Average coherence and entropy

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Quantum coherence is one of the characteristic features of quantum mechanics and underpins many quantum mysteries. To eliminate the influence of the reference basis on the coherence of a quantum state and uncover its intrinsic properties, it is common to study coherence by averaging over different reference bases. Using the metric-adjusted skew information, we explore three natural approaches to average coherence of a state: average over all orthonormal bases, average over all elements of an operator orthonormal base, and average over a complete family of mutually unbiased bases. We establish the equivalence among these three types of average coherence and interpret the unified average coherence as the coherence of a quantum state relative to a depolarizing channel. Additionally, we employ the unified average coherence to introduce a notion of quantum f entropy (where f is an operator monotone function associated with the metric-adjusted skew information) and demonstrate that quantum f entropy possesses properties analogous to the ubiquitous von Neumann entropy. Furthermore, we illuminate some connections between quantum f entropy and quasientropy, and compare f entropy with von Neumann entropy, Rényi entropy, and Tsallis entropy for some typical states.

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I. INTRODUCTION

Two fundamental and prominent features of quantum mechanics are coherence and entropy, which play crucial roles in quantum formalism and experiments. Quantum coherence, arising from the quantum superposition principle, is a significant feature of quantum mechanics that distinguishes quantum mechanics from classical physics [1–5]. With the rapid development of quantum information, the study of coherence has also evolved from the category of fundamental problems in quantum mechanics to the paradigm of physical resource that can be exploited. In fact, coherence is the cause of quantum interference, quantum nonlocality, and quantum entanglement. Moreover, with the aid of coherence, many tasks that are impossible or difficult to complete by classical methods can now be achieved via quantum information processing. For example, coherence can greatly improve the accuracy of parameter estimation in quantum metrology [6,7], and is crucial for quantum algorithms [8–10], quantum thermodynamics [11–16], and quantum biology [17,18]. These results inspired people to establish various resource theories of coherence from many different angles [19–31]. Several quantifiers of coherence based on relative entropy [20], Tsallis relative α entropy [32,33], max-relative entropy [34], Wigner-Yanase skew information [35,36], metric-adjusted skew information [37], etc., have been introduced, subjected to physical requirements such as monotonicity under certain types of free operations in the resource theory of coherence. In particular, the relative entropy of coherence is a prominent measure of coherence and plays an important role in the process of coherence distillation [38].

As a measure of (missing) information, entropy is a key concept in thermodynamics and information theory [39–42], and is now playing an increasingly important role in quantum information theory [43–48]. Even though Shannon entropy is the most commonly used quantity in information theory, there are still several generalized entropies such as Rényi entropy [49], Havrda-Charvat entropy [50], and Tsallis entropy [51], which have found interesting and important applications in some situations. In quantum theory, von Neumann entropy [40,43], as the analog of classical Shannon entropy, is widely used to quantify correlations and quantum entanglement [43–48], while the corresponding generalized quantum entropies are also powerful in various physical contexts such as entanglement detection [52,53], steering detection [54], quantum key distribution [55], and Bell inequalities [56–58].

Recently researchers take much effort to connect quantum coherence with entanglement [22], quantum correlations [26,59], and quantum uncertainty [29,60]. Considering the fundamental importance and ubiquity of both coherence and entropy, it is desirable to investigate their connections. The present paper is devoted to this issue. Since the canonical coherence quantifiers, such as the relative entropy of coherence, the l_1 norm of coherence, and the coherence based on the Hilbert-Schmidt norm depend on the choice of a reference basis [20], and the coherence measure based on the Wigner-Yanase skew information and a family of coherence measures

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based on the metric-adjusted skew information are all relative to a general channel [31,37], it is natural to study the average coherence in order to get rid of the dependence of the basis or channel and to reveal intrinsic features of coherence encoded in the state. The average coherence based on the Hilbert-Schmidt norm is related to the linear entropy, and the average coherence based on the Wigner-Yanase skew information is related to the unified-(r, s) quantum entropy for $r = 1/2$ and $s = 2$ [61]. These observations and the enquiry on the relations between coherence and entropy inspire us to introduce three versions of average coherence via the metric-adjusted skew information and employ them to introduce quantum f entropy. We reveal basic properties of quantum f entropy, illuminate its connections with quasientropy, and compare quantum f entropy with von Neumann entropy, Rényi entropy, and Tsallis entropy through several typical examples.

The remainder of the paper is structured as follows. In Sec. II, we use the metric-adjusted skew information to investigate three versions of average coherence: the average over all orthonormal bases, the average over all elements of an operator orthonormal basis, and the average over a complete family of mutually unbiased bases (MUBs). In Sec. III, we introduce quantum f entropy related to an operator monotone function f by means of average coherence, discuss its fundamental properties, and show that it indeed possesses many desirable properties intuitively required for a measure of entropy. We further establish a relation between quantum f entropy and quasientropy. We make a comparative study among quantum f entropy, von Neumann entropy, Rényi entropy, and Tsallis entropy in Sec. IV. Finally we conclude with a summary in Sec. V. For simplicity, we assume that the quantum systems are finite dimensional, although many results can be readily extended to infinite dimensional cases. The detailed proofs of the main results are put in the Appendix.

II. AVERAGE COHERENCE IN TERMS OF METRIC-ADJUSTED SKEW INFORMATION

In this section, after recalling the definition of metric-adjusted skew information and some coherence measures of a state (relative to an operator, a reference basis, and a quantum channel, respectively) in terms of metric-adjusted skew information [62], we evaluate three versions of average coherence and establish their equivalence.

Metric-adjusted skew information is a considerable generalization of the Wigner-Yanase-Dyson skew information along the line of quantum Fisher information [62–64]. To illuminate this, we review some basic notions. A function $f : (0, +\infty) \to \mathbb{R}$ is said to be operator monotone if for any natural number n and $A, B \in M_{n,+}(\mathbb{C})$ (the set of nonnegative definite $n \times n$ complex matrices), $0 \leqslant A \leqslant B$ implies $0 \leqslant f(A) \leqslant f(B)$. An operator monotone function is said to be symmetric if $f(x) = xf(x^{-1})$, normalized if $f(1) = 1$, and regular if $f(0) \equiv \lim_{x\to 0} f(x) \neq 0$. We denote the set of all symmetric normalized regular operator monotone functions by $\mathcal{F}_{\rm r}$ [65,66]. Following Refs. [65–68], for $f \in \mathcal{F}_{\rm r}$, let

$$\tilde{f}(x) = \frac{1}{2}\left((x+1) - (x-1)^2\frac{f(0)}{f(x)}\right), \quad x > 0, \tag{1}$$

and the associated (numerical) mean is defined as

$$m_f(x, y) = yf(xy^{-1}), \quad x, y > 0. \tag{2}$$

There exists a bijective correspondence between monotone metrics (quantum Fisher information) and $f \in \mathcal{F}_{\rm r}$, which is given by [69]

$$\langle A, B\rangle_{\rho,f} = {\rm tr}[A^\dagger m_f(L_\rho, R_\rho)^{-1}(B)] \tag{3}$$

for operators A and B (not necessarily Hermitian) on the system Hilbert space. Here $L_X(B) = XB$ and $R_X(B) = BX$ are the left and right multiplication by an operator X, respectively.

Notice that the right-hand side of Eq. (3) can be calculated explicitly as follows. Let ρ be a state on a d-dimensional system with the spectral decomposition $\rho = \sum_i \lambda_i|\phi_i\rangle\langle\phi_i|$, where $\lambda_i > 0$ are the eigenvalues with corresponding eigenvectors $|\phi_i\rangle$, and then for any reasonable function $s : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$, by use of functional calculus, the corresponding operator function of L_ρ and R_ρ can be expressed as [66]

$$s(L_\rho, R_\rho) = \sum_{i,j} s(\lambda_i, \lambda_j)L_{\Pi_i}R_{\Pi_j},$$

where $\Pi_i = |\phi_i\rangle\langle\phi_i|$, $i = 1, 2, \ldots, m$. In particular,

$$m_f(L_\rho, R_\rho) = \sum_{i,j} m_f(\lambda_i, \lambda_j)L_{\Pi_i}R_{\Pi_j},$$

$$m_f(L_\rho, R_\rho)^{-1} = \sum_{i,j} \frac{1}{m_f(\lambda_i, \lambda_j)}L_{\Pi_i}R_{\Pi_j}.$$

Consequently,

$$\langle A, B\rangle_{\rho,f} = \sum_{i,j} \frac{1}{m_f(\lambda_i, \lambda_j)}{\rm tr}(A^\dagger\Pi_i B\Pi_j).$$

Metric-adjusted skew information of ρ relative to an operator A (not necessarily Hermitian) is defined as [62]

$$I_f(\rho, A) = \frac{f(0)}{2}\langle[\rho, A], [\rho, A]\rangle_{\rho,f}, \tag{4}$$

which can be viewed as a coherence measure of ρ relative to the operator A. By simple calculation, we have

$$I_f(\rho, A) = \tfrac{1}{2}{\rm tr}[\rho(A^\dagger A + AA^\dagger)] - {\rm tr}[A^\dagger m_{\tilde{f}}(L_\rho, R_\rho)(A)].$$

Metric-adjusted skew information $I_f(\rho, A)$ defined by Eq. (4) has the following properties, which follow from Refs. [37,62,63], or can be directly verified.

(i) $0 \leqslant I_f(\rho, A) \leqslant V(\rho, A)$. Moreover, $I_f(\rho, A) = 0$ if and only if $[\rho, A] = 0$, and $I_f(\rho, A) = V(\rho, A)$ if ρ is a pure state, where $V(\rho, A) = \frac{1}{2}{\rm tr}(\rho(A^\dagger A + AA^\dagger)) - |{\rm tr}(\rho A)|^2$ is the generalized variance. In particular, for any pure state ρ, $I_f(\rho, A)$ is independent of f.

(ii) $I_f(U\rho U^\dagger, UAU^\dagger) = I_f(\rho, A)$, and $I_f(U\rho U^\dagger, A) = I_f(\rho, U^\dagger AU)$ for any unitary operator U.

(iii) $I_f(\rho, A)$ is convex in ρ.

(iv) $I_f(\rho, A)$ is additive under tensoring in the sense that

$$I_f(\rho\otimes\sigma, A\otimes\mathbf{1}^b + \mathbf{1}^a\otimes B) = I_f(\rho, A) + I_f(\sigma, B) \tag{5}$$

for any quantum states ρ and σ, and any operators A and B on parties a and b, respectively. Here $\mathbf{1}^a$ and $\mathbf{1}^b$ are the identity operator on parties a and b, respectively. In particular,

$$I_f(\rho\otimes\sigma, A\otimes \mathbf{1}^b) = I_f(\rho, A). \quad (6)$$

(v) $I_f(\rho, A)$ is additive under direct sum in the sense that

$$I_f\left(\bigoplus_j p_j\rho_j, \bigoplus_j A_j\right) = \sum_j p_j I_f(\rho_j, A_j), \quad (7)$$

for any quantum states ρ_j, any operators A_j, and any probability distribution $\{p_j\}$.

(vi) For any bipartite state ρ^{ab} shared by parties a and b and any operator A^a on party a,

$$I_f(\rho^{ab}, A^a\otimes \mathbf{1}^b) \geqslant I_f(\rho^a, A^a), \quad (8)$$

with $\rho^a = \mathrm{tr}_b\rho^{ab}$ the reduced state on party a of the bipartite state ρ^{ab}.

Let $\{|i\rangle : i = 1, 2, \dots, d\}$ be an orthonormal basis of the system Hilbert space. A simple and natural measure of coherence of ρ relative to the reference basis $\{|i\rangle : i = 1, 2, \dots, d\}$, or equivalently, relative to the corresponding von Neumann measurement $\Pi = \{\Pi_i = |i\rangle\langle i| : i = 1, 2, \dots, d\}$, can be defined as [37]

$$C_f(\rho|\Pi) = \sum_i I_f(\rho, \Pi_i) = \sum_i I_f(\rho, |i\rangle\langle i|), \quad (9)$$

which can be naturally generalized to the coherence of ρ relative to a quantum channel $\mathcal{E}$ as [37]

$$C_f(\rho|\mathcal{E}) = \sum_j I_f(\rho, E_j). \quad (10)$$

Here E_j are Kraus operators of $\mathcal{E}$, i.e., $\mathcal{E}(\rho) = \sum_j E_j\rho E_j^\dagger$, $\sum_j E_j^\dagger E_j = \mathbf{1}$ (identity operator).

The coherence measure $C_f(\rho|\mathcal{E})$ enjoys many desirable properties required for a reasonable coherence quantifier. However, it depends crucially on the channel $\mathcal{E}$. To reveal intrinsic features of coherence encoded in the state ρ, one may take certain average of $C_f(\rho|\mathcal{E})$ with respect to the channel $\mathcal{E}$. We will consider the following three averaging procedures: Average coherence over all orthonormal bases, average coherence over all elements of an operator orthonormal basis, and average coherence over a complete family of MUBs. We will further demonstrate their equivalence, and employ them to study relations between coherence and entropy.

A. Average coherence over all orthonormal bases

The first averaging procedure is to consider channels induced by von Neumann measurements (or equivalently, orthonormal bases) and take the average of $C_f(\rho|\Pi)$ with respect to all von Neumann measurements (reference bases) Π. Since any reference basis can be obtained by a unitary operation on a fixed basis, this averaging procedure amounts to integration over the unitary orbit of a fixed basis, and is equivalent to the integration over the unitary group equipped with the normalized Haar measure. Thus let us define

$$C_f^{\mathcal{U}}(\rho) = \int_{\mathcal{U}} C_f(\rho|U\Pi U^\dagger)dU, \quad (11)$$

where dU denotes the normalized Haar measure on the full unitary group $\mathcal{U}$ of the system Hilbert space, and $U\Pi U^\dagger = \{U|i\rangle\langle i|U^\dagger : i = 1, 2, \dots, d\}$.

Proposition 1. For any state ρ in a d-dimensional system and $f\in\mathcal{F}_{\mathrm{r}}$, it holds that

$$C_f^{\mathcal{U}}(\rho) = \frac{d - \mathrm{tr}[m_{\tilde f}(L_\rho, R_\rho)]}{d+1}. \quad (12)$$

Let λ_i be the eigenvalues of ρ, then $\mathrm{tr}[m_{\tilde f}(L_\rho, R_\rho)] = \sum_{ij} m_{\tilde f}(\lambda_i, \lambda_j)$.

For the proof, see Appendix A 1.

Equation (12) has two interesting interpretations: The first is as an exact uncertainty relation concerning all von Neumann measurements, and the second is as an intrinsic measure of quantum uncertainty of ρ. In contrast to the results involving the l_1 norm and relative entropy of coherence obtained in Ref. [70], we have derived an exact complementary relation for coherence based on the metric-adjusted skew information. This indicates that the coherence measure based on metric-adjusted skew information enjoys some nice geometric features.

B. Average coherence over all elements of an operator orthonormal basis

The second natural averaging procedure is the average coherence

$$C_f^{\mathrm{ob}}(\rho) = \frac{1}{d+1}\sum_{\alpha=1}^{d^2} I_f(\rho, X_\alpha) \quad (13)$$

of ρ with respect to all elements of an operator orthonormal basis. Here $\{X_\alpha : \alpha = 1, 2, \dots, d^2\}$ is a family of d^2 operators, which constitutes an operator orthonormal basis of the Hilbert space $L(H)$, the set of all bounded linear operators on H with the Hilbert-Schmidt inner product $\langle A, B\rangle = \mathrm{tr}(A^\dagger B)$. This quantity, as a measure of information content in the state ρ, is actually independent of the choice of the operator orthonormal basis, as can be proved in a similar fashion to Ref. [71].

Proposition 2. For any state ρ in a d-dimensional system and $f\in\mathcal{F}_{\mathrm{r}}$, it holds that

$$C_f^{\mathrm{ob}}(\rho) = \frac{d - \mathrm{tr}[m_{\tilde f}(L_\rho, R_\rho)]}{d+1}. \quad (14)$$

In particular, this average coincides with $C_f^{\mathcal{U}}(\rho)$ given by Eq. (12).

For the proof, see Appendix A 2.

Equation (14) relates coherence of ρ relative to elements of an operator orthonormal basis, and can also be interpreted as an exact uncertainty relation for operators. In particular, for any pure state $\rho = |\psi\rangle\langle\psi|$ in a d-dimensional system, noting that $\mathrm{tr}(m_{\tilde f}(L_\rho, R_\rho)) = 1$, we have

$$C_f^{\mathrm{ob}}(|\psi\rangle\langle\psi|) = \frac{d-1}{d+1}.$$

C. Average coherence over a complete family of MUBs

The third averaging procedure is over a complete family of MUBs. Recall that two orthonormal bases $B_1 = \{|b_{1i}\rangle :$

$i = 1, 2, \ldots, d\}$ and $B_2 = \{|b_{2i}\rangle : i = 1, 2, \ldots, d\}$ of a d-dimensional system are mutually unbiased if [72,73]

$$|\langle b_{1i}|b_{2j}\rangle|^2 = \frac{1}{d}, \quad i, j = 1, 2, \ldots, d.$$

When the dimension d is a prime power (i.e., $d = p^k$ for a prime number p and a positive integer k), there always exists a complete family of $d+1$ MUBs $\{B_\nu : \nu = 1, 2, \ldots, d+1\}$ with $B_\nu = \{|b_{\nu i}\rangle : i = 1, 2, \ldots, d\}$ [72,73]. In general, the existence of a complete family of MUBs in any dimension is an outstanding open issue, even in dimension 6 [74].

Let

$$C_f^{\mathrm{mub}}(\rho) = \frac{1}{d+1}\sum_{\nu=1}^{d+1} C_f(\rho|B_\nu) \tag{15}$$

be the average coherence of ρ with respect to a complete family of MUBs $\{B_\nu : \nu = 1, 2, \ldots, d+1\}$, which is actually independent of the choice of the complete family of MUBs, as will be seen in the proof of the following result.

Proposition 3. For any state ρ in a d-dimensional system with d a prime power and $f \in \mathcal{F}_{\mathrm{r}}$, it holds that

$$C_f^{\mathrm{mub}}(\rho) = \frac{d - \mathrm{tr}[m_{\tilde{f}}(L_\rho, R_\rho)]}{d+1}. \tag{16}$$

For the proof, see Appendix A 3.

We remark that the above result holds for any dimension as long as a complete family of MUBs exists in this dimension.

D. Equivalence

By inspecting Eqs. (12), (14), and (16), we have the following result.

Proposition 4. For any state ρ of any prime power dimensional system and any operator monotone function $f \in \mathcal{F}_{\mathrm{r}}$, we have

$$C_f^{\mathcal{U}}(\rho) = C_f^{\mathrm{ob}}(\rho) = C_f^{\mathrm{mub}}(\rho).$$

This shows that average coherences over the unitary group, over all elements of any operator orthonormal basis, and over any complete family of MUBs are equivalent. Due to the equivalence among the three kinds of average coherence, we henceforth denote the unified average coherence by $C_f(\rho)$ for simplicity. To emphasize this notation, we write the following equation for the convenience of reference:

$$C_f(\rho) = C_f^{\mathcal{U}}(\rho) = C_f^{\mathrm{ob}}(\rho) = C_f^{\mathrm{mub}}(\rho). \tag{17}$$

Next, we explain this equivalence from the perspective of the quantum channel. According to the proof of Proposition 3, we find that

$$C_f^{\mathrm{mub}}(\rho) = C_f(\rho|\mathcal{E}_{\mathrm{De}}),$$

where the depolarizing quantum channel $\mathcal{E}_{\mathrm{De}}$ is defined as

$$\mathcal{E}_{\mathrm{De}}(X) = \frac{1}{d+1}X + \frac{1}{d+1}\mathrm{tr}(X)\mathbf{1}. \tag{18}$$

By substituting Eq. (9) into Eq. (11), we have

$$C_f^{\mathcal{U}}(\rho) = \int_{\mathcal{U}} \sum_i I_f(\rho, U\Pi_i U^\dagger)dU.$$

Notice that

$$\int_{\mathcal{U}} \sum_i (U\Pi_i U^\dagger)X(U\Pi_i U^\dagger)dU = \frac{1}{d+1}X + \frac{1}{d+1}\mathrm{tr}(X)\mathbf{1}$$

by using Eq. (A1) which is given in Appendix A 1. Hence, the average coherence over the unitary group is precisely the coherence relative to the channel $\mathcal{E}_{\mathrm{De}}$ defined by Eq. (18). By the definition of average coherence over all elements of an operator orthonormal basis, it follows that

$$C_f^{\mathrm{ob}}(\rho) = \sum_{\alpha=1}^{d^2} I_f\left(\rho, \frac{1}{\sqrt{d+1}}X_\alpha\right) + I_f\left(\rho, \frac{\mathbf{1}}{\sqrt{d+1}}\right),$$

which is also the coherence of ρ relative to the channel $\mathcal{E}_{\mathrm{De}}$ since $\{\mathbf{1}/\sqrt{d+1}, X_\alpha/\sqrt{d+1} : \alpha = 1, 2, \ldots, d^2\}$ is a family of Kraus operators of $\mathcal{E}_{\mathrm{De}}$.

To sum up, the average coherences over the unitary group, over elements of any operator orthonormal basis, and over any complete family of MUBs are all equal to the coherence relative to the depolarizing channel, which is a mixture of the identity channel with probability $1/(d+1)$ and the completely depolarizing channel with probability $d/(d+1)$.

We further specialize to two important cases.

(a) For

$$f(x) = f_{\mathrm{WY}}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2, \tag{19}$$

we have

$$C_{f_{\mathrm{WY}}}(\rho) = \frac{d - (\mathrm{tr}\sqrt{\rho})^2}{d+1}, \tag{20}$$

which is precisely the average coherence based on the Wigner-Yanase skew information [61].

(b) For

$$f(x) = f_{\mathrm{WYD}}(x) = \frac{\alpha(1-\alpha)(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha} - 1)} \tag{21}$$

with $\alpha \in (0, 1)$, we have the average coherence

$$C_{f_{\mathrm{WYD}}}(\rho) = \frac{d - \mathrm{tr}(\rho^\alpha)\mathrm{tr}(\rho^{1-\alpha})}{d+1},$$

which is related to the Wigner-Yanase-Dyson skew information.

A general coherence measure $C_f(\rho|\mathcal{E})$ of a quantum state ρ relative to a quantum channel $\mathcal{E}$ is studied in Ref. [37], which has many nice properties such as unitary invariance, convexity, and monotonicity. Since these three versions of average coherence are equivalent and equal to the coherence $C_f(\rho|\mathcal{E}_{\mathrm{De}})$ of the quantum state ρ relative to the depolarizing channel $\mathcal{E}_{\mathrm{De}}$, the average coherence also has the above nice properties and thus is indeed a coherence measure.

III. QUANTUM f ENTROPY IN TERMS OF OPERATOR MONOTONE FUNCTIONS

In this section, we first discuss a complementary relation between the average coherence $C_{\mathrm{HS}}^{\mathcal{U}}(\rho)$ based on the Hilbert-Schmidt norm and the linear entropy $S_2(\rho)$ as well as that between the average coherence $C_{f_{\mathrm{WY}}}(\rho)$ based on

Wigner-Yanase skew information and the unified-(r, s) quantum entropy $S_r^s(\rho)$ for $r = 1/2$ and $s = 2$. These observations, together with the desire to connect coherence and entropy, inspire us to employ the average coherence $C_f(\rho)$ to introduce a family of quantum entropies which we call quantum f entropy. Furthermore, we prove that quantum f entropy possesses several desirable properties analogous to the ubiquitous von Neumann entropy and establish a relation between quantum f entropy and quasientropy [75–77].

Let us start by reviewing two complementary relations between average coherence and entropy. First, recall that the coherence of ρ relative to a reference basis $\{|i\rangle : i = 1, 2, \ldots, d\}$, or equivalently, the corresponding von Neumann measurement $\Pi = \{\Pi_i = |i\rangle\langle i| : i = 1, 2, \ldots, d\}$, is defined as [20]

$$C_{\rm HS}(\rho|\Pi) = \left\| \rho - \sum_i \Pi_i \rho \Pi_i \right\|^2.$$

Similar to the first averaging procedure in Sec. II. A, the average coherence of the state ρ based on the Hilbert-Schmidt norm with respect to all von Neumann measurements can be directly calculated as

$$C_{\rm HS}^{\mathcal{U}}(\rho) = \int_{\mathcal{U}} C_{\rm HS}(\rho|U\Pi U^\dagger)dU = \frac{d{\rm tr}\rho^2 - 1}{d+1}.$$

This implies that the linear entropy defined by $S_2(\rho) = 1 - {\rm tr}\rho^2$ is related to $C_{\rm HS}^{\mathcal{U}}(\rho)$ by the following equality:

$$dS_2(\rho) + (d+1)C_{\rm HS}^{\mathcal{U}}(\rho) = d - 1. \quad (22)$$

Secondly, recall that the unified-(r, s) quantum entropy is defined as [78]

$$S_r^s(\rho) = \frac{1}{(1-r)s}\big[({\rm tr}\rho^r)^s - 1\big], \quad r \neq 1,\ s \neq 0.$$

In particular, when $r = 1/2$, $s = 2$, it reduces to

$$S_{1/2}^2(\rho) = ({\rm tr}\sqrt{\rho})^2 - 1,$$

which turns out to be related to the average coherence $C_{f_{\rm WY}}(\rho)$, i.e., Eq. (20), by the following equality:

$$S_{1/2}^2(\rho) + (d+1)C_{f_{\rm WY}}(\rho) = d - 1. \quad (23)$$

Notice that both Eqs. (22) and (23) imply that there exist complementary relations between average coherence and entropy. So, it is natural to generalize the entropy

$$S_{1/2}^2(\rho) = (d-1) - (d+1)C_{f_{\rm WY}}(\rho)$$

related to the operator monotone function $f_{\rm WY}$ to that related to an arbitrary operator monotone function $f \in \mathcal{F}_{\rm r}$. That is, given a function $f \in \mathcal{F}_{\rm r}$, using the average coherence $C_f(\rho)$, we can introduce the corresponding entropy

$$S_f(\rho) \equiv (d-1) - (d+1)C_f(\rho), \quad (24)$$

which we call quantum f entropy. By use of Eq. (12), it can be reexpressed as

$$S_f(\rho) = {\rm tr}[m_{\tilde{f}}(L_\rho, R_\rho)] - 1. \quad (25)$$

From Eq. (24), it follows that

$$S_f(\rho) + (d+1)C_f(\rho) = d - 1, \quad (26)$$

which signifies a complementary relation between quantum f entropy and the average coherence $C_f(\rho)$.

Next, we show that $S_f(\rho)$ is indeed a bona fide measure of entropy, as consolidated by the following properties.

Proposition 5. The following hold.

(1) $S_f(\rho) \geqslant 0$, and the equality holds if and only if ρ is a pure state.

(2) For a d-dimensional system, the f entropy of a quantum state is at most $d - 1$. The f entropy is equal to $d - 1$ if and only if the system is in the completely mixed state $\mathbf{1}/d$.

(3) $S_f(\rho)$ is concave in ρ in the sense that

$$S_f\left(\sum_j p_j\rho_j\right) \geqslant \sum_j p_j S_f(\rho_j),$$

where $\{p_j\}$ is a probability distribution and ρ_j are states.

(4) Suppose a composite system shared by parties a and b is in a pure state $|\Psi\rangle\langle\Psi|$, then $S_f(\rho^a) = S_f(\rho^b)$, where $\rho^a = {\rm tr}_b|\Psi\rangle\langle\Psi|$, and $\rho^b = {\rm tr}_a|\Psi\rangle\langle\Psi|$.

(5) $S_f(\rho \otimes \mathbf{1}/d) = S_f(\mathbf{1}/d \otimes \rho) = dS_f(\rho) + S_f(\mathbf{1}/d)$.

(6) $S_f(\sum_j p_j|j\rangle\langle j| \otimes \rho_j) \geqslant \sum_j p_j S_f(\rho_j)$ where $\{|j\rangle\}$ is an orthonormal basis of one system, ρ_j are quantum states on another system, and $\{p_j\}$ is a probability distribution.

(7) $S_f(U\rho U^\dagger) = S_f(\rho)$ for any unitary operator U.

(8) For any random unitary channel $\mathcal{E}_{\rm RU}(\rho) = \sum_k p_k U_k \rho U_k^\dagger$ with U_k unitary operators and $\{p_k\}$ a probability distribution, we have $S_f(\mathcal{E}_{\rm RU}(\rho)) \geqslant \sum_k p_k S_f(\rho)$.

(9) Let $f \in \mathcal{F}_{\rm r}$ be a bounded continuous operator monotone function. Assume that ρ_n such that $\rho_n \to \rho$ in norm, then $S_f(\rho_n) \to S_f(\rho)$.

For the proof, see Appendix A 4.

From Eqs. (13) and (17), we have another representation of quantum f entropy:

$$S_f(\rho) = d - 1 - \sum_{\alpha=1}^{d^2} I_f(\rho, X_\alpha), \quad (27)$$

which is independent of the choice of the operator orthonormal basis $\{X_\alpha : \alpha = 1, 2, \ldots, d^2\}$. According to item 1 in Proposition 5, $S_f(\rho) = 0$ if and only if ρ is a pure state, which is equivalent to the average coherence $C_f(\rho)$ reaching its maximum $(d-1)/(d+1)$. According to item 2 in Proposition 5, the quantum f entropy reaches its maximum $d - 1$ if and only if the system is in the completely mixed state $\mathbf{1}/d$, which is equivalent to the average coherence reaching its minimum zero.

In order to relate quantum f entropy to quasientropy, we now recall the concept of quasientropy. Let $M_n(\mathbb{C})$ denote the algebra of $n \times n$ complex matrices. For positive definite matrices (thus invertible) ρ_1, ρ_2, and $A \in M_n(\mathbb{C})$ and a function $f : \mathbb{R}^+ \equiv [0, \infty) \to \mathbb{R}$, the quasientropy is defined as [75–77]

$$S_f^A(\rho_1|\rho_2) = \big\langle A\rho_2^{1/2}, f[\Delta(\rho_1/\rho_2)]\big(\rho_2^{1/2}\big)\big\rangle,$$

where $\langle B, C\rangle = {\rm tr}(B^\dagger C)$ is the Hilbert-Schmidt inner product and $\Delta(\rho_1/\rho_2) : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is the linear mapping defined by $\Delta(\rho_1/\rho_2)(B) = \rho_1 B\rho_2^{-1}$. Quasientropy can be expressed in terms of the mean as

$$S_f^A(\rho_1|\rho_2) = \big\langle A, m_f\big(L_{\rho_1}, R_{\rho_2}\big)(A)\big\rangle.$$

In particular, when $\rho_1 = \rho_2 = \rho$, we get

$$S_f^A(\rho|\rho) = \langle A, m_f(L_\rho, R_\rho)(A)\rangle. \tag{28}$$

From the proof of item 9 in Proposition 5 (see Appendix A 4), we obtain

$$S_f(\rho) = \sum_{\alpha=1}^{d^2} S_{\tilde{f}}^{X_\alpha}(\rho|\rho) - 1, \tag{29}$$

which connects quantum f entropy and quasientropy.

Combining Eqs. (27) and (29), we have the tradeoff relation

$$\sum_{\alpha=1}^{d^2} I_f(\rho, X_\alpha) + \sum_{\alpha=1}^{d^2} S_{\tilde{f}}^{X_\alpha}(\rho|\rho) = d, \tag{30}$$

which can be viewed as a complementary relation for metric-adjusted skew information and quasientropy. In terms of the average coherence, Eq. (30) can be expressed as

$$(d+1)C_f(\rho) + \sum_{\alpha=1}^{d^2} S_{\tilde{f}}^{X_\alpha}(\rho|\rho) = d. \tag{31}$$

From the tradeoff relations (26) and (31), we find that the average coherence $C_f(\rho)$ characterizes the information content in the state ρ and is associated with the quantum f entropy or quasientropy of ρ. This indicates that the entropy of a state cannot be arbitrarily small when the average coherence is very small.

We now evaluate quantum f entropy for some typical operator monotone functions.

(a) For $f(x) = f_{\rm WY}(x)$ defined by Eq. (19), the quantum f entropy can be evaluated as

$$S_{f_{\rm WY}}(\rho) = (\mathrm{tr}\sqrt{\rho})^2 - 1, \tag{32}$$

which is just $S^2_{1/2}(\rho)$ according to Eqs. (23) and (26).

(b) For $f(x) = f_{\rm WYD}(x)$ defined by Eq. (21), the quantum f entropy can be evaluated as

$$S_{f_{\rm WYD}}(\rho) = \mathrm{tr}\rho^\alpha \mathrm{tr}\rho^{1-\alpha} - 1, \quad \alpha \in (0,1). \tag{33}$$

(c) For $f(x) = f_{\rm SLD}(x) = (1+x)/2$, we have

$$I_{f_{\rm SLD}}(\rho, A) = \tfrac{1}{4}\mathrm{tr}(\rho L^2),$$

which is the quantum Fisher information defined via symmetric logarithmic derivative L determined by the operator equation $(L\rho + \rho L)/2 = i[\rho, A]$ [79–81]. The quantum f entropy can be evaluated as

$$S_{f_{\rm SLD}}(\rho) = \sum_{ij} \frac{2\lambda_i\lambda_j}{\lambda_i + \lambda_j} - 1, \tag{34}$$

where λ_i are the nonzero eigenvalues of ρ.

It is easy to prove that if $\tilde{f} \leqslant \tilde{g}$, then $I_f(\rho, A) \geqslant I_g(\rho, A)$, which yields $S_f(\rho) \leqslant S_g(\rho)$. In addition, by a similar proof given in Refs. [67,81], we have

$$I_f(\rho, A) \leqslant I_{f_{\rm SLD}}(\rho, A) \leqslant \frac{1}{2f(0)} I_f(\rho, A),$$

and the constant $1/(2f(0))$ is optimal. Thus,

$$S_{f_{\rm SLD}}(\rho) \leqslant S_f(\rho) \leqslant \bar{S}_f(\rho), \tag{35}$$

where

$$\bar{S}_f(\rho) = 2f(0)S_{f_{\rm SLD}}(\rho) + (1 - 2f(0))S_f(\mathbf{1}/d)$$

is an average of quantum entropy $S_{f_{\rm SLD}}(\rho)$ and quantum f entropy of the maximally mixed state. Thus the quantum f entropy $S_f(\rho)$ is lower bounded by the quantum entropy related to operator monotone function $f_{\rm SLD}$, and upper bounded by the average of quantum entropy $S_{f_{\rm SLD}}(\rho)$ and quantum f entropy of the maximally mixed state.

IV. COMPARISON

In order to gain a more concrete and intuitive understanding of quantum f entropy and its relations with other entropies, we compare it with several typical quantum entropies in the literature.

For a qubit state expressed in the computational basis $\{|0\rangle, |1\rangle\}$ as

$$\rho = \frac{1}{2}\left(\mathbf{1} + \sum_{i=1}^{3} r_i\sigma_i\right) = \frac{1}{2}\begin{pmatrix} 1+r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3\end{pmatrix}, \tag{36}$$

where $r_i \in \mathbb{R}$, $\mathbf{1}$ is the identity operator, and σ_i are the Pauli matrices, it can be directly verified that it has the following spectral decomposition:

$$\rho = \lambda_1|\psi_1\rangle\langle\psi_1| + \lambda_2|\psi_2\rangle\langle\psi_2|, \tag{37}$$

with eigenvalues

$$\lambda_1 = \tfrac{1}{2}(1+r), \quad \lambda_2 = \tfrac{1}{2}(1-r),$$

and the corresponding eigenvectors

$$|\psi_1\rangle = \frac{(r_1 - ir_2)|0\rangle - (r_3 - r)|1\rangle}{\sqrt{2r(r - r_3)}},$$

$$|\psi_2\rangle = \frac{(r_1 - ir_2)|0\rangle - (r_3 + r)|1\rangle}{\sqrt{2r(r + r_3)}}.$$

Here $r = \sqrt{r_1^2 + r_2^2 + r_3^2} \leqslant 1$ is the modulus of the Bloch vector of ρ.

Before comparing entropies of ρ, let us first focus on the connections among the linear entropy, the purity, and the average coherence as well as quantum f entropy through this example. From the spectral decomposition of ρ, we readily obtain Tsallis-2 entropy (i.e., the linear entropy)

$$S_2(\rho) = 1 - \mathrm{tr}\rho^2 = \tfrac{1}{2}(1 - r^2),$$

and the purity [82]

$$P(\rho) = \mathrm{tr}\rho^2 = \tfrac{1}{2}(1 + r^2),$$

which implies that

$$S_2(\rho) + P(\rho) = 1. \tag{38}$$

The average coherence of the state ρ can be directly evaluated as

$$C_f(\rho) = \frac{2r^2 f(0)}{3(1+r)f\left(\frac{1-r}{1+r}\right)}, \tag{39}$$

TABLE I. Comparison among the linear entropy, purity, f entropy, and average coherence of qubit states ρ defined by Eq. (36) with $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$.

	$S_2(\rho)$	$P(\rho)$	$S_f(\rho)$	$C_f(\rho)$
Expression	$\frac{1-r^2}{2}$	$\frac{1+r^2}{2}$	$1 - \frac{2r^2 f(0)}{(1+r)f(\frac{1-r}{1+r})}$	$\frac{2r^2 f(0)}{3(1+r)f(\frac{1-r}{1+r})}$
Maximum	$\frac{1}{2}$	1	1	$\frac{1}{3}$
Minimum	0	$\frac{1}{2}$	0	0
Convex or concave	Concave	Convex	Concave	Convex

and the quantum f entropy of the state ρ is

$$S_f(\rho) = 1 - \frac{2r^2 f(0)}{(1+r)f\left(\frac{1-r}{1+r}\right)}.$$

Consequently,

$$S_f(\rho) + 3C_f(\rho) = 1, \tag{40}$$

which is consistent with Eq. (26).

In order to compare Eqs. (38) and (40) more clearly, we explain the difference in Table I. From the table, we see that when the quantum state is the maximally mixed state, the linear entropy reaches the maximum $1/2$ and the f entropy also reaches the maximum 1, while the purity reaches its minimum $1/2$, and the average coherence reaches the minimum zero. When the quantum state is a pure state, the linear entropy reaches its minimum zero, the f entropy also reaches the minimum zero, while the purity reaches the maximum 1, and the average coherence reaches the maximum $1/3$.

Now, we proceed to discuss relations and difference between quantum f entropy and some commonly used entropies such as von Neumann entropy, Rényi entropy, and Tsallis entropy. Recall that for a quantum state ρ, the von Neumann entropy was defined as [40]

$$S(\rho) = -\mathrm{tr}\,\rho \log_2 \rho,$$

and Rényi-2 entropy was defined as [49]

$$R_2(\rho) = -\log_2 \mathrm{tr}\rho^2.$$

For a qubit state ρ, these entropies can be readily evaluated as

$$S(\rho) = 1 - \frac{1}{2}\log_2(1-r^2) + \frac{r}{2}\log_2\frac{1-r}{1+r},$$
$$R_2(\rho) = 1 - \log_2(1+r^2).$$

Using Eqs. (32) and (34), we can evaluate quantum f entropy of the qubit state ρ by specifying f to some typical operator monotone functions as

$$S_{f_{\mathrm{WY}}}(\rho) = \sqrt{1-r^2}, \quad S_{f_{\mathrm{SLD}}}(\rho) = 1 - r^2.$$

We depict the graphs of the above entropies in Fig. 1. We see that all the five entropies behave similarly in some sense. That is, they are all concave and decrease with respect to r. When $r = 0$, which corresponds to $\rho = \mathbf{1}/2$, all the entropies achieve the maximum and all the maxima of these entropies are 1 except for Tsallis-2 entropy whose maximum is $1/2$. When $r = 1$, which means that ρ is a pure state, all the entropies achieve the minimum zero. Furthermore, we notice that $S_{f_{\mathrm{SLD}}}(\rho)$ is the smallest in the family of f entropy, which is consistent with Eq. (35).

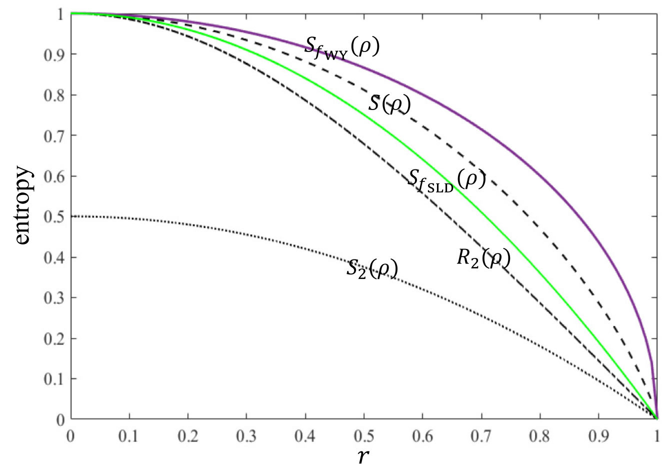

FIG. 1. Comparison among von Neumann entropy $S(\rho)$, Rényi-2 entropy $R_2(\rho)$, Tsallis-2 entropy $S_2(\rho)$, and quantum f entropy including $S_{f_{\mathrm{WY}}}(\rho)$ and $S_{f_{\mathrm{SLD}}}(\rho)$ as functions of r for qubit state ρ defined by Eq. (36) with $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$.

The qubit systems provide illuminating examples to illustrate that the coherence measure $C_f(\rho|\Pi)$ defined by Eq. (9) is dependent on the choice of the reference basis, while the average coherence $C_f(\rho)$ does not since it can be directly verified that if we choose $\Pi = \{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, then

$$C_f(\rho|\Pi) = \frac{f(0)\left(r_1^2 + r_2^2\right)}{(1+r)f\left(\frac{1-r}{1+r}\right)}. \tag{41}$$

If we choose another orthonormal basis

$$\left\{|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),\ \ |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right\}$$

with the corresponding von Neumann measurement being $\Pi' = \{|+\rangle\langle +|, |-\rangle\langle -|\}$, then

$$C_f(\rho|\Pi') = \frac{f(0)\left(r_2^2 + r_3^2\right)}{(1+r)f\left(\frac{1-r}{1+r}\right)}, \tag{42}$$

which is obviously different from $C_f(\rho|\Pi)$ when $r_1 \neq r_3$.

Next, we consider the Werner states

$$\mathbf{w} = \frac{d-x}{d^3-d}\mathbf{1}_d \otimes \mathbf{1}_d + \frac{dx-1}{d^3-d}F, \quad x \in [-1, 1]$$

on $\mathbb{C}^d \otimes \mathbb{C}^d = \mathbb{C}^{d^2}$ with $\{|\mu\rangle : \mu = 1, 2, \ldots, d\}$ an orthonormal basis of the d-dimensional Hilbert space $\mathbb{C}^d$ and $F = \sum_{\mu,\nu=1}^d |\mu\rangle\langle\nu| \otimes |\nu\rangle\langle\mu|$ the swap operation. The spectral decomposition of $\mathbf{w}$ reads [83]

$$\mathbf{w} = \frac{2p}{d^2+d}\Pi_s + \frac{2(1-p)}{d^2-d}\Pi_a, \quad p \in [0, 1] \tag{43}$$

with eigenvalues

$$\lambda_1 = \frac{2p}{d^2+d}, \quad \lambda_2 = \frac{2(1-p)}{d^2-d}$$

of multiplicities $(d^2+d)/2$ and $(d^2-d)/2$, respectively. Here Π_s and Π_a are projections onto the symmetric and

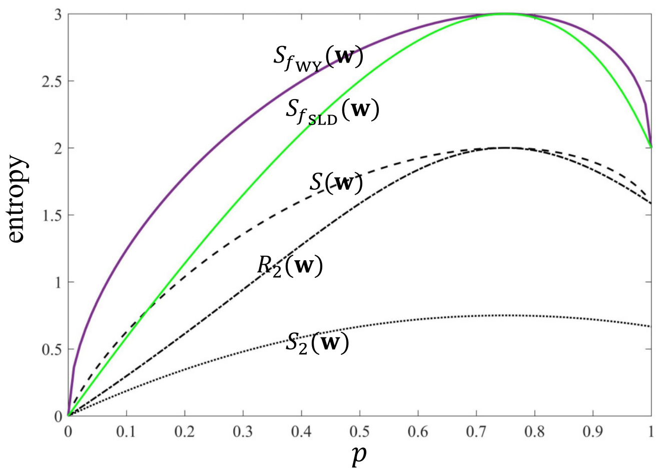

FIG. 2. Comparison among von Neumann entropy $S(\mathbf{w})$, Tsallis-2 entropy $S_2(\mathbf{w})$, Rényi-2 entropy $R_2(\mathbf{w})$, and quantum f entropy including $S_{f_{\rm WY}}(\mathbf{w})$ and $S_{f_{\rm SLD}}(\mathbf{w})$ as functions of $p \in [0, 1]$ for Werner states $\mathbf{w}$ defined by Eq. (43) with $d = 2$.

antisymmetric subspaces of $\mathbb{C}^d \otimes \mathbb{C}^d$, respectively, and $p = \mathrm{tr}(\mathbf{w}\Pi_s)$. It can be directly evaluated that

$$S(\mathbf{w}) = -p\log_2 \frac{2p}{d^2+d} - (1-p)\log_2 \frac{2(1-p)}{d^2-d},$$
$$S_2(\mathbf{w}) = 1 - \left(\frac{2p^2}{d^2+d} + \frac{2(1-p)^2}{d^2-d}\right),$$
$$R_2(\mathbf{w}) = -\log_2\left(\frac{2p^2}{d^2+d} + \frac{2(1-p)^2}{d^2-d}\right),$$
$$S_{f_{\rm WY}}(\mathbf{w}) = \frac{1}{2}(\sqrt{p(d^2+d)} + \sqrt{(1-p)(d^2-d)})^2 - 1,$$
$$S_{f_{\rm SLD}}(\mathbf{w}) = dp + \frac{d^2-d}{2} + \frac{2p(1-p)d(d^2-1)}{d+1-2p} - 1.$$

We depict the graphs of the above entropies of Werner states with $d = 2$ in Fig. 2 (hence in this case the Werner states live in $\mathbb{C}^4$). From the figure we see that all the five entropies behave similarly in some sense. We see that von Neumann entropy $S(\mathbf{w})$, Rényi-2 entropy $R_2(\mathbf{w})$, and Tsallis-2 entropy $S_2(\mathbf{w})$ may be less than $f_{\rm SLD}$ entropy (which is the smallest in the family of f entropy) for some Werner states, so none of these entropies belongs to the family of f entropy. Therefore, these well-known entropies are all different from quantum f entropy.

V. SUMMARY

For the average coherence, we have considered three natural and seemingly different yet equivalent procedures: The first is taking the average over all orthonormal bases, the second is taking the average over all elements of an operator orthonormal basis, and the third is taking the average over a complete family of mutually unbiased bases. We have employed average coherence to study entropy. We have introduced quantum f entropy related to operator monotone functions by means of the average coherence, and revealed basic properties of quantum f entropy. We have shown that it is indeed a bona fide measure of entropy. We have established some tradeoff relations among average coherence, quantum f entropy, and quasientropy. To illustrate f entropy, we have evaluated and compared various entropies, and have found that they exhibit qualitatively similar behaviors. In particular, we have made a rather detailed comparison among f entropy for $f = f_{\rm WY}, f_{\rm SLD}$ and von Neumann entropy, Rényi-2 entropy, and Tsallis-2 entropy. We have illuminated some similarities and differences between them.

Different entropies capture uncertainties of states from different angles and have their own advantages in different contexts. Quantum f entropy has the advantage that it is a sufficiently large family related to many important operator monotone functions and is intrinsically related to average coherence.

In view of the information-theoretic meaning of quantum f entropy and its intricate connections with average coherence, the foundational and practical implications of quantum f entropy deserve further investigations and exploitation.

ACKNOWLEDGMENTS

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APPENDIX

Here we present the detailed proofs of Propositions 1, 2, 3, and 5. Notice that Proposition 4 follows readily from Propositions 1–3.

1. Proof of Proposition 1

By using Eq. (11), the definition of the metric-adjusted skew information, and $(U|i\rangle\langle i|U^\dagger)^2 = U|i\rangle\langle i|U^\dagger$, Eq. (12) follows from

$$C_f^{\mathcal{U}}(\rho) = \sum_{i=1}^d \int_{\mathcal{U}} \{\mathrm{tr}(\rho U|i\rangle\langle i|U^\dagger) - \mathrm{tr}[U|i\rangle\langle i|U^\dagger m_{\tilde{f}}(L_\rho, R_\rho)(U|i\rangle\langle i|U^\dagger)]\}dU$$
$$= 1 - \sum_{i=1}^d \mathrm{tr}\left(|i\rangle\langle i| \int_{\mathcal{U}} U^\dagger m_{\tilde{f}}(L_\rho, R_\rho)(U|i\rangle\langle i|U^\dagger)U\,dU\right)$$

$$= 1 - \sum_{i=1}^{d} \mathrm{tr}\left\{|i\rangle\langle i|\left[\frac{d\,\mathrm{tr}(m_{\tilde f}(L_\rho, R_\rho)(\mathbf{1})) - \mathrm{tr}(m_{\tilde f}(L_\rho, R_\rho))}{d(d^2-1)}\mathbf{1} + \frac{d\,\mathrm{tr}(m_{\tilde f}(L_\rho, R_\rho)) - \mathrm{tr}(m_{\tilde f}(L_\rho, R_\rho)(\mathbf{1}))}{d(d^2-1)}|i\rangle\langle i|\right]\right\}$$

$$= 1 - \sum_{i=1}^{d} \frac{d - \mathrm{tr}(m_{\tilde f}(L_\rho, R_\rho) + d\,\mathrm{tr}(m_{\tilde f}(L_\rho, R_\rho)) - 1}{d(d^2-1)}$$

$$= \frac{d - \mathrm{tr}(m_{\tilde f}(L_\rho, R_\rho))}{d+1}.$$

In the above derivation, we have used the fact that $\mathrm{tr}(m_{\tilde f}(L_\rho, R_\rho)(\mathbf{1})) = 1$ and the relation [84]

$$\int_{\mathcal{U}} U^\dagger \Phi(UXU^\dagger)U\,dU = \frac{d\,\mathrm{tr}(\Phi(\mathbf{1})) - \mathrm{tr}(\Phi)}{d(d^2-1)}\mathrm{tr}(X)\mathbf{1} + \frac{d\,\mathrm{tr}(\Phi) - \mathrm{tr}(\Phi(\mathbf{1}))}{d(d^2-1)}X. \tag{A1}$$

2. Proof of Proposition 2

Notice that $C_f^{\mathrm{ob}}(\rho)$ is independent of the choice of the operator orthonormal basis and thus is well defined. In the following we choose the operator orthonormal basis $\{E_{ij} = |i\rangle\langle j| : i, j = 1, 2, \ldots, d\}$, where E_{ij} is the matrix with entry 1 at the (i, j) site and zero elsewhere. Let $\rho = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$ be the spectral decomposition of ρ, then

$$\sum_{i,j=1}^{d} I_f(\rho, E_{ij}) = d - \sum_{i,j=1}^{d}\sum_{k,l=1}^{m} m_{\tilde f}(\lambda_k, \lambda_l)\mathrm{tr}(|j\rangle\langle i|\phi_k\rangle\langle\phi_k|i\rangle\langle j|\phi_l\rangle\langle\phi_l|) = d - \sum_{k,l=1}^{m} m_{\tilde f}(\lambda_k, \lambda_l)\sum_{i=1}^{d}\langle i|\phi_k\rangle\langle\phi_k|i\rangle\sum_{j=1}^{d}\langle j|\phi_l\rangle\langle\phi_l|j\rangle$$

$$= d - \sum_{k,l=1}^{m} m_{\tilde f}(\lambda_k, \lambda_l)\mathrm{tr}(|\phi_k\rangle\langle\phi_k|)\mathrm{tr}(|\phi_l\rangle\langle\phi_l|) = d - \sum_{k,l=1}^{m} m_{\tilde f}(\lambda_k, \lambda_l) = d - \mathrm{tr}[m_{\tilde f}(L_\rho, R_\rho)],$$

from which we obtain Eq. (14).

3. Proof of Proposition 3

First we compute the trace of the superoperator $m_{\tilde f}(L_\rho, R_\rho)$. Let $\rho = \sum_i \lambda_i|\phi_i\rangle\langle\phi_i|$ be the spectral decomposition of the state ρ. For any operator X,

$$m_{\tilde f}(L_\rho, R_\rho)(X) = \sum_{ij} m_{\tilde f}(\lambda_i, \lambda_j)|\phi_i\rangle\langle\phi_i|X|\phi_j\rangle\langle\phi_j| = \sum_j\left(\sum_i m_{\tilde f}(\lambda_i, \lambda_j)|\phi_i\rangle\langle\phi_i|\right)X|\phi_j\rangle\langle\phi_j|.$$

By the fact that $\mathrm{tr}(\Phi) = \sum_j \mathrm{tr}(A_j)\mathrm{tr}(B_j^\dagger)$ for the superoperator Φ represented as $\Phi(X) = \sum_j A_j X B_j^\dagger$ [84], we have

$$\mathrm{tr}[m_{\tilde f}(L_\rho, R_\rho)] = \sum_j \mathrm{tr}\left(\sum_i m_{\tilde f}(\lambda_i, \lambda_j)|\phi_i\rangle\langle\phi_i|\right)\mathrm{tr}(|\phi_j\rangle\langle\phi_j|) = \sum_{i,j} m_{\tilde f}(\lambda_i, \lambda_j).$$

Next by the definition of $C_f^{\mathrm{mub}}(\rho)$, we can express it as

$$C_f^{\mathrm{mub}}(\rho) = \frac{1}{d+1}\sum_{\nu=1}^{d+1}\sum_{i=1}^{d}\{\mathrm{tr}(\rho|b_{\nu i}\rangle\langle b_{\nu i}|) - \mathrm{tr}[|b_{\nu i}\rangle\langle b_{\nu i}|m_{\tilde f}(L_\rho, R_\rho)(|b_{\nu i}\rangle\langle b_{\nu i}|)]\}$$

$$= 1 - \frac{1}{d+1}\sum_{k,l} m_{\tilde f}(\lambda_k, \lambda_l)\sum_{\nu=1}^{d+1}\sum_{i=1}^{d}\mathrm{tr}(|b_{\nu i}\rangle\langle b_{\nu i}|\phi_k\rangle\langle\phi_k|b_{\nu i}\rangle\langle b_{\nu i}|\phi_l\rangle\langle\phi_l|)$$

$$= 1 - \frac{1}{d+1}\sum_{k,l} m_{\tilde f}(\lambda_k, \lambda_l)\mathrm{tr}\left(\sum_{\nu=1}^{d+1}\sum_{i=1}^{d}(|b_{\nu i}\rangle\langle b_{\nu i}| \otimes |b_{\nu i}\rangle\langle b_{\nu i}|)(|\phi_k\rangle\langle\phi_k| \otimes |\phi_l\rangle\langle\phi_l|)\right).$$

In order to evaluate the average coherence $C_f^{\mathrm{mub}}(\rho)$, we need the following identity:

$$\frac{1}{d+1}\sum_{\nu=1}^{d+1}\sum_{i=1}^{d}|b_{\nu i}\rangle\langle b_{\nu i}| \otimes |b_{\nu i}\rangle\langle b_{\nu i}| = \frac{\mathbf{1}\otimes\mathbf{1} + F}{d+1}, \tag{A2}$$

where $F = \sum_\alpha X_\alpha \otimes X_\alpha^\dagger = \sum_{i,j} |i\rangle\langle j| \otimes |j\rangle\langle i|$ is the swap operator. We invoke the relation

$$\rho = \sum_{\nu=1}^{d+1} \rho(B_\nu) - \mathbf{1}, \tag{A3}$$

proved by Ivanovic [85], where $\rho(B_\nu) = \sum_{j=1}^{d} \langle b_{\nu j}|\rho|b_{\nu j}\rangle |b_{\nu j}\rangle\langle b_{\nu j}|$. Plugging $\rho(B_\nu)$ into Eq. (A3) we get $\rho = \sum_{\nu=1}^{d+1}\sum_{i=1}^{d} |b_{\nu i}\rangle\langle b_{\nu i}|\rho|b_{\nu i}\rangle\langle b_{\nu i}| - \mathbf{1}$, which leads to the following equation:

$$X = \sum_{\nu=1}^{d+1}\sum_{i=1}^{d} |b_{\nu i}\rangle\langle b_{\nu i}|X|b_{\nu i}\rangle\langle b_{\nu i}| - \mathrm{tr}(X)\mathbf{1} \tag{A4}$$

for any operator X. Notice that

$$\sum_{\nu=1}^{d+1}\sum_{i=1}^{d} |b_{\nu i}\rangle\langle b_{\nu i}|b_{\nu i}\rangle\langle b_{\nu i}| = \sum_{\nu=1}^{d+1}\sum_{i=1}^{d} |b_{\nu i}\rangle\langle b_{\nu i}| = (d+1)\mathbf{1}.$$

Thus if we take $E_\gamma = |b_{\nu i}\rangle\langle b_{\nu i}|/\sqrt{d+1}$ with $\gamma = (\nu, j)$, then they are Kraus operators of the channel $\mathcal{E}$, and

$$\mathcal{E}(X) = \frac{1}{d+1}\sum_{\nu=1}^{d+1}\sum_{j=1}^{d} |b_{\nu j}\rangle\langle b_{\nu j}|X|b_{\nu j}\rangle\langle b_{\nu j}|. \tag{A5}$$

Combining Eqs. (A4) and (A5), we get

$$\mathcal{E}(X) = \frac{1}{d+1}(X + \mathrm{tr}(X)\mathbf{1}).$$

Hence if we put $F_0 = \mathbf{1}/\sqrt{d+1}$, $F_\alpha = X_\alpha/\sqrt{d+1}$, $\alpha = 1, 2, \ldots, d^2$, where $\{X_\alpha : \alpha = 1, 2, \ldots, d^2\}$ is an operator orthonormal basis of $L(H)$, then $\{F_\beta : \beta = 0, 1, \ldots, d^2\}$ is a set of Kraus operators of $\mathcal{E}$. Here we have used the fact that $\sum_{\alpha=1}^{d^2} X_\alpha Y X_\alpha^\dagger = \mathrm{tr}(Y)\mathbf{1}$ for any operator Y [86]. Now we have constructed two sets of Kraus operators $\{E_\gamma\}$ and $\{F_\beta\}$ of the channel $\mathcal{E}$. There exist complex numbers $u_{\gamma\beta}$ such that $E_\gamma = \sum_\beta u_{\gamma\beta}F_\beta$ for any γ and $\sum_\gamma u_{\gamma\beta}u_{\gamma\beta'}^\dagger = \delta_{\beta\beta'}$. Consequently,

$$\sum_\gamma E_\gamma \otimes E_\gamma^\dagger = \sum_\gamma\sum_\beta u_{\gamma\beta}F_\beta \otimes \left(\sum_{\beta'} u_{\gamma\beta'}F_{\beta'}\right)^\dagger = \sum_{\beta,\beta'}\sum_\gamma u_{\gamma\beta}u_{\gamma\beta'}^\dagger F_\beta \otimes F_{\beta'}^\dagger = \sum_\beta F_\beta \otimes F_\beta^\dagger,$$

which implies Eq. (A2). Therefore, we obtain

$$\begin{aligned} C_f^{\mathrm{mub}}(\rho) &= 1 - \frac{1}{d+1}\sum_{i,j} m_{\tilde{f}}(\lambda_i, \lambda_j)\mathrm{tr}[(\mathbf{1}\otimes\mathbf{1} + F)(|\phi_i\rangle\langle\phi_i| \otimes |\phi_j\rangle\langle\phi_j|)] = 1 - \frac{1}{d+1}\sum_{i,j} m_{\tilde{f}}(\lambda_i, \lambda_j)(1+\delta_{ij}) \\ &= 1 - \frac{1}{d+1}\left(\mathrm{tr}(m_{\tilde{f}}(L_\rho, R_\rho)) + \sum_i m_{\tilde{f}}(\lambda_i, \lambda_i)\right) = 1 - \frac{1}{d+1}\{\mathrm{tr}[m_{\tilde{f}}(L_\rho, R_\rho)] + 1\} = \frac{d - \mathrm{tr}[m_{\tilde{f}}(L_\rho, R_\rho)]}{d+1}, \end{aligned}$$

where the second equality follows from

$$\mathrm{tr}[F(A\otimes B)] = \sum_{i,j} \mathrm{tr}(|i\rangle\langle j|A \otimes |j\rangle\langle i|B) = \sum_{i,j}\langle j|A|i\rangle\langle i|B|j\rangle = \mathrm{tr}(AB).$$

4. Proof of Proposition 5

To prove item 1, for any state ρ with the spectral decomposition $\rho = \sum_i \lambda_i|\phi_i\rangle\langle\phi_i|$, where the eigenvalues $\lambda_i > 0$, we have

$$\sum_{i,j} m_{\tilde{f}}(\lambda_i, \lambda_j) \geqslant \sum_{i=j} m_{\tilde{f}}(\lambda_i, \lambda_j) = \sum_i m_{\tilde{f}}(\lambda_i, \lambda_i) = \sum_i \lambda_i = 1,$$

thus $S_f(\rho) \geqslant 0$. When ρ is a pure state, $S_f(\rho) = 0$. On the other hand, if ρ is not a pure state, then ρ has at least two eigenvalues, and

$$\sum_{i,j} m_{\tilde{f}}(\lambda_i, \lambda_j) > \sum_i m_{\tilde{f}}(\lambda_i, \lambda_i) = 1.$$

Therefore $S(\rho) = 0$ if and only if ρ is a pure state.

For item 2, $\forall f \in \mathcal{F}_{\rm r},\ x, y > 0$, it holds that

$$m_{\tilde{f}}(x, y) = x\tilde{f}\left(\frac{y}{x}\right) = \frac{1}{2}\left(x + y - (y - x)^2 \frac{f(0)}{xf\left(\frac{y}{x}\right)}\right).$$

Thus $m_{\tilde{f}}(x, y) \leqslant (x + y)/2$ and $m_{\tilde{f}}(x, y) = (x + y)/2$ if and only if $x = y$. It follows that

$$\sum_{i,j} m_{\tilde{f}}(\lambda_i, \lambda_j) \leqslant \sum_{i,j} \frac{\lambda_i + \lambda_j}{2} \leqslant d. \tag{A6}$$

Thus $S_f(\rho) \leqslant d - 1$ for any state ρ, and the equality holds if and only if $\sum_{i,j} m_{\tilde{f}}(\lambda_i, \lambda_j) = d$, which is equivalent to $\sum_{i,j} m_{\tilde{f}}(\lambda_i, \lambda_j) = \sum_{i,j}(\lambda_i + \lambda_j)/2 = d$. Hence, $m_{\tilde{f}}(\lambda_i, \lambda_j) = (\lambda_i + \lambda_j)/2$ for any $i, j = 1, 2, \ldots, d$, which implies $\lambda_i = \lambda_j$ for all $i, j = 1, 2, \ldots, d$, and the state is the completely mixed state $\mathbf{1}/d$. On the other hand, by direct calculation, we have $S_f(\mathbf{1}/d) = d - 1$.

Item 3 follows from Eq. (27) and the convexity of $I_f(\rho, K)$ in ρ.

For item 4, from the Schmidt decomposition we know that the eigenvalues of the density operators ρ^a and ρ^b are the same. Quantum f entropy is determined completely by the eigenvalues, so the desired result follows.

For item 5, by the symmetry of $m_{\tilde{f}}(x, y)$, for any states ρ and σ, we have $S_f(\rho \otimes \sigma) = S_f(\sigma \otimes \rho)$. Let $\{|\phi_i\rangle : i = 1, 2, \ldots, d\}$ be a complete set of eigenvectors of $\rho = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$, which thus constitute an orthonormal basis of the system Hilbert space. Consequently, $\mathbf{1} = \sum_{i=1}^d |\phi_i\rangle\langle\phi_i|$, and

$$\rho \otimes \frac{\mathbf{1}}{d} = \sum_{i,j} \frac{\lambda_i}{d} |\phi_i\rangle\langle\phi_i| \otimes |\phi_j\rangle\langle\phi_j|,$$

then

$$S_f\left(\rho \otimes \frac{\mathbf{1}}{d}\right) = \sum_{i,i'} \sum_{j,j'} m_{\tilde{f}}\left(\lambda_i \frac{1}{d}, \lambda_{i'} \frac{1}{d}\right) - 1 = \frac{1}{d} \sum_{j,j'} \sum_{i,i'} m_{\tilde{f}}(\lambda_i, \lambda_{i'}) - 1 = \frac{1}{d} \sum_{j,j'} (S_f(\rho) + 1) - 1$$
$$= dS_f(\rho) + S_f\left(\frac{\mathbf{1}}{d}\right).$$

Item 6 follows from item 3 and

$$S_f\left(\sum_j p_j |j\rangle\langle j| \otimes \rho_j\right) \geqslant \sum_j p_j S_f(|j\rangle\langle j| \otimes \rho_j) = \sum_j p_j S_f(\rho_j).$$

For item 7, let $\{X_\alpha : \alpha = 1, 2 \ldots, d^2\}$ be an operator orthonormal basis for $L(H)$, then $\{UX_\alpha U^\dagger : \alpha = 1, 2, \ldots, d^2\}$ is also an operator orthonormal basis for $L(H)$, and

$$S_f(U\rho U^\dagger) = d - 1 - \sum_{\alpha=1}^{d^2} I_f(U\rho U^\dagger, X_\alpha) = d - 1 - \sum_{\alpha=1}^{d^2} I_f(\rho, U^\dagger X_\alpha U) = S_f(\rho).$$

In view of the convexity of $I_f(\rho, K)$ in ρ and Eq. (27), we have

$$S_f(\mathcal{E}_{\rm RU}(\rho)) = d - 1 - \sum_{\alpha=1}^{d^2} I_f(\mathcal{E}_{\rm RU}(\rho), X_\alpha) = d - 1 - \sum_{\alpha=1}^{d^2} I_f\left(\sum_k p_k U_k \rho U_k^\dagger, X_\alpha\right) \geqslant d - 1 - \sum_k p_k \sum_{\alpha=1}^{d^2} I_f(U_k \rho U_k^\dagger, X_\alpha)$$
$$= d - 1 - \sum_k p_k \sum_{\alpha=1}^{d^2} I_f(\rho, U_k^\dagger X_\alpha U_k) = d - 1 - \sum_k p_k [d - 1 - S_f(\rho)] = S_f(\rho),$$

from which item 8 follows.

To prove item 9, by Eqs. (27) and (28), for any operator orthonormal basis $\{X_\alpha : j = 1, 2, \ldots, d^2\}$ (self-adjoint operators) of $L(H)$, we have

$$S_f(\rho) = d - 1 - \sum_{\alpha=1}^{d^2} I_f(\rho, X_\alpha) = d - 1 - \sum_{\alpha=1}^{d^2} {\rm tr}\left(\rho X_\alpha^2\right) + \sum_{\alpha=1}^{d^2} {\rm tr}[X_\alpha m_{\tilde{f}}(L_\rho, R_\rho)(X_\alpha)]$$
$$= \sum_{\alpha=1}^{d^2} {\rm tr}[X_\alpha m_{\tilde{f}}(L_\rho, R_\rho)(X_\alpha)] - 1 = \sum_{\alpha=1}^{d^2} S_{\tilde{f}}^{X_\alpha}(\rho|\rho) - 1,$$

where the third equation holds in view of $\sum_{\alpha=1}^{d^2} X_\alpha^2 = d\mathbf{1}$. If $\rho_n \to \rho$ in norm, then for any bounded continuous operator monotone function f, $S_f^A(\rho_n|\rho_n) \to S_f^A(\rho|\rho)$ for any operator A [77]. $f \in \mathcal{F}_r$ implies that $\widetilde{f}$ is also a bounded continuous operator monotone function [66], so $S_f(\rho_n) \to S_f(\rho)$.

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