

$$\partial\gamma_{\theta} = \gamma_0 \circ S, \quad \partial(\cdot) \equiv \left. \frac{\partial(\cdot)}{\partial\theta} \right|_{\theta=0}. \tag{2.17}$$

The Helstrom bound $H_{\boldsymbol{\gamma}}$ for the submodel of the Poisson states is then [19]

$$H_{\boldsymbol{\gamma}} \equiv \frac{[\partial\beta(G_{\theta})]^2}{\|S\|_{\gamma_0}^2}. \tag{2.18}$$

H_{γ} is a lower bound on the mean-square error E of any unbiased estimator and any measurement for the semiparametric problem; the precise definition of E is in Appendix A. Appendix A also shows that the generalized Helstrom bound \tilde{H} for the semiparametric problem [8] can be expressed as the supremum of all the submodel bounds, viz.,

$$\tilde{H} = \sup_{\gamma} H_{\gamma}, \quad (2.19)$$

such that, for any submodel γ ,

$$E \geq \tilde{H} \geq H_{\gamma}. \quad (2.20)$$

C. Quantum bounds for moment estimation

Suppose that the point-spread function $\psi(x)$ has a normalized width 1, $G(x)$ is centered at $x = 0$, and the support of G is infinite with $\sup\{|x| : G(x) > 0\} \equiv \Delta$. The subdiffraction regime is defined by $\Delta \ll 1$. Let the parameter of interest be a generalized moment of order $\mu \in \mathbb{N}_0 \equiv \{0, 1, 2, \dots\}$, defined as

$$\beta(G) = \int_{-\infty}^{\infty} G(x)b(x)dx, \quad b(x) = x^{\mu} + o(\Delta^{\mu}), \quad (2.21)$$

where $o[f(\Delta)]$ denotes an order smaller than $f(\Delta)$ as $\Delta \rightarrow 0$. Other asymptotic notions $O[f(\Delta)]$ [order at most $f(\Delta)$], $\Omega[f(\Delta)]$ [order at least $f(\Delta)$], and $\Theta[f(\Delta)]$ [order exactly $f(\Delta)$] may also be used in this paper. \tilde{H} is hard to compute, so Paper II seeks an unfavorable submodel such that the submodel bound H_{γ} gives the desired scaling law. Assuming $\beta = \int_{-\infty}^{\infty} F(x)b(x)dx$ in terms of the normalized F and a submodel with a fixed N , Paper II proves a submodel bound given by

$$H_{\gamma} = \frac{\Omega(\Delta^{2\lfloor \mu/2 \rfloor})}{N}, \quad \mu \geq 1. \quad (2.22)$$

For a β given by Eqs. (2.21) in terms of the unnormalized G , on the other hand, the submodel bound in Paper II remains valid after a simple rescaling and can be expressed as

$$H_{\gamma} = \Omega(\Delta^{2\lfloor \mu/2 \rfloor}), \quad (2.23)$$

where we omit the scaling with N for brevity. It can also be shown that Eq. (2.23) remains valid for $\mu = 0$ by considering a submodel with N being a function of θ .

To improve upon the submodel used in Paper II, here we consider a p -dimensional submodel,

$$\Upsilon = \{\gamma_{\theta} = \Gamma(G_{\theta}) : \theta \in \Theta \subseteq \mathbb{R}^p\} \subset \Gamma(\mathbf{G}), \quad (2.24)$$

with $\theta = (\theta_0, \theta_1, \dots, \theta_{p-1})$. Its Helstrom bound is

$$H_{\Upsilon} = u^{\top} K^{-1} u, \quad (2.25)$$

where u is the $p \times 1$ column vector given by

$$u_j \equiv \partial_j \beta(G_{\theta}), \quad \partial_j(\cdot) \equiv \left. \frac{\partial(\cdot)}{\partial \theta_j} \right|_{\theta=0}, \quad (2.26)$$

\top is the transpose, and K is the $p \times p$ Helstrom information matrix given by

$$K_{jk} \equiv \langle S_j, S_k \rangle_{\gamma_0}, \quad (2.27)$$

$$\partial_j \gamma_{\theta} = \gamma_0 \circ S_j. \quad (2.28)$$

H_{Υ} coincides with the maximum Helstrom bound for all the one-dimensional submodels within Υ [8,20]. It follows that

$$\tilde{H} \geq H_{\Upsilon}. \quad (2.29)$$

Instead of guessing an unfavorable submodel, the computation of H_{Υ} allows us to obtain directly the bound for the least favorable submodel within Υ .

The parametrization of the object intensity $G_{\theta}(x)$ affects the bound only through the true intensity $G_0(x)$ and the derivatives $\partial_j G_{\theta}(x)$ at $\theta = 0$, since

$$u_j = \int_{-\infty}^{\infty} \partial_j G_{\theta}(x)b(x)dx, \quad (2.30)$$

$$\partial_j \gamma_{\theta} = \int_{-\infty}^{\infty} \partial_j G_{\theta}(x)e^{-ikx}|\psi\rangle\langle\psi|e^{ikx}dx. \quad (2.31)$$

Taking advantage of this fact, the submodel approach in Paper II picks a $\partial G_{\theta}(x)$ first and then constructs a submodel based on the derivative. The derivative chosen there is

$$\partial G_{\theta}(x) = G_0(x)a_{\mu}(x), \quad (2.32)$$

where $a_{\mu}(x)$ is an orthonormal polynomial with respect to G_0 in the sense of

$$a_j(x) = \sum_{k=0}^j A_{jk}x^k, \quad \langle a_j, a_k \rangle_{G_0} = \delta_{jk}, \quad (2.33)$$

and μ in Eq. (2.32) is chosen to match the order of the moment parameter of interest. We are guaranteed to do no worse than the submodel bound in Paper II if we assume a p -dimensional submodel that obeys

$$\partial_j G_{\theta}(x) = G_0(x)a_j(x), \quad j = 0, \dots, \mu, \dots, p-1, \quad (2.34)$$

such that it includes the one-dimensional submodel used in Paper II. An example parametrization that satisfies all the desired properties is

$$G_{\theta}(x) = \left\{ 1 + \tanh \left[\sum_{j=0}^{p-1} \theta_j a_j(x) \right] \right\} G_0(x). \quad (2.35)$$

The tanh function ensures that each derivative of $G_{\theta}(x)$ agrees with Eq. (2.34) and also $G_{\theta}(x)$ is a valid intensity function as per Eq. (2.14).

To demonstrate the relations among all the aforementioned bounds more explicitly, we write the u in the submodel Helstrom bound H_{Υ} given by Eq. (2.25) as

$$u_j = \int_{-\infty}^{\infty} G_0(x)a_j(x)b(x)dx = \langle a_j, b \rangle_{G_0}, \quad (2.36)$$

and we write the relation between a_j and the score operator S_j through Eqs. (2.28), (2.31), and (2.34) in terms of a linear map Γ_* as

$$S_j = \Gamma_* a_j. \quad (2.37)$$

Γ_* is called a pushforward [5] or a generalized conditional expectation [21–23]. As shown in Appendix B, H_{Υ} can be expressed as

$$H_{\Upsilon} = \max_{s \in \text{span}\{a_0, \dots, a_{p-1}\}} \frac{\langle s, b \rangle_{G_0}^2}{\|\Gamma_* s\|_{\Gamma(G_0)}^2}, \quad (2.38)$$

where span denotes the linear span. This expression makes it clear that H_{Υ} for the p -dimensional submodel cannot be lower than the submodel bound H_{Υ} considered in Paper II, given by

$$H_{\Upsilon} = \frac{\langle a_{\mu}, b \rangle_{G_0}^2}{\|\Gamma_{*} a_{\mu}\|_{\Gamma(G_0)}^2}. \quad (2.39)$$

On the other hand, following Eq. (2.19), the generalized Helstrom bound can be expressed as

$$\tilde{H} = \sup_{s: \|\Gamma_{*} s\|_{\Gamma(G_0)} < \infty} \frac{\langle s, b \rangle_{G_0}^2}{\|\Gamma_{*} s\|_{\Gamma(G_0)}^2}. \quad (2.40)$$

Then

$$E \geq \tilde{H} \geq H_{\Upsilon} \geq H_{\Upsilon} \quad (2.41)$$

holds for Eqs. (2.38)–(2.40). We are, however, unable to prove $H_{\Upsilon} \rightarrow \tilde{H}$ in the $p \rightarrow \infty$ limit, because it is unknown whether $\text{span}\{a_j : j \in \mathbb{N}_0\}$ is dense in the space $\{s : \|\Gamma_{*} s\|_{\Gamma(G_0)} < \infty\}$ needed for Eq. (2.40).

III. NUMERICAL ANALYSIS

A. Method

Assuming a parameter of interest given by Eqs. (2.21), the submodel Helstrom bound H_{Υ} can be computed from Eqs. (2.25), (2.27), (2.28), (2.31), (2.34), and (2.36). The coefficients A_{jk} of the orthonormal polynomials $\{a_j\}$ can be computed by standard linear algebra; see, for example, Appendix A in Paper II. Note that

$$\langle a_j, x^{\mu} \rangle_{G_0} = 0 \text{ if } j > \mu. \quad (3.1)$$

If the parameter of interest β is a simple moment with $b(x) \propto x^{\mu}$, then $u_j = 0$ for $j > \mu$, meaning that β has no local sensitivity to such a θ_j , and it suffices to compute the submodel bound with $j \leq p-1 = \mu$. For a generalized moment with order μ , the diminished sensitivity of β to the higher-order parameters should similarly make our bound somewhat insensitive to a restriction to the parameter dimension p as long as $p-1 \geq \mu$.

We numerically solve the Lyapunov equation in Eq. (2.28) and compute the Helstrom information matrix K in Eq. (2.27) by working in the point-spread-function-adapted (PAD) basis [10,24]

$$\{|\psi_n\rangle = (-i)^n \tilde{a}_n(\hat{k})|\psi\rangle : n = 0, 1, \dots, q-1\}, \quad (3.2)$$

where $\{\tilde{a}_n\}$ are the orthonormal polynomials with respect to $|\langle k|\psi\rangle|^2$. We choose the PAD basis because the matrix entries of γ_0 and $\partial_j \gamma_{\theta}$ in this basis decay quickly for subdiffraction objects, thus helping us reduce the errors due to the necessary truncation of the matrices in the numerics. We use MATLAB on a personal computer to perform the computation and its `lyap` function to solve the Lyapunov equation.

To model the performance of SPADE, we follow Paper II and take $b(x)$ for an even moment order to be

$$b(x) = \frac{1}{D_n^2} |C_n(x)|^2, \quad (3.3)$$

$$C_n(x) \equiv \langle \psi_n | e^{-ikx} | \psi \rangle = D_n x^n + o(\Delta^n). \quad (3.4)$$

D_n can be shown to be real and positive. β is then a generalized moment of order $\mu = 2n$ as per Eqs. (2.21) and can be

estimated by a measurement of the PAD mode $|\psi_n\rangle$. Let the detected photon number in the mode be m_n , which is Poisson under the limit given by Eqs. (2.6). Under the Poisson limit, the expected value of m_n is

$$\mathbb{E}(m_n) = \langle \psi_n | \Gamma(G_0) | \psi_n \rangle \quad (3.5)$$

$$= \int_{-\infty}^{\infty} G_0(x) |C_n(x)|^2 dx = D_n^2 \beta. \quad (3.6)$$

An unbiased estimator is then

$$\check{\beta} = \frac{m_n}{D_n^2}, \quad (3.7)$$

and its mean-square error is

$$E = \frac{\mathbb{E}(m_n)}{D_n^4} = \frac{\beta}{D_n^2} = O(\Delta^{\mu}). \quad (3.8)$$

For an odd moment order, we take

$$b(x) = \frac{1}{D_n D_{n+1}} \text{Re}[C_n(x) C_{n+1}^*(x)], \quad (3.9)$$

such that β is a generalized moment of order $\mu = 2n+1$ and can be estimated by a measurement of the interferometric PAD modes [9,10]

$$|\psi_n^+\rangle \equiv \frac{1}{\sqrt{2}} (|\psi_n\rangle + |\psi_{n+1}\rangle), \quad (3.10)$$

$$|\psi_n^-\rangle \equiv \frac{1}{\sqrt{2}} (|\psi_n\rangle - |\psi_{n+1}\rangle). \quad (3.11)$$

Let the detected photon number in the plus mode be m_n^+ and that in the minus mode be m_n^- and assume again the Poisson limit. Their expected values are respectively given by

$$\mathbb{E}(m_n^+) = \langle \psi_n^+ | \Gamma(G_0) | \psi_n^+ \rangle \quad (3.12)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} G_0(x) |C_n(x) + C_{n+1}(x)|^2 dx, \quad (3.13)$$

$$\mathbb{E}(m_n^-) = \langle \psi_n^- | \Gamma(G_0) | \psi_n^- \rangle \quad (3.14)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} G_0(x) |C_n(x) - C_{n+1}(x)|^2 dx. \quad (3.15)$$

An unbiased estimator is then

$$\check{\beta} = \frac{m_n^+ - m_n^-}{2D_n D_{n+1}}, \quad (3.16)$$

and its error is

$$E = \frac{\mathbb{E}(m_n^+) + \mathbb{E}(m_n^-)}{4D_n^2 D_{n+1}^2} = O(\Delta^{\mu-1}). \quad (3.17)$$

Given the quantum bound H_{Υ} for a certain submodel $\{G_{\theta}\}$ and the SPADE error E at a certain G_0 , the bound for the submodel $\{cG_{\theta}\}$ and the error at cG_0 , where c is a known constant, are scaled simply by c . Since our goal here is to study the ratio between H_{Υ} and E and their scalings with Δ , there is no loss of generality if we assume that G_0 happens to be normalized. Assume, furthermore, that G_0 is the rectangle function given by

$$G_0(x) = \begin{cases} 1/(2\Delta), & |x| \leq \Delta, \\ 0, & \text{otherwise.} \end{cases} \quad (3.18)$$

H_{Υ} and E can now be computed numerically.

We note that the superiority of SPADE over direct imaging for moment estimation has been shown many times before [5,9–12], so here we do not study direct imaging and focus only on a comparison between SPADE and the quantum limit.

B. Results

We first assume the Gaussian optical transfer function

$$\langle k|\psi\rangle = \left(\frac{2}{\pi}\right)^{1/4} \exp(-k^2), \quad (3.19)$$

such that

$$\psi(x) = \frac{1}{(2\pi)^{1/4}} \exp\left(-\frac{x^2}{4}\right), \quad (3.20)$$

$$C_n(x) = D_n x^n \exp\left(-\frac{x^2}{8}\right), \quad D_n = \frac{1}{2^n \sqrt{n!}}. \quad (3.21)$$

With the method described in Sec. III A, we compute the quantum bound H_Υ and the SPADE error E as a function of the object size Δ . The code assumes a submodel with $p = 10$ parameter dimensions and truncates the matrices of γ_0 and $\partial_j \gamma_\theta$ in the PAD basis $\{|\psi_0\rangle, \dots, |\psi_{q-1}\rangle\}$ by taking $q = 6$. Figure 1 plots the results in log-log scale for moment orders $\mu = 0, \dots, 7$, together with the straight-line fits by

$$\log_{10} H_\Upsilon \approx \log_{10} H_\Upsilon^{(0)} + H_\Upsilon^{(1)} \log_{10} \Delta, \quad (3.22)$$

$$\log_{10} E \approx \log_{10} E^{(0)} + E^{(1)} \log_{10} \Delta, \quad (3.23)$$

where the superscript (0) denotes the prefactor and the superscript (1) denotes the exponent in the scaling with Δ . We obtain $\log_{10} H_\Upsilon^{(0)}$, $H_\Upsilon^{(1)}$, $\log_{10} E^{(0)}$, and $E^{(1)}$ by passing $\log_{10} \Delta$ and $\log_{10} H_\Upsilon$ or $\log_{10} E$ to the `polyfit` function in MATLAB with degree 1; the resulting coefficients are reported in Table I. These numbers change by at most 1% even if we increase the parameter dimension p to 16 and the Hilbert-space dimension q to 12.

The results in Fig. 1 and Table I demonstrate three significant features. The first is that the exponents $H_\Upsilon^{(1)}$ and $E^{(1)}$ follow closely the theoretical exponent $2\lfloor\mu/2\rfloor$ for the considered range of Δ , hence verifying that the scaling law works well for nonzero Δ . The second feature is that the SPADE errors are all very close to the quantum bounds up to moment order $\mu = 3$, as quantified by the ratios $E^{(0)}/H_\Upsilon^{(0)}$ in Table I being close to 1 and the gaps in Fig. 1 being close to 0. The small gaps mean that SPADE is almost quantum-optimal for those moments. The third feature is that, for higher moment orders $\mu \geq 4$, the gaps begin to widen, and the results are less conclusive regarding the optimality of SPADE.

In Fig. 2 and Table II, we also report the numerical results for the rectangle transfer function

$$\langle k|\psi\rangle = \begin{cases} 1/\sqrt{2}, & |k| \leq 1, \\ 0, & \text{otherwise,} \end{cases} \quad (3.24)$$

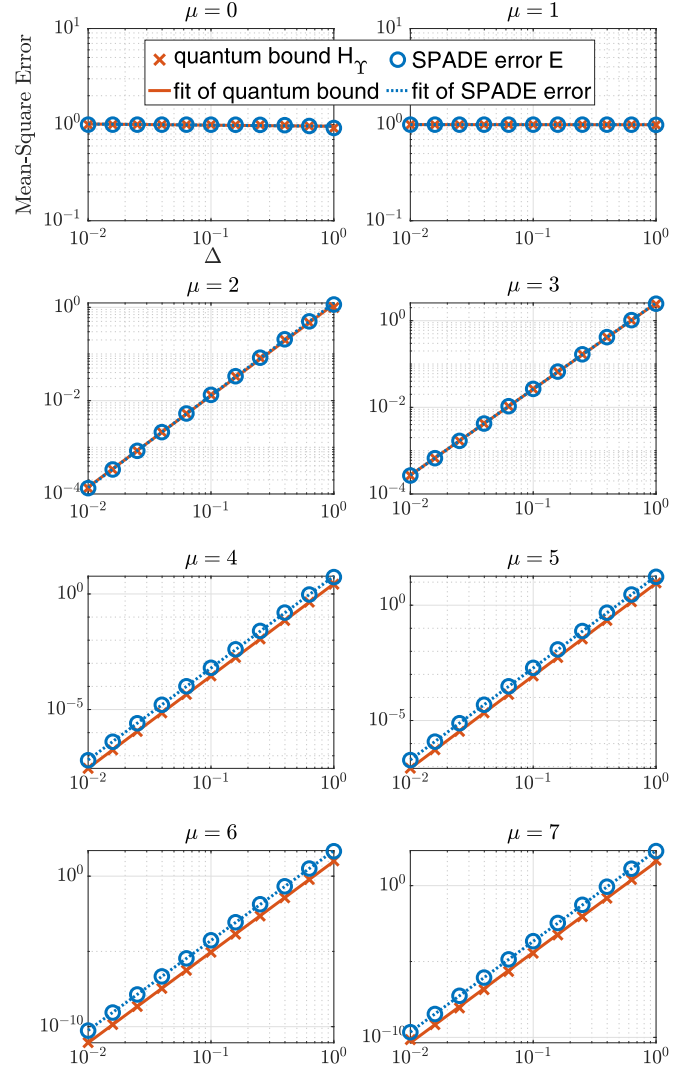


FIG. 1. Plots of the numerically computed Helstrom bound H_Υ (crosses) and the SPADE mean-square error E (circles) versus the object size Δ in log-log scale, assuming the Gaussian transfer function given by Eq. (3.19). μ is the order of the moment parameter of interest. The lines are straight-line fits as per Eqs. (3.22) and (3.23) and the fitting coefficients are given in Table I. Both axes are dimensionless.

which leads to

$$\psi(x) = \frac{1}{\sqrt{\pi}} j_0(x) = \frac{1}{\sqrt{\pi}} \begin{cases} (\sin x)/x, & x \neq 0, \\ 1, & x = 0, \end{cases} \quad (3.25)$$

$$C_n(x) = \sqrt{2n+1} j_n(x), \quad D_n = \frac{2^n n! \sqrt{2n+1}}{(2n+1)!}, \quad (3.26)$$

where j_n is the spherical Bessel function of the first kind [25]. The assumptions and formats of Fig. 2 and Table II otherwise follow those of Fig. 1 and Table I. These results exhibit the same features described earlier for the Gaussian transfer function.

C. Limitations

We point out a few limitations of our numerical approach. First, we are unable to prove that our submodel bound H_Υ

TABLE I. Coefficients from the straight-line fits of the Helstrom bound H_Υ and the SPADE error E versus the object size Δ in log-log scale, as plotted in Fig. 1, assuming the Gaussian transfer function given by Eq. (3.19). μ is the order of the moment parameter of interest, and the fitting coefficients are defined by Eqs. (3.22) and (3.23). For brevity, the prefactors $H_\Upsilon^{(0)}$ and $E^{(0)}$ and their ratios $E^{(0)}/H_\Upsilon^{(0)}$ are rounded to two significant figures, while the exponents $H_\Upsilon^{(1)}$ and $E^{(1)}$ are rounded to one decimal place.

μ	0	1	2	3	4	5	6	7
$H_\Upsilon^{(0)}$	0.96	1.0	1.2	2.5	2.8	8.9	9.6	36
$E^{(0)}$	0.96	1.0	1.2	2.6	5.9	18	50	200
$E^{(0)}/H_\Upsilon^{(0)}$	1.0	1.0	1.1	1.0	2.1	2.0	5.2	5.6
$H_\Upsilon^{(1)}$	0.0	0.0	2.0	2.0	4.0	4.0	6.0	6.0
$E^{(1)}$	0.0	0.0	2.0	2.0	4.0	4.0	6.0	6.0

approaches the generalized Helstrom bound \tilde{H} even in the $p \rightarrow \infty$ limit, as discussed at the end of Sec. II C. The potential existence of a higher quantum bound means that, when the gap between the error and the quantum bound shown in Fig. 1 or Fig. 2 is large, we cannot tell whether it is because the measurement is suboptimal or because the quantum bound is loose. Second, even \tilde{H} may not be tight if multiple scalar parameters are to be estimated simultaneously, and there exist better—albeit harder-to-compute—bounds for that problem [8,26–30]. Third, the numerical approach must assume a specific true object intensity G_0 and a specific moment order μ in computing each bound, so even if the numerical results demonstrate the scaling proved in Papers I and II, the analytic result in Papers I and II remains more general in the sense that the scaling is proved there for a much more general class of G_0 and an arbitrary μ . Fourth, while the numerical approach can produce the prefactor $H_\Upsilon^{(0)}$ for a given G_0 and μ , no rigorous analytic result is yet available regarding the prefactor as a function of μ , and the general behavior of the prefactor remains an interesting open problem.

IV. POTENTIAL EXTENSIONS

For completeness, here we discuss some straightforward potential extensions of our theory. First, it is easy to use our method to compute the quantum bound for other types of the parameter of interest $\beta(G)$ by changing u in Eqs. (2.25) and (2.26); $\beta(G)$ can even be nonlinear with respect to G . A simple example is the normalized moment

$$\beta(G) = \frac{1}{N(G)} \int_{-\infty}^{\infty} G(x)b(x)dx, \tag{4.1}$$

$$N(G) \equiv \int_{-\infty}^{\infty} G(x)dx. \tag{4.2}$$

To compute H_Υ for the normalized moment, one only needs to change u_j to

$$u_j = \partial_j \beta(G_\theta) = \langle a_j, b \rangle_{F_\theta} - \langle a_j, 1 \rangle_{F_\theta} \langle b, 1 \rangle_{F_\theta}, \tag{4.3}$$

where $F_\theta \equiv G_\theta/N(G_\theta)$. Of all the orthonormal polynomials, only a_0 has a nonzero $\langle a_j, 1 \rangle_{F_\theta}$.

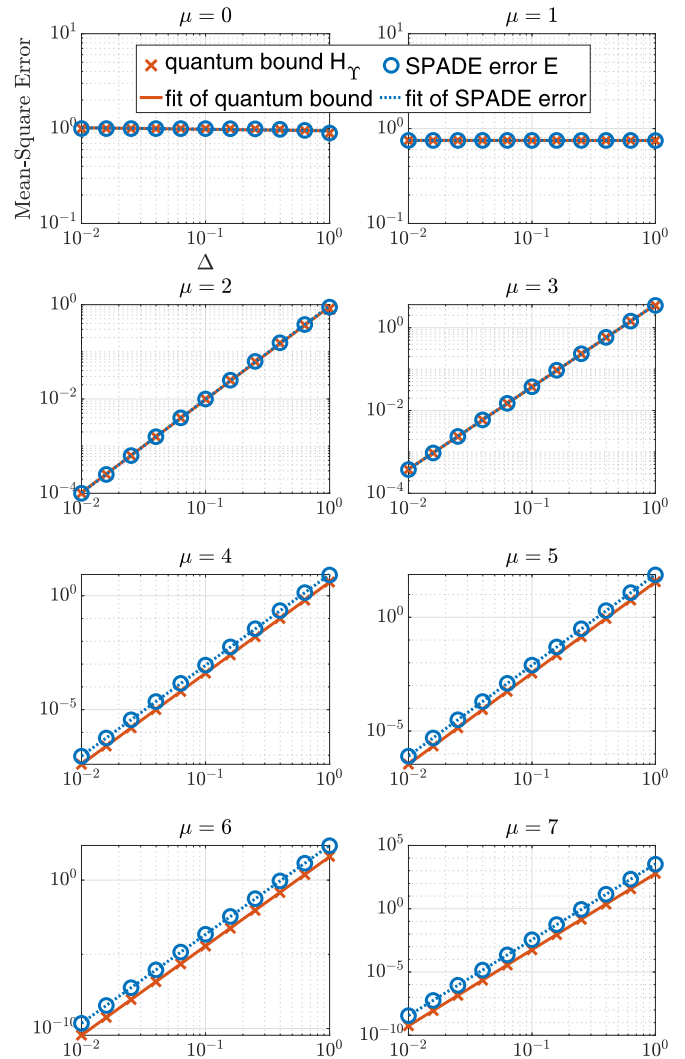


FIG. 2. Same as Fig. 1, except that the rectangle transfer function given by Eq. (3.24) is assumed.

Second, if N is known, as assumed in Papers I and II, then we can parametrize the normalized F only and keep N fixed, such that $G_\theta = NF_\theta$. The normalization implies that all scores should obey

$$\partial_j \Gamma(G_\theta) = \langle S_j, 1 \rangle_{\Gamma(G_\theta)} = \langle a_j, 1 \rangle_{G_\theta} = 0, \tag{4.4}$$

TABLE II. Same as Table I, except that the rectangle transfer function given by Eq. (3.24) is assumed.

μ	0	1	2	3	4	5	6	7
$H_\Upsilon^{(0)}$	0.95	0.75	0.91	3.6	4.0	36	38	590
$E^{(0)}$	0.95	0.75	0.94	3.6	8.6	76	220	3400
$E^{(0)}/H_\Upsilon^{(0)}$	1.0	1.0	1.0	1.0	2.1	2.1	5.7	5.8
$H_\Upsilon^{(1)}$	0.0	0.0	2.0	2.0	4.0	4.0	6.0	6.0
$E^{(1)}$	0.0	0.0	2.0	2.0	4.0	4.0	6.0	6.0

meaning that θ_0 and a_0 should be excluded from the submodel. A valid submodel with N fixed is

$$F_\theta(x) = \frac{1 + \tanh \left[\sum_{j=1}^p \theta_j a_j(x) \right] F_0(x)}{\int_{-\infty}^{\infty} (\text{numerator}) dx}. \quad (4.5)$$

Assuming this submodel and a normalized moment given by Eq. (4.1), the u_j given by Eq. (4.3) becomes $\langle a_j, b \rangle_{F_0}$, and we obtain back the theory assumed in Paper II.

Third, when the parameter of interest is a normalized moment given by Eq. (4.1), the estimator with SPADE should be changed depending on the situation. If N is known, then one can simply divide the estimators in Sec. III A by N , and the mean-square errors in Sec. III A are then scaled by $1/N^2$. If N is unknown, an unbiased estimator can still be constructed if one can detect the total photon number L in all the modes. Consider the estimation of an even-order β given by Eqs. (3.3) and (4.1) and a measurement in the PAD basis. Conditioned on L , the photon numbers $\{m_0, m_1, \dots\}$ detected in the modes now follow the multinomial distribution, and an unbiased estimator is simply Eq. (3.7) divided by L . The error conditioned on L is now

$$E = \frac{\beta}{D_n^2 L} (1 - D_n^2 \beta) = \frac{O(\Delta^\mu)}{L}, \quad (4.6)$$

which is slightly different from Eq. (3.8) because of the multinomial statistics but still has the same optimal scaling with Δ . The theory for an odd-order β is similar.

V. CONCLUSION

Our numerical analysis has verified the object-size scaling law for the moment estimation error in incoherent optical imaging. Papers I and II have proved the law only in the asymptotic limit of vanishing object size, and our results here show that the law also works well for nonzero sizes in reality. We have also shown that SPADE is close to quantum optimal for moment orders up to $\mu = 3$. For higher orders, the results are less conclusive, as significant gaps are observed between the SPADE errors and the quantum bounds. It remains an open question whether a better measurement can be found or a tighter bound can be derived.

In terms of future directions, apart from addressing the limitations discussed in Sec. III C and the potential extensions in Sec. IV, an important problem is to find the quantum limit to the reconstruction of the object intensity G , rather than just its moments, as well as the measurement to approach it. A multidimensional parameter of interest β should then be assumed, and a quantum bound beyond the Helstrom family [8,26–30] may be needed to produce a tighter bound. Finding the measurement to achieve it will also be a difficult, but rewarding, problem.

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APPENDIX A: GENERALIZED HELSTROM BOUND

Let the set of all possible density operators in a statistical problem be $\rho = \{\rho_\theta : \theta \in \Theta\}$, and let the true density operator be $\rho \in \rho$. Let the parameter of interest be $\beta : \Theta \rightarrow \mathbb{R}$. Consider the density operator $\omega = \rho^{\otimes M}$ and a measurement modeled by a positive operator-valued measure \mathcal{E} , such that the probability of an event Λ upon the measurement is

$$P(\Lambda) = \text{tr} \mathcal{E}(\Lambda) \omega = \text{tr} \mathcal{E}(\Lambda) \rho^{\otimes M}. \quad (A1)$$

Let a measurement outcome be λ and an estimator be $\check{\beta}(\lambda)$. The mean-square error is

$$E = \int [\check{\beta}(\lambda) - \beta]^2 \text{tr} \mathcal{E}(d\lambda) \omega. \quad (A2)$$

To derive a lower bound on E , consider the one-dimensional parametric submodel

$$\sigma = \{\sigma_\theta : \theta \in \Phi \subseteq \mathbb{R}\} \subseteq \rho. \quad (A3)$$

We require σ to contain the true ρ . Without loss of generality, assume a parametrization that gives $\sigma_0 = \rho$. Define the score operator S for the submodel by

$$\partial \sigma_\theta = \sigma_0 \circ S, \quad (A4)$$

let $\{S\}$ be the scores of all submodels with $\|S\|_\rho < \infty$, and define the tangent space $\mathcal{T} \equiv \overline{\text{span}}\{S\}$ as the closed linear span. Assume also that there exists a so-called influence operator δ , defined by the properties

$$\text{tr} \rho \delta = 0, \quad \langle \delta, S \rangle_\rho = \partial \beta \quad (A5)$$

for all submodels. For any measurement and any unbiased estimator, the generalized Helstrom bound \tilde{H} is given by (see Ref. [8] and Lemma 2 in Paper II)

$$E \geq \tilde{H} = \frac{1}{M} \|\Pi(\delta|\mathcal{T})\|_\rho^2 = \sup_{u \in \text{span}\{S\}} \frac{\langle \delta, u \rangle_\rho^2}{M \|u\|_\rho^2}, \quad (A6)$$

where $\Pi(\delta|\mathcal{T})$ is the orthogonal projection of δ into \mathcal{T} and span denotes the linear span. We can also make the benign assumption that any linear combination of scores is the score of a submodel, or $\text{span}\{S\} = \{S\}$. Then \tilde{H} is the supremum of all the submodel bounds, viz.,

$$\tilde{H} = \sup_\sigma H_\sigma. \quad (A7)$$

For Poisson states, each submodel bound is given by Eq. (2.18) according to Ref. [19], and we can take Eq. (2.19) to be the generalized Helstrom bound in this paper.

APPENDIX B: ALTERNATIVE EXPRESSION FOR THE HELSTROM BOUND

The goal here is to show that Eq. (2.38) is the same as the Helstrom bound given by Eq. (2.25). Any $s \in \text{span}\{a_0, \dots, a_{p-1}\}$ can be expressed as

$$s = \sum_j v_j a_j, \quad v \in \mathbb{R}^p. \quad (B1)$$

Treating v as a column vector and using Eq. (2.36), the right-hand side of Eq. (2.38) becomes

$$\max_v \frac{(v^\top u)^2}{v^\top K v}. \quad (\text{B2})$$

The Cauchy-Schwartz inequality yields

$$(v^\top u)^2 = (v^\top K^{1/2} K^{-1/2} u)^2 \leq (v^\top K v)(u^\top K^{-1} u), \quad (\text{B3})$$

which makes Eq. (B2) equal to Eq. (2.25).

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