Radiation of a variable charge flying into a medium

Xingzhen Fan and Andrey V. Tyukhtin [*]
Saint Petersburg State University, 7/9 Universitetskaya Embankment, Saint Petersburg 199034, Russia

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We study an electromagnetic radiation of a small bunch having a variable charge value and crossing a flat interface between two media. Both media are linear, homogeneous, stationary, and isotropic. They may have frequency dispersion but no spatial dispersion. Cherenkov radiation can be generated in the second medium only. It is assumed that the bunch charge value decreases exponentially starting from a certain time moment after the charge enters the second medium. This means that we take into account the scatter of the particle path lengths connected with the statistical nature of energy losses of particles. It is taken into account that a filamentous "trace" consisting of immobile charges is formed in the second medium. We obtain a general solution of the problem, which is the sum of the forced field, i.e., the charge field in the unbounded medium, and the free field connected with the influence of the boundary. An asymptotic study for the far-field zone is carried out. We obtain expressions for the spherical wave as well as for the cylindrical wave generated in the second medium if the charge velocity is sufficiently high. The spherical wave is radically different from the usual transition radiation, since it consists of two parts: the transition radiation wave and the wave generated due to the bunch decay and the formation of its trace. We describe the main properties of radiation. If the process of the bunch decay starts exactly at the boundary between media, then the angular distribution of the radiation energy has a single maximum in each of two regions. In the second medium, the radiation, as a rule, is greater than in the vacuum area, and this difference increases with increasing the charge velocity. If the charge velocity is larger than the speed of light in the second medium, then, along with the appearance of Cherenkov radiation, the properties of the spherical wave change radically. In particular, the main maximum of the angular distribution of the radiation energy increases essentially. If the distance from the charge entry point to the region of the bunch decay is sufficiently large, then a complicated interference pattern with many extremes arises due to the summation of different spherical waves.

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I. INTRODUCTION

Problems of electromagnetic radiation of charged particle bunches moving in media are studied in a series of monographs and in a huge number of journal papers (see, for instance, [1–9]). Usually, in such problems, it is assumed that the particle bunch has some constant velocity, and the value of its charge does not change during the motion. An exception is problems of radiation in the dielectric waveguide structures where the bunch moves in a vacuum channel: bunch dynamics associated with interaction of particles of the bunch is often taken into account in such a situation [10,11].

However, if the bunch moves in the medium then its particles interact with the medium particles, which leads to certain changes of the bunch. In more or less dense media, this interaction is usually the main factor determining the bunch evolution. Various variants of this evolution have been described in many monographs and papers (see [12–20] and references therein).

Depending on the particle mass and velocity, the medium density, and other factors, it is possible to have both a rapid deviation of particles from a rectilinear trajectory, leading to beam scattering, and an almost rectilinear motion of the bunch. The latter variant is typical for bunches of heavy particles (protons and ions). It should also be noted that these particles usually lose the majority of their energy in a relatively small section of the trajectory (this effect is referred to as the "Bragg peak") [12–16]. Because of this property, bunches of heavy particles are widely used in medicine (proton and ion therapy).

The features characterizing the passage of the charged particles bunches through the matter are quite complex and depend on many factors (energy, mass, and charge of particles; matter density; etc.) [12–18]. In this paper, we restrict ourselves to the following model. We assume that each particle of the bunch moves at some constant velocity until some moment when it stops instantly. The velocities of all particles until the stopping moment are the same. However, the moments of stopping are different for different particles. This means that we take into account the scatter of the particle path lengths, connected with the statistical nature of energy losses of particles [12–16].

Thus, the number of bunch particles decreases: the moving particles turn into nonmoving ones. We will assume that a certain fraction of particles stops per the unit length of the bunch path. This means that the bunch population and therefore the bunch charge decrease exponentially with time. This model of the bunch evolution is one of the simplest ones. It allows performing a complete analytical calculation

[*]a.tyuhtin@spbu.ru

of the generated radiation and describing the main physical effects.

Such a model is close to the actually observed dynamics of bunches of heavy particles (protons and ions). A decrease of the particle number according to the law close to the exponential is presented, in particular, in [13–16]. It should be noted that the described model allows one to advance quite far in the analytical description of the radiation process. At the same time, the more complicated models of bunch dynamics will hardly demonstrate qualitatively new effects.

It should also be noted that the described model automatically includes the special case of an instantaneous charge change. This case is of particular interest in connection with the problem of ion recharge due to the exchange of electrons with the medium. One such problem is considered in [20] where the ion charge instantly changes when it passes through the interface. The model considered here is much more general. It allows taking into account an arbitrary distance from the boundary to the charge change region, as well as the finiteness of the time interval of the charge change.

Previously, we analyzed the radiation of the described bunch with variable charge in a homogeneous infinite medium [21]. For applications, it is more important to consider the case when the bunch flies from the first medium (whose properties are usually close to the vacuum) into the second medium having some relatively high refractive index. We will focus on this problem in the present paper.

We assume that the bunch charge does not change in the first medium due to its low density. In the second medium, the charge value remains practically constant until a certain time moment, starting from which the bunch decay occurs. The particular case of coincidence of the charge entry moment and the moment of the bunch decay start can be also considered within the framework of this model. In the situation described, the generated wave field is a combination of three types of radiation: transition radiation, radiation due to the charge decreasing process, and Cherenkov radiation (CR) (if the bunch velocity is sufficiently high).

II. FORMULATION OF THE PROBLEM

We will analyze a radiation with wavelengths significantly exceeding the particle bunch size. In this case, we can consider a point charge. The charge speed $\vec{v} = c\vec{\beta}$ is assumed to be constant, and the charge value q is variable. In order to satisfy the charge conservation law (continuity equation), a certain "additional" nonmoving source with the charge density ρ_1 (the "trace" of the initial bunch in the medium) has to be introduced. Combining the axis z with the line of the charge motion, we can write the total charge density (ρ_Σ) and the current density ($\vec{j}_\Sigma$) in the following form:

$$\rho_\Sigma = \rho + \rho_1, \qquad \rho = q(t)\delta(x, y, z - vt),$$
$$\rho_1 = -\frac{dq(z/v)}{dz}\delta(x, y)\Theta(vt - z),$$
$$\vec{j}_\Sigma = \vec{j} = v\rho\vec{e}_z, \tag{1}$$

where $\Theta(\zeta) = 0$ for $\zeta < 0$ and $\Theta(\zeta) = 1$ for $\zeta > 0$. It can be easily verified that these expressions satisfy the continuity equation $\mathrm{div}\vec{j}_\Sigma + \partial\rho_\Sigma/\partial t = 0$.

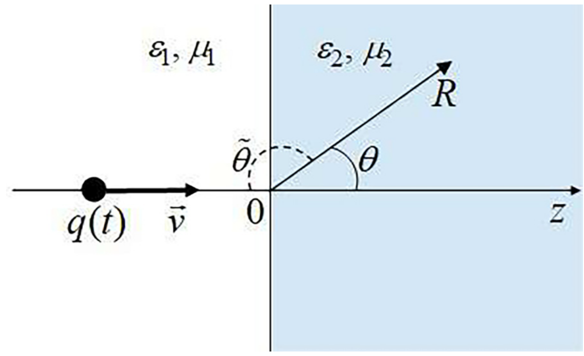

FIG. 1. The problem geometry.

The formation of the trace means that the bunch particles stop due to the interaction with the medium particles, i.e., they turn from moving particles to particles at rest (therefore $\vec{j}_1 = 0$). From the point of view of macroscopic electrodynamics, the detailed description of this process is of no importance. For example, this can be the recombination of the bunch electrons with ions (if the medium is a plasma), the stopping of particles due to collisions with neutral molecules, etc. For us, only the fact of the formation of the filamentous additional charge is important.

It is assumed that the particle bunch moves perpendicularly to the interface between two media, flying from the medium 1 ($z < 0$) into the medium 2 ($z > 0$) (Fig. 1). Both media are considered to be homogeneous, linear, stationary, and isotropic, and do not have any spatial dispersion (frequency dispersion can take place). The media are characterized by permittivities $\varepsilon_{1,2}$ and permeabilities $\mu_{1,2}$. Accordingly, the wave numbers are equal to $k_{1,2} = k_0 n_{1,2}$, where $k_0 = \omega/c$ is the wave number in vacuum, and $n_{1,2} = \sqrt{\varepsilon_{1,2}\mu_{1,2}}$ are the complex refractive indices of the media. Initially, we will assume that the media have some conductivity, i.e., $\mathrm{Im}\,\varepsilon_{1,2} > 0$ at $\omega > 0$. Ultimately, we will be interested in the case when the conductivity is negligible, and absorption of radiation in the medium is insignificant.

We will consider here only ordinary ("right-handed") media, i.e., we assume that the real values of the refractive indices take place only for real positive values of permittivity and permeability ($\mathrm{Re}\,\varepsilon_{1,2} > 0$ and $\mathrm{Re}\,\mu_{1,2} > 0$). The case of a "left-handed" medium (where both constants are negative in the same frequency range) can be considered similarly. Here we exclude this case from consideration for the sake of the relative compactness of formulas. Note that radiation generated by flying of a constant charge into a left-handed medium was studied in [22,23].

We assume that the bunch charge is constant ($q = q_0$) up to the moment $t = t_0 > 0$, i.e., in the medium 1 and, possibly, on some part of the trajectory in the medium 2. At $t > t_0$, the charge value decreases. Let us assume that for any small time interval dt the bunch loses the same fraction of the charge dq, that is, $dq/dt = q/\tau$, where $\tau = \text{const} > 0$. Solving this equation we find

$$q(t) = \begin{cases} q_0 & \text{for } t < t_0, \\ q_0 \exp\left[-(t - t_0)/\tau\right] & \text{for } t > t_0. \end{cases} \tag{2}$$

In this case, the motionless trace of charges has the density

$$\rho_1 = \frac{q_0}{v\tau}\exp\left(-\frac{z-vt_0}{v\tau}\right)\delta(x,y)\Theta(vt-z)\Theta(z-vt_0), \quad (3)$$

where $\Theta(\xi)=1$ for $\xi>0$ and $\Theta(\xi)=0$ for $\xi<0$.

III. FORCED FIELD

The total charge field in both media can be represented in the form

$$\vec{E}^{(1,2)} = \vec{E}^{q(1,2)} + \vec{E}^{b(1,2)},$$
$$\vec{H}^{(1,2)} = \vec{H}^{q(1,2)} + \vec{H}^{b(1,2)}, \quad (4)$$

where indices 1 and 2 refer to the first and second media, respectively. The field with the superscript q is a “forced” one, i.e., the charge field in the corresponding homogeneous infinite medium. The field with the superscript b is a “free” one, i.e., the field arising due to the presence of the interface between two media (we use here the terminology introduced in [4]).

Note that the forced field was analyzed by us in [21]. Here we will focus only on obtaining the expression for it in the form that is convenient for solving the boundary problem under consideration.

Let us use the vector ($\vec{A}$) and scalar (Φ) potentials connecting with the field components by the formulas $\vec{E}=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-\nabla\Phi$, $\vec{B}=\mathrm{rot}\vec{A}$ (a Gaussian system of units is used). We apply the Lorentz gauge (modified to the media), and then the time Fourier transforms of the potentials obey the Helmholtz equation:

$$\left(\Delta + k_{1,2}^2\right)\begin{Bmatrix}\vec{A}_\omega^{(1,2)}\\ \Phi_\omega^{(1,2)}\end{Bmatrix} = -4\pi\begin{Bmatrix}\mu_{1,2}\vec{j}_\omega/c\\ \rho_{\Sigma\omega}/\varepsilon_{1,2}\end{Bmatrix}. \quad (5)$$

The forced field potentials $\vec{A}_\omega^{q(1,2)}$ and $\Phi_\omega^{q(1,2)}$ also obey Eqs. (5), and the free field potentials $\vec{A}_\omega^{b(1,2)}$ and $\Phi_\omega^{b(1,2)}$ obey the same equations with zero right-hand side.

We will solve Eqs. (5) by the Fourier method. First of all, it is necessary to find the four-dimensional Fourier transforms of the charge and current densities. For the charge and current densities of the bunch itself, we have

$$\begin{Bmatrix}\rho_{\omega,\vec{k}}\\ \vec{j}_{\omega,\vec{k}}\end{Bmatrix} = \frac{1}{(2\pi)^4}\int_{R^4}\begin{Bmatrix}\rho\\ \vec{j}\end{Bmatrix}e^{i\omega t - i\vec{k}\vec{R}}dV\,dt = \frac{q_\Omega}{(2\pi)^3}\begin{Bmatrix}1\\ \vec{v}\end{Bmatrix}, \quad (6)$$

where $q_\Omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}q(t)e^{i\Omega t}dt$, $\Omega=\omega-vk_z$. Calculating the Fourier transform of the charge density of the trace, we obtain

$$\rho_{1\,\omega,\vec{k}} = -\frac{\Omega\, q_\Omega}{8\pi^3(\omega+i0)}. \quad (7)$$

The validity of this expression can be checked by calculating the inverse Fourier integral, which leads to (3). Summing up (6) and (7), for the total source we obtain

$$\rho_{\Sigma\,\omega,\vec{k}} = \frac{vk_z q_\Omega}{8\pi^3(\omega+i0)}, \quad \vec{j}_{\Sigma\,\omega,\vec{k}} = \frac{vq_\Omega\vec{e}_z}{8\pi^3}. \quad (8)$$

In the case of bunch with the exponentially decreasing charge (2), for the Fourier transform of the bunch charge we have

$$q_\Omega = q_0\delta(\Omega) + \frac{q_0}{2\pi}\left[\frac{1}{i(\Omega+i0)} + \frac{\tau}{1-i\Omega\tau}\right]e^{i\Omega t_0}. \quad (9)$$

Note that the term “$+i0$” in the denominator in (7) and (9) provides the required bypass of the pole.

Writing $\vec{A}_\omega^{q(1,2)}$ and $\Phi_\omega^{q(1,2)}$ as the inverse Fourier integrals over the wave-vector components, substituting them into Eqs. (5), and equating the integrands, we obtain the fourfold Fourier transforms of the potentials. After that, we write the inverse Fourier transform over the wave-vector components and switch to the cylindrical coordinate system both in the physical space (r,φ,z) and in the space of the wave vectors (k_r,φ_k,k_z). For the time Fourier transforms of the potentials, we obtain

$$\begin{Bmatrix}\vec{A}_\omega^{q(1,2)}\\ \Phi_\omega^{q(1,2)}\end{Bmatrix} = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}dk_z\int_0^\infty dk_r\int_0^{2\pi}d\varphi_k\begin{Bmatrix}\mu_{1,2}\vec{\beta}\\ \dfrac{c\beta k_z}{(\omega+i0)\varepsilon_{1,2}}\end{Bmatrix}\frac{k_r q_\Omega\exp\left[ik_z z + ik_r r\cos(\varphi_k-\varphi)\right]}{k_r^2+k_z^2-k_{1,2}^2}. \quad (10)$$

Since the integral over φ_k reduces to the tabular one [24], Eq. (10) can be written as

$$\begin{Bmatrix}\vec{A}_\omega^{q(1,2)}\\ \Phi_\omega^{q(1,2)}\end{Bmatrix} = \frac{1}{\pi}\int_{-\infty}^{\infty}dk_z\int_0^\infty dk_r\begin{Bmatrix}\mu_{1,2}\vec{\beta}\\ \dfrac{c\beta k_z}{(\omega+i0)\varepsilon_{1,2}}\end{Bmatrix}\frac{k_r q_\Omega J_0(k_r r)\exp(ik_z z)}{k_r^2+k_z^2-k_{1,2}^2}, \quad (11)$$

where $J_0(k_r r)$ is the Bessel function.

In contrast to [21], here it is convenient to represent the result as an integral over the tangential component k_r, therefore we take the integrals over k_z. Let us calculate the integral for the potential $\vec{A}$. Substituting (9) into (11) and calculating the integral with the delta function, we obtain

$$\vec{A}_\omega^{q(1,2)} = \frac{q_0\mu_{1,2}\vec{\beta}}{\pi}\int_0^\infty k_r J_0(k_r r)\left\{\frac{e^{i\omega z/v}}{v\left(k_r^2-s_{1,2}^2\right)} + \frac{e^{i\omega t_0}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ik_z(z-vt_0)}}{k_z^2-\kappa_{1,2}^2}\left[\frac{1}{i(\Omega+i0)}+\frac{\tau}{1-i\Omega\tau}\right]dk_z\right\}dk_r, \quad (12)$$

where $s_{1,2}=\sqrt{\omega^2v^{-2}(n_{1,2}^2\beta^2-1)}$, $\kappa_{1,2}=\sqrt{k_{1,2}^2-k_r^2}$. We set that these roots are defined by the rules $\mathrm{Im}\,s_{1,2}>0$, $\mathrm{Im}\,\kappa_{1,2}>0$. For the real parts, this leads to the rule $\mathrm{sgn}\,\mathrm{Re}\,s_{1,2}=\mathrm{sgn}\,\mathrm{Re}\,\kappa_{1,2}=\mathrm{sgn}\omega$.

The contour of integration over k_z in (12) can be supplemented with the closed infinite semicircle located in the upper half plane ($\mathrm{Im}\, k_z > 0$) at $z - vt_0 > 0$ and in the lower half plane ($\mathrm{Im}\, k_z < 0$) at $z - vt_0 < 0$. After that, the integral over k_z is found by calculating the residues at the poles $k_z = \kappa_{1,2}$, $\Omega = -i0$, $i\Omega\tau = 1$, if $z - vt_0 > 0$, and at the pole $k_z = -\kappa_{1,2}$, if $z - vt_0 < 0$. As a result, we obtain

$$\vec{A}^{q(1,2)}_{\omega} = \frac{q_0\mu_{1,2}\vec{\beta}}{\pi}\int_0^{\infty} k_r J_0(k_r r)\left\{\frac{U_{1,2}}{2\omega\kappa_{1,2}}e^{i\omega t_0}e^{i\kappa_{1,2}|z-vt_0|} + \frac{e^{i\omega z/v}}{v}\left[\frac{\Theta(vt_0 - z)}{k_r^2 - s_{1,2}^2} + V_{1,2}\Theta(z - vt_0)\right]\right\}dk_r, \tag{13}$$

where

$$U_{1,2} = \frac{\omega}{[\omega - v\kappa_{1,2}\mathrm{sgn}(z - vt_0)]\{1 - i\tau[\omega - v\kappa_{1,2}\mathrm{sgn}(z - vt_0)]\}},$$
$$V_{1,2} = \frac{\exp[-(z - vt_0)/(v\tau)]}{k_r^2 - s_{1,2}^2 - v^{-2}\tau^{-2} + 2iv^{-2}\omega\tau^{-1}}. \tag{14}$$

Note that it is advisable to transform the integral along the semiaxis into an integral along the entire real axis using the formula [25]

$$J_0(x) = \frac{H_0^{(1)}(x) - H_0^{(1)}(e^{i\pi}x)}{2}, \tag{15}$$

where $H_0^{(1)}(x)$ is Hankel function. Similarly, one can obtain the corresponding expression for $\Phi^{q(1,2)}_{\omega}$.

As a result, using the formulas $\vec{E}_{\omega} = \frac{i\omega}{c}\vec{A}_{\omega} - \nabla\Phi_{\omega}$, $\vec{B}_{\omega} = \mathrm{rot}\vec{A}_{\omega}$, we obtain the following expressions for the nonzero components of the forced electromagnetic field in each of the two media:

$$E^{q(1,2)}_{r\omega} = \frac{q_0\omega}{2\pi c^2\varepsilon_{1,2}}\int_{e^{i\pi}\infty}^{\infty} k_r^2 H_1^{(1)}(k_r r)\left\{\mathrm{sgn}(z - vt_0)\frac{\beta U_{1,2}}{2k_0^3}e^{i\omega t_0}e^{i\kappa_{1,2}|z-vt_0|} + \frac{e^{i\omega z/v}}{k_0\beta}\left[\frac{\Theta(vt_0 - z)}{k_r^2 - s_{1,2}^2} + \left(1 + \frac{i}{\omega\tau}\right)V_{1,2}\Theta(z - vt_0)\right]\right\}dk_r, \tag{16}$$

$$E^{q(1,2)}_{z\omega} = \frac{iq_0\omega}{2\pi c^2}\int_{e^{i\pi}\infty}^{\infty} k_r H_0^{(1)}(k_r r)\left\{\frac{\beta k_r^2 U_{1,2}}{2k_0^3\varepsilon_{1,2}\kappa_{1,2}}e^{i\omega t_0}e^{i\kappa_{1,2}|z-vt_0|} + e^{i\omega z/v}\left[\frac{n_{1,2}^2\beta^2 - 1}{\varepsilon_{1,2}\beta^2}\frac{\Theta(vt_0 - z)}{k_r^2 - s_{1,2}^2} + \left(\mu_{1,2} - \frac{(1 + i\omega^{-1}\tau^{-1})^2}{\varepsilon_{1,2}\beta^2}\right)V_{1,2}\Theta(z - vt_0)\right]\right\}dk_r, \tag{17}$$

$$H^{q(1,2)}_{\varphi\omega} = \frac{q_0\omega\beta}{2\pi c^2}\int_{e^{i\pi}\infty}^{\infty} k_r^2 H_1^{(1)}(k_r r)\left\{\frac{U_{1,2}}{2k_0^2\kappa_{1,2}}e^{i\omega t_0}e^{i\kappa_{1,2}|z-vt_0|} + \frac{e^{i\omega z/v}}{k_0\beta}\left[\frac{\Theta(vt_0 - z)}{k_r^2 - s_{1,2}^2} + V_{1,2}\Theta(z - vt_0)\right]\right\}dk_r. \tag{18}$$

IV. FREE FIELD

The Fourier transform of the free field component $E^{b(1,2)}_{z\omega,\vec{k}_r} \equiv E^{b(1,2)}_{z\omega,k_x,k_y}$ satisfies the equation

$$\frac{\partial^2}{\partial z^2}E^{b(1,2)}_{z\omega,\vec{k}_r} + \left(k_{1,2}^2 - k_r^2\right)E^{b(1,2)}_{z\omega,\vec{k}_r} = 0, \tag{19}$$

where $k_r^2 = k_x^2 + k_y^2$. Its solutions are the functions

$$E^{b(1,2)}_{z\omega,\vec{k}_r} = \frac{q_0\omega}{c^2}C_{\mp}e^{\mp i\kappa_{1,2}z}, \tag{20}$$

where the dimensional factor $q_0\omega/c^2$ is introduced for convenience. The signs "−" or "+" in (20) correspond to the waves propagating either in the negative (in medium 1) or in the positive (in medium 2) direction of the z axis. The Fourier transforms of other field components can be easily found using Maxwell equations.

Next, we write the corresponding inverse Fourier integrals:

$$F_{\omega} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_{\omega,\vec{k}_r}e^{i\vec{k}_r\vec{r}}dk_x dk_y = \int_0^{\infty} dk_r\cdot k_r\int_0^{2\pi} d\varphi_k\cdot F_{\omega,\vec{k}_r}e^{ik_r r\cos(\varphi_k - \varphi)}. \tag{21}$$

After calculating the table integrals over φ_k [24], the following representations for the nonzero field components are obtained:

$$\begin{Bmatrix} E^{b(1,2)}_{r\omega} \\ E^{b(1,2)}_{z\omega} \\ H^{b(1,2)}_{\varphi\omega} \end{Bmatrix} = 2\pi \frac{q_0\omega}{c^2}\int_0^\infty \begin{Bmatrix} \pm i\kappa_{1,2}J_1(k_r r) \\ k_r J_0(k_r r) \\ -ik_0\varepsilon_{1,2}J_1(k_r r) \end{Bmatrix} C_\mp e^{\mp i\kappa_{1,2}z}dk_r. \tag{22}$$

Using (15) and a similar relation for $J_1(x)$ [24], we write the field components as integrals over the entire real axis:

$$\begin{Bmatrix} E^{b(1,2)}_{r\omega} \\ E^{b(1,2)}_{z\omega} \\ H^{b(1,2)}_{\varphi\omega} \end{Bmatrix} = \pi \frac{q_0\omega}{c^2}\int_{e^{i\pi}\infty}^\infty \begin{Bmatrix} \pm i\kappa_{1,2}H^{(1)}_1(k_r r) \\ k_r H^{(1)}_0(k_r r) \\ -ik_0\varepsilon_{1,2}H^{(1)}_1(k_r r) \end{Bmatrix} C_\mp e^{\mp i\kappa_{1,2}z}dk_r. \tag{23}$$

To find the coefficients $C_\mp$, we use the usual boundary conditions:

$$\varepsilon_1 E^{(1)}_{z\omega}\big|_{z=0} = \varepsilon_2 E^{(2)}_{z\omega}, \quad E^{(1)}_{r\omega}\big|_{z=0} = E^{(2)}_{r\omega}. \tag{24}$$

They result in the system of algebraic equations which has the following solution:

$$C_+ = C_+^0 - \frac{i\beta k_r^2(\varepsilon_2\kappa_1 - \varepsilon_1\kappa_2)\tilde{U}_2}{4\pi^2 k_0^3\varepsilon_2\kappa_2(\varepsilon_2\kappa_1+\varepsilon_1\kappa_2)}e^{i(\omega+v\kappa_2)t_0}, \tag{25}$$

$$C_- = C_-^0 - \frac{i\beta k_r^2\tilde{U}_1}{4\pi^2k_0^3\varepsilon_1\kappa_1}e^{i(\omega+v\kappa_1)t_0} + \frac{i\beta k_r^2\tilde{U}_2}{2\pi^2k_0^3(\varepsilon_1\kappa_2+\varepsilon_2\kappa_1)}e^{i(\omega+v\kappa_2)t_0}, \tag{26}$$

where

$$C_+^0 = \frac{i}{2\pi^2(\varepsilon_2\kappa_1+\varepsilon_1\kappa_2)}\left[\left(\frac{n_1^2\beta^2-1}{\beta^2}\kappa_1 + \frac{k_r^2}{k_0\beta}\right)\frac{1}{k_r^2-s_1^2} - \left(\frac{n_2^2\beta^2-1}{\beta^2}\kappa_1 + \frac{\varepsilon_1k_r^2}{k_0\beta\varepsilon_2}\right)\frac{1}{k_r^2-s_2^2}\right], \tag{27}$$

$$C_-^0 = -\frac{i}{2\pi^2(\varepsilon_1\kappa_2+\varepsilon_2\kappa_1)}\left[\left(\frac{n_1^2\beta^2-1}{\beta^2}\kappa_2 - \frac{\varepsilon_2k_r^2}{k_0\beta\varepsilon_1}\right)\frac{1}{k_r^2-s_1^2} - \left(\frac{n_2^2\beta^2-1}{\beta^2}\kappa_2 - \frac{k_r^2}{k_0\beta}\right)\frac{1}{k_r^2-s_2^2}\right], \tag{28}$$

where

$$\tilde{U}_{1,2} = \frac{\omega}{(\omega+v\kappa_{1,2})[1-i\tau(\omega+v\kappa_{1,2})]}. \tag{29}$$

V. RADIATION IN MEDIUM 1

Further we investigate the field in the wave (far-field) area $|k_1|R \gg 1$. In this area, the main part of the electromagnetic field is the radiation field which is of most interest to us.

We assume that, in medium 1, the charge velocity does not exceed the phase velocity of the waves: $v < c/\operatorname{Re} n_1$. For the practically important case, when the medium 1 is close to vacuum, this condition is satisfied automatically.

Let us introduce the spherical coordinates R, θ $(r = R\sin\theta,\, z = R\cos\theta)$ and the new integration variable χ such that $k_r = k_1\sin\chi$. The total field, consisting of the forced and free fields, can be written as

$$\begin{Bmatrix} E^{(1)}_{r\omega} \\ E^{(1)}_{z\omega} \\ H^{(1)}_{\varphi\omega} \end{Bmatrix} = \pi\frac{q_0k_0^3n_1^2}{c}\int_\Gamma \begin{Bmatrix} i\cos\chi H^{(1)}_1(k_1R\sin\theta\sin\chi) \\ \sin\chi H^{(1)}_0(k_1R\sin\theta\sin\chi) \\ -i\sqrt{\varepsilon_1/\mu_1}H^{(1)}_1(k_1R\sin\theta\sin\chi) \end{Bmatrix} \times \tilde{C}_-\cos\chi\exp(-ik_1R\cos\theta\cos\chi)d\chi, \tag{30}$$

where

$$\tilde{C}_- = C_-^0 + \frac{i\beta k_r^2\tilde{U}_2}{2\pi^2k_0^3(\varepsilon_1\kappa_2+\varepsilon_2\kappa_1)}e^{i(\omega+v\kappa_2)t_0}. \tag{31}$$

The integration contour Γ is shown in Fig. 2.

Further we will apply the steepest descent method [26,27]. In addition to the condition $|k_1|R \gg 1$ we will assume that the distance to the observation point is large compared to the distance to the charge decay region: $R \gg vt_0$. Then the function $\tilde{C}_(\chi)$ is a slow function compared to the fast function $\exp(-ik_1R\cos\theta\cos\chi)$.

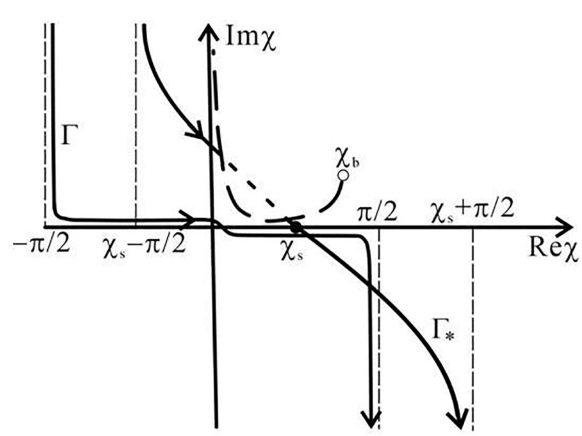

FIG. 2. The initial contour of integration Γ and the steepest descent path Γ_* on the complex plane χ; the dashed line shows the cut.

First, we need to transform the original integration contour Γ to the steepest descent contour Γ_* (Fig. 2). It can be shown that the singularities of the integrand cannot be intersected while transforming the contour. Using the asymptotic of the Hankel functions [25], we obtain from (30) the following approximate integrals over the steepest descent contour:

$$\begin{Bmatrix} E_{r\omega}^{(1)} \\ E_{z\omega}^{(1)} \\ H_{\varphi\omega}^{(1)} \end{Bmatrix} \approx \sqrt{2\pi} e^{-i\pi/4} \frac{q_0 k_0^3 n_1^2}{c} \int_{\Gamma_*} \frac{\tilde{C}_- \cos\chi}{\sqrt{k_1 R \sin\theta \sin\chi}} \begin{Bmatrix} \cos\chi \\ \sin\chi \\ -\sqrt{\varepsilon_1/\mu_1} \end{Bmatrix} \times \exp\{ik_1 R \cos[\chi - (\pi - \theta)]\} d\chi. \tag{32}$$

The saddle point of the integrand is located at $\chi = \chi_s = \pi - \theta$. The integrals (32) are calculated with the help of the well-known formula of the steepest descent method [26,27]. Using the spherical coordinates R, θ, and φ, one can show that the radiation field has only components $E_\theta^{(1)}$ and $H_\varphi^{(1)}$:

$$E_{\theta\omega}^{(1)} \approx \sqrt{\frac{\mu_1}{\varepsilon_1}} H_{\varphi\omega}^{(1)} \approx -2\pi i \frac{q_0 k_0^2 n_1}{c \tan\theta} \frac{e^{ik_1 R}}{R} \tilde{C}_-\big|_{\chi=\chi_s}. \tag{33}$$

Radiation emerging from a limited region of space is usually characterized by the spectral-angular density of the radiated energy. The expression for it can be easily obtained using the formula for the energy flux density of the spherical wave:

$$\vec{S} = \frac{c}{4\pi} E_\theta^{(1)} H_\varphi^{(1)} \vec{e}_R. \tag{34}$$

The total energy passing through the unit area orthogonal to this vector for the entire time is equal to

$$\int_{-\infty}^{\infty} S_R dt = \int_0^{\infty} c \left|\sqrt{\varepsilon_1/\mu_1}\right| \left|E_{\theta\omega}^{(1)}\right|^2 d\omega. \tag{35}$$

The integrand in (35), multiplied by R^2, is the spectral-angular radiation energy density:

$$\frac{d^2 W}{d\Omega d\omega} = cR^2 \left|\sqrt{\varepsilon_1/\mu_1}\right| \left|E_{\theta\omega}^{(1)}\right|^2 = \frac{q_0^2}{4\pi^2 c} |F|^2, \tag{36}$$

where

$$F = \frac{2(\varepsilon_1^3 \mu_1)^{1/4}}{(\varepsilon_1 N_{21} - \varepsilon_2 n_1 \cos\theta)\tan\theta} \left[-\frac{(n_1^2\beta^2 - 1)N_{21} - \varepsilon_2 \mu_1 \beta \sin^2\theta}{1 - n_1^2 \beta^2 \cos^2\theta} + \frac{(n_2^2\beta^2 - 1)N_{21} - n_1^2 \beta \sin^2\theta}{1 - \beta^2 N_{21}^2} + \frac{n_1^2 \beta \sin^2\theta}{(1 + \beta N_{21})[1 - i\omega\tau(1 + \beta N_{21})]} e^{i\omega t_0 (1 + \beta N_{21})} \right]. \tag{37}$$

Here $N_{21} = \sqrt{n_2^2 - n_1^2 \sin^2\theta}$.

In the case of the homogeneous medium ($\varepsilon_2 = \varepsilon_1$, $\mu_2 = \mu_1$) the formula (37) goes to the following expression:

$$F = F_h = -\frac{(\varepsilon_1 \mu_1^3)^{1/4} \beta \sin\theta}{(1 - n_1 \beta \cos\theta)[1 - i\omega\tau(1 - n_1 \beta \cos\theta)]} \times e^{i\omega t_0 (1 - n_1 \beta \cos\theta)}. \tag{38}$$

This result coincides with that obtained in [21] if we put $t_0 = 0$.

In the nonrelativistic case ($\beta \ll 1$), keeping only quantities of order β, we obtain

$$F = \frac{(\varepsilon_1^7 \mu_1^5)^{1/4} \beta \sin(2\theta)}{\varepsilon_1 N_{21} - \varepsilon_2 n_1 \cos\theta} \times \left[\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1} + \frac{1}{1 - i\omega\tau} e^{i\omega t_0 (1 + \beta N_{21})} \right]. \tag{39}$$

We also note the special case when the second medium has a very large real permittivity or it is a good conductor (anyway $|\varepsilon_2| \gg |\varepsilon_1|$). Then it can be shown that

$$F \approx -\frac{2(\varepsilon_1 \mu_1^3)^{1/4} \beta \sin\theta}{1 - n_1^2 \beta^2 \cos^2\theta}. \tag{40}$$

This result coincides with that for the case of the unchanging charge flying into an ideal conductor (which is expected since processes inside an ideal conductor do not affect the electromagnetic field outside it).

VI. RADIATION IN MEDIUM 2

The asymptotic of the field in the medium 2 under conditions $|k_2|R \gg 1$ and $R \gg vt_0$ is calculated in a similar way. For this area, it is useful to introduce the new integration variable χ as follows: $k_r = k_2 \sin\chi$. Without going into details of the calculation, we note only the most important points.

Unlike the medium 1, for the medium 2, it is possible that $|n_2|\beta > 1$. In such a situation, the pole determined by the equation $s_2 = k_2 \sin\chi_p$, where $\chi_p = \arccos[(n_2\beta)^{-1}]$, can be intersected during the transformation of the initial integration contour to the steepest descent path, and it can make some significant contribution. Physically, this means the presence of CR, i.e., the cylindrical wave diverging from the charge trajectory. This pole contributes to the forced field under condition $0 < z < vt_0 + r\cot\chi_p$, and to the free field under condition $0 < z < r\cot\chi_p$ (we assume here that the imaginary parts of ε_2 and μ_2 are negligible and, respectively, n_2 and χ_p are real). However, it turns out that these contributions mutually compensate each other so that, in the total field, the pole contribution is present only in the region $r\cot\chi_p < z < vt_0 + r\cot\chi_p$ (Fig. 3). This region is the area where CR exists.

However, it should be noted that, on the boundaries of this region ($z = r\cot\chi_p$ and $z = vt_0 + r\cot\chi_p$), the saddle point coincides with the pole and the asymptotic presented below is incorrect. Note that a more precise ("uniform") asymptotic for the case of an infinite homogeneous medium is given in our paper [21] (this asymptotic is valid in the vicinity of the boundary $z = vt_0 + r\cot\chi_p$).

Spherical waves emerging from the point where the charge enters the medium 2 and from the region where the bunch

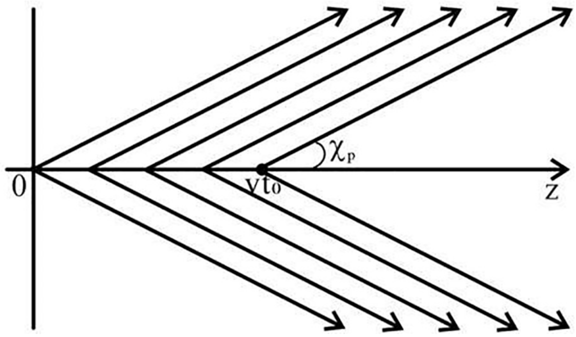

FIG. 3. The area of Cherenkov radiation.

loses the charge are determined by the contribution of the saddle point $\chi = \chi_s = \theta$.

Omitting calculations, we present the final result for the field asymptotic in medium 2. It can be written as

$$\vec{E}^{(2)} = \vec{E}^{(I)} + \vec{E}^{(II)}, \quad \vec{H}^{(2)} = \vec{H}^{(I)} + \vec{H}^{(II)}. \tag{41}$$

Here, the term with the index (I) is the contribution of the saddle point. In the spherical coordinates, it has only components $E_\theta^{(1)}$ and $H_\varphi^{(1)}$:

$$E_{\theta\omega}^{(I)} \approx \sqrt{\frac{\mu_2}{\varepsilon_2}} H_{\varphi\omega}^{(I)} \approx \frac{q_0}{c}\left[\frac{2\pi i k_0^2 n_2}{\tan\theta}\tilde{C}_+\bigg|_{\chi=\theta} - \frac{\beta\mu_2}{2\pi}\sin\theta\, U_2\bigg|_{\chi=\theta} e^{i\omega(1-n_2\beta\cos\theta)t_0}\right]\frac{e^{ik_2R}}{R}. \tag{42}$$

The term with the index (II) is the contribution of the pole. If $n_2\beta < 1$ then it is insignificant (exponentially decreases with increasing in r). If $n_2\beta > 1$ then it is a cylindrical wave of CR with components

$$\begin{Bmatrix} E_{r\omega}^{(II)} \\ E_{z\omega}^{(II)} \\ H_{\varphi\omega}^{(II)} \end{Bmatrix} = \frac{q_0}{c\sqrt{2\pi s_2 r}} \begin{Bmatrix} s_2/(\beta\varepsilon_2) \\ -cs_2^2/(\varepsilon_2\omega) \\ s_2 \end{Bmatrix} \times \exp\left(i\omega z/v + is_2 r - i\pi/4\right). \tag{43}$$

This radiation exists only in the region $r\cot\chi_p < z < vt_0 + r\cot\chi_p$ (Fig. 3). Note that, in this region, the cylindrical wave (43), which decreases proportionally to $r^{-1/2}$, dominates the spherical wave, which decreases proportionally to R^{-1}.

It should be noted that the conditions $|k_2R| \gg 1$ and $R \gg vt_0$ are necessary but not sufficient for the applicability of the asymptotic (41)–(43). In addition, it is necessary that the saddle point is sufficiently far from the branch point $\chi_b = \arcsin(n_1/n_2)$ and from the pole $\chi_p = \arccos[(n_2\beta)^{-1}]$.

The first of these conditions reduces to the inequality $|k_2R(\theta - \chi_b)| \gg 1$. This means that the observation angle θ is not close to the limiting angle of the total internal reflection χ_b.

The second condition is essential only in the case of $|n_2|\beta > 1$ (i.e., in the presence of Cherenkov radiation), when the pole χ_p is real or almost real. This condition is reduced to the requirement that the observation point is not close to the boundaries of the region of existence of the cylindrical wave: $|k_2(z\tan\chi_p - r)| \gg 1$ and $|k_2[(z - vt_0)\tan\chi_p - r]| \gg 1$.

The spectral-angular density of energy of the spherical wave (42) is determined by the expression

$$\frac{d^2W}{d\Omega d\omega} = cR^2\left|\sqrt{\varepsilon_2/\mu_2}\right|\left|E_{\theta\omega}^{(2)}\right|^2 = \frac{q_0^2}{4\pi^2 c}|F|^2 \tag{44}$$

where

$$F = \frac{2\varepsilon_2^{3/4}\mu_2^{1/4}}{(\varepsilon_2 N_{12} + \varepsilon_1 n_2\cos\theta)\tan\theta}\left[-\frac{(n_1^2\beta^2 - 1)N_{12} + n_2^2\beta\sin^2\theta}{1 - \beta^2 N_{12}^2} + \frac{(n_2^2\beta^2 - 1)N_{12} + \varepsilon_1\mu_2\beta\sin^2\theta}{1 - \beta^2 n_2^2\cos^2\theta}\right] - \beta\varepsilon_2^{1/4}\mu_2^{3/4}\sin\theta\left(W_- e^{-i\omega t_0 n_2\beta\cos\theta} - W_+\frac{\varepsilon_2 N_{12} - \varepsilon_1 n_2\cos\theta}{\varepsilon_2 N_{12} + \varepsilon_1 n_2\cos\theta}e^{i\omega t_0 n_2\beta\cos\theta}\right)e^{i\omega t_0}. \tag{45}$$

Here

$$N_{12} = \sqrt{n_1^2 - n_2^2\sin^2\theta}, \quad W_\pm = (1 \pm n_2\beta\cos\theta)^{-1}[1 - i\omega\tau(1 \pm n_2\beta\cos\theta)]^{-1}.$$

For nonrelativistic motion of charge, simplifying (44), one can obtain the following approximate expression:

$$F = \frac{\beta\varepsilon_2^{7/4}\mu_2^{5/4}\sin(2\theta)}{\varepsilon_2 N_{12} + \varepsilon_1 n_2\cos\theta}\frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_2} - \frac{\beta\varepsilon_2^{1/4}\mu_2^{3/4}\sin\theta}{1 - i\omega\tau}\left(e^{-i\omega t_0 n_2\beta\cos\theta} - \frac{\varepsilon_2 N_{12} - \varepsilon_1 n_2\cos\theta}{\varepsilon_2 N_{12} + \varepsilon_1 n_2\cos\theta}e^{i\omega t_0 n_2\beta\cos\theta}\right)e^{i\omega t_0}. \tag{46}$$

VII. NUMERICAL RESULTS AND DISCUSSION

The results of calculation of the spectral-angular density of the spherical wave energy depending on the angle $\tilde{\theta} = \pi - \theta$ (see Fig. 1) for various values of β, ωt_0, and $\omega\tau$ are shown in Figs. 4–6. The first medium (the region $0 < \tilde{\theta} < 90^\circ$) is assumed to be vacuum, and the second medium (the region $90^\circ < \tilde{\theta} < 180^\circ$) is characterized by the real constants $\varepsilon_2 = 1.5$ and $\mu_2 = 1$.

Recall that, in accordance with (38), in the unbounded homogeneous medium, radiation is emitted mainly "forward" [21]. The maximum of the angular density is achieved at some acute angle with respect to the motion direction. As the velocity β increases, the maximum increases and tends to $\theta = 0$ if $\beta \to 1$.

In the presence of the interface, the angular dependence changes dramatically. First, we note that the radiation in the second medium is more intense than in the first one (in

Figs. 4–6, the different scales on the vertical axis are used for regions 1 and 2). This difference increases with increasing the velocity. For ultrarelativistic bunches, the difference between the main maxima in regions 1 and 2 can be several orders of magnitude (Fig. 6).

Note that the radiation does not propagate along the z axis and along the interface ($\theta = 90^\circ$). The last fact is explained by the fact that the wave falls on the boundary at the "sliding" angle; in this case, the reflection coefficient is equal to -1, and the total field is equal to zero.

Let us consider in detail Fig. 4, illustrating the case of the relatively low velocity ($\beta = 0.1$). If the process of the bunch decay starts exactly at the interface ($t_0 = 0$), then both in the first (vacuumlike) region and in the second region there is only one maximum (Fig. 4, top). Both maxima have a relatively large width and decrease with increasing the bunch decay time τ. The latter fact is explained by the fact that the number of particles stopping per unit time decreases, which leads to the decrease of the corresponding part of the spherical wave.

Note that, in the region $90^\circ < \tilde{\theta} < 180^\circ$, the maximum has the character of a "fracture." It is observed when the angle θ is equal to the limiting angle of the total internal reflection: $\theta = \chi_b = \arcsin(n_1/n_2) \approx 55^\circ$. This fracture is related to the fact that, at such an angle, the saddle point of the integrand coincides with its branch point χ_b (see Sec. VI). The obtained asymptotic is not valid at $|\theta - \chi_b| \leqslant (k_2R)^{-1}$, but, due to the condition $k_2R \gg 1$, this angular region is insignificant.

If the process of the bunch decay begins at some significant distance from the interface, then the complex dependency consisting from several maxima and minima can take place in both regions (Fig. 4, bottom). This effect is explained by the fact that the waves emitted from the different zones are added from the point of entry of the charge into medium 2 and from the zone of the bunch decay (in the second medium there is also the wave reflected from the boundary). This effect is observed if the wave amplitudes are comparable in magnitude. If the bunch decay is sufficiently slow, then the role of the corresponding spherical wave is insignificant, and the effect noted above disappears (the dotted curves).

It should be underlined that the radiation under study differs radically from the usual transition radiation of a charge flying into the medium and not changing in any way. The angular dependencies for such radiation are close to the dotted black curves in Fig. 4 since, for them, the charge decay time τ is very long, and the usual transition radiation plays the main role. As we see, at $t_0 = 0$ (the top plot in Fig. 4), with an increase in τ, the radiation increases significantly, although the form of the curves remains approximately the same. If $\omega t_0 = 100$ (the bottom plot), then, with an increase in τ, the angular dependence becomes much more complicated due to the interference of waves radiated from different regions of the bunch trajectory.

It should be also emphasized that the radiation in the situation under consideration is radically different from the radiation of the decreasing charge in the unbounded homogeneous medium: the angular dependencies for such

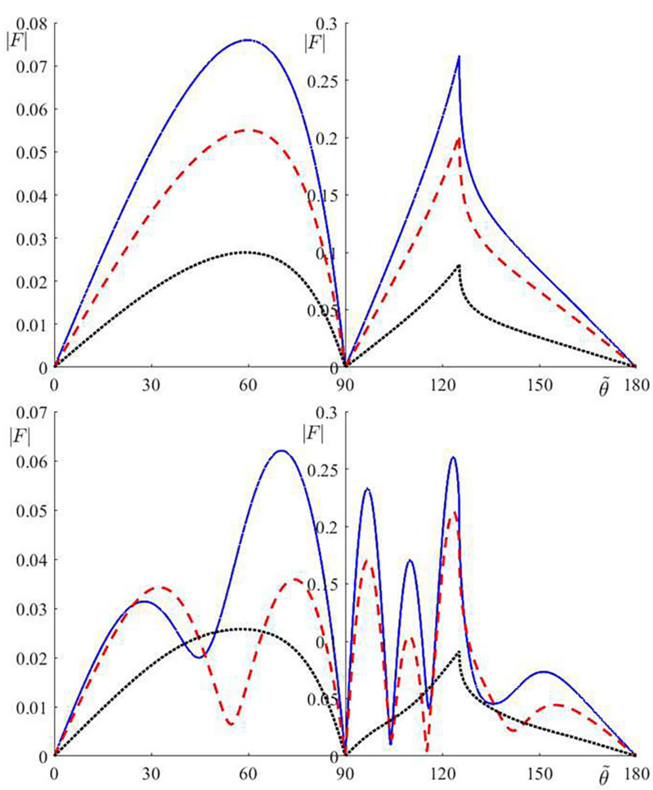

FIG. 4. Dependence of F on the angle $\tilde{\theta} = \theta - \pi$ in the case $\beta = 0.1$. Other parameters: $\varepsilon_1 = \mu_1 = \mu_2 = 1$, $\varepsilon_2 = 1.5$; $\omega\tau = 0$ (blue solid curve), $\omega\tau = 1$ (red dashed curve), $\omega\tau = 50$ (black dotted curve); for the top plot $\omega t_0 = 0$; for the bottom plot $\omega t_0 = 100$.

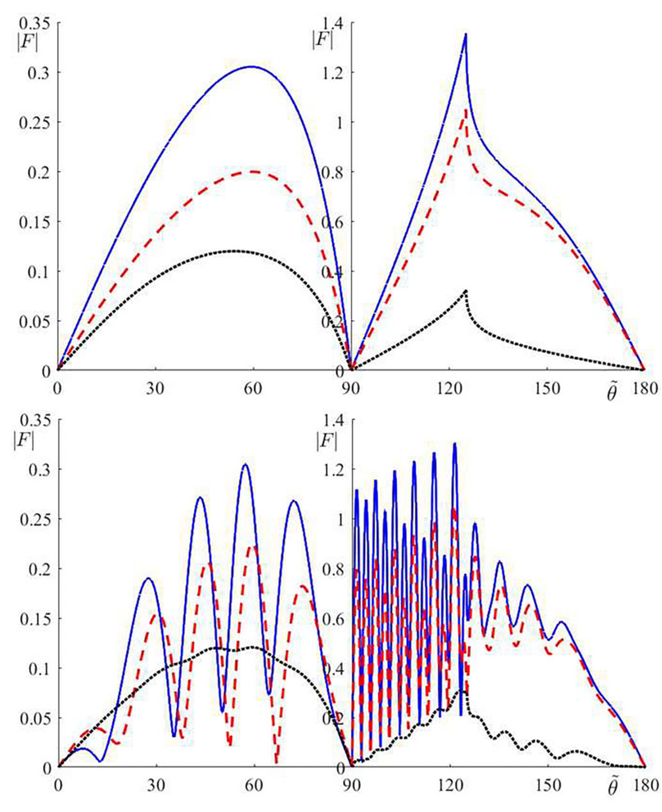

FIG. 5. The same as in Fig. 4 for $\beta = 0.5$.

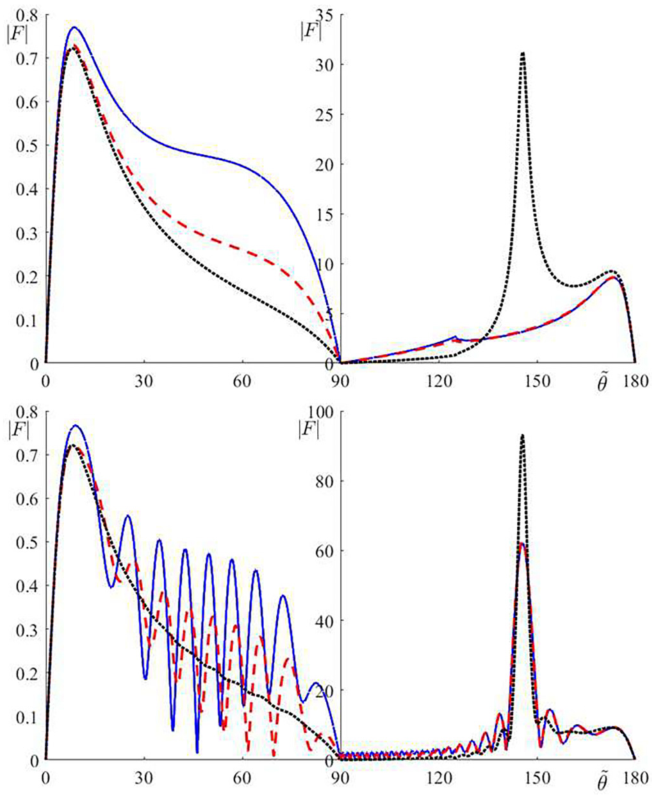

FIG. 6. The same as in Fig. 4 for $\beta = 0.99$.

radiation always have a simple form with a single maximum ([21], Fig. 3).

In the case of $\beta = 0.5$ (Fig. 5), the features noted above do not change qualitatively. We note only that with an increase in the velocity the general increase in the radiation energy occurs, and the number of extremes at $\omega t_0 \gg 1$ increases also.

Figure 6 illustrates the case when the charge velocity is larger than the wave velocity in the second medium ($n_2\beta > 1$), and therefore Cherenkov radiation is generated. It should be emphasized that CR itself is not shown in these graphs. Its field decreases proportionally to $r^{-1/2}$, therefore it is much larger than the field of spherical waves. However, it exists only in the area $r \cot \chi_p < z < vt_0 + r \cot \chi_p$, which is small if $R \gg vt_0$ (note that $\chi_p \approx 34^\circ$ for Fig. 6). The plots are incorrect in this narrow angular region, since the saddle point coincides with the pole at $\theta = \chi_p$ (see Sec. VI).

In the case $n_2\beta > 1$, the following features of the spherical wave can be noted. First, it is much more intensive compared to the case $n_2\beta < 1$ which can be seen from comparison of Fig. 6 with Figs. 4 and 5. In the vacuum region, the main maximum shifts towards the smaller angles $\tilde{\theta}$ with increasing the velocity, and it is close to 0° at $\beta \approx 1$. In the region $90^\circ < \tilde{\theta} < 180^\circ$, if the values ωt_0 and/or $\omega\tau$ are sufficiently large, then there is a sharp maximum at the angle $\tilde{\theta} = \pi - \chi_p$. Thus, the transition from the "subluminal" ($n_2\beta < 1$) regime of the charge motion to the "superluminal" one ($n_2\beta > 1$) leads not only to the appearance of Cherenkov radiation, but also to radical changes of the spherical wave properties.

VIII. CONCLUSION

We have studied the electromagnetic radiation of a charged bunch of small size crossing the flat interface between two media. It was assumed that the charge magnitude decreases exponentially starting from a certain time moment after the charge enters the second medium. It was taken into account that a filamentous trace consisting of immobile charges is formed in the second medium. It was assumed that Cherenkov radiation could be generated in the second medium only.

We have obtained a general solution, which is the sum of the forced field (the charge field in the unbounded medium) and the free field occurring due to the influence of the interface. An asymptotic study for the far-field zone has been carried out. We have found the expressions for both the spherical wave and the cylindrical wave generated in the second medium at a sufficiently high charge velocity.

It should be emphasized that the angular dependencies for spherical waves differ radically both from those occurring in the case of an unbounded medium [21] and from corresponding dependencies for ordinary transition radiation. The spherical wave consists of two parts: the wave of transition radiation and the wave resulting from the bunch decay and formation of its trace. Depending on the distance from the boundary to the region of the bunch decay, the time of the decay, the bunch velocity, and other parameters, radiation patterns of very different types are formed.

A series of computations has been performed for the case when the charge flies from vacuum into the optically denser medium. If the process of the bunch decay starts at the interface, then the angular distribution of the radiation energy has one maximum in each of two regions. In the second medium, the radiation as a rule is more intensive than that in the vacuum area, and this difference increases with increasing the charge velocity. If the charge velocity exceeds the Cherenkov threshold in the second medium, then, along with the appearance of Cherenkov radiation, the properties of the spherical wave change dramatically. In particular, the main maximum of the angular distribution of the radiation energy strongly increases. If the distance from the charge entry point to the region of the bunch decay is sufficiently large, then a complicated interference pattern with many extremes arises.

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