# Decoherence reduction via continuous dynamical decoupling: Analytical study of the role of the noise spectrum

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We analyze the robust character against the nonstatic noise of *clock transitions* implemented via a method of continuous dynamical decoupling (CDD) in a hyperfine Zeeman multiplet in $^{87}$Rb. The emergence of features specific to the quadratic corrections to the linear Zeeman effect is evaluated. Our analytical approach, which combines methods of stochastic analysis with time-dependent perturbation theory, allows tracing the decoherence process for generic noise sources. Working first with a basic CDD scheme, it is shown that the amplitude and frequency of the (sinusoidal driving) field of control can be appropriately chosen to force the nonstatic random input to have a (time-dependent) perturbative character. Moreover, in the dressed-state picture, the effect of noise is described in terms of an operative random variable whose properties, dependent on the driving field, can be analytically characterized. In this framework, the relevance of the spectral density of the fluctuations to the performance of the CDD technique is precisely assessed. In particular, the range of noise correlation times where the method of decoherence reduction is still efficient is identified. The results obtained in the basic CDD framework are extrapolated to concatenated schemes. The generality of our approach allows its applicability beyond the specific atomic system considered.

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## I. INTRODUCTION

Decoherence in a quantum system, i.e., the loss of *purity* generated by the coupling to noncontrollable environments, is a fundamental difficulty in the realization of intrinsically quantum effects. Curbing the effect of the interactions of the system components with the environment, and consequently, extending the coherence times is a basic requirement for the advances in the implementation of quantum technologies [1–5]. Indeed, a primary objective of the research in this field is the development of technical schemes to steer the system evolution while protecting the relative phases. Apart from technical importance, preserving the coherence has central relevance to fundamental areas of research. In this sense, it is worth pointing out its crucial role in the realization of fundamental effects with ultracold atoms [6].

Different methods for decoherence reduction were proposed and applied in the last decades. Actually, a variety of strategies have been designed to cope with the specific characteristics of the different sources of noise. Significant objectives have been achieved: in some cases, the coherence times have been enlarged by orders of magnitude. Among the methods applied, the techniques of dynamical decoupling stand out as particularly effective. They basically consist of strategies to effectively disconnect the system from the environment that generates the fluctuations. Their original design incorporated sequences of pulses of control devised to average out the effect of noise [7–10]. In subsequent variations of the original proposals [11,12], the pulses were replaced by continuous-wave driving fields, aimed at facilitating the integration of the information protocols and at simplifying the experimental realization. For those methods to be operative it is necessary to minimize their (unavoidable) invasive effect on the system whose control is intended. In this sense, concatenation schemes set up to deal with the extra noise introduced by the auxiliary fields were developed [13–15]. The applicability of those techniques to qubits realized with trapped ions and atoms, nitrogen vacancies (NV) centers in diamond, or quantum dots has been extensively reported (see, for instance, [16–19] and references therein).

Here, we will focus on a recent application of a continuous dynamical decoupling (CDD) scheme to atoms of $Rb^{87}$ which resulted in a significant reduction of the effect of magnetic noise on transitions associated to a hyperfine Zeeman multiplet [20]. Indeed, a system of *clock transitions* almost immune to the presence of noise was generated. The attained stability against fluctuations has played a key role in subsequent research on the implementation of different fundamental effects with Bose-Einstein condensates of $Rb^{87}$ [21]. The technique applied to build up the CDD scheme was based on using a radio-frequency driving field orthogonal to the original Zeeman component. The objective was to force the magnetic-field fluctuations to play a secondary role in the dynamics. To cope with the additional noise introduced via stochastic variations in the driving intensity, a concatenation scheme was incorporated: a second field of control was designed to mitigate the effect of that extra noisy input. In the analysis of the experimental realization, the noise was assumed to be static: no time dependence of the fluctuations was contemplated. Hence, the random changes in the magnetic

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field were considered to merely affect the reproducibility of the initial conditions for the different experimental realizations. Here, it is worth stressing that it is in slow-noise setups where the performance of the CDD techniques have been mainly evaluated. Moreover, in the studies where nonstatic noise has been contemplated, its effect has been frequently analyzed via numerical simulation or through approximations valid only in specific regimes (the limit of large observation times or the adiabatic scenario have been usually tackled). It is pertinent to add that the majority of those studies dealt with the pulsed variant of the dynamical-decoupling technique [22]. In our work, we will go beyond that scenario: the potential applicability of the CDD method to deal with generic fluctuations will be analytically evaluated. In particular, the robustness against nonstatic noise of the *clock transitions* implemented in [20] will be assessed. Our approach starts with a description of the system dynamics previous to the application of the CDD method: the dephasing effects of generic magnetic fluctuations on the Zeeman multiplet will be analytically described, the emphasis being put on the differential features associated to the spectrum and correlation time of noise. Then, to analyze how the system dynamics is modified in the CDD scheme, we will work with the basis of eigenstates of the driving term (the so-called *dressed-state* representation). It will be apparent that the magnetic-noise component, diagonal in the original Zeeman-state basis, becomes off-diagonal in the new representation. Moreover, for a sufficiently large driving intensity, and consequently, for a large dressed-energy separation, the (nonstatic) noise contribution to the dynamics can be regarded as a (time-dependent) perturbative term. The use of methods of stochastic analysis combined with the application of time-dependent perturbation theory will allow us to characterize the efficiency of the CDD method. In particular, the relevance of the noise spectral density to the performance of the decoherence-reduction technique will be studied. Some practical conclusions on extending the range of applicability of the method will be extracted from our results.

The outline of the paper is as follows. In Sec. II, we analyze some fundamental aspects of the decohering effects of generic noise on a hyperfine Zeeman multiplet. An approach of complete validity will allow us to trace general dephasing features emergent in the asymptotic regimes. Additionally, we will describe the loss of *purity* in any time regime for fluctuations potentially relevant to different experimental setups. In Sec. III, the system dynamics in the CDD scheme will be tackled. The analytical characterization of the noise-induced transfer of population between dressed states will be used to scrutinize the stability of the *clock transitions* implemented in [20]. First, we will concentrate on the linear Zeeman regime associated to weak magnetic fields. Then, in Sec. IV, it will be shown that quadratic corrections to the linear Zeeman effect do not alter the operative character of the CDD technique. As a proof of consistency, we will recover the findings of previous work on slow-noise by taking the limit of large correlation time in our results. The connection with previous predictions on the role of the noise spectrum in the CDD-method performance will be also established. Finally, the general conclusions are summarized in Sec. V.

## II. EFFECT OF MAGNETIC FLUCTUATIONS ON THE COHERENT EVOLUTION IN A ZEEMAN MULTIPLET: RELEVANCE OF SPECIFIC NOISE PROPERTIES

As in [20], we consider here the system formed by the three $m_F$ states of the $F = 1$ ground-state manifold of $Rb^{87}$ (electronic configuration $[Kr]5S_{1/2}$ and nuclear spin $I = 3/2$). The practical interest of this system is clear: the Zeeman multiplet resulting from the application of a static magnetic field is a basic component of arrangements used in a variety of lines of research on Bose-Einstein condensates. The associated achievements are numerous, from synthetic spin-orbit coupling [6] to the emulation of non-Abelian gauge fields [21], or the generation of nonlinear Landau-Zener transitions [23,24]. The applicability of the system demands dealing with the deleterious effect of magnetic-field noise on the control of the dynamics. Actually, the fluctuations lead to the broadening of the spectral lines of the interstate transitions and induce decoherence in the system evolution. The study of the noisy dynamics of that system has also general implications as it exemplifies how the protocols proposed in quantum technologies can become inefficient due to dephasing in the evolution of the system components.

In [20], quadratic corrections to the linear Zeeman effect were considered. Here, as we focus on the analytical description of the role of generic noise in the dynamics, we will initially concentrate on the linear scenario: the differential effects of the noise characteristics can be already traced in that basic version of the model. Further on, in Sec. IV, the implications of the quadratic Zeeman effect will be analyzed. Hence, we first deal with the Hamiltonian

$$H = [\omega_0 + \delta\omega_0(t)]F_z, \tag{1}$$

where $\omega_0$ denotes the mean value of the characteristic frequency of the multiplet, $\delta\omega_0(t)$ is the shift induced by the fluctuations, and $\vec{F}$ is the angular momentum operator. In terms of the system parameters, the mean frequency is expressed as $\omega_0 = g_F(g_s|\mu_B| - g_I\mu_N)B_0/\hbar$, where $g_F$ is the Landé factor of the multiplet, $\mu_B$ is the Bohr magneton, $\mu_N$ is the nuclear magneton, $g_s$ and $g_N$ are, respectively, the $g$ factors of the spin and nuclear gyromagnetic ratios. The applied magnetic field $\vec{B}$ will be expressed as $\vec{B} = [B_0 + \delta B_0(t)]\vec{k}$: $B_0$ denotes the mean value and $\delta B_0(t)$ stands for the fluctuations. The noisy displacement in the frequency $\delta\omega_0(t)$ is given by $\delta\omega_0(t) = g_F(g_s|\mu_B| - g_I\mu_N)\delta B_0(t)/\hbar$. As any deterministic shift can be included in $\omega_0$, $\delta\omega_0(t)$ will be considered to have a zero mean value. In the theory developed to account for the experimental results of [20], the fluctuations were considered to be static. Hence, it was assumed that the different realizations of the (time-independent) stochastic variable $\delta B_0$ simply lead to a variation of the initial conditions for each experimental run. In the present work, we tackle the case of generic noise; no restrictions on the magnitude of the correlation time of $\delta\omega_0(t)$ are assumed. Our general description incorporates the static-noise setting as a particular case.

### A. Decay of the coherences: General characteristics in the asymptotic regimes

In the study of the systems proposed to implement quantum-information protocols, a quantum description of the environments where decoherence originates is frequently necessary. In those cases, the theoretical framework incorporates standard techniques developed in the study of open quantum systems. In the present case, as noise enters the system via a classical field, it is feasible to consider the stochastic variable as a driving element in the system evolution. Accordingly, the approach used to characterize the dynamics includes as a first step the analysis of the (unitary) evolution for each *noisy trajectory*, i.e., for each set of values realized by the random variable along a time sequence. Subsequently, the statistical average over noise realizations is carried out [25].

Our procedure starts by applying the unitary transformation

$$U(t) = e^{-i\omega_0 t F_z/\hbar}. \tag{2}$$

In the associated rotating frame, the system, prepared in the state $|\psi(0)\rangle$, evolves, for each stochastic trajectory, as

$$|\psi(t)\rangle = e^{-i\zeta(t)F_z/\hbar}|\psi(0)\rangle, \tag{3}$$

where $\zeta(t)$ is the nonstationary random variable defined by

$$\zeta(t) = \int_0^t \delta\omega_0(t')dt'. \tag{4}$$

Correspondingly, the density matrix in the representation of states $\{|k; F, m_F\rangle\}$ ($k$ stands for additional quantum numbers characterizing the ground-state configuration) is given by

$$\rho_{m_F,m'_F}(t) = \rho_{m_F,m'_F}(0)e^{i(m_F-m'_F)\zeta(t)}. \tag{5}$$

Now, the stochastic character of the system is incorporated by making the average over fluctuations. The resulting (reduced) density matrix reads

$$\rho_{m_F,m'_F}(t) = \rho_{m_F,m'_F}(0)\big\langle e^{i(m_F-m'_F)\zeta(t)}\big\rangle, \tag{6}$$

where $\langle\rangle$ stands for the average over noise realizations (no confusion with the standard quantum average will be possible throughout the text).

From the above equation, it is apparent that the populations do not change. It is also evident that, to obtain the precise evolution of the coherences, the statistical characterization of $\zeta(t)$ is necessary. At this point, a first general difficulty is noticeable: since $\zeta(t)$ is the sum of elementary increments $\delta\omega_0(t)dt$, which, for finite correlation time $\tau_c$, are statistically dependent, its characterization, and in turn, the description of the coherence evolution are not trivial for a generic stochastic variable $\delta\omega_0(t)$. Despite this fundamental limitation, it is possible to identify important properties of the dephasing, valid for a generic random input $\delta\omega_0(t)$, in the following regimes.

(i) In the limit of large correlation times, i.e., for $t \ll \tau_c$, which corresponds to the slow-noise scenario of the *clock transitions* implemented in [20], the phase shift can be approximated as

$$\zeta(t) \simeq \delta\omega_0(0)t.$$

Consequently, the average in Eq. (6) is completely determined by the probability distribution $W_D[\delta\omega_0(0)]$, i.e.,

$$\rho_{m_F,m'_F}(t) = \rho_{m_F,m'_F}(0)\int d(\delta\omega_0)W_D(\delta\omega_0)e^{i(m_F-m'_F)\delta\omega_0 t}. \tag{7}$$

In particular, for a Gaussian input $\delta\omega_0(0)$ with variance var$[\delta\omega_0]$ we obtain

$$\rho_{m_F,m'_F}(t) \propto e^{-\frac{1}{2}(m_F-m'_F)^2\mathrm{var}[\delta\omega_0]t^2}, \tag{8}$$

which corresponds to Gaussian decay with characteristic time

$$\tau_d = [(m_F - m'_F)^2\mathrm{var}([\delta\omega_0)/2]^{-1/2}.$$

(ii) In the limit of short correlation times, i.e., for $t \gg \tau_c$, which is eventually reached as longer evolution times are attained in the monitoring of the system, it is possible to write $\zeta(t)$ in the form

$$\zeta(t) = \int_0^{\Delta t}\delta\omega_0(t)dt + \int_{\Delta t}^{2\Delta t}\delta\omega_0(t)dt + \cdots + \int_{(n-1)\Delta t}^{t}\delta\omega_0(t)dt,$$

with a large $n$, and, still with an interval $\Delta t$ larger than $\tau_c$, which guarantees that the different summands are uncorrelated. Hence, applying the Central Limit Theorem [26], one concludes that, since $\zeta(t)$ can be expressed as the sum of a large number of statistically independent variables, it presents an approximate normal distribution. Therefore, one simply needs to evaluate the mean $\langle\zeta(t)\rangle$ and the variance $\langle\zeta^2(t)\rangle - \langle\zeta(t)\rangle^2$. Accordingly, we proceed as

$$\langle\zeta(t)\rangle = \left\langle\int_0^t \delta\omega_0(t')dt'\right\rangle = \int_0^t \langle\delta\omega_0(t')\rangle dt' = \langle\delta\omega_0\rangle t = 0, \tag{9}$$

where we use the notation $\langle\delta\omega_0(t)\rangle \equiv \langle\delta\omega_0\rangle$ since a stationary input $\delta\omega_0(t)$ is being considered. Now, aiming at the practical applicability of the analysis, we will evaluate $\langle\zeta^2(t)\rangle$ in terms of a magnitude of operative use in the characterization of noise, namely, the spectral density. To this end, we first recall the Wiener-Khinchin theorem [28], which connects the Fourier transform of the autocorrelation function $G(\tau) = \langle\delta\omega_0(0)\delta\omega_0(\tau)\rangle$ with the spectrum $S(\omega)$, namely,

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau e^{-i\omega\tau}G(\tau), \tag{10}$$

and the associated inverse expression

$$G(\tau) = \int_{-\infty}^{\infty} d\omega e^{i\omega\tau}S(\omega). \tag{11}$$

Hence, we use Eq. (10) to calculate the variance as

$$\begin{aligned}\langle\zeta^2(t)\rangle &= \left\langle\int_0^t \delta\omega_0(\tau)d\tau \times \int_0^t \delta\omega_0(\tau')d\tau'\right\rangle \\ &= \int_0^t d\tau\int_0^t d\tau'\langle\delta\omega_0(\tau)\delta\omega_0(\tau')\rangle \\ &= \int_{-t}^t d\tau(t-|\tau|)\int_{-\infty}^{\infty} d\omega e^{i\omega\tau}S(\omega) \\ &= 2\int_0^{\infty} d\omega\left[\frac{\sin(\omega t/2)}{\omega/2}\right]^2 S(\omega).\end{aligned} \tag{12}$$

(An appropriate change of variables has been implemented). Furthermore, in the considered limit $t \gg \tau_c$, the function $[\frac{\sin(\omega t/2)}{\omega/2}]^2$ can be approximated in terms of the Dirac delta function $\delta(\omega)$ ($[\frac{\sin(\omega t/2)}{\omega/2}]^2 \rightarrow 2\pi t\delta(\omega)$), and the integral can be analytically evaluated. Specifically,

$$\langle\zeta^2(t)\rangle \simeq 2\pi S(0)t. \quad (13)$$

It is then concluded that, in the regime considered, the coherences present an exponential decay, namely,

$$\rho_{m_F,m'_F}(t) \propto e^{-(m_F-m'_F)^2\pi S(0)t}, \quad (14)$$

the $1/e$ scaling time being

$$\tau_d = [(m_F - m'_F)^2\pi S(0)]^{-1}.$$

Hence, it is the noise spectrum at zero frequency that determines the magnitude of the dephasing time. The emergence, irrespective of the noise properties, of a universal exponential-decay regime in the limit of long observation times was analyzed in previous studies on dephasing in different physical contexts [19,27]. In particular, the dependence of the decay rate on the zero-frequency spectrum was reported in systems where $1/f$ noise is relevant. It is evident that to identify the type of noise present in a particular setup, the results extracted from the analysis of the asymptotic regimes are not sufficient. Advances in tracking the fluctuations demand a more complete description of the coherence decay. A detailed modeling of the noise characteristics is needed to establish the origin of features emergent in the decoherence process. In the following, we will proceed along this line.

### B. Tracing the dephasing process in a generic time regime

To describe the system evolution in any time regime, the complete statistical characterization of $\zeta(t)$ is required, and consequently, the properties of $\delta\omega_0(t)$ must be specified. Here, to have a good predictive power in different contexts, a quite general model with wide practical applicability is assumed. Namely, we consider that $\delta\omega_0(t)$ corresponds to a zero-mean stationary Ornstein-Uhlenbeck process [28], i.e., it is a Gaussian variable whose mean value and correlation function are respectively given by

$$\langle\delta\omega_0(t)\rangle = 0, \quad (15)$$

and

$$G(t-t') = \langle\delta\omega_0(t)\delta\omega_0(t')\rangle = \mathrm{var}[\delta\omega_0]e^{-\alpha|t-t'|}, \quad (16)$$

where $\alpha$ is a positive real coefficient which represents the inverse of the correlation time, i.e., $\tau_c = \alpha^{-1}$. From Eq. (10), the spectrum is found to be given by

$$S(\omega) = \frac{\alpha\,\mathrm{var}[\delta\omega_0]}{\pi(\alpha^2+\omega^2)}. \quad (17)$$

This modeling of noise asw used in previous studies on related systems [14,15,29]. In particular, it was employed in a numerical simulation of the effect of noise on *clock states* implemented in NV centers in diamond [30].

The characterization of $\zeta(t)$ follows from the application of techniques of stochastic analysis [26]. Specifically, for the mean value, we have

$$\langle\zeta(t)\rangle = \left\langle\int_0^t \delta\omega_0(t')dt'\right\rangle = \int_0^t \langle\delta\omega_0(t')\rangle dt' = 0. \quad (18)$$

Additionally, $\langle\zeta^2(t)\rangle$ is obtained as

$$\begin{aligned}\langle\zeta^2(t)\rangle &= \left\langle\int_0^t \delta\omega_0(\tau)d\tau \times \int_0^t \delta\omega_0(\tau')d\tau'\right\rangle \\ &= \mathrm{var}[\delta\omega_0]\int_{-t}^t d\tau(t-|\tau|)e^{-\alpha|\tau|} \\ &= \frac{2\mathrm{var}[\delta\omega_0]}{\alpha^2}(\alpha t + e^{-\alpha t} - 1) \quad (19)\\ &= 2\pi S(0)\left(t + \frac{e^{-\alpha t}-1}{\alpha}\right). \quad (20)\end{aligned}$$

Notice that by fixing $\alpha$ and taking the limits $t \rightarrow 0$ and $t \rightarrow \infty$ in this expression, we consistently recover the results previously obtained using general arguments in the limits of large correlation time ($t \ll \alpha^{-1}$) and small correlation time ($t \gg \alpha^{-1}$). In particular, it is shown that, at large times, $\langle\zeta^2(t)\rangle$ is correctly expressed as a function of the zero-frequency value of the spectrum $S(0)$. In the crossover, a complex time dependence, determined by the specific value of the correlation time, is observed.

Now, once $\langle\zeta(t)\rangle$ and $\langle\zeta^2(t)\rangle$ are known, the (Gaussian) probability distribution $W_D[\zeta(t)]$ is completely determined, and the evolution of the reduced density matrix is evaluated to give

$$\begin{aligned}\rho_{m_F,m'_F}(t) &= \rho_{m_F,m'_F}(0)\int d[\zeta(t)]W_D[\zeta(t)]e^{i(m_F-m'_F)\zeta(t)} \\ &= \rho_{m_F,m'_F}(0)e^{-\frac{1}{2}(m_F-m'_F)^2\langle\zeta^2(t)\rangle}. \quad (21)\end{aligned}$$

A general remark on the entire system evolution is pertinent. The decay of the coherences, observed in any time regime and traced when the statistical average is carried out, reflects the loss of *purity* in the system evolution. If no coherence-preservation strategies are implemented, the system is of no use to realize protocols where specific quantum characteristics are required in large time intervals. We stress also that the present context corresponds to a phase-fluctuation scenario: since noise only affects the energy splittings of the used diagonal representation, it has a purely dephasing effect. There is no loss of population. This is in contrast with setups where the entrance of noise occurs through nondiagonal terms (i.e., via terms which do not commute with the Hamiltonian). There, one speaks of *relaxation* of the system, instead of pure dephasing. As we will see in the next section, the present dephasing setting is converted into a relaxation scenario when the driving field of the CDD is connected.

## III. APPLICATION OF DYNAMICAL-DECOUPLING METHODS TO NONSTATIC FLUCTUATIONS

The basis of the implementation of the CDD method of [20] was the inclusion in the experimental setup of a driving field orthogonal to the (static) Zeeman component. To deal with that extra term, a dressed-state representation, which incorporates the time dependence of the driving, was used. (See [31] for an alternative scheme which incorporates a continuous-observation scheme). In that scenario, the eigenvalues and eigenstates of the complete Hamiltonian can be exactly obtained as the fluctuations are time independent. Here, to build up a framework where the effect of non-static noise can be tackled, we will resort to a perturbative picture. In passing, our approach will allow us to clearly identify the basic mechanism responsible for the effectiveness of the CDD method, and in particular, for its functioning in the realization of [20]. In this sense, we point out that, in the dressed-state picture, the term in the Hamiltonian corresponding to magnetic noise (fluctuations in $B_0$) becomes off-diagonal. Furthermore, for a sufficiently large separation of the diagonal elements in the new basis, which can be implemented by increasing the driving intensity, a perturbative scheme with characteristic parameter given by the quotient between the noise magnitude and the driving intensity can be set up. Notice that this procedure is applicable irrespective of the time properties of the fluctuations. In the case of static noise, the random component leads to a second-order correction to the eigenvalues. Hence, the frequencies of the dressed-state transitions become noise immune to first order. There is a shortcoming in the practical arrangement: since stochastic variations in the driving intensity cannot be avoided, the scheme introduces additional fluctuations in the system. Furthermore, since that extra noisy term enters the diagonal elements, it is a first-order component of the dressed-state picture. To cope with this additional random input, a second driving field (the probe field) orthogonal to the first one is incorporated. The procedure can be continued: additional concatenated probe fields can be included till the magnitude of the remnant noise, entering the system through the last driving field, can be considered to be negligible compared with the final splitting.

Let us address now the case of nonstatic fluctuations. We will concentrate on a model system that incorporates the basic components of the CDD schemes. Namely, we will consider the Hamiltonian given by

$$H = [\omega_0 + \delta\omega_0(t)]F_z + 2\Omega_d \cos(\omega_d t)F_x, \tag{22}$$

where $\Omega_d$ and $\omega_d$ are the characteristic parameters of the control field ($\Omega_d$ is proportional to the Landé factor of the hyperfine multiplet and to the field intensity). Only one driving term is considered: the results obtained for this primary scenario are straightforwardly generalized to more elaborate arrangements. (See [32] for a proposal of bichromatic dressing).

### A. Setting up the perturbative scheme

Because of the nonstatic character of $\delta\omega_0(t)$, the method used in previous work [20] to analytically characterize the system dynamics is not applicable: it is not possible to obtain exact eigenvalues of $H$. Still, an alternative procedure to evaluate the performance of the CDD scheme can be devised. Namely, by choosing the driving frequency as $\omega_d = \omega_0$, and working in the rotating frame defined by the unitary transformation

$$U_1(t) = e^{-i\omega_d t F_z/\hbar}, \tag{23}$$

the Hamiltonian in Eq. (22) is rewritten as

$$H = \delta\omega_0(t)F_z + \Omega_d F_x, \tag{24}$$

where the rotating wave approximation (RWA) is applied ($\Omega_d$ is assumed to be much smaller than $\omega_d$) and the same notation $H$ is being used for the rotated Hamiltonian $U_1^\dagger H U_1 - i\hbar U_1^\dagger \dot{U}_1$. The additional transformation

$$U_2(t) = e^{-i\frac{\pi}{2}F_y/\hbar} \tag{25}$$

leads to

$$H = \Omega_d F_z + \delta\omega_0(t)F_x. \tag{26}$$

From the form of $H$, it is apparent that, if the driving-field intensity is much stronger than the noise magnitude, i.e., for $\Omega_d \gg |\delta\omega_0(t)|$, the Hamiltonian can be split as the sum of a zero-order term

$$H_0 = \Omega_d F_z$$

(with eigenstates $|F, \tilde{m}_F\rangle$, $\tilde{m}_F = 0, \pm 1$, and associated eigenvalues $E(\tilde{m}_F) = \tilde{m}_F \Omega_d \hbar$) and a (time-dependent) perturbative contribution

$$W(t) = \delta\omega_0(t)F_x. \tag{27}$$

In this approach, the effect of $W(t)$ for each noisy trajectory can be characterized. From time-dependent perturbation theory, it is known that, for the system prepared in one of the states, let us say the state $|\tilde{m}_F\rangle$, the probability of transition to other state ($|\tilde{m}'_F\rangle$) is given to first order by

$$P_{\tilde{m}_F,\tilde{m}'_F}(t) = \frac{1}{\hbar^2}\left|\int_0^t dt' W_{\tilde{m}'_F,\tilde{m}_F}(t') e^{i(E_{\tilde{m}'_F} - E_{\tilde{m}_F})t'/\hbar}\right|^2$$
$$= \frac{|(F_x)_{\tilde{m}'_F,\tilde{m}_F}|^2}{\hbar^2}\left|\int_0^t dt' \delta\omega_0(t') e^{i(\tilde{m}'_F - \tilde{m}_F)\Omega_d t'}\right|^2, \tag{28}$$

where it is taken into account that, since $E_{\tilde{m}'_F}$ and $E_{\tilde{m}_F}$ are zero-order eigenvalues, their difference is given by $E_{\tilde{m}'_F} - E_{\tilde{m}_F} = (\tilde{m}'_F - \tilde{m}_F)\Omega_d \hbar$ (see the form of $H_0$). Additionally, the expression of the matrix element

$$W_{\tilde{m}'_F,\tilde{m}_F}(t') = \delta\omega_0(t')(F_x)_{\tilde{m}'_F,\tilde{m}_F}$$

is used. The next step is dealing with the stochastic character of the evolution. Let us see that the statistical average

$$\langle P_{\tilde{m}_F,\tilde{m}'_F}(t)\rangle = \frac{|(F_x)_{\tilde{m}'_F,\tilde{m}_F}|^2}{\hbar^2}\left\langle\left|\int_0^t dt' \delta\omega_0(t') e^{i(\tilde{m}'_F - \tilde{m}_F)\Omega_d t'}\right|^2\right\rangle \tag{29}$$

is a useful indicator of the efficiency of the CDD method. From a first qualitative evaluation, we can conclude that, when the driving intensity is increased, more rapid does become the oscillation resulting from the exponential $e^{i(\tilde{m}'_F - \tilde{m}_F)\Omega_d t'}$. Consequently, provided that $\delta\omega_0(t')$ does not have harmonic components in resonance with $e^{i(\tilde{m}'_F - \tilde{m}_F)\Omega_d t'}$, an effective averaging out of the integral value can be predicted. This consideration can also be formulated from the statistical analysis of the stochastic variable defined as

$$\chi(t) = \int_0^t dt' \delta\omega_0(t') e^{i(\tilde{m}'_F - \tilde{m}_F)\Omega_d t'}, \tag{30}$$

present in Eq. (29). It is shown that, if the spectral density of $\delta\omega_0(t')$ does not reach a significant value at the frequency $(\tilde{m}'_F - \tilde{m}_F)\Omega_d$, the exponential factor $e^{i(\tilde{m}'_F - \tilde{m}_F)\Omega_d t'}$ leads to a reduction in the variance of $\chi(t)$ with respect to that of the nonmodulated variable $\zeta(t)$ given by Eq. (4). Consequently, in that case, the inhibition of the population transfer as $\Omega_d$ grows can be conjectured. In the following, we will see that a quantitative analysis confirms these predictions.

### B. Incorporating the noise characteristics

Taking into account the Ornstein-Uhlenbeck characteristics of $\delta\omega_0(t)$, the average of the population transfer given by Eq. (29) is evaluated as follows:

$$\begin{aligned}\langle P_{\tilde{m}_F,\tilde{m}'_F}(t)\rangle &= \frac{|(F_x)_{\tilde{m}'_F,\tilde{m}_F}|^2}{\hbar^2}\left\langle \int_0^t d\tau\,\delta\omega_0(\tau)e^{i\Omega_e\tau}\times\int_0^t d\tau'\,\delta\omega_0(\tau')e^{-i\Omega_e\tau'}\right\rangle\\ &\propto \mathrm{var}[\delta\omega_0]\int_0^t d\tau\int_0^t d\tau' e^{-\alpha|\tau-\tau'|}e^{i\Omega_e(\tau-\tau')}\\ &= \frac{2\mathrm{var}[\delta\omega_0]}{\alpha^2+\Omega_e^2}\left[\alpha t+\frac{\Omega_e^2-\alpha^2}{\alpha^2+\Omega_e^2}(1-e^{-\alpha t}\cos\Omega_e t)-\frac{2\alpha\Omega_e e^{-\alpha t}}{\alpha^2+\Omega_e^2}\sin\Omega_e t\right], \end{aligned} \tag{31}$$

where we use the effective frequency $\Omega_e = (\tilde{m}'_F - \tilde{m}_F)\Omega_d$ and the integral has been calculated via an adequate change of variables.

From the above expression some preliminary conclusions can be drawn.

(i) A crucial aspect of the applicability of the CDD method is uncovered by the analysis of the regime $\alpha \gg \Omega_e$. It is apparent from Eq. (31) that the role of the field intensity $\Omega_d$ loses relevance as the noise correlation time decreases, i.e., for a growing $\alpha$. Furthermore, for $\alpha \gg \Omega_e$, the dependence of the population transfer on $\Omega_d$ vanishes. This finding can be understood using arguments relative to the spectral decomposition of the fluctuations. In this sense, it is convenient to work with the Fourier transform of the stochastic variable $\delta\omega_0(t)$. Accordingly, we write

$$\delta\omega_0(t) = \int d\omega c(\omega)e^{i\omega t}, \tag{32}$$

where the harmonic components are given by

$$c(\omega) = \frac{1}{2\pi}\int dt\,\delta\omega_0(t)e^{-i\omega t}, \tag{33}$$

and are distributed according to the (Lorentzian) spectral density given by Eq. (17). [See Ref. [28] for a complete statistical characterization of $c(\omega)$]. Notice that, as the correlation time decreases, the spectrum becomes wider.

Hence, using the harmonic components, the perturbation can be rewritten as

$$W(t) = \left(\int d\omega c(\omega)e^{i\omega t}\right)F_x,$$

and the Hamiltonian in Eq. (26) can be regarded as representing the driving of the (dressed) triplet system by a pulse of harmonic signals which are effective in inducing interstate transitions only when the quasiresonance condition $\omega \simeq \Omega_e$ is fulfilled. Equation (17) makes it evident that, for small values of $\alpha$ (large correlation times), i.e., for a narrow spectrum, there are no harmonic components of noise in resonance with the interstate transition frequency $\Omega_e$. Hence, the noise-induced transfer of population is blocked. On the other hand, for a sufficiently large value of $\alpha$, and in turn, for a large spectral width, the variation of $\Omega_e$ does not reduce the fraction of noisy components in resonance with that frequency. In that stage, the value of the energy splitting is not longer a limiting element of the population transfer. It is then understood that in the range defined by $\alpha \gg \Omega_e$, the CDD method is not longer effective for mitigating the effect of the fluctuations. These findings are illustrated in Figs. 1 and 2, where the transition probability is represented as a function of time for different sets of parameters $\alpha$ and $\Omega_e$. Notice that the CDD scheme is highly efficient for small values of $\alpha$ (Fig. 1): the transition probability is significantly reduced as the driving intensity is increased. In contrast, the differential effect of the CDD scheme for growing driving intensity is hardly noticeable for a wide spectral density, i.e., for a large $\alpha$ (Fig. 2). (To focus on the combined role of the noise spectral width and the driving intensity, we use the scale factor $A = \frac{|(F_x)_{\tilde{m}'_F,\tilde{m}_F}|^2}{\hbar^2}\mathrm{var}[\delta\omega_0]$ in the representation of the transition probability).

(ii) Additional arguments in the same line are extracted by expressing the population transfer, given by Eq. (31), as a function of the spectrum, i.e.,

$$\begin{aligned}\langle P_{\tilde{m}_F,\tilde{m}'_F}(t)\rangle \propto S(\Omega_e)\bigg[t+\frac{1}{\alpha}\frac{\Omega_e^2-\alpha^2}{\alpha^2+\Omega_e^2}(1-e^{-\alpha t}\cos\Omega_e t)\\ -\frac{2\Omega_e e^{-\alpha t}}{\alpha^2+\Omega_e^2}\sin\Omega_e t\bigg].\end{aligned} \tag{34}$$

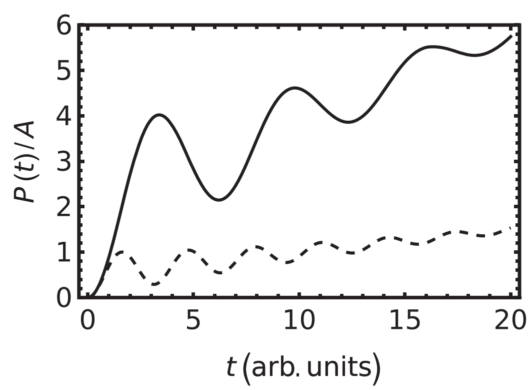


FIG. 1. Noise-induced transition probability $P$ between two dressed states as a function of time (in arbitrary units). The used parameters are $\alpha = 0.1$ and $\Omega_e = 1$ (continuous line) and $\alpha = 0.1$ and $\Omega_e = 2$ (dashed line). (We use the scale factor $A = \frac{|(F_x)_{\tilde{m}'_F,\tilde{m}_F}|^2}{\hbar^2}\mathrm{var}[\delta\omega_0]$ in the representation of the probability).

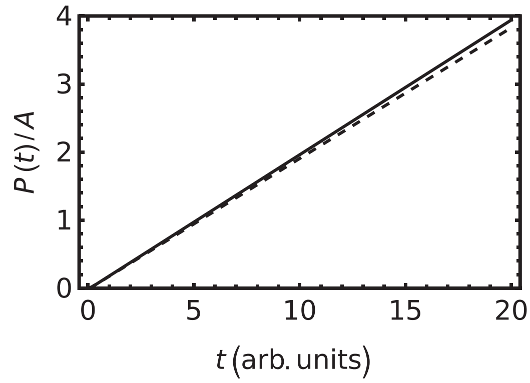


FIG. 2. Same caption as that of Fig. 1, with $\alpha = 10$ and $\Omega_e = 1$ (continuous line) and $\alpha = 10$ and $\Omega_e = 2$ (dashed line).

We stress that no divergence emerges from the term that incorporates the factor $\frac{1}{\alpha}$: as can be seen in Eq. (17), the spectral density $S(\Omega_e)$ includes a factor $\alpha$.

The central role played by the spectral component corresponding to the effective frequency $S(\Omega_e)$ is evident in Eq. (34). For a narrow spectrum, it is possible, by increasing the driving intensity to force $\Omega_e$ out of the spectral range, i.e., to make $S(\Omega_e) \simeq 0$. In contrast, for a flat spectrum, no significant variations in $S(\Omega_e)$ take place as $\Omega_e$ grows.

(iii) Although it is already patent in the above arguments, it is worth stressing that, except in the regime $\alpha \gg \Omega_e$, the CDD method can be considered to be efficient. Indeed, as the driving intensity, proportional to $\Omega_d$, and consequently, to $\Omega_e$, is increased, the effect of the stochastic input is reduced: the averaged probability of transition diminishes as $\Omega_e$ grows, and therefore, the zero-order eigenstates (the dressed states $|F, \tilde{m}_F\rangle$, $\tilde{m}_F = 0, \pm 1$) better approximate the eigenstates of the complete noisy Hamiltonian.

(iv) Another proof of consistency of the entire approach is obtained by checking that the results corresponding to static fluctuations are recovered in the limit of large correlation times. For $\tau_c \to \infty$ ($\alpha \to 0$), the transfer of population is given by

$$\langle P_{\tilde{m}_F,\tilde{m}'_F}(t)\rangle = \frac{|(F_x)_{\tilde{m}'_F,\tilde{m}_F}|^2}{\hbar^2}\frac{2\mathrm{var}[\delta\omega_0]}{\Omega_e^2}(1-\cos\Omega_e t), \quad (35)$$

which matches the average over noise realizations of the probability of transition between two dressed states induced by a static random perturbation $W = \delta\omega_0 F_x$. Note that the order of magnitude of this first perturbative correction is determined by the quotient $\mathrm{var}[\delta\omega_0]/\Omega_e^2$, in agreement with the precision reached in the application of CDD methods to static noise [20]. Therefore, the zero-order eigenstates (the dressed states of the used representation) are approximate eigenstates of the complete (noisy) Hamiltonian, with precision given by $\mathrm{var}[\delta\omega_0]/\Omega_e^2$. Using the terminology introduced in [20], we recover the conclusion that the *clock states* are noise immune to first order in the quotient $\delta\omega_0/\Omega_e$.

(v) The above conclusions, extracted from the study of the basic CDD method, i.e., for the scheme incorporating one driving field, are straightforwardly extrapolated to more elaborate setups. As, in any stage in the CDD scheme, the last noisy component entering the system is transferred to an off-diagonal term through an appropriate change of representation, its effect on the dynamics can always be characterized in terms of a population transfer between effective zero-order eigenstates similar to that given by Eqs. (29). Therefore, the effectiveness of the decoherence-reduction method is guaranteed provided that the *final* interstate transition frequencies are out of the dominant part of the spectral range of the residual noise. Note that controlling the frequencies of transition, in particular, the effective frequency $\Omega_e$, to avoid the occurrence of resonances with the noise spectral components has the limitations associated to the application of the RWA and to the system reduction employed in the description of the model system. A careful analysis of each experimental setup is needed: since the consecutive application of the RWA as different drivings are incorporated implies a reduction in the magnitude of the splittings, keeping the last $\Omega_e$ outside the spectral range of the corresponding final noise is not trivial.

(vi) It is worth pointing out that, since the application of time-dependent perturbation theory to first-order requires only up to the second moment of noise, the used framework embodies in fact a Gaussian approximation.

## IV. ROLE OF THE QUADRATIC ZEEMAN EFFECT IN THE DYNAMICAL-DECOUPLING SCHEME

Given the high precision required for the characterization of noise effects, the system description must go beyond the strictly linear regime corresponding to the Zeeman effect in the weak magnetic-field limit. The next-order contribution to the Zeeman shift accounts for the field-induced coupling between hyperfine multiplets. For alkali gases, that correction can be analytically evaluated (the simultaneous matrix representation of both hyperfine and Zeeman terms can be decomposed into blocks characterized by the value of the quantum number $m_F$ which can be analytically solved). Indeed, using the Breit-Rabi formula [33,34], the net quadratic Zeeman shift can be approximated as

$$\hbar\epsilon\left(\frac{F_z^2}{\hbar^2} - I\right), \quad (36)$$

where the parameter $\epsilon$ is given by

$$\epsilon = \frac{(g_s\mu_B - g_I\mu_N)^2 B^2}{4\Delta W_{hf}},$$

with $\Delta W_{hf}$ being the hyperfine energy splitting between the $F = 2$ and $F = 1$ terms. (We stress that a more accurate approximation to the quadratic corrections does not alter the conclusions of this section.) Accordingly, the Hamiltonian that describes the original Zeeman multiplet is rewritten as

$$H = [\omega_0 + \delta\omega_0(t)]F_z + \hbar\epsilon\left(\frac{F_z^2}{\hbar^2} - I\right). \quad (37)$$

Important for the evaluation of the efficiency of the decoupling scheme is to keep in mind that the dominant noisy contribution comes from the linear dependence of $\omega_0$ on the random magnetic field. Given its second-order character, the stochastic variation of $\epsilon$ will not be included in the model.

The system evolution for each stochastic trajectory is given by

$$|\psi(t)\rangle = e^{-i[\zeta(t)F_z/\hbar+\epsilon t(F_z^2/\hbar^2-I)]}|\psi(0)\rangle, \tag{38}$$

which, in terms of the density matrix, and incorporating the statistical average, reads

$$\rho_{m_F,m'_F}(t) = \rho_{m_F,m'_F}(0)e^{i(m_F^2-m_F'^2-2)\epsilon t}\langle e^{i(m_F-m'_F)\zeta(t)}\rangle, \tag{39}$$

where $\zeta(t)$ is still given by Eq. (4). As the considerations made in Sec. II on the statistical average are still applicable, it is concluded that the quadratic corrections to the linear Zeeman effect merely leads to an oscillation of the coherences as their decay proceeds. The possibility of observing that oscillation in the experiments depends on the relative magnitude of the parameter $\epsilon$ and the dephasing rate evaluated in Sec. II.

When the CDD scheme is applied and the field of control is connected, the Hamiltonian that governs the dynamics reads

$$H = [\omega_0 + \delta\omega_0(t)]F_z + \hbar\epsilon\left(\frac{F_z^2}{\hbar^2} - I\right) + 2\Omega_d\cos(\omega_d t)F_x, \tag{40}$$

which, through the sequential application of the unitary transformations given by Eqs. (23) and (25), is cast into the form

$$H = \Omega_d F_z + \hbar\epsilon\left(\frac{F_x^2}{\hbar^2} - I\right) + \delta\omega_0(t)F_x.$$

Hence, the previous perturbative scheme must be redefined. Whereas the zero-order Hamiltonian is now given by

$$H_0 = \Omega_d F_z + \hbar\epsilon\left(\frac{F_x^2}{\hbar^2} - I\right),$$

the perturbation still corresponds to Eq. (27). The zero-order eigenvalues $E_\xi$, where $\xi = x, y, z$, are straightforwardly obtained

$$\frac{E_x}{\hbar} = \omega_x = 0,$$
$$\frac{E_y}{\hbar} = \omega_y = \frac{-\epsilon + \sqrt{\epsilon^2 + 4\Omega_d^2}}{2},$$
$$\frac{E_z}{\hbar} = \omega_z = -\frac{\epsilon + \sqrt{\epsilon^2 + 4\Omega_d^2}}{2}. \tag{41}$$

Moreover, the associated eigenstates $|x\rangle$, $|y\rangle$, and, $|z\rangle$ (the notation refers to the analogy existent with the states of the Cartesian basis [20]) can be written as

$$|\xi\rangle = c_{\xi,1}|1,1\rangle + c_{\xi,0}|1,0\rangle + c_{\xi,-1}|1,-1\rangle, \tag{42}$$

with

$$c_{\xi,1} = \left[2 + \frac{4\omega_\xi^2}{\Omega_d^4}(\omega_\xi+\epsilon)^2 - \frac{4\omega_\xi}{\Omega_d^2}(\omega_\xi/2+\epsilon)\right]^{-1/2},$$
$$c_{\xi,0} = \frac{\sqrt{2}\omega_\xi}{\Omega_d}c_{\xi,1}, \tag{43}$$
$$c_{\xi,-1} = -\left[1 - \frac{2\omega_\xi}{\Omega_d^2}(\omega_\xi+\epsilon)\right]c_{\xi,1}. \tag{44}$$

Now, time-dependent perturbation theory can be directly applied and the probability of the noise-induced transition between two of the dressed states (let us say $|\xi\rangle$ and $|\xi'\rangle$) is given by

$$\langle P_{\xi,\xi'}(t)\rangle = \frac{|(F_x)_{\xi',\xi}|^2}{\hbar^2}\left\langle\left|\int_0^t dt'\delta\omega_0(t')e^{i(\omega_{\xi'}-\omega_\xi)t'}\right|^2\right\rangle. \tag{45}$$

From the analogy of this expression with Eq. (29), it follows that the considerations of the previous section on the reduction of the noise effects achieved in the linear-Zeeman regime by using the CDD method are still applicable. The quadratic corrections modify the zero-order eigenvalues of the perturbative scheme, and consequently, the resonance condition between the transition frequencies and the noisy harmonic components which determines the effectiveness of the method. However, they do not affect the functioning of the dynamical-decoupling mechanism. Note that, as now the zero-order energy levels are not equally spaced, the efficiency of the fluctuations to induce transitions can be dependent on the specific two states involved: the values of the noise spectrum at the different transition frequencies can present a nonnegligible variation.

## V. CONCLUDING REMARKS

Given the variety of sources of noise that can be relevant to the experimental setups, a realistic consideration of the applicability of CDD methods should contemplate the potential role of finite correlation times. Indeed, it is sensible to go beyond a scenario where all the fluctuations (the original input and those resulting from random variations of the different auxiliary fields) are considered to be static. The present study provides some clues to deal with that issue: we rigorously showed that the use of simplified static-noise models is appropriate as far as the interstate transition frequencies are outside the dominant spectral ranges of the fluctuations. Whereas, previous to the application of the CDD method, it is the zero-frequency value of the noise spectrum that determines the asymptotic dephasing rate; in the CDD setup, the decoherence time is basically determined by the noise spectrum at the final effective frequency $\Omega_e$. Decoherence is significantly reduced if $\Omega_e$ does not enter the relevant part of the spectrum. This is the case of the arrangement of [20]. However, in general, the feasibility of extending the coherence times by controlling $\Omega_e$ is not guaranteed. Since the application of the RWA at the different stages of the CDD method implies a reduction in orders of magnitude of $\Omega_e$, reaching a negligible value of the noise spectrum at $\Omega_e$ can be problematic as the concatenation scheme proceeds.

The applicability of the study beyond the considered atomic context can be envisaged. Indeed, the developed approach is appropriate to any system that can be effectively described in terms of a zero-order Hamiltonian and a generic noisy off-diagonal perturbation. Particularly interesting can be the inclusion of $1/f$ noise in this framework. We recall that decoherence in qubits implemented with solid-state devices is frequently studied via a description of the fluctuations as $1/f$ noise. The compact modeling of the associated correlation function [26], which accounts for the spectrum form and introduces cutoff frequencies in the spectral range, can facilitate the application of our approach. Actually, tracing noise-induced transitions at any time would imply a significant advance in the description of the decoherence processes in that environment. Another significant objective is the comparison to the experimental findings to check the validity of the assumed Gaussian approximation or the mere stationary character of noise. One can conjecture that the relative magnitude of the effective transition frequency and the spectral cutoffs must be central in the performance of the CDD methods with $1/f$ noise.

Finally, it is worth depicting some lines of potential applicability of the study in noise spectroscopy [35–38]. A first general consideration refers to the practical use of the analytical descriptions of the decoherence processes in the design of proposals for identifying the fluctuation characteristics. Some specific objectives can be outlined: the prospect of obtaining the noise spectrum by varying the effective transition frequency and measuring the asymptotic value of the decoherence rate seems plausible; indeed, the combination of that strategy with the information extracted from the analysis of different time regimes can serve to improve the scrutiny of the noise properties. The analyticity of the approach allows also the use of designed random signals to check the precision of the proposals. The high level of control achieved in the considered experimental setup, and in fact, in more general contexts, makes it advisable to employ the techniques proposed for the realization of quantum information protocols as elements of noise identification methods.

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