Quantum illumination strategy for parameter estimation

L. Davidovich [1,2,3] and R. L. de Matos Filho [3]
[1] *Institute for Quantum Science and Engineering, Texas A&M University, College Station, Texas 77843, USA*
[2] *Department of Physics and Astronomy, Texas A&M University, College Station, Texas 77843, USA*
[3] *Instituto de Física, Universidade Federal do Rio de Janeiro, Rio de Janeiro, RJ 21941-972, Brazil*

(Received 23 March 2023; accepted 29 September 2023; published 19 October 2023)

Establishing the limits of precision in the estimation of parameters for noisy quantum channels probed by qubits is important for many areas of quantum information, such as quantum sensing, computation, and communication. Here we consider the estimation of parameters characterizing a general class of noisy Pauli channels. We show that two entangled qubits, such that only one of them probes the channel, may lead, under an entangling measurement, to strong enhancement of the precision in the estimation, as compared to the precision corresponding to sending the pair, entangled or not, through the channel. We prove that entanglement plays an essential role, as does the entangling detection procedure, consisting in projecting the final state onto a Bell-state basis. We also prove that quantum advantage is obtained only when the output state, after interaction with the sample, is not entangled anymore. This behavior has striking similarities with quantum illumination, where initial entanglement of probe and ancilla beams, followed by an entangling measurement, lead to enhancement of the sensitivity of photodetection, even after the output beams are disentangled. Similarities and differences with ghost imaging are also discussed.

DOI: 10.1103/PhysRevA.108.042612

I. INTRODUCTION

Quantum illumination [1–6] uses entangled light to illuminate objects in the presence of high levels of ambient noise and loss. Each signal sent out is entangled with a retained ancilla, and detection is made via an entangling measurement of the returning signal together with the ancilla. Interestingly, enhancing of the sensitivity of photodetection is achieved even when the two beams are not entangled anymore. Here we show that quantum sensing of parameters of open systems may have remarkable similarities with quantum illumination. This is illustrated through the estimation of parameters characterizing a broad class of quantum channels acting on qubits: the Pauli channels.

We show that, for Pauli channels probed by qubits, the use of a single probe maximally entangled with an ancilla may lead to strong enhancement of the precision in the estimation of a parameter that describes the probability of errors on a qubit, as compared to the use of two independent probes or even of two probes in a maximally entangled state, when each one probes the channel independently. With the further benefit that, for the same number of resources, only half of them go through the sample, which could be advantageous for fragile samples.

Quantum advantage is obtained only when the final state, after interaction with the sample, is not entangled anymore, corresponding to a high-noise limit and the channel becoming entanglement breaking. There is a similar but not identical effect in quantum illumination [1], where the enhancement of signal-to-noise ratio persists when no entanglement survives at the detector, but is also present for entangled final states, while here loss of entanglement is essential for quantum advantage.

Interestingly, each of the probes of the entangled pair has null information on the parameter, which is fully retrieved only by implementing an entangling measurement, consisting in projecting the final state on the Bell-state basis. This is similar to ghost imaging, which uses two correlated light beams to get an image of an object, in such a way that only one of them interacts with the object to be imaged, reaching a single-pixel detector, while the other one reaches the imaging detector [7–14]. Neither of the beams has complete information on the image, which is retrieved from the classical correlation between the two beams. In that case, even though the initial demonstration used an entangled initial state [7–9], many of the properties of a ghost imaging system can be demonstrated using classical correlations [10–14]. Here, however, entanglement plays an essential role in increasing the precision of estimation over the one corresponding to sending the two probes, entangled or not, through the sample. And the entangling measurement is essential for getting the enhancement in the precision.

Parameter estimation is closely related to the quantum channel identification problem [15–20]. There is, however, an important difference between the two tasks. While for parameter estimation the total amount of resources should be considered, including probe and ancilla, in channel identification only the number of uses of the channel is counted, that is, the ancillas are not counted when one compares the role of a single probe with the one corresponding to an entangled probe-ancilla input. The consequences of these different strategies are actually very interesting, and are discussed in this paper.

II. PARAMETER ESTIMATION AND THE CRAMÉR-RAO BOUND

The estimation of parameters characteristic of a physical process usually presupposes the indirect procedure of extracting information on these parameters from a probe that has undergone the process. The uncertainty $\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2}$ in the estimation of a parameter X satisfies the quantum version of the Cramér-Rao relation [21–25],

$$\Delta X \geqslant 1/\sqrt{\mathcal{N}\mathcal{F}_Q(X)}, \tag{1}$$

where $\mathcal{N}$ is the number of independent measurements, $\mathcal{F}_Q(X)$ is the quantum Fisher information (QFI), and it is assumed that the measurement is unbiased, that is, the averaged value of the measurements of X is equal to the true value $X_{\rm true}$ of the parameter for a neighborhood of $X_{\rm true}$, so that $d\langle X\rangle/dX|_{X=X_{\rm true}} = 1$. The quantum Fisher information is defined as the maximization, over all possible measurements, of the classical Fisher information $F(X) = \sum_j [dP_j(X)/dX]^2/P_j(X)$, where $P_j(X)$ is the probability of getting an experimental result j if the value of the parameter is X. The inequality (1) is saturated when $\mathcal{N} \gg 1$ or for Gaussian probability distributions. Therefore, the crux for getting a lower bound for the uncertainty is the evaluation of the quantum Fisher information.

A general expression for the QFI corresponding to measurements on a probe described by a parameter-dependent density operator $\hat\rho(X)$ is [23]

$$\mathcal{F}_Q(X) = {\rm tr}[\hat\rho(X)\hat L^2(X)], \tag{2}$$

where the operator $\hat L$, the symmetric logarithmic derivative, is defined by

$$\frac{d\hat\rho(X)}{dX} = \frac{1}{2}[\hat\rho(X)\hat L(X) + \hat L(X)\hat\rho(X)]. \tag{3}$$

This implies that the matrix elements of $\hat L$ in the basis of eigenstates of $\hat\rho(X)$ are

$$\hat L_{ij} = \frac{2}{p_i + p_j}\left[\frac{\partial\hat\rho(X)}{\partial X}\right]_{ij}, \tag{4}$$

where $p_{i,j}$ are eigenvalues of $\hat\rho(X)$.

The eigenstates of $\hat L_{ij}$ constitute an optimal measurement basis leading to the quantum Fisher information, which can therefore be obtained from the diagonalization of the density matrix. For unitary evolution of the probe, this yields simple analytical expressions. For open systems, analytic solutions can be obtained for low-dimensional systems or for Gaussian processes. Otherwise, procedures that lead to upper bounds for the QFI but not necessarily to exact results have been proposed [26–30]. For the class of noisy channels considered here, exact solutions can be easily found.

III. CHANNEL DISTINGUISHABILITY, PAULI CHANNELS, AND ANCILLAE

A quantum channel Λ is a completely positive trace-preserving map acting on operators in a Hilbert space $\mathcal{H}_1$. This implies that, when applied to a density operator, it yields another density operator, and also that $\Lambda \otimes \mathbf{1}$ is positive when acting on all possible extensions $\mathcal{H}_1 \otimes \mathcal{H}_2$ of $\mathcal{H}_1$.

A measure of the distinguishability between two channels, occurring with probabilities p_1, p_2, is the minimal error probability p'_E, given by [17]

$$p'_E = \tfrac{1}{2}[1 - \max_{\hat\rho\in\mathcal{H}} ||p_1\Lambda_1(\hat\rho) - p_2\Lambda_2(\hat\rho)||], \tag{5}$$

where $||\dots||$ denotes the trace norm.

As shown in Ref. [17], use of ancillae that do not interact with the channels may enhance the distinguishability between them. That is, by allowing entangled input states, and defining

$$p_E = \tfrac{1}{2}[1 - \max_{\hat\rho\in\mathcal{H}} ||(p_1\Lambda_1\otimes\mathbf{I})\hat\rho - p_2(\Lambda_2\otimes\mathbf{I})\hat\rho||], \tag{6}$$

then one may have $p_E < p'_E$.

Parameter estimation is intimately related to the distinguishability of two channels, given with the same probability, and which differ by an infinitesimal variation of one or more parameters. For two Pauli channels, given with the same probabilities, and defined as

$$\Lambda_i[\hat\rho] = \sum_{\alpha=0}^{3} q_i^{(\alpha)}\hat\sigma_\alpha\hat\rho\hat\sigma_\alpha,\ \ (i = 1, 2), \tag{7}$$

where $\sum_\alpha q_i^{(\alpha)} = 1$, $\hat\sigma_0 = \mathbf{1}$, and $\hat\sigma_i$, $i = 1, 2, 3$ are the Pauli matrices, the use of ancillae will increase the distinguishability between the two channels if and only if [17]

$$\prod_\alpha r_\alpha < 0,\ \ r_\alpha = \frac{1}{2}\big(q_1^{(\alpha)} - q_2^{(\alpha)}\big). \tag{8}$$

We are interested here in the estimation of the parameter Θ characterizing the class of Pauli channels Λ_Θ, acting on a two-dimensional Hilbert space, defined by

$$\Lambda_\Theta[\hat\rho] = (1-\Theta)\hat\rho + \Theta\sum_{\alpha=1}^{3} q_\alpha\hat\sigma_\alpha\hat\rho\hat\sigma_\alpha, \tag{9}$$

with $\sum_\alpha q_\alpha = 1$. The three Pauli operators stand for the following transformations on a probe interacting with the channel, which reflect possible errors in the transmission: spin-flip ($\hat\sigma_1$), phase flip ($\hat\sigma_3$), and spin and phase flip ($\hat\sigma_2$).

The parameter Θ determines the probability of occurrence or not of an error: $1-\Theta$ is the probability that no error occurs, while Θq_α is the probability that the error associated with $\hat\sigma_\alpha$ occurs. When $q_1 = q_2 = q_3 = 1/3$ this channel reduces to the isotropic depolarizing channel.

The precision of estimation of Θ is related to the distinguishability of Λ_Θ and $\Lambda_{\Theta+d\Theta}$. From (8), it follows that use of ancillae for parameter estimation—when they are counted as resources—may be advantageous only if, in (9), $q_\alpha > 0$. This will be assumed from now on.

If one applies the result in Ref. [17] to the estimation of the channel parameter Θ, it is straightforward to see that the use of ancillae, in conjunction with a probe system sent trough the channel, increases the quantum Fisher information about the parameter encoded in the joint output state of probe plus anciillae. In fact, the results obtained in Ref. [31] imply that, when sending a single probe trough the channels described by (9), with allowance for ancillae, corresponding to a single use of channel, the maximum quantum Fisher information about the parameter Θ is obtained with the use of a single two-level ancilla maximally entangled with the probe. In order to evaluate the enhancement of the precision due to the use of ancillae, we discuss in Sec. III A the sensing of Pauli channels with a single probe, without ancillae.

A. Sensing Pauli channels with single probes

We consider now the quantum Fisher information of a single probe, and discuss the corresponding initial pure state that produces the largest quantum Fisher information on Θ, after being sent through the channel, for different values of the q_α's. Aiming at simplifying this analysis, we introduce some symmetry into the channels represented by (9). We consider the situation where two of the coefficients q_α are equal. For example, $q_1 = q_2 = q$ and $q_3 = 1 - 2q$.

If an arbitrary pure state of a single probe, given by

$$\hat{\rho}_{\text{in}} = \frac{1}{2}\left(\mathbb{1} + \sum_{\alpha=1}^{3} < \hat{\sigma}_\alpha > \hat{\sigma}_\alpha\right), \tag{10}$$

where $\hat{\sigma}_\alpha, \alpha = 1, 2, 3$ are the Pauli operators, is sent through the channel, the output state is

$$\hat{\rho}_{\text{out}}(\Theta) = \frac{1}{2}\left(\mathbb{1} + \sum_{\alpha=1}^{3}[1 + 2(q_\alpha - 1)\Theta] < \hat{\sigma}_\alpha > \hat{\sigma}_\alpha\right). \tag{11}$$

In the Bloch sphere, the point corresponding to the state $\hat{\rho}_{\text{out}}(\Theta)$ may be represented by the vector $\mathbf{V}(\Theta)$, with components

$$V_\alpha(\Theta) = [1 + 2(q_\alpha - 1)\Theta]\langle\hat{\sigma}_\alpha\rangle, \tag{12}$$

where $\alpha = 1, 2, 3$. The output state with the largest QFI moves with the largest speed in the Bloch sphere when the parameter Θ changes infinitesimally. This state can be found via maximizing the absolute value of the vector $\dot{\mathbf{V}}(\Theta) \equiv \frac{d}{d\Theta}[\mathbf{V}(\Theta)]$, which has components $\dot{V}_\alpha = 2(q_\alpha - 1)\langle\hat{\sigma}_\alpha\rangle$. The squared modulus of $\dot{\mathbf{V}}(\Theta)$ is

$$|\dot{\mathbf{V}}(\Theta)|^2 = \sum_{\alpha=1}^{3} 4(q_\alpha - 1)^2\langle\hat{\sigma}_\alpha\rangle^2. \tag{13}$$

For definiteness, we consider here the channels for which $q_1 = q_2 = q$ and $q_3 = 1 - 2q$. Furthermore, since $\hat{\rho}_{\text{in}}$ is a pure state, one must have $\langle\hat{\sigma}_1\rangle^2 + \langle\hat{\sigma}_2\rangle^2 + \langle\hat{\sigma}_3\rangle^2 = 1$. Under these conditions, one gets

$$\begin{aligned} |\dot{\mathbf{V}}(\Theta)|^2 &= 4(q-1)^2(\langle\hat{\sigma}_1\rangle^2 + \langle\hat{\sigma}_2\rangle^2) \\ &\quad + 16q^2[1 - (\langle\hat{\sigma}_1\rangle^2 + \langle\hat{\sigma}_2\rangle^2)] \\ &= 4(1 - 2q - 3q^2)(\langle\hat{\sigma}_1\rangle^2 + \langle\hat{\sigma}_2\rangle^2) + 16q^2. \end{aligned} \tag{14}$$

Notice that the allowed values of q are $q \in [0, 0.5]$. As q changes from 0–0.5, the locuses of optimal pure states of the probe on the surface of the Bloch sphere change in an interesting way: they lie initially on the equator of the Bloch sphere, then at precisely $q = 1/3$, corresponding to the depolarizing channel, all states on the surface of the Bloch sphere become optimal, and finally, for $q > 1/3$, the optimal states lie on the poles of the Bloch sphere. Indeed, it is easy to see that the root of $1 - 2q - 3q^2$ that lies inside the interval $\in [0, 0.5]$ is $q = 1/3$. Consequently, it turns out, from Eq. (14), that, for this value of q, $|\dot{\mathbf{V}}(\Theta)| = 16q^2$ and does not depend on the initial state $\hat{\rho}_{\text{in}}$, so that, in this situation, all pure states of the probe produce the same QFI. For $q \in [0, 1/3)$, $1 - 2q - 3q^2 > 0$ and, in order to maximize $|\dot{\mathbf{V}}(\Theta)|$, one has to maximize the value of $\langle\hat{\sigma}_1\rangle^2 + \langle\hat{\sigma}_2\rangle^2$. This means that the pure input states $\hat{\rho}_{\text{in}}$ that produce the largest QFI are those which have $\langle\hat{\sigma}_1\rangle^2 + \langle\hat{\sigma}_2\rangle^2 = 1$. Consequently, for these values of q, all the pure states on the equator of the Bloch sphere produce the same and largest QFI. For $q \in (1/3, 0.5]$, $1 - 2q - 3q^2 < 0$. Now, the states that maximize the QFI must have $\langle\hat{\sigma}_1\rangle^2 + \langle\hat{\sigma}_2\rangle^2 = 0$ and lie on the poles of the Bloch sphere.

The quantum Fisher information for each value of q and the corresponding best initial state of the probe can be found with the help of (4) and (9). It reads

$$\mathcal{F}_Q^{(\text{p})}(\Theta) = \frac{1-q}{\Theta[1 - (1-q)\Theta]}, \tag{15}$$

for $q \leqslant 1/3$. When $q > 1/3$, one has to replace $q \to 1 - 2q$ in the above expression. This information can be retrieved via a measurement of the output state of the probe in the basis consisting of the initial state and the state orthogonal to it.

B. Ancilla-assisted sensing of Pauli channels

Here we show a further advantage of the ancilla strategy: for a range of values of Θ and q_α, the use of a single probe maximally entangled with an ancilla may lead to better estimation of the parameter Θ than the use of two independent probes or even of two probes in a maximally entangled state, when each one probes the channel independently. As Θ increases from 0, first the two entangled probes produce a larger QFI than the two independent probes and the probe entangled with an ancilla. This happens until Θ reaches a value where the output state of the two initially entangled probes, after testing the channel, has no entanglement at all. From this point, as Θ increases further, the two independent probes produce the largest QFI. This changes when Θ reaches a value where the joint output state of probe plus ancilla becomes completely disentangled. From that point, up to the value $\Theta = 1$, the probe plus ancilla produce the largest QFI. Remarkably, in order to fully recover the QFI of their joint output state, probe plus ancilla have to be measured in a maximally entangled basis, although no entanglement remains in that state. Indeed, we prove that the largest information one can recover about the parameter, via joint measurements of local observables of probe and ancilla, is equal to the QFI of a single probe, so the ancilla is not useful in this case.

The quantum Fisher information $\mathcal{F}_Q^{(\text{p+a})}(\Theta)$ of the output state of probe plus ancilla, initially in a maximally entangled state, is obtained from the output state of the channel extension $\mathbb{1} \otimes \Lambda_\Theta$. For instance, for an initial singlet Bell state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$, where $\{|+\rangle, |-\rangle\}$ are eigenstates of the Pauli operator $\hat{\sigma}_3$, the density operator for the outgoing state is

$$\begin{aligned} \hat{\rho}_{\text{out}}^{(\text{p+a})}(\Theta) &= \mathbb{1} \otimes \Lambda_\Theta[|\Psi^-\rangle\langle\Psi^-|] = (1-\Theta)|\Psi^-\rangle\langle\Psi^-| \\ &\quad + \Theta[q_1|\Phi^-\rangle\langle\Phi^-| + q_2|\Phi^+\rangle\langle\Phi^+| \\ &\quad + q_3|\Psi^+\rangle\langle\Psi^+|], \end{aligned} \tag{16}$$

where $\{|\Psi^+\rangle, |\Psi^-\rangle, |\Phi^+\rangle, |\Phi^-\rangle\}$ are the four Bell states. Note that $\hat{\rho}_{\text{out}}^{(\text{p+a})}(\Theta)$ is diagonal in the Bell basis. Since these eigenstates do not depend on Θ, it follows from (4) that the

Bell basis is an optimal measurement basis and, therefore, the Fisher information connected to the measurement in this basis equals the quantum Fisher information of the output state. The corresponding quantum Fisher information can be easily calculated from (4) and (16), giving

$$\mathcal{F}_Q^{(\mathrm{p+a})}(\Theta) = \frac{1}{\Theta(1-\Theta)}, \tag{17}$$

where we used the relation $\sum_\alpha q_\alpha = 1$. Note that the above quantity is independent of the values of q_α. This means that probe plus ancilla in a maximally entangled initial state produce the same QFI on Θ for the whole class of channels given by (9). The quantum Fisher information in (17) is always larger than the QFI of a single probe given in (15), when $q \in (0, 0.5)$. For $q = 0$ and $q = 0.5$, the two QFI's coincide, consistently with the condition (8) for the usefulness of the ancilla-assisted strategy.

One may wonder how important is the initial entanglement of probe plus ancilla, and also the final entangling measurement, in achieving the superiority of the ancilla-based protocol. This is discussed in the following.

1. Ancilla is not useful for separable initial states

It is important to note that if the initial state is separable, the ancilla plays no role in the estimation of Θ. Indeed, for an initial state $\hat\rho_0 = \sum_k p_k \hat\rho_a^k \otimes \hat\rho_p^k$, with $\sum_k p_k = 1$ and $p_k \geqslant 0$, where $\hat\rho_a^k$ and $\hat\rho_p^k$ are density operators for the ancilla and probe, respectively, one has $\hat\rho_\Theta = \mathbb{1} \otimes \Lambda_\Theta(\hat\rho_0) = \sum_k p_k \hat\rho_a^k \otimes \Lambda_\Theta(\hat\rho_p^k)$. Therefore, using the convexity and additive properties of the quantum Fisher information,

$$\begin{aligned}\mathcal{F}_Q^{(\mathrm{a+p})}(\hat\rho_\Theta) &\leqslant \sum_k p_k\, \mathcal{F}_Q\big[\hat\rho_a^k \otimes \Lambda_\Theta\big(\hat\rho_p^k\big)\big]\\ &= \sum_k p_k\, \mathcal{F}_Q\big[\Lambda_\Theta\big(\hat\rho_p^k\big)\big],\end{aligned} \tag{18}$$

where the equality follows from the fact that $\hat\rho_a^k$ does not depend on Θ and $\mathcal{F}_Q[\hat\rho_a^k \otimes \Lambda_\Theta(\hat\rho_p^k)] = \mathcal{F}_Q[\hat\rho_a^k] + \mathcal{F}_Q[\Lambda_\Theta(\hat\rho_p^k)] = \mathcal{F}_Q[\Lambda_\Theta(\hat\rho_p^k)]$.

Consequently, for separable initial states of probe plus ancilla, the quantum Fisher information reduces to the average of QFIs corresponding to single probes testing the channel, and cannot be higher than the QFI for a single probe in an optimal (pure) state. Therefore, classical correlations in the initial state of probe plus ancilla play no role in any possible advantage of the use of ancillae for the estimation of Θ.

2. Ancilla is not useful when joint local measurements are performed

Entangling measurement is essential for the superiority of the ancilla-assisted protocol. Indeed, we show now that the maximization over all local observables of the Fisher information produced by measurement of local observables of ancilla plus probe in the output state $\hat\rho_{\mathrm{out}}^{(\mathrm{p+a})}(\Theta)$, given in Eq. (16), leads to the QFI of an optimal input state of a single probe testing the channel.

Any pure state in a 2×2 Hilbert space is eigenstate of an observable $\hat\sigma_n = (\hat\sigma_1\hat{\mathbf{i}} + \hat\sigma_2\hat{\mathbf{j}} + \hat\sigma_3\hat{\mathbf{k}})\cdot\hat{\mathbf{n}}$, where $\hat\sigma_i$, $i = 1, 2, 3$, are Pauli operators, $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ is a basis in tridimensional Euclidean space, and $\hat{\mathbf{n}} = \sin(\theta)\cos(\varphi)\hat{\mathbf{i}} + \sin(\theta)\sin(\varphi)\hat{\mathbf{j}} + \cos(\theta)\hat{\mathbf{k}}$ is an unitary vector pointing in the direction determined by the polar angle θ and the azimuthal angle φ.

Consequently, the measurement of any two local observables of the system ancilla plus probe may be represented by measurement in a product basis $\{|\hat\sigma_{n,+}^{(a)}, \hat\sigma_{n',+}^{(p)}\rangle, |\hat\sigma_{n,+}^{(a)}, \hat\sigma_{n',-}^{(p)}\rangle, |\hat\sigma_{n,-}^{(a)}, \hat\sigma_{n',+}^{(p)}\rangle, |\hat\sigma_{n,-}^{(a)}, \hat\sigma_{n',-}^{(p)}\rangle\}$. Here the states $|\hat\sigma_{n,\pm}^{(a)}\rangle$ are the two eigenstates of the operator $\hat\sigma_n^{(a)}$, acting on the Hilbert space of the ancilla, with $\hat{\mathbf{n}}$ pointing in the direction given by the angles (θ, φ), and $|\hat\sigma_{n',\pm}^{(p)}\rangle$ are the two eigenstates of the operator $\hat\sigma_{n'}^{(p)}$, acting on the Hilbert space of the probe, with $\hat{\mathbf{n}}'$ pointing in the direction given by the angles (θ', φ'). These eigenstates are given by

$$\big|\hat\sigma_{n,+}^{(a,p)}\big\rangle = \frac{1}{\sqrt2}\big[\cos(\theta/2)\big|\hat\sigma_{3,+}^{(a,p)}\big\rangle + e^{i\varphi}\sin(\theta/2)\big|\hat\sigma_{3,-}^{(a,p)}\big\rangle\big],$$

$$\big|\hat\sigma_{n,-}^{(a,p)}\big\rangle = \frac{1}{\sqrt2}\big[\sin(\theta/2)\big|\hat\sigma_{3,+}^{(a,p)}\big\rangle - e^{i\varphi}\cos(\theta/2)\big|\hat\sigma_{3,-}^{(a,p)}\big\rangle\big].$$

The probabilities P_{++}, P_{+-}, P_{-+}, and P_{--} of the four possible measurement results can be obtained from the output state $\hat\rho_{\mathrm{out}}^{(\mathrm{p+a})}(\Theta)$ via the relations

$$P_{ij} = \mathrm{tr}\big[\big|\hat\sigma_{\mathrm{n,i}}^{(\mathrm{a})}, \hat\sigma_{\mathrm{n',j}}^{(\mathrm{p})}\big\rangle\big\langle\hat\sigma_{\mathrm{n,i}}^{(\mathrm{a})}, \hat\sigma_{\mathrm{n',j}}^{(\mathrm{p})}\big|\hat\rho_{\mathrm{out}}^{(\mathrm{p+a})}(\Theta)\big]. \tag{19}$$

For simplicity, we put $q_1 = q_2 = q_3 = 1/3$. For this situation, using the above relation and Eq. (16), it is straightforward to show that

$$\begin{aligned}P_{++}(\Theta) = P_{--}(\Theta) &= \frac14(1-A) + \frac{A}{3}\Theta\\ P_{+-}(\Theta) = P_{-+}(\Theta) &= \frac14(1+A) - \frac{A}{3}\Theta,\end{aligned} \tag{20}$$

with $A = \cos(\theta)\cos(\theta') + \cos(\varphi - \varphi')\sin(\theta)\sin(\theta')$. Notice that $|A| \leqslant 1$. Using these quantities and the expression for the resulting Fisher information

$$F(\Theta) = \sum_{ij} \frac{1}{P_{ij}}\left(\frac{d}{dP_{ij}}\Theta\right)^2, \tag{21}$$

one obtains

$$F(\Theta) = \frac{16A^2/9}{[(1-A) + 4A\Theta/3][(1+A) - 4A\Theta/3]}. \tag{22}$$

The maximum value of $F(\Theta)$ is reached when $|A| = 1$ and is given by

$$F_{\max}(\Theta) = \frac23\frac{1}{\Theta(1-2\Theta/3)}. \tag{23}$$

This is precisely the QFI for a single probe testing the channel in the optimal input state, given in Eq. (15), when $q = 1/3$. Therefore, the measurement in an entangled basis is essential for the usefulness of the ancilla-based strategy.

C. Two entangled probes sensing Pauli channels independently

The quantum Fisher information $\mathcal{F}_Q^{(\mathrm{p+p})}(\Theta)$ of the output state of two probes initially in a maximally entangled state $|\Psi^-\rangle$, with each probe testing the channel independently, is obtained from the output state of the channel extension

$\Lambda_\Theta \otimes \Lambda_\Theta$, corresponding to two independent uses of the channel:

$$\hat{\rho}_{\text{out}}^{(p+p)} = \Lambda_\Theta \otimes \Lambda_\Theta[|\Psi^-\rangle\langle\Psi^-|] = [(1-\Theta)^2 + (1-4q+6q^2)\Theta^2]|\Psi^-\rangle\langle\Psi^-| + 2q\Theta(1-2q\Theta)|\Phi^-\rangle\langle\Phi^-| + 2q\Theta(1-2q\Theta)|\Phi^+\rangle\langle\Phi^+| + 2[\Theta(1-\Theta)(1-2q) + q^2\Theta^2]|\Psi^+\rangle\langle\Psi^+|. \tag{24}$$

Note that this state is also diagonal in the Bell basis. Therefore a measurement of the output state in that basis retrieves the QFI contained in that state, which is given by

$$\mathcal{F}_Q^{(\text{p+p})}(\Theta) = \frac{4[(6q^2-4q+2)\Theta-1]^2}{1+2\Theta[(3q^2-2q+1)\Theta-1]} + \frac{4q^2(1-4q\Theta)^2}{q\Theta(1-2q\Theta)} + \frac{2[2\Theta(q^2+2q-1)+1-2q]^2}{\Theta(1-2q)+(q^2+2q-1)\Theta^2}. \tag{25}$$

We note that since the output states (16) and (24) are Bell diagonal, it is very simple to determine whether they are entangled or not [32]: the necessary and sufficient condition for entanglement is that the largest eigenvalue of those states is larger than $1/2$.

IV. COMPARISON OF THE DIFFERENT STRATEGIES FOR PAULI-CHANNEL PARAMETER ESTIMATION

We compare now the three different strategies for the estimation of Θ, corresponding to three two-qubit initial states: the initial two-qubit probe plus ancilla and two probes independently testing the channel, either in a product state or in a maximally entangled state. The QFI for two probes in a product state is twice the one for a single probe.

Figure 1 compares the respective quantum Fisher information for the depolarizing channel ($q = 1/3$). Several properties of this channel were investigated in Ref. [15] from the viewpoint of channel identification. For the parameter estimation considered here, the conjunction of its symmetry with proper counting of resources leads to very interesting results,

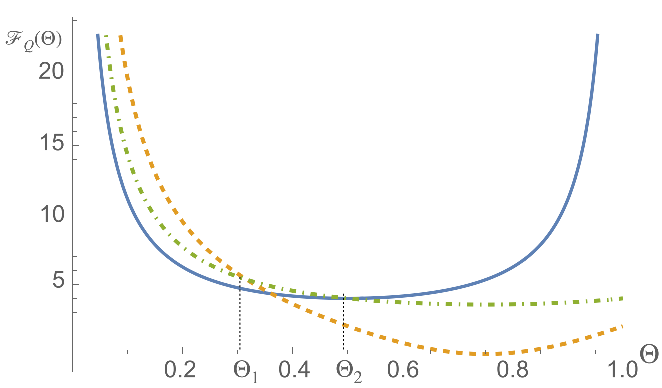

FIG. 1. Quantum Fisher information (QFI) for the estimation of Θ, for the depolarizing channel ($q = 1/3$), and for three different two-qubit initial states: (a) probe plus ancilla (full blue line), and two probes independently testing the channel, either in (b) a product state (dot-dashed green line) or in (c) an entangled state (dashed orange line). At $\Theta = \Theta_1 = (3-\sqrt{3})/4$ the quantum Fisher information of (c) becomes smaller than the one for (b). At $\Theta = \Theta_2 = 0.5$, the output state corresponding to probe plus ancilla becomes disentangled and the respective QFI becomes the largest one.

which relate quantum advantage with the entanglement of the outgoing qubits, as described in the following.

The two probes initially in a maximally entangled state lead to the best precision for Θ ranging from 0 to $\Theta_1 = (3-\sqrt{3})/4$, the value for which the corresponding output state becomes disentangled. From this point, up to $\Theta = 1$, the corresponding QFI becomes worse than the one for two probes in an initial product state. Interestingly, the probe plus ancilla initial state leads to the largest QFI when $\Theta > \Theta_2 = 0.5$, with Θ_2 being the value of the parameter for which the channel becomes entanglement breaking and the output state becomes disentangled. In this case, the region of supremacy of the probe plus ancilla setup is precisely the one where the output state becomes disentangled, in strong contrast with the two initially entangled probes. However, in order to recover the full information on the parameter, it is essential to implement a global measurement in the Bell-state basis. As shown before, a measurement of local observables of probe and ancilla leads, at most, to the quantum Fisher information of a single probe in an optimal initial state. In the strong-noise limit, the advantage of the probe plus ancilla setup is outstanding: For $\Theta \gtrsim 0.9$ (strong depolarization), the QFI for the probe plus ancilla becomes much larger then those for two probes, entangled or not.

The close connection here between usefulness of the ancilla and disentanglement, as well as the need for a Bell-basis measurement, has a striking similarity with the behavior of quantum illumination [1]. In both cases, it is necessary to perform the final detection in the Bell basis in order to get maximal quantum advantage. In quantum illumination, enhanced sensitivity of photodetection persists even after disentanglement, but it also occurs while the output state is entangled. Here, loss of entanglement of the output state is strictly necessary for quantum advantage.

Figure 2 compares the three strategies for an anisotropic channel, with $q = 0.25$. This implies that in (9) $q_1 = q_2 = 0.25$ and $q_3 = 0.5$. In this case, the output state corresponding to probe plus ancilla becomes disentangled before its quantum Fisher information is the largest one. Disentanglement now is still necessary for quantum advantage, but it is not sufficient anymore. Also, the precision in the estimation of Θ for two initially entangled probes becomes smaller than the one for the optimal initial product state shortly before the output state becomes disentangled. These results hold for a range of values of q from approximately 0.25–0.37.

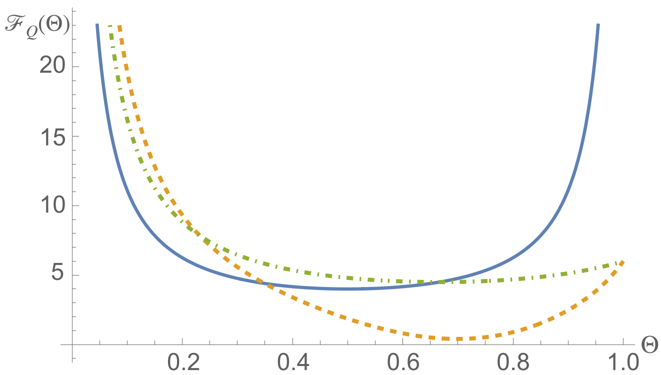

FIG. 2. Quantum Fisher information for the estimation of the parameter Θ, for the anisotropic channel with $q = 0.25$, corresponding to three different two-qubit initial states: (a) probe plus ancilla (full blue line), and two probes independently testing the channel, either in (b) a product state (dot-dashed green line) or in (c) an entangled state (dashed orange line).

V. CONCLUSION

We have considered here a very general class of quantum channels acting on qubits and demonstrated the relevance of setups consisting of probe plus ancilla for parameter estimation, even when taking the same resource counting for all possible setups involving an incoming pair of qubits, as is usually the case in parameter estimation. Several interesting features of this approach have been discussed. In particular, we have shown that an isotropic depolarizing channel leads, for an initial maximally entangled state of probe and ancilla, to a precision in the estimation of the depolarization parameter better than other setups involving two probes testing the channel, entangled or not, precisely when the output state becomes disentangled. Interestingly, this behavior is opposite to the one for two entangled probes, when quantum advantage disappears after entanglement is lost. Similar features can be found for anisotropic channels.

The optimal measurement on the probe plus ancilla pair is a global one, on the Bell basis, even when the output state is disentangled. These results are similar but not identical to those for quantum illumination, which also requires an entangling measurement for maximal sensitivity, and for which enhancement of the sensitivity of photodetection is still present even when the output is disentangled, but it also occurs while the output state is entangled.

We have shown the remarkable result that, in the strong-noise limit, the probe plus ancilla setup may lead to strong enhancement of the precision of estimation, as compared to two qubits, entangled or not, probing the channel. On the other hand, for weak depolarization, entangled probes, testing independently the channels, lead to the best precision. In view of the ubiquity of Pauli channels in dynamical processes involving qubits, the results demonstrated in this paper should be useful for quantum information tasks that hinge on the precise assessment of parameters.

ACKNOWLEDGMENTS

The authors acknowledge the support of the Brazilian agencies CNPq, CAPES, and the Rio de Janeiro State Foundation for Research Support (FAPERJ). R.L.M.F. acknowledges the support of the John Templeton Foundation (Grant No. 62424).

[1] S. Lloyd, Science **321**, 1463 (2008).
[2] S.-H. Tan, B. I. Erkmen, V. Giovannetti, S. Guha, S. Lloyd, L. Maccone, S. Pirandola, and J. H. Shapiro, Phys. Rev. Lett. **101**, 253601 (2008).
[3] S. Guha and B. I. Erkmen, Phys. Rev. A **80**, 052310 (2009).
[4] Z. Zhang, S. Mouradian, F. N. C. Wong, and J. H. Shapiro, Phys. Rev. Lett. **114**, 110506 (2015).
[5] M. Sanz, U. Las Heras, J. J. Garcia-Ripoll, E. Solano, and R. Di Candia, Phys. Rev. Lett. **118**, 070803 (2017).
[6] T. Gregory, P.-A. Moreau, E. Toninelli, and M. J. Padget, Sci. Adv. **6**, eaay2652 (2020).
[7] D. V. Strekalov, A. V. Sergienko, D. N. Klyshko, and Y. H. Shih, Phys. Rev. Lett. **74**, 3600 (1995).
[8] T. B. Pittman, Y. H. Shih, D. V. Strekalov, and A. V. Sergienko, Phys. Rev. A **52**, R3429(R) (1995).
[9] C. H. Monken, P. H. Souto Ribeiro, and S. Pádua, Phys. Rev. A **57**, 3123 (1998).
[10] R. S. Bennink, S. J. Bentley, and R. W. Boyd, Phys. Rev. Lett. **89**, 113601 (2002).
[11] A. Gatti, E. Brambilla, M. Bache, and L. A. Lugiato, Phys. Rev. Lett. **93**, 093602 (2004).
[12] R. S. Bennink, S. J. Bentley, R. W. Boyd, and J. C. Howell, Phys. Rev. Lett. **92**, 033601 (2004).
[13] J. H. Shapiro and R. W. Boyd, Quantum. Inf. Process. **11**, 949 (2012).
[14] M. J. Padgett and R. W. Boyd, Phil. Trans. R. Soc. A. **375**, 20160233 (2017).
[15] A. Fujiwara, Phys. Rev. A **63**, 042304 (2001).
[16] A. Fujiwara, Phys. Rev. A **70**, 012317 (2004).
[17] M. F. Sacchi, Phys. Rev. A **71**, 062340 (2005).
[18] M. F. Sacchi, Phys. Rev. A **72**, 014305 (2005).
[19] G. Wang and M. Ying, Phys. Rev. A **73**, 042301 (2006).
[20] S. Pirandola, R. Laurenza, C. Lupo, and J. L. Pereira, npj Quantum Inf. **5**, 50 (2019).
[21] H. Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, 1946), p. 500.
[22] R. Fisher, Math. Proc. Camb. Philos. Soc. **22**, 700 (1925).
[23] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976), Chap. VIII.4.
[24] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1982).
[25] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[26] B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Nat. Phys. **7**, 406 (2011).

[27] B. M. Escher, L. Davidovich, N. Zagury, and R. L. de Matos Filho, Phys. Rev. Lett. **109**, 190404 (2012).
[28] R. Demkowicz-Dobrzański, J. Kolodyński, and M. Gută, Nature Commun. **3**, 1063 (2012).
[29] C. L. Latune, B. M. Escher, R. L. de Matos Filho, and L. Davidovich, Phys. Rev. A **88**, 042112 (2013).
[30] M. M. Taddei, B. M. Escher, L. Davidovich, and R. L. de Matos Filho, Phys. Rev. Lett. **110**, 050402 (2013).
[31] S. Pirandola and C. Lupo, Phys. Rev. Lett. **118**, 100502 (2017).
[32] M. D. Lang and C. M. Caves, Phys. Rev. Lett. **105**, 150501 (2010).