Exact and lower bounds for the quantum speed limit in finite-dimensional systems

Mattias T. Johnsson,[1] Lauritz van Luijk,[2] and Daniel Burgarth[1,3]
[1]School of Mathematical and Physical Sciences, Macquarie University, North Ryde, New South Wales 2109, Australia
[2]Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstrasse 2, 30167 Hannover, Germany
[3]Physics Department, Friedrich-Alexander Universität of Erlangen-Nuremberg, Staudtstrasse 7, 91058 Erlangen, Germany

(Received 18 April 2023; accepted 6 September 2023; published 2 November 2023)

A fundamental problem in quantum engineering is determining the lowest time required to ensure that all possible unitaries can be generated with the tools available, which is one of a number of possible quantum speed limits. We examine this problem from the perspective of quantum control, where the system of interest is described by a drift Hamiltonian and set of control Hamiltonians. Our approach uses a combination of Lie algebra theory, Lie groups, and differential geometry and formulates the problem in terms of geodesics on a differentiable manifold. We provide explicit lower bounds on the quantum speed limit for the case of an arbitrary drift, requiring only that the control Hamiltonians generate a topologically closed subgroup of the full unitary group, and formulate criteria as to when our expression for the speed limit is exact and not merely a lower bound. These analytic results are then tested and confirmed using a numerical optimization scheme. Finally, we extend the analysis to find a lower bound on the quantum speed limit in the common case where the system is described by a drift Hamiltonian and a single control Hamiltonian.

DOI: 10.1103/PhysRevA.108.052403

I. INTRODUCTION

The emergence of quantum technologies such as quantum information processing, quantum engineering, and quantum sensing has relied on our increasing ability to manipulate quantum systems with high levels of precision. Such manipulation requires the ability to carry out quantum operations and state preparation with high fidelity, in the presence of noisy environments, as quickly as possible, and potentially subject to a number of real-world constraints.

These requirements are the province of the field of quantum control, which is primarily concerned with methods of steering a quantum system using a set of classical control inputs to the system [1,2]. Two major topics within this field are characterizing the operations that can be carried out and the states that can be reached with a given set of controls, as well as determining the specific time dependence of those controls that will steer the system to the intended goal. The questions regarding the gates that can be implemented and state reachability are approached using the methods of bilinear control theory [3,4], which usually involve a Lie theoretic framework [3,5,6]. The questions regarding the determination of the time-dependent control fields (pulses), on the other hand, have no good general strategy and are generally difficult. Analytic methods of optimal control theory can be employed [3,7,8], but usually numerical optimization is used, typically involving gradient-based search strategies with some fidelity cost functional [9–12].

While these aspects of quantum control have been extensively studied, less attention has been given to the question of the speed at which specific unitaries can be generated or specific states can be reached. Given that decoherence is present in all quantum information processing, it is important to minimize the time taken to perform quantum operations. The time taken to reach specific targets given the set of controls available is known as the quantum speed limit [13–15] or, more precisely, there are a number of different speed limits, some for the transformation of states, some for unitary transformation, some for uncontrolled dynamics, and some for controlled dynamics [14].

We will be more precise later, but in general terms the quantum speed limit we will consider in this paper is the following: Assuming we have a set of controls that allows us to achieve all possible unitaries in a finite-dimensional system, what is the minimum time by which we can guarantee we can produce all possible unitaries? In other words, how much time must we allow to be certain that we can accomplish everything that can be done with the system?

The exact time for this type of quantum speed limit is generally very difficult to determine for a specific quantum system, unless that system is very low dimensional or possesses a very high degree of symmetry. Nonetheless, in some special cases the limit can be computed; see, for example, [16–22]. This difficulty means that work has concentrated on finding lower bounds for the speed limit rather than exact results. Various bounds have been obtained for closed finite-dimensional systems as well as for open systems [23–36]. While these bounds are not tight, they can provide information on how the speed limit is likely to scale with regard to quantities of interest, such as system dimension or total energy. It is notable that many of these approaches make use of energy uncertainty of the system, applying the original results of Mandelstam and Tamm [37], as well as the more modern interpretation of Margolus and Levitin [23].

Given this background, we can state a generic quantum control problem and investigate its speed limit as follows. We consider a Hamiltonian given by

$$H = H_d + \sum_{j=1}^{m} f_j(t) H_j, \tag{1}$$

where H_d and H_j are time-independent Hamiltonians acting on a finite-dimensional Hilbert space and $f_j(t)$ are of real time-dependent scalar functions. The term H_d is called the drift Hamiltonian and is always present. The H_j are the control Hamiltonians and we assume that we have arbitrary control over the $f_j(t)$, as even in this case the quantum speed limits are very difficult to determine.

The system evolves according to the Schrödinger equation

$$i\frac{d}{dt}U(t) = \left(H_d + \sum_{j=1}^{m} f_j(t)H_j\right)U(t), \quad U(0) = \mathbb{1}, \tag{2}$$

where $U(t)$ is the unitary time-evolution operator. In an n-dimensional system $U(t)$ can be represented as a unitary $n \times n$ matrix. Further, as unitary operators are physically indistinguishable up to a phase, we can choose to remove this excessive phase degree of freedom by demanding that $U(t)$ have unit determinant, making it a special unitary matrix. This is accomplished by choosing the drift and control Hamiltonians to be traceless, and we will assume this is the case throughout this paper.

The system is called controllable if it is possible to find control functions $f_j(t)$ such that, given enough time, we can achieve any possible unitary (up to a phase) or, equivalently, if we can generate all possible members of the Lie group SU(n). There is a beautiful Lie-algebraic result that states that this is the case if and only if [3,5] the dynamical Lie algebra $\{iH_d, iH_1, iH_2, \ldots, iH_m\}_{\text{LA}}$ has dimension $n^2 - 1$, i.e., the dynamical Lie algebra generated by the control Hamiltonians and drift Hamiltonian is the Lie algebra su(n).

The next natural question is, if a quantum system is controllable, how long will it take to produce a specific unitary in the worst case or, equivalently [4,6], in the case of compact groups such as SU(n), since the system is controllable, what is the minimum time by which we can guarantee we can produce all possible unitaries? This is what we will refer to as the quantum speed limit in this paper.

We note that some authors make a distinction between quantum control systems which are fully controllable only in the presence of a drift term (i.e., removing the drift Hamiltonian would cause the system to no longer be fully controllable) from those systems for which this is not the case. Systems of the latter type are known as strongly controllable [38] and are fully controllable with control Hamiltonians alone regardless of the presence or absence of any drift term. Due to our assumption that the control strengths $f_j(t)$ can be arbitrarily large, strongly controllable systems can reach any unitary in an arbitrarily short amount of time, rendering the concept of a speed limit irrelevant. For that reason we consider only systems of the first type, where the drift is required to ensure the system is controllable.

In this paper our goal is to derive lower bounds on the speed limits of controllable quantum systems that are as general as possible. We do not restrict the system to a specific number of dimensions, demand it describes a set of qubits, or require the drift Hamiltonian to be of a specific form, as is common in other speed limit calculations (e.g., [28]; see [14] for a review). We require no knowledge of the quantum energy uncertainty of the system. We will require only that the control subgroup is topologically closed, where the control subgroup is the set of all unitaries that can be reached by application of the control Hamiltonians alone, and will thus form a subgroup of SU(n). However, the resulting speed limits can be hard to analytically compute explicitly as they require determining the diameter of rather abstract manifolds, so we examine in more detail cases where the manifolds are symmetric spaces [39], which can arise, for example, if the Lie algebra associated with the control subgroup forms a Cartan decomposition [3] of the full dynamical Lie algebra.

This will allow us to derive explicit analytic lower bounds for the quantum speed limit for a number of control schemes corresponding to cases where the control group is one of SO(n), Sp($n/2$), or S[U(p) × U(q)], with $p + q = n$, and investigate when this bound will be tight. We also consider the case where the number of control Hamiltonians is not enough to span the full Lie algebra corresponding to these groups and give the minimum number of control Hamiltonians required to generate the algebra. Due to the fact that many control problems will not have enough controls to generate these groups, we also derive a bound for the common general case where there is a only a single control Hamiltonian. In all cases, our results are completely general and valid for arbitrary dimension. Finally, in order to test our analytic results, we carry out an exploration of quantum speed limits for a variety of low-dimensional systems using numerical simulations. This not only provides a check on our results, but allows an investigation of the efficiency of numerical optimal control algorithms for bilinear systems.

The structure of the paper is as follows. We begin in Sec. II by formulating the quantum speed limit problem in terms of Lie algebras and Lie groups and introduce concepts we will require such as cosets, quotient spaces, and adjoint orbits, as well as laying out our basic approach. We introduce the idea that the problem can be treated as movement on a manifold, with the movement direction and speed given by the drift Hamiltonian. Since the mathematical machinery will not be familiar to some readers, we provide illustrative examples.

In Sec. III we explain how one can obtain a speed limit by determining the diameter of a manifold (i.e., the two points farthest apart) and dividing by the speed at which the system moves on the manifold. We describe the conditions on the manifold required for this to work and give a way of computing the speed of movement from the system's drift Hamiltonian. We establish that symmetric spaces provide manifolds meeting the criteria, give their diameters, and use them to compute explicit expressions for the lower bound on the quantum speed limit.

Section IV examines when the lower bound developed in the preceding section is actually tight. It develops a criterion based on the dimension of the adjoint orbit and commutation relations between the drift Hamiltonian and the matrix representation of the Lie algebra corresponding to the controls.

As this criterion is sufficient but not necessary, in Sec. V we investigate what else can be said about the tightness of the bound if the controls arise from a Cartan decomposition.

This allows understanding the control problem in terms of root systems, and we illustrate the results by considering the case where the control group is SO(n).

In Sec. VI we treat the problem of finding the quantum speed limit numerically and compare the simulations to our analytic results. This allows both a test of our bounds and an examination of how well standard optimization techniques used in quantum control work.

Finally, in Sec. VII we consider the case where we have only a limited set of controls so that we do not have a symmetric space and derive a bound on the speed limit for the common case where the system has only a single control Hamiltonian.

II. PROBLEM FORMULATION IN TERMS OF LIE GROUPS AND ALGEBRAS

The calculation of quantum speed limits is often approached using Lie group-theoretic techniques. We will also make use of these mathematical structures, so we briefly provide the relevant background here. Good explanations of this material can be found in, for example, [3,39,40].

In what follows, we will denote groups with an uppercase letter, e.g. $G = \mathrm{SU}(n)$, and algebras with a lowercase letter in Fraktur font ($\mathfrak{g} = \mathrm{su}(n)$). Multiple letter algebras are in lowercase roman font.

Let $\mathfrak{g}$ be the full Lie algebra generated by the drift Hamiltonian and the control Hamiltonians, i.e., $\mathfrak{g} = \{iH_d, iH_1, iH_2, \ldots, iH_m\}_{\mathrm{LA}}$, and let the Lie algebra generated by the control Hamiltonians alone be given by $\mathfrak{k} = \{iH_1, iH_2, \ldots, iH_m\}_{\mathrm{LA}}$. Clearly $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$. The system is said to be controllable if $\mathfrak{g} = \mathrm{su}(n)$.

We denote the control group, i.e., the group of unitaries generated by exponentiating $\mathfrak{k}$, by K and the dynamical Lie group generated by $\mathfrak{g}$ by G. Clearly, $K \subseteq G$ is a subgroup and $G \subseteq \mathrm{SU}(n)$ with equality if the system is controllable.

At any given time, the system evolves according to (2). Since the control amplitudes $f_j(t)$ can be arbitrarily large, we can generate any unitary $U \in K$ in an arbitrarily short time to arbitrarily good precision (see [3] for a rigorous justification of this point). Now suppose our control problem is to produce a unitary U_{target} that moves us between the two unitaries U_1 and U_2, i.e., $U_2 = U_{\mathrm{target}} U_1$. Since we can move between elements of K arbitrarily quickly, all elements of K are equivalent, meaning if we apply any controls after we have generated the specific unitary U_{target}, all resulting unitaries KU_{target} are equivalent in terms of how quickly we can generate them. Because of this, we can view our control problem as actually asking how to move between the right cosets KU_1 and KU_2, where the right coset is $KU = \{kU \mid k \in K\}$. Furthermore, as the system evolves in time, the unitary at any point in time, given by (2), is equivalent to any other element in its coset, because it can be moved within the coset arbitrarily quickly.

Alternatively, one could define equivalence in terms of left cosets, where now we consider how to move between the left cosets U_1K and U_2K. Again, these cosets are equivalent in terms of the minimum time it takes to use controls to move between them, but now the controls are being applied before the unitary rather than after.

From a quantum control perspective, the most natural approach would be to consider equivalence under left and right multiplication with unitaries in K, leading to two-sided cosets. Mathematically, it is easier to stick to a one-sided coset, and we will from now on consider right cosets. To "divide out" the degree of freedom associated with each coset, one defines the quotient space G/K as the set of right cosets Kg. We denote each coset by $[g] = Kg$ with $g \in G$, since the element g indexes the coset. The cosets can also be seen as the orbits of the natural left action of K on G and the space of orbits is G/K.

If K is a normal subgroup of G, then G/K is itself a Lie group [39,40]. However, even if this is not the case, provided G is a Lie group and K is a closed subgroup (in the topological sense,[1]) then G/K is a differentiable manifold [40] that is also a (right) homogeneous space, meaning that it carries a (right) transitive G-action, which is given by $[g'] \cdot g = [g'g]$. Specifically, G/K can be given the structure of a smooth manifold with dimension $\dim(G/K) = \dim G - \dim K$. Movement within a coset does not result in movement in G/K, but movement between cosets does. Movement within a coset is produced by the control Hamiltonians and movement between the cosets requires the drift Hamiltonian.

As the system evolves via (2) it traces out a continuous path in G/K space, and the quantum speed limit is governed by how fast we can move between the two points corresponding to U_1 and U_2. Clearly, we cannot move arbitrarily in G/K. Our movement on G/K is determined by the drift Hamiltonian, with the direction of the movement determined by where we are within a coset at any given time, allowing us some degree of steering.

In particular, we have the following: If G is a compact and connected Lie group [e.g., SU(n)] and K is a closed Lie subgroup of G, with associated Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, then we can decompose the Lie algebra $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ with

$$[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \tag{3}$$

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \tag{4}$$

where $\mathfrak{p} = \mathfrak{k}^{\perp}$ with respect to an Ad-invariant inner product on $\mathfrak{g}$ [39], e.g., the Hilbert-Schmidt inner product. Note that while $\mathfrak{k}$ is a Lie algebra, $\mathfrak{p}$ is in general not closed under the Lie bracket. The Ad-invariant inner product on $\mathfrak{g}$ induces a bi-invariant Riemannian geometry on G which in turn induces a G-invariant Riemannian geometry on G/K (see the next section for details). This equips the manifold G/K with the structure of a so-called reductive space, which is a more restricted variety of a homogeneous space.

Any evolution purely under the action of the controls, without the drift, will produce motion only within a coset. Without loss of generality, we can assume $iH_d \in \mathfrak{p}$, since any contribution that lies in $\mathfrak{k}$ can be removed by application of the controls. Since $\mathfrak{p}$ is orthogonal to $\mathfrak{k}$, this means that any evolution under the drift alone moves purely in G/K, with no movement within a coset. Specifically, for a reductive space, the inner product lets us identify the tangent $T_o(G/K)$ at the origin $o = [\mathbb{1}]$ with $\mathfrak{p}$.

[1]For counterexamples see, e.g., Sec. 1.1 in [40].

To show how the action of the control steers the direction of motion in G/K, we need the concept of the adjoint orbit. The adjoint orbit of $A \in \mathfrak{g}$ is given by

$$\mathcal{O}(A) = \{k^{-1}Ak \mid k \in K\}. \tag{5}$$

By Eq. (3), we have $\mathcal{O}(A) \subset \mathfrak{p}$ for $A \in \mathfrak{p}$. We can see how this steers the evolution in G/K space as follows [16]: Take elements $k_1, k_2 \in K$ that belong to the coset containing the identity and consider where they move under the action of the drift after a short time Δt. We obtain

$$k_1 \to e^{-iH_d\Delta t}k_1 = k_1 e^{-ik_1^{-1}H_d k_1 \Delta t}, \tag{6}$$

showing that after the evolution it is now a member of the coset $[e^{-ik_1^{-1}H_d k_1\Delta t}]$. Similarly, k_2 moves to a coset $[e^{-ik_2^{-1}H_d k_2\Delta t}]$. Since we can choose to be anywhere in a coset arbitrarily quickly due to the action of the controls, we see that the adjoint orbit represents the directions we are able to move from the origin of G/K.

This mathematical machinery can be somewhat opaque, so we present a simple example that illustrates these concepts. We consider computing the quantum speed limit of a controllable quantum system in a two-dimensional Hilbert space, i.e., the group associated with the unitary evolution operator is SU(2). This is one of the few cases where the speed limit is explicitly known.

We take our Hamiltonian to be

$$H = \sigma_z + f(t)\sigma_x \tag{7}$$

and the Schrödinger equation is given by

$$-i\frac{d}{dt}U(t) = [\sigma_z + f(t)\sigma_x]U(t), \quad U(0) = \mathbb{1}. \tag{8}$$

The Lie algebra associated with the single control is just $\text{span}\{i\sigma_x\}$, while the full dynamical Lie algebra associated with the drift and controls is $\text{span}\{i\sigma_x, i\sigma_y, i\sigma_z\}$. Since this algebra is three dimensional and this matches $n^2 - 1$, where n is the Hilbert space dimension, the system is controllable. Our Lie-algebra decomposition is $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, with $\mathfrak{k} = \text{span}\{i\sigma_x\}$ and $\mathfrak{p} = \text{span}\{i\sigma_y, i\sigma_z\}$. We have $\mathfrak{g} = \text{su}(2)$, $\mathfrak{k} = \mathfrak{u}(1)$, $G = \text{SU}(2)$, and $K = \text{U}(1)$. The manifold corresponding to the quotient space G/K can in general be quite complicated, but in this case it is particularly simple; the manifold $G/K = \text{SU}(2)/\text{U}(1)$ is isomorphic to the two-sphere S^2.

Since the control algebra is one dimensional, the control group subgroup K generated by $\mathfrak{k}$ can be parametrized by a single parameter α as $e^{i\alpha\sigma_x}$, $\alpha \in [0, 2\pi]$, and the adjoint orbit is given by the set

$$\begin{aligned}\mathcal{O}(iH_d) &= \{e^{-i\alpha\sigma_x}i\sigma_z e^{i\alpha\sigma_x} \mid \alpha \in [0, 2\pi]\} \\ &= \{i\cos(2\alpha)\sigma_z + i\sin(2\alpha)\sigma_y \mid \alpha \in [0, 2\pi]\}.\end{aligned} \tag{9}$$

Here S^2 is two dimensional and the tangent space at the origin is defined by $\text{span}\{i\sigma_y, i\sigma_z\} = \mathfrak{p}$. Since Eq. (9) allows any direction in the tangent space by a suitable choice of α, we can move in any direction in G/K we wish. As we will show in later sections, the speed of movement in G/K is constant and determined purely by the drift Hamiltonian. This means that the speed limit is achieved by moving on a great circle geodesic between two antipodal points, as this yields the maximum possible evolution time between any two unitaries for the system.

The concepts of speeds and distances on the G/K manifold are determined by the Riemannian metric on G/K which depends on the inner product chosen on $\mathfrak{g}$. As will be shown later, if we choose the Killing form for the inner product, then for this particular example the speed of movement is $2\sqrt{2}$ and the distance between two antipodal points is $\sqrt{2}\pi$, giving the time for the quantum speed limit as $t = \pi/2$, which agrees with the standard result [3].

We note that this is an unusual way to look at this problem. The normal approach is to apply the maxim "algebra is easier than geometry" and use Lie algebra, Lie groups and results such as the maximal torus theorem, rather than considering geodesics on a manifold. Nonetheless, the idea of obtaining a speed limit by dividing the diameter of the G/K manifold by the drift velocity will prove extremely useful. In the case where the adjoint orbit allows us to move on a geodesic connecting the two points farthest apart on the manifold, we can obtain an exact speed limit, and if it does not allow movement on such a geodesic, such a method will still provide a lower bound.

III. QUANTUM SPEED LIMITS FROM MANIFOLD DIAMETER AND DRIFT VELOCITY

As discussed in the preceding section, in order to obtain speed limits from the structure of the G/K manifold, we need some way of assigning distances to the space. This involves bridging the two descriptions of the problem: The control and drift Hamiltonians defining the system are described by the Lie algebra, while the unitaries corresponding to the system evolution are described by the Lie group and associated manifold.

To see the issue, consider the group SU(2). The associated manifold is the three-sphere, which describes the topology, but there is no metric associated with it (yet); for example, there is no concept of the size of its diameter. The way the metric is imposed is to define an inner product on the Lie algebra which is then pushed around the group to define an inner metric on all tangent spaces. For the inner product on the Lie algebra $\mathfrak{g}$ we will take

$$\langle X, Y\rangle_K = -2n\,\text{Tr}(XY), \quad X, Y \in \mathfrak{g}. \tag{10}$$

This inner product is Ad-invariant since the group G consists of unitary operators. The dimension-dependent factor of $2n$ is chosen such that the inner product is equal to the Killing inner product in the case $\mathfrak{g} = \text{su}(n)$; it will, however, drop out in the quantum speed limits. We now obtain the inner product at the tangent space of a general element $g \in G$ from

$$\langle X, Y\rangle_g = \langle g^{-1}X, g^{-1}Y\rangle_K = \langle Xg^{-1}, Yg^{-1}\rangle_K, \tag{11}$$

where $X, Y \in T_gG$ are tangent vectors at g. The second equality holds by Ad-invariance of the inner product on $\mathfrak{g}$. This equips G with a bi-invariant Riemannian geometry (meaning that both left and right multiplication act isometrically). For such groups the geodesics through an element g are precisely the curves of the form $t \mapsto ge^{vt}$, where $v \in \mathfrak{g}$ (see [41], Lemma 21.2).

The quotient space G/K inherits a G-invariant Riemannian geometry from G: At the origin o the inner product $\langle X, Y\rangle_o$ is defined as $\langle X, Y\rangle_K$ using that $T_oG/K \cong \mathfrak{p} \subset \mathfrak{g}$. This is extended to arbitrary points $[g]$ by the (differential of the) G action just as in (11): $\langle X, Y\rangle_{[g]} = \langle X \cdot g^{-1}, Y \cdot g^{-1}\rangle_o$ (this is indeed well defined, i.e., independent of the choice of g within the coset). In particular, the resulting Riemannian metric is automatically G invariant, meaning that G acts isometrically on G/K. It now holds by construction that the natural projection $\pi : G \to G/K$ induces an isometry between $(\ker d\pi|_g)^\perp$ and $T_{\pi(g)}(G/K)$ for all g. Such a map is called a Riemannian submersion [39]. Since $\ker(d\pi|_{\mathbb{1}}) = \mathfrak{k}$, this just follows from $T_o(G/K) \cong \mathfrak{p}$ and our definition of the metric (in general, we have $\ker d\pi|_g = g^{-1}\mathfrak{k}g$ and hence $T_{[g]}G/K \cong g^{-1}\mathfrak{p}g$). The notation $d\pi|_g$ means that we take the differential of π at the point g, which is a linear map $T_gG \to T_{\pi(g)}G/K$. A concrete description of this geometry for basic examples like SU(2)/SO(3) is given in Sec. V.

The crucial point for us is the following: From π being a Riemannian submersion it follows that geodesics in G/K running through a coset $x = [g]$ are precisely curves of the form $[g\exp(ut)] = x \cdot \exp(ut)$ with $u \in g^{-1}\mathfrak{p}g$ and that they have the same length as their corresponding lifts of G (see [39], Proposition 18.8).

Let us summarize the relevant structure: We have a quantum control problem with dynamical Lie algebra $\mathfrak{g}$ and control algebra $\mathfrak{k}$, associated Lie groups $G = e^{\mathfrak{g}}$ and $K = e^{\mathfrak{k}}$, and K is a closed subgroup of G. We use the Killing form as an inner product on $\mathfrak{g}$ and take the decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ with $\mathfrak{p} = \mathfrak{k}^\perp$. We can always ensure $iH_d \in \mathfrak{p}$ by removing any part not in $\mathfrak{p}$ via the controls. We know G/K is a reductive space and we know precisely which form the geodesics on G/K have.

We now compute the speed at which the system moves through G/K as it evolves. We know the possible directions of travel at the origin are given by the adjoint orbit of the drift, $k^\dagger iH_dk \in \mathcal{O}(iH_d)$ with $k \in K$, so in a time dt we move to the coset of $\exp(ik^\dagger H_dkdt)$. To determine the distance ds this corresponds to in G/K we use the metric on G/K and because we have a Riemannian submersion we can employ (10) to obtain

$$\begin{aligned} ds &= \sqrt{\langle ik^{-1}H_dkdt, ik^{-1}H_dkdt\rangle_K} \\ &= dt\sqrt{2n\,\mathrm{Tr}(H_d^2)}, \end{aligned} \tag{12}$$

where we have used the fact that the Killing form is Ad invariant. Using the G invariance of the metric on G/K, the same argument shows this result also holds at other points $x \neq o \in G/K$. This means the speed at which the system moves in G/K is constant and is given by

$$v = \sqrt{2n\,\mathrm{Tr}(H_d^2)}. \tag{13}$$

TABLE I. Diameter of various compact symmetric spaces arising from the quotient G/K, when using the Killing inner product on $\mathfrak{g}$ in order to obtain a Riemannian metric on G/K.

G/K	diam(G/K)
SU(n)/SO(n)	$\frac{\sqrt{2}}{2}\pi n$ for n even
SU(n)/SO(n)	$\frac{\sqrt{2}}{2}\pi(n^2-1)^{1/2}$ for n odd
SU(n)/Sp($\frac{n}{2}$)	$\frac{\pi}{2}n$ for $n/2$ even
SU(n)/Sp($\frac{n}{2}$)	$\frac{\pi}{2}(n^2-4)^{1/2}$ for $n/2$ odd
SU($p+q$)/S[U(p) × U(q)]	$\pi(p+q)^{1/2}p^{1/2}$ for $p \leqslant q$

Now that we know the form a geodesic in G/K must take and speed with which a quantum system moves along it, the task is to find the diameter of the G/K space, that is, the farthest distance possible pairs of points can have. Given the fact that motion in G/K is at constant speed, this will give us a lower bound on the quantum speed limit, that is, the time taken to produce the most difficult unitary. This proves the following theorem.

Theorem 1. Let G be the dynamical Lie group of the control problem (1) and assume that the subgroup $K \subset G$ generated by the controls alone is closed. Let T_{QSL} be the minimum time in which all unitaries of G can be reached. Then

$$T_{\mathrm{QSL}} \geqslant \frac{\mathrm{diam}(G/K)}{\sqrt{2n\,\mathrm{Tr}(H_d^2)}}. \tag{14}$$

The practical usefulness of this result of course relies on an explicit computation of the diameter (or at least a lower bound). The diameter of the Riemannian manifold G/K is

$$\mathrm{diam}(M) = \sup\{d(x, y) : x, y \in G/K\}, \tag{15}$$

where $d(x, y)$, the Riemannian distance between x and y, is the infimum over the lengths of curves connecting these points as measured by the metric. Since G/K is homogeneous the definition is equivalent to $\mathrm{diam}(M) = \sup_{x\in G/K} d(x, o)$.

That Eq. (14) is only a lower bound in general is due to the restricted movement on G/K: The possible directions are given by the adjoint orbit $\mathcal{O}(iH_d)$. If the adjoint orbit does not allow for the needed directions, the time taken to generate some unitaries will be longer than the lower bound given in Eqs. (16)–(20).

Finding the diameter of the homogeneous space G/K is in general difficult. However, the diameter of all symmetric spaces arising from classical compact groups has been calculated by Yang [42]. [We note there appears to be an error in Yang's paper; the results given for the diameters of SU($2n$)/Sp(n) should be divided by $\sqrt{2}$.] If we consider only symmetric spaces arising from quotient groups of the form G/K, where $G = \mathrm{SU}(n)$, there are only three possibilities, which we list in Table I. Note that the group Sp(n) refers to the compact symplectic group and we have chosen to use the Killing form as the inner product on the Lie algebra $\mathfrak{g}$ to obtain a metric on G/K. Consequently, if the Lie group K generated by the controls is one of SO(n), Sp(n), or S[U(p) × U(q)]

[the matrices of unit determinant in $\mathrm{U}(p) \times \mathrm{U}(q)$], we obtain the quantum speed limits

$$\mathrm{SO}(n): \quad T_{\mathrm{QSL}} \geqslant \begin{cases} \frac{\sqrt{n}\pi}{2\sqrt{\mathrm{Tr}(H_d^2)}} & \text{for } n \text{ even} \quad (16) \\ \frac{\pi(n^2-1)^{1/2}}{2\sqrt{n\,\mathrm{Tr}(H_d^2)}} & \text{for } n \text{ odd}, \quad (17) \end{cases}$$

$$\mathrm{Sp}(n/2): \quad T_{\mathrm{QSL}} \geqslant \begin{cases} \frac{\sqrt{n}\pi}{2\sqrt{2\,\mathrm{Tr}(H_d^2)}} & \text{for } n/2 \text{ even} \quad (18) \\ \frac{\pi(n^2-4)^{1/2}}{2\sqrt{2n\,\mathrm{Tr}(H_d^2)}} & \text{for } n/2 \text{ odd}, \quad (19) \end{cases}$$

$$\mathrm{S}[\mathrm{U}(p)\times\mathrm{U}(q)]: \quad T_{\mathrm{QSL}} \geqslant \frac{\sqrt{p}\pi}{\sqrt{2\,\mathrm{Tr}(H_d^2)}} \quad \text{for } p \leqslant q. \quad (20)$$

The result for the case where the control group is $\mathrm{Sp}(\frac{n}{2})$ is particularly interesting. It is known that this control group provides complete state controllability even in the absence of a drift Hamiltonian [3]. As we have assumed arbitrarily strong controls, this means that one can find controls that move from any state to any other state arbitrarily quickly, that is, the speed limit for state control in this case is zero. The emergence of a finite speed limit as given by (18) and (19) highlights the difference between unitary control and state control.

It is also worth noting the appearance of explicit dependence of the Hilbert space dimension in these bounds, as existing speed limits in the literature are usually not able to include this factor.

IV. BOUND TIGHTNESS IN TERMS OF DIMENSION COUNTING

Let us discuss the tightness of our speed limit bounds from the perspective of the dimensions of the control group. Our bound was obtained from the observations that the speed of movement in G/K is constant and that the largest distance between two points (the diameter) is finite. While the existence of a length minimizing geodesic connecting the origin o with any other point $x \in G/K$ is guaranteed (by the Hopf-Rinow theorem), it is not clear that such a geodesic is available by choice of suitable controls.

Denote by D the set of points maximizing the distance from the origin, i.e., the points $x \in G/K$ with $d(o,x) = \mathrm{diam}(G/K)$, where d denotes the Riemannian length on G/K. As both inversion and the K action are isometries that fix the origin, we know that they also leave D invariant, i.e., $D^{-1} = D$ and $D \cdot k = D$ for all $k \in K$. For the bounds to be tight, it is necessary that for each $x \in D$, there is a minimal geodesic connecting the origin o with x which is of the form $[\exp(vt)]$ with $v \in \mathcal{O}(iH_d)$. This trivially holds if $\mathcal{O}(iH_d)$ is equal to the sphere $S = \partial B_r(0)$ in $\mathfrak{p}$ of radius $r = \sqrt{\langle H_d, H_d\rangle_K}$ (note that all directions in the adjoint orbit have the same length by Ad invariance). The adjoint orbit itself is a closed manifold which is a subset of S. In the case that the dimension of $\mathcal{O}(iH_d)$ is maximal (i.e., equal to $\dim\mathfrak{p} - 1$), it follows that $\mathcal{O}(iH_d)$ is equal to S and thus contains every direction in $\mathfrak{p}$.

TABLE II. Dimensions of the Lie algebras associated with the three symmetric spaces associated with $\mathrm{SU}(n)$. Here $d_k = \dim(\mathfrak{k})$ is the dimension of the control algebra and $d_p = \dim(\mathfrak{p})$ is the dimension of the symmetric space G/K. If $\dim(\{A \in \mathfrak{k} \mid [H_d, A] = 0\}) = 1 + \dim\mathfrak{k} - \dim\mathfrak{p}$, the adjoint orbit from the controls is guaranteed to have enough degrees of freedom to choose any single-parameter geodesic from the origin to a point corresponding to the diameter of the space.

G/K	d_p	d_k	$d_k - d_p + 1$
$\mathrm{SU}(n)/\mathrm{SO}(n)$	$\frac{1}{2}(n^2+n-2)$	$\frac{n}{2}(n-1)$	$2-n$
$\mathrm{SU}(n)/\mathrm{Sp}(\frac{n}{2})$	$\frac{1}{2}(n^2-n-2)$	$\frac{n}{2}(n+1)$	$2+n$
$\mathrm{SU}(p+q)/\mathrm{S}[\mathrm{U}(p)\times\mathrm{U}(q)]$	$2pq$	p^2+q^2-1	$(p-q)^2$

The dimension of the adjoint orbit is

$$\dim\mathcal{O}(iH_d) = \dim\mathfrak{k} - \dim(\{A \in \mathfrak{k} \mid [H_d, A] = 0\}) \quad (21)$$

because $T_A\mathcal{O}(A) \cong \mathfrak{p}/\ker[A,\cdot]$. This means that the bound is tight if we have equality in

$$\dim(\{A \in \mathfrak{k} \mid [H_d, A] = 0\}) \geqslant 1 + \dim\mathfrak{k} - \dim\mathfrak{p}. \quad (22)$$

This inequality always holds and equality is equivalent to the ability to move into every possible direction in G/K.

We stress that this is a sufficient condition, but not a necessary one. Even if the adjoint orbit does not have enough directions to access all dimensions of $\mathfrak{p}$, that does not rule out the possibility that, for a specific drift Hamiltonian, a single-parameter geodesic from the origin to the locus corresponding to the diameter with an initial direction lying in the adjoint orbit does not exist.

Table II lists the relevant dimensions for $\mathfrak{k}$ and $\mathfrak{p}$ for the symmetric spaces we are considering, as well the quantity corresponding to the right-hand side of (22). For the symmetric spaces $\mathrm{SU}(n)/\mathrm{Sp}(\frac{n}{2})$ and $\mathrm{SU}(p+q)/\mathrm{S}[\mathrm{U}(p)\times\mathrm{U}(q)]$ the number degrees of freedom in the control group exceeds that of the quotient space, so naive dimension counting arguments suggest the bound is likely to be tight, although one must test for equality in Eq. (22) to be sure. Combined with the numerical results in the Sec. VI, we make the following conjecture.

Conjecture 1. Let $\mathrm{SU}(n)$ be the dynamical Lie group of the control problem (1) and $K = \mathrm{Sp}(\frac{n}{2})$. Then Eqs. (18) and (19) are tight. If $K = \mathrm{S}[\mathrm{U}(p)\times\mathrm{U}(q)]$, with $p + q = n$, then Eq. (20) is tight.

However, it is clear that for the case $\mathrm{SU}(n)/\mathrm{SO}(n)$ with $n > 2$ it is never possible to achieve equality in (22) as the dimension of a space can never be less than zero. Nonetheless, as we will see in our numerical tests of the speed limit in Sec. VI, for some drift Hamiltonians the bounds (16) and (17) are still tight. To investigate this in more detail, we consider case where the control algebra is $\mathfrak{k} = \mathfrak{so}(n)$. We wish to determine the size of $\dim(\{A \in \mathfrak{k} \mid [H_d, A] = 0\})$. To begin, we note that any drift H_d can be moved into the Cartan subalgebra by some controls. This subalgebra is diagonal with trace zero, meaning we need only consider the case where H_d is diagonal. Let $H_d = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where the λ_i are the eigenvalues of H_d.

We choose the basis of $\mathfrak{k}$ to be the set of $n \times n$ matrices given by $B_{ij} = |e_i\rangle\langle e_j| - |e_j\rangle\langle e_i|$, $i < j$, where $|e_i\rangle$ is the column vector with a 1 in row i and 0 everywhere else. The size of this basis is $\dim\mathfrak{k} = n(n-1)/2$.

The commutator of H_d with the basis elements of $\mathfrak{k}$ is given by

$$[H_d, B_{ij}] = (\lambda_i - \lambda_j)(|e_i\rangle\langle e_j| + |e_j\rangle\langle e_i|), \tag{23}$$

demonstrating that to ensure $[H_d, B_{ij}] = 0$ we require $\lambda_i = \lambda_j$. This means that $\dim(\{A \in \mathfrak{k} \mid [H_d, A] = 0)$ is given by the number of pairs M of eigenvalues of H_d that are degenerate, giving $\dim\mathcal{O}(iH_d) = \dim\mathfrak{k} - M$.

So, for example, if all eigenvalues are distinct, $M = 0$, meaning $\dim\mathcal{O}(iH_d) = \dim\mathfrak{k}$. If all eigenvalues are identical, then $M = \frac{1}{2}n(n-1) = \dim\mathfrak{k}$, meaning $\dim\mathcal{O}(iH_d) = 0$. This shows that the more eigenvalues that are degenerate, the smaller the chance the adjoint orbit allows us to choose a direction that makes the bound tight.

As an example, consider the case SU(2)/SO(2) discussed in the preceding section. Here $d_k = 1$ and $d_p = 2$, so the equality in Eq. (22) is achieved when the two eigenvalues of H_d are not degenerate. Specifically, in this case the adjoint orbit is one dimensional, and since G/K is the two-sphere, this single degree of freedom for the adjoint orbit suffices to choose arbitrary directions on the two-dimensional manifold, meaning achieving a minimal geodesic from the origin to the diameter is always possible.

V. EXAMINATION OF THE TIGHTNESS OF OUR BOUNDS WITH CARTAN CONTROLS

In the preceding section we developed a criterion that was sufficient to show our speed limit bounds were tight, based on determining the dimension of the adjoint orbit. As this criterion is not necessary, however, this section examines what else can be said about the tightness of the bounds. We do this mostly for the controllable case $\mathfrak{g} = \mathrm{su}(n)$ by using the root system of $(\mathfrak{g}, \mathfrak{k})$, and we illustrate the approach using $\mathfrak{k} = \mathrm{so}(n)$.

We begin by considering the symmetric spaces described in the preceding section as arising from the situation where the controls form a Cartan decomposition of the full Lie algebra. As before, the control algebra is denoted by $\mathfrak{k}$ and the associated control group is $K = e^{\mathfrak{k}}$. This decomposition is often used in quantum control problems. The main point is that a Cartan decomposition provides a decomposition of the full Lie algebra of the form $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$, with $\mathfrak{p} = \mathfrak{k}^{\perp}$, which satisfies the relations

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \tag{24}$$

$$[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}, \tag{25}$$

$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}. \tag{26}$$

These conditions include those required for a reductive space plus the additional condition (26). Here the Lie algebra is again equipped with the inner product (10) in order to match the speeds and manifold diameters computed in the preceding section.

There are precisely three Cartan decompositions of su(n) [3]. They are $\mathfrak{k} = \mathrm{so}(n)$, $\mathfrak{k} = \mathrm{sp}(\frac{n}{2})$, and $\mathfrak{k} = \mathfrak{s}[\mathfrak{u}(p) \oplus \mathfrak{u}(q)]$, with $p + q = n$, where

$$\mathfrak{s}[\mathfrak{u}(p) \oplus \mathfrak{u}(q)] = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| A \in \mathfrak{u}(p), B \in \mathfrak{u}(q), \mathrm{Tr}A = -\mathrm{Tr}B \right\}. \tag{27}$$

These three decompositions are associated with the three possible symmetric spaces of SU(n) introduced before.

To proceed we need the following notion: A Cartan subalgebra of $\mathfrak{g}$ (with respect to a Cartan decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$) is a maximal Abelian subalgebra $\mathfrak{a}$ contained in $\mathfrak{p}$ [3] [subalgebras contained in $\mathfrak{p}$ are Abelian because of Eq. (26)]. All Cartan subalgebras are conjugate via an element $k \in K$ and every element of $\mathfrak{p}$ is contained in a Cartan subalgebra [3]. In particular, for every $X \in \mathfrak{p}$ there are $k \in K$ and $A \in \mathfrak{a}$ so that

$$X = kAk^{-1}. \tag{28}$$

From now on we assume that $\mathfrak{g} = \mathrm{su}(n)$. It is possible to use the maximal torus theorem to show [16] that the fastest way to generate any target unitary U_{targ} is to find the smallest τ such that it is possible to write

$$U_{\mathrm{targ}} = k_1 \exp(v\tau) k_2, \tag{29}$$

with $k_1, k_2 \in K$ and $v \in \mathfrak{p}$ of the form

$$v = \sum_{i=1}^{m} \beta_i X_i, \quad \beta \geqslant 0, \ \sum \beta_i = 1, \ X_i \in \mathcal{W}(iH_d), \tag{30}$$

where $\mathcal{W}(iH_d) = \mathfrak{a} \cap \mathcal{O}(iH_d)$ is the Weyl orbit of iH_d. Note that Eq. (29) does not actually give a specific minimal time solution; it merely states the form it must take and reduces the difficulty of the (usually numerical) optimization problem.

Clearly, $v \in \mathfrak{p}$ and gives the direction of the geodesic connecting the identity and U_{targ} in G/K, so (29) shows the correct control strategy is to apply strong controls initially to pick the correct direction in G/K provided the adjoint orbit allows the direction, drift for a time with all controls at zero, and then apply strong controls again to move to the final desired U_{targ} within the coset. If v lies in $\mathcal{O}(iH_d)$, we can generate it and will always be capable of moving on a geodesic between any two points in G/K, including from the identity to the point the farthest away corresponding to the diameter of G/K. Since all elements of the Weyl orbit commute, $\exp(v\tau)$ can be written

$$\exp(v\tau) = \exp(\beta_1 X_1)\exp(\beta_2 X_2)\cdots\exp(\beta_m X_m), \tag{31}$$

with β_i and X_i as in (30). Because the elements of the Weyl orbit $\mathcal{W}(iH_d)$ are a subset of the adjoint orbit $\mathcal{O}(iH_d)$, we are clearly capable of implementing $\exp(v\tau)$ through the action of the drift and arbitrarily strong controls.

It is important to note, however, that the fastest way to implement a unitary by using the available controls, i.e., the path described by (31), is not necessarily a minimal geodesic between the initial and final points even though it is a piecewise geodesic. Only if the right-hand side of (31) consists of single exponential is it possible that the time this fastest path takes coincides with our lower bound given by Eqs. (16)–(20).

We now examine the question as to when the v in Eq. (31) lies within the adjoint orbit, making our speed limit lower bounds tight. As said in the preceding section, the fact that G/K is homogeneous implies that $\mathrm{diam}(G/K) =$

$\sup_{x\in G/K} d(o,x)$, meaning we need only to look for the point x corresponding to the target unitary that is farthest from the group identity along a single-parameter geodesic. This point has the property that a geodesic starting at the origin stops being length minimizing after running through x. The set of points where geodesics starting at the origin o stop being length minimizing is known as the cut locus (of the origin). By the Hopf-Rinow theorem [39], there is for every $x \in G/K$ a minimizing geodesic joining it with the origin. If G is simply connected [as is the case for $\mathrm{SU}(n)$], the symmetric space G/K is also simply connected (see [43], Proposition 3.6), which implies that the cut locus coincides with what is called the first conjugate locus (see [44], Theorem 3.5.4). To illustrate these ideas, we give an explicit description of the cut locus for SU(2)/SO(3) below.

The conjugate locus can be described in terms of the positive roots $\Delta^+(\mathfrak{g},\mathfrak{a})$ of the Lie algebra $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{a}$. Specifically, the exact form of the conjugate locus of a point $x \in M$ is given by [42,44,45]

$$C(x) = \{x \cdot e^A k \mid k \in K, A \in \mathfrak{a} \text{ s.t. (33) holds}\}, \tag{32}$$

with the root condition

$$\exists \alpha \in \Delta^+(\mathfrak{g},\mathfrak{a}),\ 0 \neq m \in \mathbb{Z}: \quad \alpha(A) = im\pi \tag{33}$$

and the first conjugate locus corresponds to $m = \pm 1$. Note that the locus $C(x)$ is K invariant (i.e., invariant under the right action by elements of K) and invariant under inversion. As explained previously, we only care about $C(o)$. In this case we have $o \cdot e^A k = [e^A k] = [\exp(k^{-1}Ak)]$, so the conjugate (and hence cut) locus consists precisely of the points $[\exp \mathcal{O}(A)]$, where A satisfies the root condition (33).

In order to illustrate this approach, we consider the simplest cases: SU(2)/SO(2) and SU(3)/SO(3).

For SU(2)/SO(2) we consider the control problem (7). The control algebra is $\mathfrak{k} = \mathrm{span}\{i\sigma_x\}$ and the Cartan subalgebra is given by $\mathfrak{a} = \mathrm{span}\{i\sigma_z\}$. The locus $C(o)$ consists of points $[e^{i\eta\sigma_z}] \cdot e^{i\beta\sigma_x}$, where $\beta \in [0, 2\pi]$ and $A = i\eta\sigma_z$ must satisfy the root condition. There is a single positive root α_1 in this case given by $\alpha_1(i\eta\sigma_z) = 2i\eta$. This means we require $2i\eta = \pm i\pi$. The cut locus is therefore given by the set $\{[e^{\pm i\sigma_z\pi/2}e^{i\beta\sigma_x}] \mid \beta \in [0, 2\pi]\}$, which actually only contains the single coset $[e^{i\sigma_z\pi/2}]$ because all $e^{\pm i\sigma_z\pi/2}e^{i\beta\sigma_x}$ determine the same coset. A geometrical way to understand this is that the coset $[e^{i\sigma_z\pi/2}]$ is the unique point on $S^2 = \mathrm{SU}(2)/\mathrm{SO}(3)$ that is an antipodal point to the origin. The group action of SO(2) acts on the sphere by rotating about the axis going through the origin and hence fixes this antipodal point. Since the control elements k_1 and k_2 can be applied arbitrarily quickly, our drift will hit the conjugate locus at time $t = \pi/2$, giving the expected speed limit and showing the bound is tight.

The SU(3)/SO(3) case is more complex. Generally speaking, for a Riemannian manifold the set of points corresponding to the diameter and the set of points corresponding to the cut locus are not the same. It is the case, however, that the diameter locus must be a (possibly equal) subset of the cut locus.

Consider the question whether there is single-parameter geodesic that lies in the Weyl orbit that, up to conjugation by the controls, lies on the conjugate locus at a specific time t_{QSL} given by the quantum speed limit bound in Eqs. (16) and (17), that is, whether for a given drift H_d we can find a solution for a specific $A \in \mathfrak{a}$, $k_1, k_2 \in K$, satisfying

$$\exp(A) = k_1 \exp(iH_d t_{\mathrm{QSL}}) k_2, \tag{34}$$

with $\alpha(A) = \pm i\pi$. If we can find such an A, then we know we can move to the cut locus on a single minimal geodesic, but this final point may not lie on the set of diameter points. If it does, our bound is clearly tight, since in order to reach the diameter, our geodesic must fail to be distance minimizing for the first time at that point. This means that the condition given by Eq. (34) is necessary but not sufficient. To make it sufficient, it would be necessary to be able find a solution to Eq. (34) for all $A \in \mathfrak{a}$, which is generally not possible.

The Cartan subalgebra $\mathfrak{a}$ has rank 2 and can be parametrized as $A = c_1 h_1 + c_2 h_2$, where c_1 and c_2 are real parameters and we use the Cartan-Weyl basis

$$h_1 = i\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h_2 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{35}$$

There are three positive roots for su(3) and their action on A is given by

$$\alpha_1(c_1 h_1 + c_2 h_2) = i(2c_1 - c_2), \tag{36}$$

$$\alpha_2(c_1 h_1 + c_2 h_2) = i(-c_1 + 2c_2), \tag{37}$$

$$\alpha_3(c_1 h_1 + c_2 h_2) = i(c_1 + c_2). \tag{38}$$

Applying these roots to (33) shows that the $A \in \mathfrak{a}$ generating the conjugate locus can be parametrized as the union of the three sets of Lie algebra elements

$$\begin{aligned} A_1 &= i\,\mathrm{diag}\{c_1, c_1 - m_1\pi, -2c_1 + m_1\pi\}, \\ A_2 &= i\,\mathrm{diag}\left\{c_1, \tfrac{1}{2}(-c_1 + m_2\pi), -\tfrac{1}{2}(c_1 + m_2\pi)\right\}, \\ A_3 &= i\,\mathrm{diag}\{c_1, -2c_1 + m_3\pi, c_1 - m_3\pi\}, \end{aligned} \tag{39}$$

where c_1 is an arbitrary real parameter and $m_i = \pm$.

This shows that the bound (17) will be exact if we can find integers m_i not equal to zero and control group elements k and k' such that

$$k \exp(A) k' = \exp(iH_d t_{\mathrm{QSL}}) \tag{40}$$

for all A satisfying (39), as this ensures we can reach the entire cut locus, of which the diameter is a subset. Since each $k \in K$ has three parameters, this is already an eight-parameter problem and is analytically difficult. Higher-dimensional groups will pose an even bigger problem.

We can however gain some partial information by making use of Eqs. (36)–(38). We note that any drift $iH_d \in \mathfrak{p}$ can be moved into a Cartan subalgebra $\mathfrak{a}$ via conjugation by some controls, i.e., $iH_d^{\mathfrak{a}} = kiH_d k^\dagger$. This is a unitary transformation which does not change the spectrum, and since the Cartan subalgebra is spanned by the real diagonal matrices with zero trace, we write $H_d^{\mathfrak{a}}$ in terms of its eigenvectors $H_d^{\mathfrak{a}} = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3\}$, where $\lambda_3 = -\lambda_1 - \lambda_2$ since $\mathrm{Tr}(H_d^{\mathfrak{a}}) = 0$.

If we apply Eqs. (36), (38), and (33) to $A = iH_d^{\mathfrak{a}} t = it\,\mathrm{diag}\{\lambda_1, \lambda_2, -\lambda_1 - \lambda_2\}$ we see that to intersect the cut locus we need one of

$$(\lambda_1 + 2\lambda_2)t = m_1\pi, \tag{41}$$

$$(\lambda_1 - \lambda_2)t = m_2\pi \tag{42}$$

to be satisfied. Since t is a continuous positive parameter, these conditions will almost always be satisfied for some t, unless λ_1 and λ_2 are chosen to make the left-hand side of one of (41) or (42) equal to zero. This occurs if $\lambda_1 = \lambda_2$ or if $\lambda_1 = -2\lambda_2$. However, these two conditions are equivalent, since if $\lambda_1 = -2\lambda_2$, then $\lambda_3 = -\lambda_1 - \lambda_2 = \lambda_2$ due to the zero-trace condition, showing that λ_1 and λ_3 are degenerate. Consequently if any two eigenvalues are degenerate, one of (41) or (42) cannot be met.

Note that this condition does not guarantee there is no element of the Weyl orbit that produces a single-parameter geodesic from the identity to the point corresponding to the diameter, but it reduces the possibility since it ensures there is a portion of the cut locus that cannot be reached. This is because each root condition corresponds to a geodesic that intersects a different portion of the cut locus, so failing one of the conditions (41) or (42) will only result in the bound not being tight if the diameter lies on that portion of the cut locus.

However, this is more powerful than might first be imagined, since if any drift Hamiltonian with degenerate eigenvalues exceeds the lower bound on the speed limit, then all drift Hamiltonians with degenerate eigenvalues will exceed the lower bound. This is because the ordering of the elements of a diagonal matrix can be arbitrarily switched by controls, and multiplying the drift by a scalar does not change whether bound is tight; it merely stretches the timescale. This means all drifts with two degenerate eigenvalues have the same behavior regarding whether the bound is tight. If this can be determined for a single case in SU(3)/SO(3), the behavior of all drift Hamiltonians is known. This is one of the questions that will be investigated numerically in the next section.

VI. NUMERICAL TESTS OF THE ANALYTIC SPEED LIMITS

In Sec. III we derived lower bounds on the quantum speed limit for various types of controls and in Sec. V we looked at evidence for when these bounds might be saturated, i.e., when the bound is actually exact. We now examine these systems to determine the speed limit via a numerical optimization procedure. The motivation is to provide checks on both analytic bounds as well as to test our dimension counting and eigenvalue degeneracy arguments for bound tightness laid out in Secs. IV and V. In addition, bilinear optimal control problems are seldom analytically tractable and are usually approached numerically, so our analytic results provide an ideal test for checking the performance of various optimization strategies.

Our approach is to determine the quantum speed limit numerically for a variety of drifts and a variety of Hilbert space dimensions, assuming the controls Hamiltonians generate one of the three Lie algebras $\mathrm{so}(n)$, $\mathrm{sp}(\frac{n}{2})$, or $\mathfrak{s}[\mathfrak{u}(p)\oplus\mathfrak{u}(q)](n)$. To do this we choose a series of Haar-random unitary targets and attempt to numerically find optimal controls that, for a specific drift Hamiltonian H_d, would achieve that unitary at a specific chosen time T. That time is divided into N discrete intervals (time slots), with the width of each time slot given by T/N, and the controls are assumed to have a constant amplitude over each interval, i.e., the controls are time dependent but piecewise constant. In the limit of a large number of time slots, arbitrary control functions are well approximated. Specifically, we solve

$$i\frac{d}{dt}U(t) = \left(H_d + \sum_{j=1}^{m} f_j(t)H_j\right)U(t), \tag{43}$$

with an initial random guess at the amplitudes in each time slot for each independent control function $f_j(t)$. We use QUTIP's optimal control package [46] with a gradient-ascent algorithm to find the control functions that maximize the overlap between the final unitary resulting from the evolution of (43) and the desired target unitary, as given by the phase-insensitive fidelity measure

$$F = \frac{1}{n}|\mathrm{Tr}[U^{\dagger}_{\mathrm{target}}U(T)]|. \tag{44}$$

This process is then repeated many times with different random initial guesses to help the optimizer becoming stuck in local minima. For each target unitary, we gradually increase the time T until a solution could be found where the fidelity error $1 - F$ is less than a cutoff of 10^{-7}. This is repeated for a large number of random unitaries, and the quantum speed limit for that particular drift is taken to be the lowest time for which we could guarantee solutions for all the unitaries with a fidelity error less than the cutoff. This is illustrated in Fig. 1 with a small sample of the results for the SU(3)/SO(3) case for a particular drift corresponding to a predicted analytic

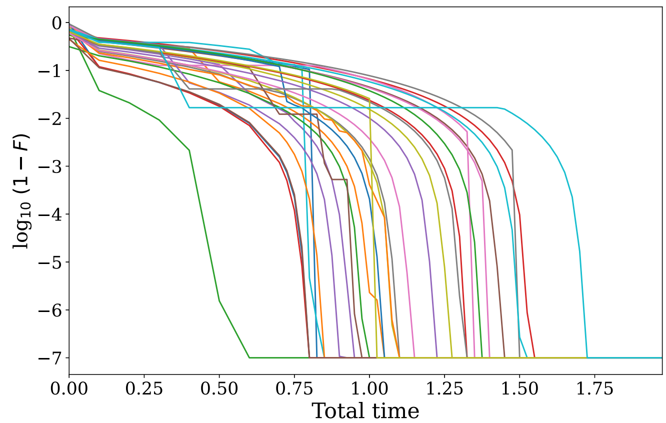

FIG. 1. Example of how the quantum speed limit is determined numerically, for the case with SO(3) controls. Each line corresponds to a random target unitary in SU(3). We attempt to find a solution for time-dependent controls for a given fixed time total time T (horizontal axis) and a specified drift H_d. As T is increased, better solutions can be found, giving a better-fidelity overlap with the target unitary. When the fidelity error is lower than some cutoff, we take this to mean we have found a solution for the control pulse that can generate the unitary. If this is repeated many times for many random unitary targets, the speed limit is taken to be the time for which we can find a control pulse for all possible targets in this time or less. This plot shows 30 Haar random unitary targets with $H_d = \mathrm{diag}\{1, 0, -1\}$ and 100 time slots.

quantum speed limit of $t_{\rm QSL} = 1.81$. It shows how the fidelity error for any given target reduces as more time is allowed, until we reach a sudden drop in the error which we interpret as the existence of a set of control functions that can achieve that unitary.

Not all drift Hamiltonians H_d need to be examined. First, if iH_d has some overlap with $\mathfrak{k}$ then this portion can be removed arbitrarily quickly by application of the controls, so we can assume $iH_d \in \mathfrak{p}$. Second, since $iH_d \in \mathfrak{p}$, due to (28) it can also be moved into a subspace of $\mathfrak{p}$ corresponding to a Cartan subalgebra $\mathfrak{a}$ by application of the controls. This means we need only consider drift Hamiltonians drawn from $\mathfrak{a}$ (multiplied by i).

There are a number of reasons that the numerical approach may provide a speed limit higher than the true one, making it difficult to determine if the lower bounds given by Eqs. (16)–(20) are truly tight. First, for a given time there may have been a better solution that the optimizer simply missed, even with many attempts with random initial conditions. Second, because we have divided the total time T into N time slots, elements of the control group cannot be performed arbitrarily fast; they take at least T/N. Both of these serve to ensure the speed limit found numerically will be slightly higher than the true speed limit. Third, since the testing is done with a set of discrete choices of time T, there may be a fast solution at a specific low T that we do not see because that value of T is not tested, giving the illusion that the speed limit for that unitary is higher than it actually is. Conversely, we draw the target unitaries from a Haar-random set. As the dimension of the Hilbert space increases, it becomes increasingly difficult to properly sample the set of possible unitaries, and this is exacerbated by the fact that higher dimensions take longer to simulate, so fewer targets can be sampled.

With these caveats in mind, we now examine the results of the numerical optimization process. We first consider the case where the controls generate the $\mathrm{Sp}(n/2)$ subgroup. As conjectured in Sec. V, we might expect the speed limit bounds given by (18) and (19) to be tight. The elements of the Lie algebra $\mathrm{sp}(\frac{n}{2})$ have the form $\begin{pmatrix} L_1 & L_2 \\ -L_2^* & L_1^* \end{pmatrix}$, with L_1 skew Hermitian and $L_2 = L_2^T$, where L_1 and L_2 are complex and $\frac{n}{2} \times \frac{n}{2}$ in size. One chooses a basis for this space, and the control Hamiltonians will be given by this basis multiplied by i.

As discussed above, we need only consider drift Hamiltonians that lie within the Cartan subalgebra, which drastically reduces the possibilities. For $\mathrm{sp}(\frac{n}{2})$ this is given by matrices of the form [3]

$$A = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}, \tag{45}$$

with D diagonal and $D \in \mathrm{su}(\frac{n}{2})$. Figure 2 shows results for the $\mathrm{SU}(4)/\mathrm{Sp}(2)$ case, with a drift Hamiltonian $H_d = \mathrm{diag}\{1, -1, 1, -1\}$. Up to a constant factor, this is in fact the only drift Hamiltonian that lies within the Cartan subalgebra. As expected, all random target unitaries chosen can be reached with a time under the speed limit given by (18), and the maximal time falls on the speed limit, showing that the bound is tight.

Next we consider the case where the controls generate the $\mathrm{S}[\mathrm{U}(p) \times \mathrm{U}(q)]$ subgroup of $\mathrm{SU}(n)$, with $p + q = n$, $p \leqslant q$.

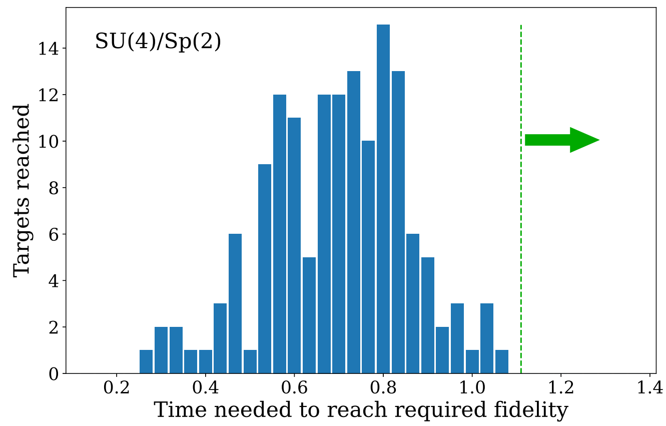

FIG. 2. Speed limit for the $\mathrm{SU}(4)/\mathrm{Sp}(2)$ case, with a drift $H_d = \mathrm{diag}\{1, -1, 1, -1\}$. The histogram shows the fastest possible times to achieve 150 randomly chosen unitary targets when using $\mathrm{Sp}(\frac{n}{2})$ controls, with the analytic lower bound of 1.11 given by (18) represented by the vertical green dashed line. As all targets can be met in a time less than the bound and some targets are at the bound, the bound is tight.

Its Lie algebra $\mathfrak{s}[\mathfrak{u}(p) \oplus \mathfrak{u}(q)]$ is given by Eq. (27). Again, we conjectured above that the bound given by (20) is tight. The Cartan subalgebra is given by matrices of the form [3]

$$A = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}, \tag{46}$$

where B is a real $p \times q$ matrix that is zero everywhere except for the first p columns, which is given by a $p \times p$ diagonal matrix. We chose our drift Hamiltonian to be given by

$$H_d = i\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{47}$$

Figure 3 shows results for the $\mathrm{SU}(5)/\mathrm{S}[\mathrm{U}(3) \times \mathrm{U}(2)]$ case with the drift Hamiltonian given by (47). As expected, all random target unitaries chosen can be reached with a time equal to or less than the speed limit given by (20). Again, we conclude that in this case the bound is tight.

We now arrive at the third and final case, $\mathrm{SU}(n)/\mathrm{SO}(n)$. Dimension counting arguments suggest that we cannot always rely on the bound being tight and at least in the $\mathrm{SU}(3)/\mathrm{SO}(3)$ case we expect the bound to fail to be tight if the drift Hamiltonian has a degenerate eigenvalue.

The Lie algebra $\mathrm{so}(n)$ associated with the $\mathrm{SO}(n)$ control group is the set of $n \times n$ traceless skew-Hermitian complex matrices, and the Cartan subalgebra is the set of real, diagonal, and traceless matrices. Our numerics are carried out for the $\mathrm{SU}(3)/\mathrm{SO}(3)$ case, where the control Hamiltonians are given by the three Gell-Mann matrices λ_2, λ_5, and λ_7, and the Cartan subalgebra is spanned by $i\lambda_3$ and $i\lambda_8$. We first consider a drift Hamiltonian $H_d = \mathrm{diag}\{1, 0, -1\}$, which clearly does not have degenerate eigenvalues. The results are shown in Fig. 4. Interestingly, we see that the speed limit lower bound is still tight. No target unitary takes longer than this lower bound.

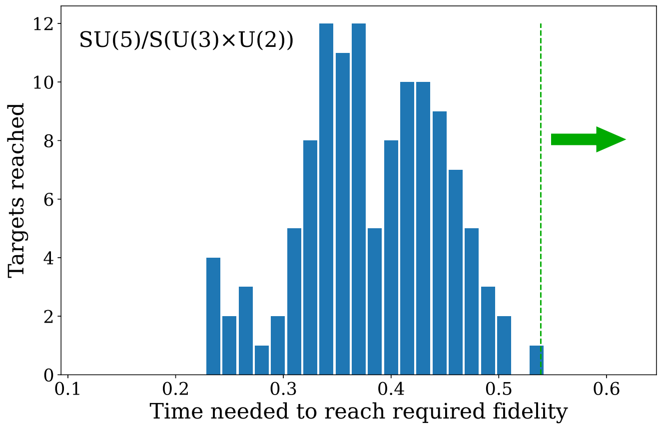

FIG. 3. Speed limit for the SU(5)/S[U(2) × U(3)] case, with a drift Hamiltonian given by (47). The histogram shows the fastest possible times to achieve 120 randomly chosen unitary targets when using S[U(2) × U(3)] controls, with the analytic lower bound of 0.539 given by (20) represented by the vertical green dashed line. As all targets can be met in a time less than the bound and some targets are at the bound, the bound is tight.

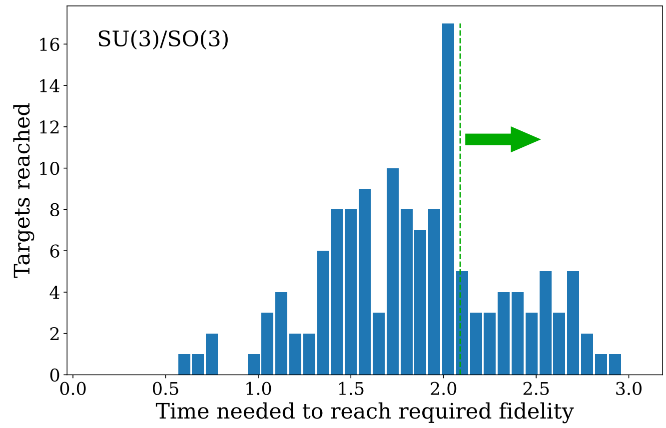

FIG. 5. Speed limit for the SU(3)/SO(3) case, with a drift $H_d = \mathrm{diag}\{1, -0.5, -0.5\}$. The histogram shows the fastest possible times to achieve 140 randomly chosen unitary targets when using SO(3) controls, with the analytic lower bound of 2.09 given by (17) represented by the vertical green dashed line. Consequently, the bound is not tight for this particular drift: The slowest unitary is around 3.0, while the lower bound is around 2.0. This is expected because H_d has two degenerate eigenvalues.

Finally, we consider the case with a drift $H_d = \mathrm{diag}\,\{1, -\frac{1}{2}, -\frac{1}{2}\}$, which *does* have a degenerate eigenvalue. The results are shown in Fig. 5 and we see that while the analytic lower bound given by (17) is still respected it is no longer tight, which is what we expect due to H_d possessing degenerate eigenvalues.

Collectively, these results provide a check on the analytic results for the lower bounds on the quantum speed limit. They confirm that the bounds (16)–(20) are accurate, showing that if we consider all possible unitaries, there will be at least one that takes at least this long to generate. These simulations also support our conjecture that for the $\mathrm{Sp}(\frac{n}{2})$ and $\mathrm{S}[\mathrm{U}(p) \times \mathrm{U}(q)]$ control schemes, the bounds are tight, meaning that there is at least one unitary that takes exactly that long to produce, but no unitaries will take longer. Furthermore, the results show that, for the SO(3) control case where the drift has a pair of degenerate eigenvalues, the bound is respected but is not tight, as expected. Interestingly, the bound with the SO(3) control case where the drift has distinct eigenvalues does appear to be tight, at least for the particular drift Hamiltonian we chose.

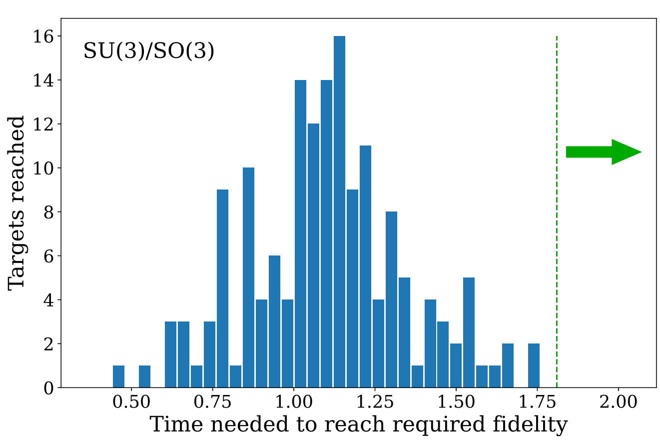

FIG. 4. Speed limit for the SU(3)/SO(3) case, with a drift $H_d = \mathrm{diag}\{1, 0, -1\}$. The histogram shows the fastest possible times to achieve 160 randomly chosen unitary targets when using SO(3) controls, with the analytic lower bound of 1.81 given by (17) represented by the vertical green dashed line. As all targets can be met in a time less than the bound and some targets are at the bound, the bound is tight.

Finally, we see that numerical optimization techniques to find optimal control pulses for quantum systems appear to work remarkably well. Optimal pulses are found that respect the analytic bounds exactly, providing evidence that such methods can be trusted for bilinear control problems.

VII. SPEED LIMITS WITHOUT A FULL SET OF LIE ALGEBRA CONTROLS

The previous sections have obtained lower bounds on the quantum speed limit for systems with arbitrary drifts and with controls that form a closed subgroup of SU(n), as well as considering in more detail the case where the control Hamiltonians are one of the Lie algebras $\mathfrak{so}(n)$, $\mathfrak{sp}(\frac{n}{2})$, or $\mathfrak{s}[\mathfrak{u}(p) \oplus \mathfrak{u}(q)]$. The number of control Hamiltonians required to span these Lie algebras is given by d_k in Table II and can be seen to scale quadratically in n. Such a situation might seem to be difficult to arrange in practice.

However, it is important to realize that the controls themselves need not provide a full basis for the algebra but rather that the dynamical Lie algebra generated through repeated application of the commutators of the controls provide such a basis. Clearly, if we have a full set of controls that already provide a basis, that is enough. However, the question is, how few control Hamiltonians do we actually need to generate these algebras?

It is known that the simple compact classical Lie algebras $\mathfrak{su}(n)$, $\mathfrak{so}(n)$, and $\mathfrak{sp}(\frac{n}{2})$ can be generated by "one and a half" elements [47,48]. This means that if we choose any element in the algebra, there exists a second element in the algebra that along with the first will generate the entire algebra, provided neither of the two is the identity. Consequently, one never needs more than two control Hamiltonians to generate the full so(n) or sp($\frac{n}{2}$) algebras, ensuring the results in previous sections are applicable.

Finally, so far we have only discussed systems where we have multiple control Hamiltonians, but the situation with a single control, i.e., where the system Hamiltonian is given by

$$H = H_d + f(t)H_c, \tag{48}$$

is very common. It is therefore useful to derive a bound on the quantum speed limit in this case.

Again, the full Lie algebra of the system is $\mathfrak{g} = \mathrm{su}(n)$ and the control subalgebra is one dimensional and is given by $\mathfrak{k} = \mathrm{span}\{iH_c\}$. This pair does not admit a Cartan decomposition unless $H_c \in \mathrm{so}(2)$. Indeed, the Lie group generated by $K = \exp(\mathfrak{k})$ is in some cases not even topologically closed. Consequently, the quotient space G/K may not be a homogeneous space, let alone a symmetric space. We can however apply the results we derived in previous sections to obtain a lower bound on the quantum speed limit in this case by "embedding" this control problem into another which does satisfy our criteria.

To obtain a bound we note that since H_c is Hermitian it can be transformed into a diagonal, purely real matrix $H_c' = UH_cU^\dagger$ via a unitary transformation. In this new basis the drift is given by $H_d' = UH_dU^\dagger$. Changing the basis of the problem via unitary transformation cannot change the speed limit since a basis change is only a mathematical convenience. We also introduce an auxiliary control problem with the same drift H_d' but with the control group given by $\mathrm{S}[\mathrm{U}(p) \times U(q)]$ with $p + q = n$ and an associated control algebra $\mathfrak{k}$. This auxiliary problem does admit a Cartan decomposition.

Since iH_c' is diagonal, purely imaginary, and traceless, it can be written

$$iH_c' = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad \mathrm{Tr}(D_1) + \mathrm{Tr}(D_2) = 0, \tag{49}$$

where D_1 and D_2 are diagonal, imaginary, and $p \times p$ and $q \times q$, respectively. Consequently, we have $D_1 \in \mathfrak{u}(p)$ and $D_2 \in \mathfrak{u}(q)$ and thus $iH_c' \subset \mathfrak{k}$. This means that the control problem

$$H = H_d' + f(t)H_c' \tag{50}$$

is the same as the auxiliary control problem, except with fewer controls, that is, it has a single control from $\mathfrak{k}$, rather the entire basis set of $p^2 + q^2 - 1$ controls. Hence whatever the lower bound on the quantum speed limit for the auxiliary control problem, the lower bound for the system described by (50) must be at least as large since it has a strict subset of the controls relative to the auxiliary problem. Since the system described by (50) is physically equivalent to (48) and since the trace is unchanged by a unitary transformation, we obtain a lower bound on the quantum speed limit of (48) given by

$$T_{\mathrm{QSL}} \geqslant \frac{\sqrt{p}\pi}{\sqrt{2\,\mathrm{Tr}\big(H_d^2\big)}}, \tag{51}$$

where we have assumed without loss of generality that $p \leqslant q$.

Since our split of the sizes of D_1 and D_2 in (49) is only constrained by $p \leqslant q$, we are free to choose the size of p and q to make the lower bound (51) as large as possible. This clearly occurs when $p = \lfloor n/2 \rfloor$, yielding

$$T_{\mathrm{QSL}} \geqslant \frac{\sqrt{\lfloor n/2 \rfloor}\pi}{\sqrt{2\,\mathrm{Tr}\big(H_d^2\big)}} \tag{52}$$

for the case where we have a single control. In the general case one would not expect (52) to be tight, but it does provide a rigorous lower bound and demonstrates how the quantum speed limit scales with dimension and how it depends on the form of the drift.

VIII. CONCLUSION

The purpose of this paper has been to develop a lower bound for the quantum speed limit of a controllable finite-dimensional system, given the assumption that the controls can be arbitrarily strong. We have also investigated the circumstances under which this lower bound is not merely a bound but is actually exact.

We have used the techniques of Lie algebras, Lie groups, and differential geometry. Mindful that these areas may not be entirely familiar to many physicists, we have provided a pedagogical development of this material, making it clear why it is relevant and constantly tying it back to the physics. We have also provided a number of examples to aid this process. Our approach has been completely general, and the basic result given by Theorem 1 holds for Hilbert spaces of arbitrary dimension, arbitrary drift Hamiltonians, and does not require specific symmetries. The only requirement is that the control group is topologically closed.

This basic result, however, does require some knowledge of the diameter of the homogeneous space corresponding to the quotient group of SU(n) with the control group. While this is generally difficult to determine, exact diameters are available for symmetric spaces, allowing us to give explicit bounds in this case. It is important to note, however, that even if the exact diameter of the quotient group is not known analytically, any ability to bound the diameter, analytically or numerically, can immediately be used in our expression for the quantum speed limit and merely results in a looser bound.

We have also examined the question of when our formula for the quantum speed limit is not merely a lower bound but is actually exact. This led us to Conjecture 1 for specific cases. In the fully general case we developed a sufficiency criterion based on the dimension of the adjoint orbit and commutation relations between the drift Hamiltonian and the matrix representation of the Lie algebra corresponding to the controls. As an illustration we showed how this can be done for the case where the control group is SO(n).

As this criterion for bound tightness is sufficient but not necessary, we also examined what could further be said in the case where the controls are not arbitrary but form a Cartan decomposition of the quantum control problem. In this case bound tightness depends on the cut locus of the quotient space, which can be described in terms of the positive roots of the Lie algebras. We were not able to provide a complete statement as to when the bounds were tight but did show how conditions on the roots would decrease the probability that the bound was tight.

Since the development of our results is somewhat abstract and mathematical, we have also examined our speed limit bounds using a numerical optimization procedure for a number of specific Hamiltonians. This purpose of this is twofold. First, it provides numerical confirmation of our explicit analytic bounds, as well as supporting our results on the link between the degree of degeneracy of the drift Hamiltonian and the tightness of the bounds. Second, it provides a general way to use numerical optimization to determine speed limits and demonstrates that gradient-descent-based techniques work well.

Finally, we have considered the quantum speed limit in the very common quantum control case where one has a drift Hamiltonian and a single control Hamiltonian. Such a system need not meet the assumptions for our main speed limit theorem; for example, the control group may not be closed or indeed form a quotient group that is a homogeneous space. Nonetheless, we showed it is possible to embed such a problem into a group that does meet our criteria, allowing us to use our previous results and thereby provide an explicit lower bound for this case.

ACKNOWLEDGMENTS

We thank the anonymous referees for their kind remarks and useful suggestions. D.B. acknowledges funding from the Australian Research Council (Project No. FT190100106, No. DP210101367, and No. CE170100009).

[1] S. J. Glaser, U. Boscain, T. Calarco, C. P. Koch, W. Köckenberger, R. Kosloff, I. Kuprov, B. Luy, S. Schirmer, T. Schulte-Herbrüggen, D. Sugny, and F. K. Wilhelm, Training Schrödinger's cat: Quantum optimal control, Eur. Phys. J. D **69**, 279 (2015).

[2] C. Brif, R. Chakrabarti, and H. Rabitz, Control of quantum phenomena: Past, present and future, New J. Phys. **12**, 075008 (2010).

[3] D. D'Alessandro, *Introduction to Quantum Control and Dynamics* (Chapman and Hall, Boca Raton, 2008).

[4] D. L. Elliot, *Bilinear Control Systems*, Applied Mathematical Science Vol. 169 (Springer, Berlin, 2009).

[5] V. Jurdjevic and H. J. Sussman, Control systems on Lie groups, J. Differ. Equ. **12**, 313 (1972).

[6] V. Jurdjevic, *Geometric Control Theory* (Cambridge University Press, Cambridge, 1996).

[7] J. Bechhoeffer, *Control Theory for Physicists* (Cambridge University Press, Cambridge, 2021).

[8] E. D. Sontag, *Mathematical Control Theory* (Springer, Berlin, 1998).

[9] N. Khaneja, T. Reiss, C. Kehlet, T. Schulte-Herbrüggen, and S. J. Glaser, Optimal control of coupled spin dynamics: Design of NMR pulse sequences by gradient ascent algorithms, J. Magn. Reson. **172**, 296 (2005).

[10] S. Machnes, U. Sander, S. J. Glaser, P. de Fouquieres, A. Gruslys, S. G. Schirmer, and T. Schulte-Herbrüggen, Comparing, optimizing, and benchmarking quantum-control algorithms in a unifying programming framework, Phys. Rev. A **84**, 022305 (2011).

[11] F. F Floether, P. de Fouquieres, and S. G. Schirmer, Robust quantum gates for open systems via optimal control: Markovian versus non-Markovian dynamics, New J. Phys. **14**, 073023 (2012).

[12] R. Sakai, A. Soeda, M. Murao, and D. Burgarth, Robust controllability of two-qubit Hamiltonian dynamics Phys. Rev. A **100**, 042305 (2019).

[13] T. Caneva, M. Murphy, T. Calarco, R. Fazio, S. Montangero, V. Giovannetti, and G. E. Santoro, Optimal control at the quantum speed limit, Phys. Rev. Lett. **103**, 240501 (2009).

[14] S. Deffner and S. Campbell, Quantum speed limits: From Heisenberg's uncertainty principle to optimal quantum control, J. Phys. A: Math. Theor. **50**, 453001 (2017).

[15] J. Lee, C. Arenz, H. Rabitz, and B. Russell, Dependence of the quantum speed limit on system size and control complexity, New J. Phys. **20**, 063002 (2018).

[16] N. Khaneja, R. Brockett, and S. J. Glaser, Time optimal control in spin systems, Phys. Rev. A **63**, 032308 (2001).

[17] N. Khaneja, S. J. Glaser, and R. Brockett, Sub-Riemannian geometry and time optimal control of three spin systems: Quantum gates and coherence transfer, Phys. Rev. A **65**, 032301 (2002).

[18] U. Boscain, T. Chambrion, and J.-P. Gauthier, On the $K + P$ problem for a three-level quantum system, J. Dyn. Control Syst. **8**, 547 (2002).

[19] A. Carlini, A. Hosoya, T. Koike, and Y. Okudaira, Time-optimal quantum evolution, Phys. Rev. Lett. **96**, 060503 (2006).

[20] N. Khaneja, B. Heitmann, A. Spörl, H. Yuan, T. Schulte-Herbrüggen, and S. J. Glaser, Shortest paths for efficient control of indirectly coupled qubits, Phys. Rev. A **75**, 012322 (2007).

[21] G. C. Hegerfeldt, Driving at the quantum speed limit: Optimal control of a two-level system, Phys. Rev. Lett. **111**, 260501 (2013).

[22] D. D'Alessandro, B. A. Sheller, and Z. Zhu, Time-optimal control of quantum lambda systems in the KP configuration, J. Math. Phys. **61**, 052107 (2020).

[23] N. Margolus and L. B. Levitin, The maximum speed of dynamical evolution, Physica D **120**, 188 (1998).

[24] P. Kosiński and M. Zych, The maximum speed of dynamical evolution, Phys. Rev. A **73**, 024303 (2006).

[25] J. P. Jones and P. Kok, Geometric derivation of the quantum speed limit, Phys. Rev. A **82**, 022107 (2010).

[26] B. Russell and S. Stepney, Zermelo navigation and a speed limit to quantum information processing, Phys. Rev. A **90**, 012303 (2014).

[27] S. Kallush, M. Khasin, and R. Kosloff, Quantum control with noisy fields: Computational complexity versus sensitivity to noise, New J. Phys. **16**, 015008 (2014).

[28] J. Lee, C. Arenz, D. Burgarth, and H. Rabitz, An upper bound on the time required to implement unitary operations J. Phys. A: Math. Theor. **53**, 125304 (2020).

[29] M. R. Lam, N. Peter, T. Groh, W. Alt, C. Robens, D. Meschede, A. Negretti, S. Montangero, T. Calarco, and A. Alberti,

Demonstration of quantum brachistochrones between distant states of an atom, Phys. Rev. X **11**, 011035 (2021).

[30] D. Burgarth, J. Borggaard, and Z. Zimborás, Quantum distance to uncontrollability and quantum speed limits, Phys. Rev. A **105**, 042402 (2022).

[31] A. del Campo, I. L. Egusquiza, M. B. Plenio, and S. F. Huelga, Quantum speed limits in open system dynamics, Phys. Rev. Lett. **110**, 050403 (2013).

[32] S. Deffner and E. Lutz, Quantum speed limit for non-markovian dynamics, Phys. Rev. Lett. **111**, 010402 (2013).

[33] M. M. Taddei, B. M. Escher, and L. Davidovich, Quantum speed limit for physical processes, Phys. Rev. Lett. **110**, 050402 (2013).

[34] C. Liu, Z. Y. Xu, and S. Zhu, Quantum-speed-limit time for multiqubit open systems, Phys. Rev. A **91**, 022102 (2015).

[35] D. P. Pires, M. Cianciaruso, L. C. Céleri, and G. Adesso, Generalized geometric quantum speed limits, Phys. Rev. X **6**, 021031 (2016).

[36] Y. Shao, B. Liu, M. Zhang, H. Yuan, and J. Liu, Operational definition of a quantum speed limit, Phys. Rev. Res. **2**, 023299 (2020).

[37] L. Mandelstam and I. Tamm, The uncertainty relation between energy and time in non-relativistic quantum mechanics, J. Phys. USSR **9**, 249 (1945).

[38] C. Arenz, B. Russell, D. Burgarth, and H. Rabitz, The roles of drift and control field constraints upon quantum control speed limits, New J. Phys. **19**, 103015 (2017).

[39] J. Gallier and J. Quaintance, in *Differential Geometry and Lie Groups: A Computational Perspective*, edited by H. Edelsbrunner, L. Kobbelt, and K. Polthier, Springer Series in Geometry and Computing Vol. 12 (Springer, Berlin, 2020).

[40] B. C. Hall, *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, edited by S. Axler and K. Ribet, Graduate Texts in Mathematics Vol. 222 (Springer, Cham, 2015).

[41] J. W. Milnor, *Morse Theory*, 3rd ed. (Princeton University Press, Princeton, 1969), Vol. 51.

[42] L. Yang, Injectivity radius and Cartan polyhedron for simply connected symmetric spaces, Chin. Ann. Math. B **28**, 685 (2007).

[43] S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic, New York, 1962).

[44] C. Gorodski, An introduction to Riemannian symmetric spaces, VII School and Workshop on Lie Theory, 2021 (unpublished), available at https://www.ime.usp.br/~gorodski/ps/symmetric-spaces.pdf.

[45] R. Crittenden, Minimum and conjugate points in symmetric spaces, Can. J. Math. **14**, 320 (1962).

[46] https://qutip.org/docs/latest/guide/guide-control.html.

[47] J.-M. Bois, Generators of simple Lie algebras in arbitrary characteristics, Math. Z. **262**, 715 (2009).

[48] J. W. Milnor, Curvatures of left invariant metrics on Lie groups, Adv. Math. **21**, 293 (1976).