

Hadronic vacuum polarization correction to the bound-electron g factor

Eugen Dizer^{1,2} and Zoltán Harman^{1,*}

¹Max Planck Institute for Nuclear Physics, Saupfercheckweg 1, 69117 Heidelberg, Germany

²Ruprecht Karl University of Heidelberg, Department of Physics and Astronomy,
Im Neuenheimer Feld 226, 69120 Heidelberg, Germany



(Received 22 February 2023; revised 27 June 2023; accepted 26 September 2023; published 12 October 2023)

The hadronic vacuum polarization correction to the g factor of a bound electron is investigated theoretically. An effective hadronic Uehling potential obtained from measured cross sections of e^-e^+ annihilation into hadrons is employed to calculate g -factor corrections for low-lying hydrogenic levels. Analytical Dirac-Coulomb wave functions, as well as bound wave functions accounting for the finite nuclear radius are used. Closed formulas for the g -factor shift in the case of a point-like nucleus are derived. In heavy ions, such effects are found to be much larger than for the free-electron g factor.

DOI: [10.1103/PhysRevA.108.042808](https://doi.org/10.1103/PhysRevA.108.042808)

I. INTRODUCTION

Precision Penning-trap experiments on the g factor of hydrogen-like and few-electron highly charged ions allow a thorough testing of quantum electrodynamics (QED), a cornerstone of the standard model describing electromagnetic interactions. The g factor of hydrogen-like silicon ($Z = 14$) has been measured with a 5×10^{-10} relative uncertainty [1,2], allowing to scrutinize bound-state QED theory (see, e.g., Refs. [3–13]). Two-loop radiative effects and shifts due to nuclear structure and recoil are observable in such measurements. The high accuracy which can be achieved on the experimental as well as theoretical side also enables the determination of fundamental physical constants such as the electron mass m_e [14–18]. Recently, it was shown that g -factor studies can also help in the search for new physics, i.e., the coupling strength of a hypothetical new interaction can be constrained through the comparison of theoretical and experimental results [12,19,20].

Further improved tests and the possible determinations of fundamental constants [21–23] call for an increasing accuracy on the theoretical side. The evaluation of two-loop terms up to order $(Z\alpha)^5$ (with Z being the atomic number and α the fine-structure constant) were finalized recently [24,25], increasing the theoretical accuracy especially in the low- Z regime. First milestones have been also reached in the calculation of two-loop corrections in stronger Coulomb fields, i.e., for larger values of $Z\alpha$ [26,27]. As the experiments are advancing towards heavy ions [28,29], featuring smaller and smaller characteristic distance scales for the interaction

between the bound electron and the nucleons, the effects of other forces may need to be considered as well.

Motivated by these prospects, in this article we investigate vacuum polarization (VP) corrections due to the virtual creation and annihilation of hadrons. The dominant VP contribution arises from virtual e^-e^+ pair creation, which has been widely investigated in the literature [5,6,30,31] and is well understood. The other leptonic VP effect is due to virtual muons, the contribution of which is suppressed by the square of the electron-to-muon mass ratio [32]. The hadronic VP effect, which arises due to a superposition of different virtual hadronic states, is comparable in magnitude to muonic VP, however, it requires a completely different description since the virtual hadrons interact via the strong force. An effective approach to take into account such effects for the free-electron g factor is described in, e.g., Ref. [33], in which hadronic VP is characterized by the cross section of hadron production via e^-e^+ annihilation. Following this treatment, we apply the known empirical parametric hadronic polarization function for the photon propagator from Ref. [34] to account for the complete hadronic contribution in case of the bound-electron g factor.

While in the case of the free electron, the hadronic correction only appears on the two-loop level, as a correction to the electrons electromagnetic self-interaction [see Fig. 1(a)], in the case of a bound electron it appears already as a one-loop effect [see Fig. 1(b)]. Furthermore, the hadronic VP is boosted by approximately $\sim Z^4$, i.e., by the fourth power of the nuclear charge number, and thus, as we will see later, for heavier ions above $Z = 14$ its contribution is larger than in the case of a free electron [16,35].

An effective potential constructed from the parametrized VP function, the hadronic Uehling potential, was derived in Ref. [36]. We calculate the perturbative correction to the g factor due to this radial potential employing analytical Dirac-Coulomb wave functions, as well as numerically calculated wave functions accounting for a finite-size nucleus. Analytical formulas are presented, and numerical results are given for hydrogenic systems from H to U^{91+} . We note that such an

*harman@mpi-hd.mpg.de

Published by the American Physical Society under the terms of the Creative Commons Attribution 4.0 International license. Further distribution of this work must maintain attribution to the author(s) and the published article's title, journal citation, and DOI. Open access publication funded by the Max Planck Society.

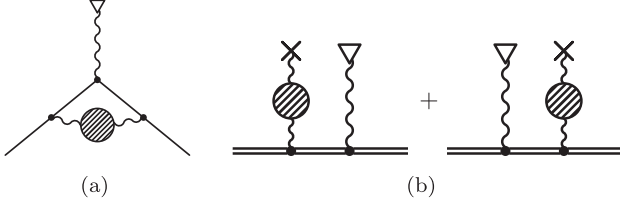


FIG. 1. Feynman diagrams representing the leading hadronic VP corrections to the (a) free-electron g factor and (b) the bound-electron g factor. Double lines represent electrons in the electric field of the nucleus and wavy lines with a triangle depict the interaction with the external magnetic field. For the free electron, it is a two-loop process where the self-interaction of the electron is perturbed by the effective hadronic polarization function (shaded bubble). For the bound electron, it is a one-loop correction where the Coulomb interaction with the nucleus (cross) is perturbed by the effective hadronic polarization function.

approach assumes an infinitely heavy nucleus, i.e., nuclear recoil effects are excluded in our treatment.

We use natural units with $\hbar = c = 1$ for the reduced Planck constant $\hbar$ and the speed of light c , and $\alpha = e^2$, where α is the fine-structure constant and e is the elementary charge. Three-vectors are denoted by bold letters.

II. g -FACTOR CORRECTIONS

Generally speaking, the g factor describes the coupling of the electron's magnetic moment to its total angular momentum $\mathbf{J}$. The corresponding first-order Zeeman splitting ΔE due to the electron's interaction with an external homogeneous magnetic field $\mathbf{B}$ is

$$\Delta E = g \mu_B \langle \mathbf{J} \cdot \mathbf{B} \rangle, \quad (1)$$

where $\mu_B = e/(2m_e)$ is the Bohr magneton of the electron and g is its g factor, which depends on the electron configuration.

On the other hand, the relativistic interaction of an electron with the external magnetic field can be derived from the minimal coupling principle in the Dirac equation. In first-order perturbation theory, this leads to the energy shift

$$\Delta E = e \langle \boldsymbol{\alpha} \cdot \mathbf{A} \rangle, \quad (2)$$

where $\boldsymbol{\alpha}$ are the usual Dirac matrices given in terms of the gamma matrices by $\alpha_i = \gamma^0 \gamma^i$ [37] and $\mathbf{A}$ is the vector potential for the magnetic field, such that $\mathbf{B} = \nabla \times \mathbf{A}$. Choosing the magnetic field to be directed along the z axis, one can see that a possible choice for the vector potential is $\mathbf{A} = [\mathbf{B} \times \mathbf{r}]/2$, where $\mathbf{r}$ is the position vector. Together with Eqs. (1) and (2), one can derive the following general expression for the g factor [31]:

$$g = \frac{2\kappa m_e}{j(j+1)} \int_0^\infty dr r G_{n\kappa}(r) F_{n\kappa}(r), \quad (3)$$

where n is the principal quantum number of the bound state, $j = |\kappa| - 1/2$ is the total angular momentum quantum number and κ is the relativistic angular momentum quantum number. The functions $G_{n\kappa}(r)$, $F_{n\kappa}(r)$ are the radial compo-

nents in the electronic Dirac wave function

$$\psi_{n\kappa m}(\mathbf{r}) = \frac{1}{r} \begin{pmatrix} G_{n\kappa}(r) \Omega_{\kappa m}(\theta, \varphi) \\ i F_{n\kappa}(r) \Omega_{-\kappa m}(\theta, \varphi) \end{pmatrix}, \quad (4)$$

where m is the magnetic quantum number and $r = |\mathbf{r}|$. The spherical spinors $\Omega_{\pm\kappa m}(\theta, \varphi)$ make up the angular components and are the same for any central potential $V(r)$ [38].

A straightforward approach for calculating the g -factor shift Δg^{VP} due to VP is to solve the radial Dirac equation numerically with the inclusion of the VP effect and then substituting the perturbed functions $G_{n\kappa}^{\text{VP}}(r)$, $F_{n\kappa}^{\text{VP}}(r)$ into Eq. (3). The difference between the perturbed and the unperturbed g factor gives the corresponding shift

$$\Delta g^{\text{VP}} = g^{\text{VP}} - g. \quad (5)$$

However, we will apply a different method to investigate the hadronic g -factor shift. As shown in Ref. [31], owing to the properties of Dirac wave functions, the g factor in Eq. (3) can be expressed through the energy eigenvalues $E_{n\kappa}$,

$$g = -\frac{\kappa}{2j(j+1)} \left(1 - 2\kappa \frac{\partial E_{n\kappa}}{\partial m_e} \right), \quad (6)$$

if the potential $V(r)$ does not depend on the electron mass m_e . This formula was used successfully, e.g., to investigate the finite nuclear size effect in Ref. [31]. We apply this new approach to investigate the vacuum polarization effect, described by an effective potential. Having a small perturbation $\delta V(r)$ to the nucleus potential (like the hadronic Uehling potential [36]), the g -factor shift can be shown to be [31]

$$\Delta g^{\text{VP}} = -\frac{\kappa^2}{j(j+1)m_e} \left\langle r \frac{\partial \delta V(r)}{\partial r} \right\rangle. \quad (7)$$

For the relativistic ground state ($1s$) and a point-like nucleus, this expectation value can be evaluated further to obtain [39]

$$\Delta g_{\text{point}}^{\text{VP}}(1s) = \frac{4(1+2\gamma)}{3m_e} \Delta E_{\text{point}}^{\text{VP}}(1s) - \frac{8Z\alpha}{3} \langle r \delta V \rangle_{1s}, \quad (8)$$

where $\gamma = \sqrt{1 - (Z\alpha)^2}$ and $\Delta E_{\text{point}}^{\text{VP}}(1s) = \langle \delta V \rangle_{1s}$ is the corresponding energy shift in first-order perturbation theory. Since the second term on the right-hand side of Eq. (8) is $Z\alpha$ times smaller than the first term, the g -factor shift can be approximated for light ions ($Z\alpha \ll 1$) with the formula

$$\Delta g_{\text{point}}^{\text{VP}}(1s) \approx \frac{4(1+2\gamma)}{3m_e} \Delta E_{\text{point}}^{\text{VP}}(1s). \quad (9)$$

A similar expression also appeared in Refs. [23,31] in a different context, studying the finite-size effect. However, we will investigate the applicability of this formula as an approximation for calculating the g -factor shift due to VP effects for light ions.

A. Leptonic vacuum polarization correction to the g factor

The leptonic VP correction to the bound-electron g factor is well known. The leading-order diagrams are shown in Fig. 2 and can be divided into two groups: the electric loop (EL) and the magnetic loop (ML) contribution. The vacuum polarization effect in the leading EL contribution [Figs. 2(a) and 2(b)] is equivalent to a perturbation in the interaction

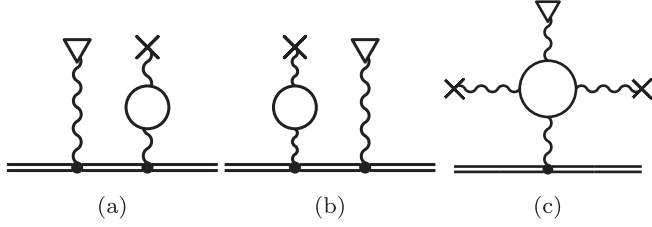


FIG. 2. Feynman diagrams representing the leading leptonic VP corrections to the bound-electron g factor. Double lines represent electrons in the electric field of the nucleus and wavy lines with a triangle (cross) depict the interaction with the external magnetic field (nucleus). Single lines represent free-particle propagators.

between the bound electron and the nucleus, and thus can be described by an effective perturbing potential $\delta V_{\text{Ue}}(\mathbf{r})$, called the Uehling potential. The corresponding Zeeman splitting ΔE_{EL} with respect to a reference state $|a\rangle$ can be derived from second-order perturbation theory [40]

$$\Delta E_{\text{EL}} = 2\langle a | \delta V_{\text{Ue}} | \delta a \rangle, \quad (10)$$

where $|\delta a\rangle$ is the first-order perturbative correction to the bound-electron wave function arising from the interaction with the external magnetic field

$$|\delta a\rangle = \sum_n \frac{E_a \neq E_n}{E_a - E_n} \frac{|n\rangle \langle n | e \boldsymbol{\alpha} \cdot \mathbf{A} | a \rangle}{E_a - E_n}. \quad (11)$$

The diagrams in Figs. 2(a) and 2(b) contribute equally to the EL correction. Instead of using the standard formulas from perturbation theory, we employ the simpler Eq. (7) from Ref. [31] to compute the VP effects to the bound-electron g factor.

In the case of leptonic vacuum loops, the well-known leptonic Uehling potential is given by [41]

$$\delta V_{\text{Ue}}(\mathbf{r}) = -\frac{2\alpha(Z\alpha)}{3\pi} \int d^3x \rho(\mathbf{x}) \frac{K_1(2m_l|\mathbf{r} - \mathbf{x}|)}{|\mathbf{r} - \mathbf{x}|}, \quad (12)$$

where $\rho(\mathbf{x})$ denotes the nuclear charge distribution normalized to unity, m_l is the mass of the virtual particle in the fermionic loop, and $K_1(x)$ is given by

$$K_1(x) = \int_1^\infty dt e^{-xt} \left(1 + \frac{1}{2t^2}\right) \frac{\sqrt{t^2 - 1}}{t^2}. \quad (13)$$

The g -factor shift of a bound electron in the ground state can be calculated analytically for a point-like nucleus and was already derived in Ref. [30]. We will show that one arrives to the same result using the approach in Eq. (7). Using the leptonic Uehling potential for a point-like nucleus [$\rho(\mathbf{x}) = \delta^{(3)}(\mathbf{x})$] [30],

$$\delta V_{\text{point}}^{\text{lept.}}(r) = -\frac{2\alpha(Z\alpha)}{3\pi r} K_1(2m_l r), \quad (14)$$

and the radial components of the electronic wave function in the ground state [32], one obtains from Eq. (7)

$$\Delta g_{\text{point}}^{\text{lept.}}(1s) = -\frac{8\alpha(Z\alpha)}{3\pi\zeta} \left[I_{133} - \frac{1}{3} I_{233} + \frac{\zeta Z\alpha}{2\gamma} \left(I_{122} - \frac{1}{3} I_{222} \right) \right]. \quad (15)$$

I_{abc} is a modification of the base integral given in Ref. [30], see Appendix A, and $\zeta = m_e/m_l$ is the ratio of the electron and the loop particle masses.

The leading-order $Z\alpha$ expansion is given by

$$\begin{aligned} \Delta g_{\text{point}}^{\text{lept.}}(1s) &= \frac{\alpha}{\pi} \left[-\frac{16\zeta^2(Z\alpha)^4}{15} + \frac{5\pi\zeta^3(Z\alpha)^5}{9} \right. \\ &+ \left(\frac{16\zeta^2}{15} \ln(2\zeta Z\alpha) - \frac{116\zeta^2}{75} - \frac{16\zeta^4}{7} \right) (Z\alpha)^6 \\ &+ \left(-\frac{5\pi\zeta^3}{9} \ln\left(\frac{\zeta Z\alpha}{2}\right) - \frac{8\pi\zeta^3}{27} + \frac{7\pi\zeta^5}{8} \right) (Z\alpha)^7 \\ &\left. + O((Z\alpha)^8) \right]. \end{aligned} \quad (16)$$

For $\zeta = 1$, this is exactly the same result as in Ref. [30], however, obtained with a different method.

The leading ML contribution [Fig. 2(c)] is equivalent to a modification of the vector potential of the external magnetic field and the Zeeman splitting ΔE_{ML} is given by [40]

$$\Delta E_{\text{ML}} = e \langle a | \boldsymbol{\alpha} \cdot \mathbf{A}_{\text{ML}} | a \rangle, \quad (17)$$

where $\mathbf{A}_{\text{ML}}$ is the modified vector potential. For a more detailed discussion and the analytic expressions for $\mathbf{A}_{\text{ML}}$, not treated in the present work, see Refs. [6,40].

For completeness, the leading-order ML contribution to the g factor of a bound electron in the ns state for a point-like nucleus is called the Delbrück contribution and is given by [5,6,42]

$$\Delta g_{\text{ML}}(ns) = \frac{7\alpha(Z\alpha)^5}{216n^3} \zeta^3. \quad (18)$$

Therefore, it is suppressed by a factor $Z\alpha$ with respect to the Uehling contribution and is not the subject of the current work.

In the next section, we will use Eq. (7) to derive analytic expressions for the leading hadronic VP correction to the bound-electron g factor.

B. Hadronic vacuum polarization correction to the g factor

As discussed in Refs. [33,34,36], the hadronic vacuum polarization function can be constructed semi-empirically from the experimental data of e^-e^+ annihilation cross sections. The entire hadronic polarization function is parametrized for seven regions of momentum transfer and is given in Ref. [34]. In Ref. [36], it was found that only the first region of parametrization is significant for the hadronic energy shift calculations. This is also clear from the physical point of view since atomic physics is dominated by low energies around eV to keV. Thus, we will use the analytic hadronic Uehling potential introduced in Ref. [36] for our calculations. For a point-like nucleus it is given by

$$\delta V_{\text{point}}^{\text{had.}}(r) = -\frac{2Z\alpha}{r} B_1 E_1 \left(\frac{r}{\sqrt{C_1}} \right), \quad (19)$$

with the coefficients $B_1 = 0.002\,3092$ and $C_1 = 3.992\,5370\text{ GeV}^{-2}$ [34,36] and the exponential integral $E_1(x)$ which can be generalized for $n = 0, 1, 2, \dots$, by [43]

$$E_n(x) = \int_1^\infty dt \frac{e^{-xt}}{t^n}. \quad (20)$$

The values for B_1 and C_1 are taken from the most recent parametrization in Ref. [34] and will be used for the calculations. The error of the numerical results is estimated by comparison with an older parametrization in Ref. [44] as was done in Ref. [36].

The corresponding hadronic Uehling potential for an extended nucleus with spherical charge distribution $\rho(\mathbf{x})$ is obtained by the convolution [36]

$$\begin{aligned} \delta V_{\text{fns}}^{\text{had.}}(\mathbf{r}) &= \int d^3x \rho(\mathbf{x}) \delta V_{\text{point}}^{\text{had.}}(\mathbf{r} - \mathbf{x}) \\ &= -\frac{4\pi Z\alpha B_1 \sqrt{C_1}}{r} \int_0^\infty dx x \rho(x) D_2^-(r, x), \end{aligned} \quad (21)$$

where $x = |\mathbf{x}|$ and

$$D_n^\pm(r, x) = E_n\left(\frac{|r-x|}{\sqrt{C_1}}\right) \pm E_n\left(\frac{|r+x|}{\sqrt{C_1}}\right). \quad (22)$$

As in our previous work [36], we will consider the homogeneously charged sphere as the model for the extended nucleus with the root-mean-square (RMS) radii taken from Ref. [45]. The charge distribution $\rho(r)$ is given by

$$\rho(r) = \frac{3}{4\pi R^3} \theta(R - r), \quad (23)$$

where $\theta(x)$ is the Heaviside step function and the effective radius R is related to the RMS nuclear charge radius R_{rms} via $R = \sqrt{5/3} R_{\text{rms}}$. The corresponding hadronic Uehling potential is given analytically in Ref. [36], see Appendix B.

Let us turn to the evaluation of the leading hadronic VP contribution to the bound-electron g factor, depicted in Fig. 1(b). In the low-energy limit, the hadronic Uehling potential is given by [46] (see also [47])

$$\delta V_{\text{non-rel.}}^{\text{had.}}(\mathbf{x}) = -4\pi Z\alpha B_1 C_1 \delta^{(3)}(\mathbf{x}). \quad (24)$$

Using Eq. (7) and the nonrelativistic expectation value of the delta function, the leading order in $Z\alpha$ of the hadronic g -factor shift for general ns states is found to be [39]

$$\begin{aligned} \Delta g_{\text{non-rel.}}^{\text{had.}}(ns) &= -\frac{4}{3m_e} \langle 12\pi Z\alpha B_1 C_1 \delta^{(3)}(\mathbf{x}) \rangle_{ns} \\ &= -\frac{16(Z\alpha)^4 m_e^2}{n^3} B_1 C_1. \end{aligned} \quad (25)$$

For the $1s$ state, a fully relativistic expression for the point-like nucleus can be given. Using the hadronic Uehling potential in Eq. (19) and the relativistic wave function of the ground state, one obtains with Eq. (7)

$$\Delta g_{\text{point}}^{\text{had.}}(1s) = \frac{4}{3m_e} \Delta E_{\text{point}}^{\text{had.}}(1s) - \frac{8B_1(Z\alpha)^2(2\lambda\sqrt{C_1})^{2\gamma}}{3\gamma(1+2\lambda\sqrt{C_1})^{2\gamma}}, \quad (26)$$

where $\lambda = Z\alpha m_e$ and $\Delta E_{\text{point}}^{\text{had.}}(1s)$ is the analytical energy shift for a point-like nucleus given in Ref. [36]

$$\begin{aligned} \Delta E_{\text{point}}^{\text{had.}}(1s) &= -\frac{Z\alpha\lambda(2\lambda\sqrt{C_1})^{2\gamma} B_1}{\gamma^2} \\ &\quad \times {}_2F_1(2\gamma, 2\gamma; 1+2\gamma; -2\lambda\sqrt{C_1}), \end{aligned} \quad (27)$$

with ${}_2F_1(a, b; c; z)$ being the hypergeometric function [43]. The expansion of this expression up to sixth order in $Z\alpha$ is given by

$$\begin{aligned} \Delta g_{\text{point}}^{\text{had.}}(1s) &= -16B_1 C_1 m_e^2 (Z\alpha)^4 \\ &\quad + \frac{512B_1 C_1^{3/2} m_e^3 (Z\alpha)^5}{9} - \frac{16B_1 C_1 m_e^2 (Z\alpha)^6}{3} \\ &\quad \times [2 + 30C_1 m_e^2 - 3 \ln(2m_e Z\alpha \sqrt{C_1})] \\ &\quad + O((Z\alpha)^7), \end{aligned} \quad (28)$$

and it coincides with the nonrelativistic approximation in Eq. (25) to order $(Z\alpha)^4$.

It is important to note that the coefficients in Eq. (28), as well as in the remaining article, arise only from the leading EL contribution depicted in Fig. 1(b), see Sec. II A.

A similar relativistic calculation for the $2s$ state yields

$$\begin{aligned} \Delta g_{\text{point}}^{\text{had.}}(2s) &= -2B_1 C_1 m_e^2 (Z\alpha)^4 + \frac{64B_1 C_1^{3/2} m_e^3 (Z\alpha)^5}{9} \\ &\quad - \frac{B_1 C_1 m_e^2 (Z\alpha)^6}{24} [41 + 420C_1 m_e^2 \\ &\quad - 48 \ln(m_e Z\alpha \sqrt{C_1})] + O((Z\alpha)^7). \end{aligned} \quad (29)$$

The leading orders of Eqs. (28) and (29) satisfy the nonrelativistic relationship in Eq. (25)

$$\Delta g_{\text{non-rel.}}^{\text{had.}}(ns) = \frac{1}{n^3} \Delta g_{\text{non-rel.}}^{\text{had.}}(1s). \quad (30)$$

C. Hadronic vacuum polarization correction to the reduced g factor

Additionally, we investigate hadronic effects on the weighted difference of the g factor and the bound-electron energy E of H-like ions, called the reduced g factor

$$\tilde{g} = g - \frac{4(1+2\gamma)}{3m_e} E, \quad (31)$$

put forward in Ref. [23] for a possible novel determination of the fine-structure constant and for testing physics beyond the standard model [20]. It was shown there that the detrimental nuclear structure contributions featuring large uncertainties can be effectively suppressed in the above combination of the g factor and level energy of the hydrogenic ground state. The question arises whether the same can be said about the hadronic VP corrections investigated in the present article.

The hadronic VP correction [Fig. 1(b)] to the reduced g factor for a point-like nucleus can be found analytically using Eqs. (26) and (27). The leading-order $Z\alpha$ expansion is

given by

$$\begin{aligned}\tilde{g}_{\text{point}}^{\text{had.}}(1s) &= \Delta g_{\text{point}}^{\text{had.}}(1s) - \frac{4(1+2\gamma)}{3m_e} \Delta E_{\text{point}}^{\text{had.}}(1s) \\ &= \frac{128}{9} B_1 C_1^{3/2} m_e^3 (Z\alpha)^5 - 64 B_1 C_1^2 m_e^4 (Z\alpha)^6 \\ &\quad + O((Z\alpha)^7).\end{aligned}\quad (32)$$

Thus, the leading term of order $(Z\alpha)^4$ in $\Delta g_{\text{point}}^{\text{had.}}(1s)$ cancels such that the hadronic VP contribution to the reduced g factor is indeed small for practical purposes. This also supports the approximation in Eq. (9). It is important to note again that the coefficients in Eq. (32) arise only from the leading EL contribution and, thus, may be altered when including higher-order corrections like the ML contribution, see Sec. II A. However, the general result that the leading term of order $(Z\alpha)^4$ cancels is unchanged since these corrections are subleading. Therefore, we may conclude that hadronic effects do not hinder the extraction of α or detailed tests of QED and standard model extensions via the measurement of $\tilde{g}$.

D. Hadronic vacuum polarization correction to the weighted g -factor difference of H- and Li-like ions

Another quantity of interest is the weighted difference of the g factors of the Li-like and H-like charge states of the same element

$$\delta_{\Xi} g = g(2s) - \Xi g(1s), \quad (33)$$

where $g(2s)$ is the g factor of the Li-like ion and $g(1s)$ is the g factor of the H-like ion. For light elements, the parameter Ξ can be calculated to great accuracy by [21,48]

$$\Xi = 2^{-2\gamma-1} \left[1 + \frac{3}{16} (Z\alpha)^2 \right] \left(1 - \frac{2851}{1000} \frac{1}{Z} + \frac{107}{100} \frac{1}{Z^2} \right). \quad (34)$$

This weighted (or specific) difference was introduced to suppress uncertainties arising from the nuclear charge radius and further nuclear structural effects [49]. Therefore, bound-state QED theory can be investigated more accurately in g -factor experiments combining H- and Li-like ions than with the individual ions alone.

As we have seen, the leading hadronic VP correction to $\delta_{\Xi} g$ for a point-like nucleus can be found analytically. We approximate $\Delta g_{\text{point}}^{\text{had.}}(2s)$ of the Li-like ion with the expression in Eq. (29) for the H-like ion. Since there are no electron-electron interactions in this approximation, we have to neglect the terms of relative orders $1/Z$ and $1/Z^2$ in Eq. (34). We note that the residual weight

$$\Xi_0 = 2^{-2\gamma-1} \left[1 + \frac{3}{16} (Z\alpha)^2 \right] = \frac{1}{8} + O((Z\alpha)^2), \quad (35)$$

exactly cancels the first two leading orders $(Z\alpha)^4$ and $(Z\alpha)^5$ of the EL contribution

$$\begin{aligned}\delta_{\Xi_0} g_{\text{point}}^{\text{had.}} &= \Delta g_{\text{point}}^{\text{had.}}(2s) - \Xi_0 \Delta g_{\text{point}}^{\text{had.}}(1s) \\ &= \frac{5}{2} B_1 C_1^2 m_e^4 (Z\alpha)^6 + O((Z\alpha)^7).\end{aligned}\quad (36)$$

Note that the leading ML contribution, which is of order $(Z\alpha)^5$, may not cancel and potentially lead to a coefficient

$(Z\alpha)^5$. However, we anticipate that a similar cancellation applies as in the leptonic ML case, see Eq. (18). The general argument still holds and we can conclude that the leading hadronic VP effects are also cancelled in the above specific difference. A similar conclusion can be drawn for the case of the specific difference introduced for a combination of H- and B-like ions [22]. This result is well understood since nuclear and hadronic VP contributions are both short-range effects with a similar behavior.

III. NUMERICAL RESULTS

As mentioned in Ref. [36], the hadronic VP contribution to the energy shift is about $1/0.665 \approx 1.5$ times smaller than the muonic VP contribution in the case of the Uehling term. This can be also confirmed for the g -factor shift. Comparing the nonrelativistic approximation for the hadronic g -factor shift $\Delta g_{\text{non-rel.,point}}^{\text{had. VP}}$ in Eq. (24) with the first term of the expression for the muonic g -factor shift $\Delta g_{\text{non-rel.,point}}^{\text{muonic VP}}$ in Eq. (16), yields for hydrogen in the ground state

$$\begin{aligned}\Delta g_{\text{non-rel.,point}}^{\text{had. VP}}(1s) &= -1.092(14) \times 10^{-16} \\ &= 0.664(9) \Delta g_{\text{non-rel.,point}}^{\text{muonic VP}}(1s).\end{aligned}\quad (37)$$

The values for the hadronic g -factor shift with an extended nucleus were calculated numerically using two different methods, both yielding the same results within the given uncertainties. The first method consists of calculating the expectation value in Eq. (7) with the FNS hadronic Uehling potential and the semi-analytic wave functions of a homogeneously charged spherical nucleus given in Ref. [50]. As a consistency check, these results were reproduced by using the approach of solving the radial Dirac equation numerically with the inclusion of the FNS potential, and substituting the resulting large and small radial wave function components into Eqs. (3) and (5). The results for the hydrogen-like systems H, Si, Ca, Xe, Kr, W, Pb, Cm, and U are given in Table I. A diagrammatic representation is shown in Fig. 3. We note that for $Z = 14$ and above, the magnitude of the hadronic vacuum polarization terms considered in this work exceed in magnitude the hadronic contribution to the free-electron g factor [16,35,51]. However, it is important to mention that the uncertainty of the leading finite nuclear size correction to the g factor is approximately an order of magnitude larger than the hadronic VP effect for all elements considered (see, e.g., Refs. [16,23]), hindering the identification of the effect. For comparison, the current value of the total bound-electron g factor for $Z = 14$ is $g(^{28}\text{Si}^{13+}) = 1.995\,348\,958\,109(584)$ [24].

The errors given in Tables I and II are based on the uncertainty of the nuclear root-mean-square radii R_{rms} given in Ref. [45], the numerical convergence of the results and an assumed uncertainty for the parameters B_1 and C_1 as described in Sec. II B. The latter was determined as the difference with respect to values obtained by using another set of parameters to describe the polarization function, stemming from Ref. [44]. We anticipate that newer updates of these parameters would not shift our results significantly (for the discussion of newer e^-e^+ annihilation data, see, e.g., Refs. [52,53]). The total error is dominated by the assumed uncertainty of

TABLE I. Results for the hadronic VP contribution to the g -factor shift of the bound electron in the ground state arising from the Uehling potential in the EL diagram [Fig. 1(b)] using different approaches: the nonrelativistic approximation $\Delta g_{\text{non-rel.,point}}^{\text{had.}}$ in Eq. (25), the relativistic formula for a point-like nucleus $\Delta g_{\text{rel.,point}}^{\text{had.}}$ in Eq. (26), the approximate formula $\Delta g_{\text{approx.,fns}}^{\text{had.}}$ using the hadronic energy shift with an extended nucleus from [36] in Eq. (9), and the full relativistic result for an extended nucleus $\Delta g_{\text{rel.,fns}}^{\text{had.}}$ using the analytical finite-size Uehling potential with numerical finite-size wave functions in Eq. (7). Root-mean-square nuclear charge radii R_{rms} are taken from Ref. [45].

Z	R_{rms} [fm]	$\Delta g_{\text{non-rel.,point}}^{\text{had.}}(1s)$	$\Delta g_{\text{rel.,point}}^{\text{had.}}(1s)$	$\Delta g_{\text{approx.,fns}}^{\text{had.}}(1s)$	$\Delta g_{\text{rel.,fns}}^{\text{had.}}(1s)$
1	0.8783(86)	$-1.092(14) \times 10^{-16}$	$-1.093(14) \times 10^{-16}$	$-1.093(13) \times 10^{-16}$	$-1.093(13) \times 10^{-16}$
14	3.1224(24)	$-4.196(53) \times 10^{-12}$	$-4.616(57) \times 10^{-12}$	$-4.490(56) \times 10^{-12}$	$-4.497(56) \times 10^{-12}$
20	3.4776(19)	$-1.748(22) \times 10^{-11}$	$-2.109(25) \times 10^{-11}$	$-1.989(25) \times 10^{-11}$	$-1.996(25) \times 10^{-11}$
36	4.1884(22)	$-1.835(23) \times 10^{-10}$	$-3.263(39) \times 10^{-10}$	$-2.664(33) \times 10^{-10}$	$-2.696(34) \times 10^{-10}$
54	4.7859(48)	$-9.29(12) \times 10^{-10}$	$-3.291(35) \times 10^{-9}$	$-2.004(25) \times 10^{-9}$	$-2.065(26) \times 10^{-9}$
74	5.3658(23)	$-3.275(41) \times 10^{-9}$	$-3.568(32) \times 10^{-8}$	$-1.261(15) \times 10^{-8}$	$-1.344(17) \times 10^{-8}$
82	5.5012(13)	$-4.938(62) \times 10^{-9}$	$-9.589(77) \times 10^{-8}$	$-2.508(31) \times 10^{-8}$	$-2.728(34) \times 10^{-8}$
92	5.8571(33)	$-7.825(98) \times 10^{-9}$	$-3.572(24) \times 10^{-7}$	$-5.705(71) \times 10^{-8}$	$-6.410(80) \times 10^{-8}$

B_1 and C_1 . Owing to the closed analytical expression for the hadronic Uehling potential, numerical uncertainties are negligible. For the results $\Delta g_{\text{approx.,fns}}^{\text{had.}}$ using the approximate formula in Eq. (9), the hadronic energy shifts $\Delta E_{\text{rel.,fns}}^{\text{approx}}$ from Ref. [36] and their respective uncertainties are utilized. For $Z = 92$, the hadronic energy shift, which is not given in Ref. [36], was calculated using the same method.

One can see that the nonrelativistic approximation in Eq. (25) represents a lower bound for the hadronic g -factor shift and is not sufficient for large atomic numbers Z . On the other hand, the analytic expression for the relativistic g -factor shift in the case of a point-like nucleus in Eq. (26) represents an upper bound and differs also significantly from the numerical results for extended nuclei. We conclude that the effects due to a finite-size nucleus need to be included in a precise calculation of the hadronic VP effect. At the present time, the uncertainty stemming from the assumed nuclear charge distribution model limits the accuracy to about 1% [36]. At

the same time, the absence of more precise parametrizations of the hadronic polarization function in the low-energy regime limits the accuracy also to about 1%, see Table I. Thus, the given errors include, to a great part, all possible limitations of the uncertainty of the hadronic g -factor shift.

The simple approximate formula in Eq. (9) is found to be a good approximation for atomic numbers below $Z = 14$. The error is less than 1% for atomic numbers up to $Z = 36$.

As shown in Secs. II C and II D, the hadronic VP contribution [Fig. 1(b)] to the reduced and the weighted g factor in the case of a point-like nucleus is at least $Z\alpha$ times smaller than the regular hadronic g -factor shift, see Eq. (28). In fact, numerical results for extended nuclei confirm that the hadronic contribution to both quantities does not differ significantly from zero for small atomic numbers below $Z = 36$ at the current level of accuracy. To see this, note that the numerical results for the finite-size reduced and weighted g factor can be obtained from Tables I and II via

$$\tilde{g}_{\text{fns}}^{\text{had.}}(1s) = \Delta g_{\text{rel.,fns}}^{\text{had.}}(1s) - \Delta g_{\text{approx.,fns}}^{\text{had.}}(1s), \quad (38)$$

$$\delta_{\Xi_0} g_{\text{fns}}^{\text{had.}} = \Delta g_{\text{rel.,fns}}^{\text{had.}}(2s) - \Xi_0 \Delta g_{\text{rel.,fns}}^{\text{had.}}(1s), \quad (39)$$

respectively. For $Z = 36$, one obtains

$$\tilde{g}_{\text{fns}}^{\text{had.}}(1s) = -32(47) \times 10^{-13}, \quad (40)$$

$$\delta_{\Xi_0} g_{\text{fns}}^{\text{had.}} = -1(64) \times 10^{-14}. \quad (41)$$

TABLE II. Results for the hadronic VP contribution to the g -factor shift of the bound electron in the $2s$ state.

Z	$\Delta g_{\text{rel.,fns}}^{\text{had.}}(2s)$
1	$-1.366(17) \times 10^{-17}$
14	$-5.673(71) \times 10^{-13}$
20	$-2.542(32) \times 10^{-12}$
36	$-3.583(45) \times 10^{-11}$
54	$-2.966(37) \times 10^{-10}$
74	$-2.189(27) \times 10^{-9}$
82	$-4.728(59) \times 10^{-9}$
92	$-1.213(15) \times 10^{-8}$

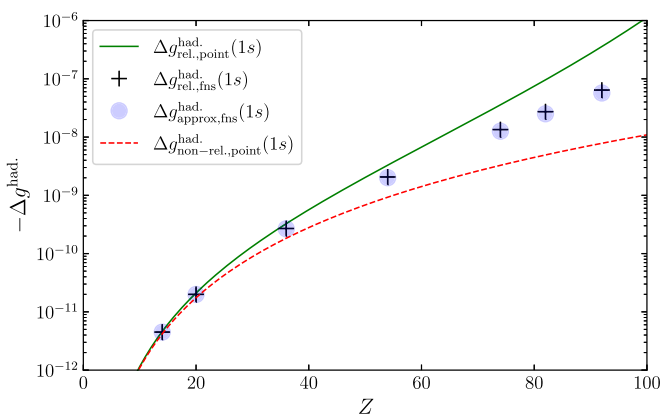


FIG. 3. Comparison of analytical and numerical results for the hadronic g -factor shift of the bound electron in the ground state of H-like ions with atomic numbers Z obtained in this work, see Table I. The green solid line represents the analytical expression for a point-like nucleus $\Delta g_{\text{rel.,point}}^{\text{had.}}$ in Eq. (25), while the red dashed line represents the nonrelativistic expression $\Delta g_{\text{non-rel.,point}}^{\text{had.}}$ in Eq. (24). The full numerical results for extended nuclei $\Delta g_{\text{rel.,fns}}^{\text{had.}}$ (crosses) are compared to the approximation $\Delta g_{\text{approx.,fns}}^{\text{had.}}$ in Eq. (9) (circles) with hadronic energy shifts for extended nuclei taken from Ref. [36].

Even for larger atomic numbers, hadronic effects do not constrain high-precision tests of QED via the measurement of the reduced and weighted g factor.

Recently, a high-precision measurement of the g -factor difference of two Ne isotopes was performed [12]. It was shown that QED effects mostly cancel, whereas nuclear effects like the nuclear recoil are well observable. In the following, we investigate hadronic VP contributions to the bound-electron g factor of the isotopes $^{20}\text{Ne}^{9+}$ and $^{22}\text{Ne}^{9+}$ in the ground state.

First, we calculate the hadronic VP correction to the g factor difference stemming from the different nuclear size of the isotopes. Nuclear recoil effects are excluded for now, and nuclear charge radii are taken from Ref. [45]. Using $R_{\text{rms}} = 3.0055(21)$ fm for $^{20}\text{Ne}^{9+}$ and $R_{\text{rms}} = 2.9525(40)$ fm for $^{22}\text{Ne}^{9+}$, the fully relativistic result for both isotopes is

$$\Delta g_{\text{rel.,fns}}^{\text{had.}}(1s, ^{20}\text{Ne}^{9+}) = -1.133(14) \times 10^{-12}, \quad (42)$$

$$\Delta g_{\text{rel.,fns}}^{\text{had.}}(1s, ^{22}\text{Ne}^{9+}) = -1.133(15) \times 10^{-12}. \quad (43)$$

This is approximately a third of the hadronic contribution of the free electron given in the Extended Table 1 in Ref. [12]. Thus, we conclude that, at the given level of accuracy, hadronic effects of the bound electron also do not hinder the precise calculation of the isotopic shift of $^{20}\text{Ne}^{9+}$ and $^{22}\text{Ne}^{9+}$.

To estimate also the hadronic VP correction stemming from the different nuclear mass of the isotopes including nuclear recoil effects, we use the nonrelativistic formula [35]

$$\Delta g_{\text{recoil}}^{\text{had.}}(1s) = \left(\frac{m_r}{m_e}\right)^2 \Delta g_{\text{non-rel.}}^{\text{had.}}(1s), \quad (44)$$

with $m_r = m_N m_e / (m_N + m_e)$ being the reduced mass for an isotope with nuclear mass m_N . This is a reasonable approximation since the nonrelativistic result for Ne ($Z = 10$), using Eq. (25), is

$$\Delta g_{\text{non-rel.}}^{\text{had.}}(1s, Z = 10) = -1.092(14) \times 10^{-12}. \quad (45)$$

Using atomic masses from Ref. [54], we obtain $m_r(^{20}\text{Ne}^{9+}) = 0.99997m_e$ and $m_r(^{22}\text{Ne}^{9+}) = 0.99998m_e$, such that, to first order,

$$\Delta g_{\text{recoil}}^{\text{had.}}(1s, ^{20}\text{Ne}^{9+}) = -1.092(14) \times 10^{-12}, \quad (46)$$

$$\Delta g_{\text{recoil}}^{\text{had.}}(1s, ^{22}\text{Ne}^{9+}) = -1.092(14) \times 10^{-12}. \quad (47)$$

Thus, also the nuclear recoil effect to the hadronic VP contribution cannot be resolved at the given level of accuracy.

IV. SUMMARY

Hadronic vacuum polarization corrections to the bound-electron g factor were calculated, employing a hadronic polarization function constructed from empirical data on electron-positron annihilation into hadrons. We found that for a broad range of H-like ions, this one-loop effect is considerably larger than hadronic VP for the free electron [see Fig. 1(a)]. Hadronic effects will be observable in future bound-electron g -factor experiments once nuclear charge radii and charge distributions are substantially better known. We also found that the hadronic effect does not pose a limitation on testing QED or physics beyond the standard model, and determining fundamental constants through specific differences of g factors for different ions, or through the reduced g factor. Finally, the analytic hadronic Uehling potential may prove to be practically applicable to further atomic observables, e.g., to transition energies in positronium, or to the hyperfine structure of HCl.

ACKNOWLEDGMENTS

E.D. would like to thank the colleagues at the Max Planck Institute for Nuclear Physics, especially the theory division lead by Christoph H. Keitel, for the hospitality during the work. We thank S. Breidenbach and H. Cakir for insightful conversations and H. Cakir for assistance with numerical computations. Supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) Project-ID 273811115 – SFB 1225.

APPENDIX A: BASE INTEGRAL I_{abc}

The base integral I_{abc} used in Eq. (15) is given by [30]

$$\begin{aligned} I_{abc} &= \int_0^1 dy \frac{(1-y^2)^{a-1/2}}{y^{b-1}} \left(\frac{\zeta Z \alpha y}{1 + \zeta Z \alpha y} \right)^{c-2\epsilon} \\ &= \frac{1}{2} (\zeta Z \alpha)^{c-2\epsilon} B\left(a + \frac{1}{2}, 1 - \frac{b-c}{2} - \epsilon\right) {}_3F_2\left(\frac{c}{2} - \epsilon, \frac{c+1}{2} - \epsilon, 1 - \frac{b-c}{2} - \epsilon; \frac{1}{2}, a + \frac{3-b+c}{2} - \epsilon; (\zeta Z \alpha)^2\right) \\ &\quad - \frac{c-2\epsilon}{2} (\zeta Z \alpha)^{c+1-2\epsilon} B\left(a + \frac{1}{2}, \frac{3-b+c}{2} - \epsilon\right) {}_3F_2 \\ &\quad \times \left(\frac{c}{2} + 1 - \epsilon, \frac{c+1}{2} - \epsilon, \frac{3-b+c}{2} - \epsilon; \frac{3}{2}, a + 2 - \frac{b-c}{2} - \epsilon; (\zeta Z \alpha)^2\right), \end{aligned} \quad (A1)$$

where $\zeta = m_e/m_l$ is the ratio of the electron and the loop particle masses, $\epsilon = 1 - \gamma$ with $\gamma = \sqrt{1 - (Z\alpha)^2}$, $B(x, y)$ is the beta function, and ${}_3F_2(a_1, a_2, a_3; b_1, b_2; z)$ is a generalized hypergeometric function [43].

APPENDIX B: HADRONIC UEHLING POTENTIAL FOR EXTENDED NUCLEI

The analytic hadronic Uehling potential for an extended nucleus with a spherical homogeneous charge distribution with effective radius R is given by [36]

$r > R$:

$$\delta V_{\text{fns,out}}^{\text{had.}}(r) = -\frac{3Z\alpha B_1 \sqrt{C_1}}{rR^3} [\sqrt{C_1} R D_3^+(r, R) - C_1 D_4^-(r, R)]. \quad (\text{B1})$$

$r \leq R$:

$$\delta V_{\text{fns,in}}^{\text{had.}}(r) = -\frac{3Z\alpha B_1 \sqrt{C_1}}{rR^3} \left[\sqrt{C_1} r + \sqrt{C_1} R E_3\left(\frac{r+R}{\sqrt{C_1}}\right) + C_1 E_4\left(\frac{r+R}{\sqrt{C_1}}\right) - \frac{1}{6} e^{\frac{r-R}{\sqrt{C_1}}} (2C_1 + \sqrt{C_1}(r+2R) + (r-R)(r+2R)) - \frac{(r-R)^2(r+2R)}{6\sqrt{C_1}} E_1\left(\frac{R-r}{\sqrt{C_1}}\right) \right]. \quad (\text{B2})$$

The parameters B_1 and C_1 characterize the hadronic polarization function and are given in Sec. II B. The functions $D_n^\pm(r, R)$ and $E_n(x)$ are defined in Eqs. (22) and (20), respectively.

-
- [1] S. Sturm, A. Wagner, B. Schabinger, J. Zatorski, Z. Harman, W. Quint, G. Werth, C. H. Keitel, and K. Blaum, *Phys. Rev. Lett.* **107**, 023002 (2011).
- [2] S. Sturm, A. Wagner, M. Kretzschmar, W. Quint, G. Werth, and K. Blaum, *Phys. Rev. A* **87**, 030501(R) (2013).
- [3] K. Pachucki, U. D. Jentschura, and V. A. Yerokhin, *Phys. Rev. Lett.* **93**, 150401 (2004).
- [4] K. Pachucki, A. Czarnecki, U. D. Jentschura, and V. A. Yerokhin, *Phys. Rev. A* **72**, 022108 (2005).
- [5] S. G. Karshenboim and A. I. Milstein, *Phys. Lett. B* **549**, 321 (2002).
- [6] R. N. Lee, A. I. Milstein, I. S. Terekhov, and S. G. Karshenboim, *Phys. Rev. A* **71**, 052501 (2005).
- [7] V. A. Yerokhin, P. Indelicato, and V. M. Shabaev, *Phys. Rev. Lett.* **89**, 143001 (2002).
- [8] V. A. Yerokhin, P. Indelicato, and V. M. Shabaev, *Phys. Rev. A* **69**, 052503 (2004).
- [9] V. M. Shabaev and V. A. Yerokhin, *Phys. Rev. Lett.* **88**, 091801 (2002).
- [10] T. Beier, *Phys. Rep.* **339**, 79 (2000).
- [11] T. Beier, I. Lindgren, H. Persson, S. Salomonson, P. Sunnergren, H. Häffner, and N. Hermanspahn, *Phys. Rev. A* **62**, 032510 (2000).
- [12] T. Sailer, V. Debierre, Z. Harman, F. Heiße, C. König, J. Morgner, B. Tu, A. V. Volotka, C. H. Keitel, K. Blaum *et al.*, *Nature (London)* **606**, 479 (2022).
- [13] A. Schneider, B. Sikora, S. Dickopf, M. Müller, N. S. Oreshkina, A. Rischka, I. A. Valuev, S. Ulmer, J. Walz, Z. Harman *et al.*, *Nature (London)* **606**, 878 (2022).
- [14] S. Sturm, F. Köhler, J. Zatorski, A. Wagner, Z. Harman, G. Werth, W. Quint, C. H. Keitel, and K. Blaum, *Nature (London)* **506**, 467 (2014).
- [15] F. Köhler, S. Sturm, A. Kracke, G. Werth, W. Quint, and K. Blaum, *J. Phys. B: At. Mol. Opt. Phys.* **48**, 144032 (2015).
- [16] J. Zatorski, B. Sikora, S. G. Karshenboim, S. Sturm, F. Köhler-Langes, K. Blaum, C. H. Keitel, and Z. Harman, *Phys. Rev. A* **96**, 012502 (2017).
- [17] H. Häffner, T. Beier, N. Hermanspahn, H.-J. Kluge, W. Quint, S. Stahl, J. Verdú, and G. Werth, *Phys. Rev. Lett.* **85**, 5308 (2000).
- [18] T. Beier, H. Häffner, N. Hermanspahn, S. G. Karshenboim, H.-J. Kluge, W. Quint, S. Stahl, J. Verdú, and G. Werth, *Phys. Rev. Lett.* **88**, 011603 (2001).
- [19] V. Debierre, C. H. Keitel, and Z. Harman, *Phys. Lett. B* **807**, 135527 (2020).
- [20] V. Debierre, C. H. Keitel, and Z. Harman, *arXiv:2202.01668*.
- [21] V. A. Yerokhin, E. Berseneva, Z. Harman, I. I. Tupitsyn, and C. H. Keitel, *Phys. Rev. Lett.* **116**, 100801 (2016).
- [22] V. M. Shabaev, D. A. Glazov, N. S. Oreshkina, A. V. Volotka, G. Plunien, H.-J. Kluge, and W. Quint, *Phys. Rev. Lett.* **96**, 253002 (2006).
- [23] H. Cakir, N. S. Oreshkina, I. A. Valuev, V. Debierre, V. A. Yerokhin, C. H. Keitel, and Z. Harman, *arXiv:2006.14261*.
- [24] A. Czarnecki, M. Dowling, J. Piclum, and R. Szafron, *Phys. Rev. Lett.* **120**, 043203 (2018).
- [25] K. Pachucki and M. Puchalski, *Phys. Rev. A* **96**, 032503 (2017).
- [26] V. A. Yerokhin and Z. Harman, *Phys. Rev. A* **88**, 042502 (2013).
- [27] B. Sikora, V. A. Yerokhin, N. S. Oreshkina, H. Cakir, C. H. Keitel, and Z. Harman, *Phys. Rev. Res.* **2**, 012002(R) (2020).
- [28] S. Sturm, I. Arapoglou, A. Egl, M. Höcker, S. Kraemer, T. Sailer, B. Tu, A. Weigel, R. Wolf, J. Crespo López-Urrutia *et al.*, *Eur. Phys. J. Spec. Top.* **227**, 1425 (2019).
- [29] H.-J. Kluge, T. Beier, K. Blaum, L. Dahl, S. Eliseev, F. Herfurth, B. Hofmann, O. Kester, S. Koszudowski, C. Kozhuharov *et al.*, *Adv. Quantum Chem.* **53**, 83 (2008).
- [30] S. G. Karshenboim, V. G. Ivanov, and V. M. Shabaev, *J. Exp. Theor. Phys.* **93**, 477 (2001).
- [31] S. G. Karshenboim, R. N. Lee, and A. I. Milstein, *Phys. Rev. A* **72**, 042101 (2005).
- [32] V. B. Berestetskii, L. P. Pitaevskii, and E. M. Lifshitz, *Quantum Electrodynamics*, 2nd ed. (Pergamon, New York, 1982).
- [33] H. Burkhardt, F. Jegerlehner, G. Penso, and C. Verzegnassi, *Z. Phys. C* **43**, 497 (1989).
- [34] H. Burkhardt and B. Pietrzyk, *Phys. Lett. B* **513**, 46 (2001).
- [35] S. G. Karshenboim and V. A. Shelyuto, *Eur. Phys. J. D* **75**, 49 (2021).
- [36] S. Breidenbach, E. Dizer, H. Cakir, and Z. Harman, *Phys. Rev. A* **106**, 042805 (2022).
- [37] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Westview, Boulder, CO, 1995).

- [38] W. R. Johnson, S. A. Blundell, and J. Sapirstein, *Phys. Rev. A* **37**, 307 (1988).
- [39] E. Dizer, Analytic Evaluation of Quantum Electrodynamical Corrections in One-Electron Ions, <http://hdl.handle.net/21.11116/0000-0006-B165-0>.
- [40] H. Cakir, V. A. Yerokhin, N. S. Oreshkina, B. Sikora, I. I. Tupitsyn, C. H. Keitel, and Z. Harman, *Phys. Rev. A* **101**, 062513 (2020).
- [41] L. W. Fullerton and G. A. Rinker, *Phys. Rev. A* **13**, 1283 (1976).
- [42] A. Czarnecki and R. Szafron, *Phys. Rev. A* **94**, 060501(R) (2016).
- [43] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (National Bureau of Standards, Washington, D.C., 1972).
- [44] H. Burkhardt and B. Pietrzyk, *Phys. Lett. B* **356**, 398 (1995).
- [45] I. Angeli and K. Marinova, *At. Data Nucl. Data Tables* **99**, 69 (2013).
- [46] J. L. Friar, J. Martorell, and D. W. L. Sprung, *Phys. Rev. A* **59**, 4061 (1999).
- [47] H. Terazawa, *Prog. Theor. Phys.* **39**, 1326 (1968).
- [48] V. A. Yerokhin, E. Berseneva, Z. Harman, I. I. Tupitsyn, and C. H. Keitel, *Phys. Rev. A* **94**, 022502 (2016).
- [49] V. M. Shabaev, D. A. Glazov, M. B. Shabaeva, V. A. Yerokhin, G. Plunien, and G. Soff, *Phys. Rev. A* **65**, 062104 (2002).
- [50] A. S. M. Patoary and N. S. Oreshkina, *Eur. Phys. J. D* **72**, 54 (2018).
- [51] D. Nomura and T. Teubner, *Nucl. Phys. B* **867**, 236 (2013).
- [52] T. Aoyama, N. Asmussen, M. Benayoun, J. Bijnens, T. Blum, M. Bruno, I. Caprini, C. Carloni Calame, M. Cè, G. Colangelo *et al.*, *Phys. Rep.* **887**, 1 (2020).
- [53] M. Davier, Z. Fodor, A. Gerardin, L. Lellouch, B. Malaescu, F. M. Stokes, K. K. Szabó, B. C. Tóth, L. Varnhorst, and Z. Zhang, [arXiv:2308.04221](https://arxiv.org/abs/2308.04221).
- [54] M. Wang, G. Audi, A. H. Wapstra, F. G. Kondev, M. MacCormick, X. Xu, and B. Pfeiffer, *Chin. Phys. C* **36**, 1603 (2012).