



## Virtual quantum error detection

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(Received 6 February 2023; revised 6 July 2023; accepted 2 October 2023; published 25 October 2023)

Quantum error correction and quantum error detection necessitate syndrome measurements to detect errors. Performing syndrome measurements for each stabilizer generator can be a significant overhead, considering the fact that the readout fidelity in the current quantum hardware is generally lower than gate fidelity. Here, by generalizing a quantum error mitigation method known as *symmetry expansion*, we propose a protocol called *virtual quantum error detection* (VQED). This method *virtually* allows for evaluating computation results corresponding to postselected quantum states obtained through quantum error detection during circuit execution, without implementing syndrome measurements. Unlike conventional quantum error detection, which requires the implementation of Hadamard test circuits for each stabilizer generator, our VQED protocol can be performed with a constant depth shallow quantum circuit with an ancilla qubit, irrespective of the number of stabilizer generators. Furthermore, for some simple error models, the computation results obtained using VQED are robust against the noise that occurred during the operation of VQED, and our method is fully compatible with other error mitigation schemes, enabling further improvements in computation accuracy and facilitating high-fidelity quantum computing.

DOI: [10.1103/PhysRevA.108.042426](https://doi.org/10.1103/PhysRevA.108.042426)

### I. INTRODUCTION

The last decade has seen the remarkable development of the noisy-intermediate quantum computing paradigm from both theoretical and experimental sides [1–8]. Nevertheless, the effect of noise lies as a crucial problem in realizing practical quantum computing. Quantum error correction (QEC) and quantum error detection (QED), which reduce computation errors through the encoding of logical qubits with many physical qubits, have been investigated for enhancing computation accuracy for a long time since the early days of quantum information science [9–14]. Syndrome measurements are performed in QEC and QED to detect physical errors by using ancilla qubits; QEC actively corrects physical errors based on the error information obtained in the decoding process while QED discards the noisy quantum states once an error is detected.

While the utility of QEC and QED has been shown theoretically in numerous previous works, they require high-fidelity syndrome measurements of stabilizer generators. Furthermore, the number of required syndrome measurements increases with the number of stabilizer generators in the QEC and QED codes. Considering the current situation of superconducting hardware, in which the measurement fidelity is lower than gate errors [6,15,16], the necessity of single-shot

measurements [17] for syndrome measurements can be a significant overhead in QEC and QED.

For the ease of error reduction in near-term quantum hardware, a class of error reduction techniques referred to as *quantum error mitigation* (QEM) has been recently studied [18–22]. In many QEM methods, the noiseless expectation values of observables are estimated via postprocessing of measurement results. This indicates that we cannot physically obtain quantum states with reduced noise; nevertheless, QEM allows for *virtually* simulating the expectation values of observables for such states. Symmetry expansion (SE) is one of the QEM methods that use symmetries inherent to the system to mitigate errors [23–26]. The noisy quantum state is virtually projected onto the symmetric subspace through random sampling of the symmetry operators, additional measurements, and classical postprocessing of measurement outcomes. As we will discuss later, SE allows for the calculation of the expectation value of an observable corresponding to the postselected quantum states through QED without implementing syndrome measurements, and hence is suitable for near-term hardware. So far, SE is theoretically formulated for error mitigation for noisy states immediately before measurement [24,25] and state preparation for rotation symmetric bosonic codes (RSBCs) [26]. Thus, SE in its current form cannot effectively suppress the accumulation of noise during computation, whereas the conventional QED can be more flexibly used during the circuit execution.

In this paper, we significantly expand the framework of SE so that it can be leveraged during the execution of quantum algorithms. Because our method enables us to obtain the

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expectation values corresponding to the postselected state via QED, we call it *virtual quantum error detection* (VQED). While the conventional SE can only detect errors immediately before the measurement of expectation values, VQED can detect errors even during the execution of the quantum circuit, enabling us to mitigate the accumulation of errors during the computation. Although VQED inherits the disadvantages of SE, i.e., we can only obtain error-mitigated expectation values, not the quantum state itself, and the required sampling complexity is quadratically worse for the success probability of QED, the significant advantages of VQED compared with the QED are as follows.

(1) The depth for QEM is constant regardless of the number of stabilizer generators of the code.

(2) We only need to measure an expectation value of an observable without the need for single-shot syndrome measurements.

(3) The obtained expectation values are robust against the noise that occurred during the operation of VQED for some simple error models.

(4) Our method is fully compatible with other QEM methods, e.g., readout error mitigation [27,28] for the ancilla qubit used in our protocol.

We numerically verify the behavior of the fidelity improvement with our VQED protocol over the conventional SE and the unencoded physical qubits. We also evaluate the required sampling costs and verify that the sampling cost for VQED does not significantly increase compared to SE. Furthermore, our method can offer virtual implementation of stabilizerlike QEM methods using spin and particle number preservation in the computation [23,29] in an even more hardware-friendly manner.

In addition, we discuss the virtual implementation of quantum error correction, which results in the computation outcome corresponding to the error-corrected quantum states. While the conventional QEC does not induce additional sampling overheads, we find that our virtual QEC generally incurs a larger sampling overhead than the virtual QED method; therefore, we conclude VQED is preferred in typical quantum computation scenarios.

## II. PRELIMINARIES

### A. Quantum error detection and quantum error correction for stabilizer codes

We first review stabilizer codes and ways to detect and correct their errors [30,31]. QED and QEC are performed by encoding quantum information into enlarged Hilbert space at the expense of multiple quantum systems. Due to its redundancy, we can detect and correct their errors during the computation.

Here, we review the stabilizer formalism, which is the most standard method to construct quantum error-correcting codes. Consider an  $n$ -qubit Pauli group as

$$\mathcal{G}_n = \{\pm 1, \pm i\} \times \{I, X, Y, Z\}^{\otimes n} \quad (1)$$

where  $I$  is the identity operator for a single qubit system and  $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ , and  $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  are Pauli operators. To encode  $k$  logical qubits into  $n$  physical qubits,

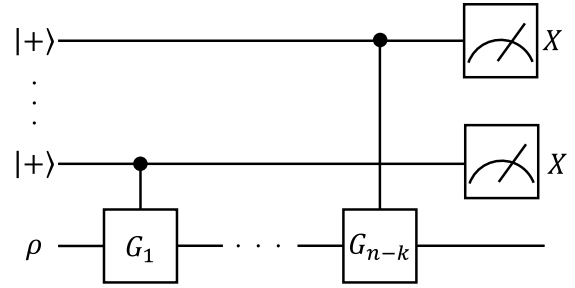


FIG. 1. Quantum circuit for quantum error detection.

we define a stabilizer group  $\mathcal{S} = \{S_1, \dots, S_{2^{n-k}}\} \subset \mathcal{G}_n$  as a commutative subgroup of the Pauli group  $\mathcal{G}_n$  with  $-I^{\otimes n} \notin \mathcal{S}$ . We denote a generator set of the stabilizer group  $\mathcal{S}$  as  $\mathcal{G} = \{G_1, \dots, G_{n-k}\}$ . Then, we can define the logical space of the stabilizer code  $\mathcal{C}$  as an eigenspace with  $+1$  eigenvalues for all the operators in the stabilizer group, i.e.,  $\mathcal{C} = \{|\psi\rangle \mid \forall S_i \in \mathcal{S}, S_i |\psi\rangle = |\psi\rangle\}$ . In the  $2^k$ -dimensional Hilbert space, we can introduce a logical basis as  $\{|0\rangle_L, |1\rangle_L\}^{\otimes k}$  and logical Pauli operators as  $\{I_L, X_L, Y_L, Z_L\}^{\otimes k}$ . The code distance  $d$  is the minimum number of physical qubits on which an arbitrary logical operator of the code nontrivially operates. We denote such stabilizer codes as  $[[n, k, d]]$  stabilizer codes.

We can detect physical errors during quantum computation by measuring the generators  $G_1, \dots, G_{n-k}$  by using the Hadamard test circuits as shown in Fig. 1, and such measurement is called *syndrome measurement*. If there exists  $G_i$  such that its measurement result is  $-1$ , then we can determine the presence of errors during the computation. Conversely, when the measurement results are  $+1$  for all  $G_i$ , we can say that there was no error with a sufficiently high probability. By continuing the computation only when the measurement results for all the generators are  $+1$ , we can project the noisy state  $\rho = \mathcal{E}(\rho_{\text{id}})$  into the code space as

$$\rho_{\text{det}} = \frac{P\rho P}{\text{tr}[\rho P]}, \quad (2)$$

where  $P$  is a projector to the code space  $\mathcal{C}$  written as

$$P = \prod_{G_i \in \mathcal{G}} \frac{I + G_i}{2} = \frac{1}{2^{n-k}} \sum_{S_i \in \mathcal{S}} S_i. \quad (3)$$

Because the probability to measure  $+1$  for all the syndrome measurements is  $\text{tr}[\rho P]$ , the effect of physical errors acting on less than  $d$  qubits can be eliminated with  $O(\text{tr}[\rho P]^{-1})$  times more execution of quantum circuits. Note that stabilizerlike QEM methods work in a similar way when the spin and electron number preservation is imposed in the variational ansatz of quantum states [23,29].

We can not only detect errors but also correct them by applying appropriate feedback operations according to the measurement results, enabling us to suppress the effect of noise without any additional execution of quantum circuits. When the measurement result for the generator  $G_i$  is  $s_i$ , and there is no measurement error, we can correct errors by applying a recovery operation  $R_s$ , which is estimated from  $s = (s_1, \dots, s_{n-k})$  to maximize the probability of correcting erroneous quantum states to the original logical state. Since

the recovery Pauli operator at least maps quantum states to a logical state,  $R_s$  commutes with  $G_i$  if  $s_i = +1$  and anticommutes if  $s_i = -1$ . In this way, the effect of physical errors acting on less than  $\lfloor (d-1)/2 \rfloor$  qubits can be corrected as

$$\rho_{\text{cor}} = \sum_{s \in \{-1,1\}^{n-k}} E_s \rho E_s, \quad (4)$$

where

$$\begin{aligned} E_s &= R_s \prod_i \frac{I + s_i G_i}{2} \\ &= P R_s. \end{aligned} \quad (5)$$

While QEC and QED can reduce the effective error rates, they impose additional difficulties in the implementation. To implement QEC and QED, we need repetitive applications of Pauli measurements for all the elements in the stabilizer generator set. Since the error rates of measurement operations are typically higher than the others [6,15,16], they induce large overheads on the process. In the case of QEC, we also need to estimate recovery operations from the observed syndrome values, and error rates must be smaller than the value called *code threshold* for a reliable estimation.

### B. Symmetry expansion

In order to combat errors on near-term devices, QEM has been developed in recent years [21,22]. SE is one of the promising QEM methods which mitigates errors by virtually projecting the noisy quantum state onto the symmetric subspace without syndrome measurements [24,25].

Suppose that we want to estimate an expectation value of an observable  $O$  for a noiseless state  $\rho_{\text{id}}$  from the measurement of the noisy state  $\rho = \mathcal{E}(\rho_{\text{id}})$ . We assume that the observable  $O$  commutes with the projector  $P$ . Then, we can mitigate errors by virtually projecting the noisy states onto the code space as

$$\begin{aligned} \text{tr}[\rho_{\text{det}} O] &= \frac{\text{tr}[\rho O P]}{\text{tr}[\rho P]} \\ &= \frac{2^{-(n-k)} \sum_{S_i \in \mathcal{S}_i} \text{tr}[\rho O S_i]}{2^{-(n-k)} \sum_{S_i \in \mathcal{S}_i} \text{tr}[\rho S_i]}, \end{aligned} \quad (6)$$

which can be calculated in the following way.

- (1) For  $s = 1, \dots, N$ , repeat the following operations.
  - (a) Uniformly sample  $S_i \in \mathcal{S}$ .
  - (b) Simultaneously measure the noisy state  $\rho$  for  $S_i$  and  $O S_i$ , and record the results as  $a_s$  and  $b_s$ .
- (2) Calculate  $a = \frac{1}{N} \sum_s a_s$  and  $b = \frac{1}{N} \sum_s b_s$ .
- (3) Output  $b/a$ .

The number of measurements needed to estimate Eq. (6) for some fixed accuracy  $\varepsilon$  is known to scale as  $N = O(\varepsilon^{-2} \text{tr}[\rho P]^{-2})$ . In this way, we can obtain an error-mitigated expectation value of the observable  $O$ , which corresponds to the virtual projection of the noisy state immediately before the measurement  $\rho$  onto  $\rho_{\text{det}}$ .

## III. VIRTUAL QUANTUM ERROR DETECTION

Symmetry expansion is only applicable to the state immediately before measurement [24,25] and state preparation

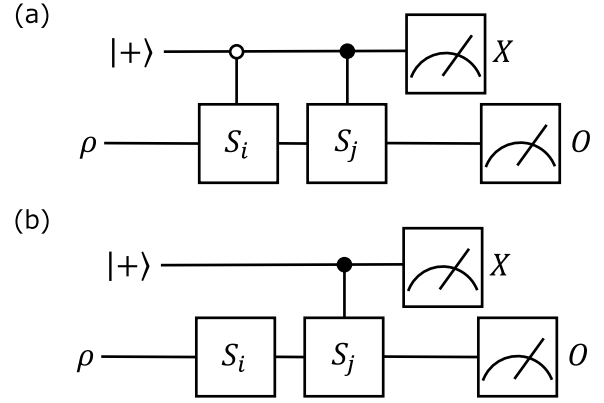


FIG. 2. Quantum circuits for virtually projecting a quantum state  $\rho$  into the code space. The white and black circles indicate the control operations which act for states 0 and 1, respectively. The circuit in panel (a) utilizes two controlled-stabilizer gates, whereas the circuit in panel (b) can virtually project quantum states using only a single controlled-stabilizer gate.

for rotation symmetric bosonic codes [26]. In this section, we introduce our VQED method, which allows for the computation of error-mitigated expectation values corresponding to the postselected states after syndrome measurements during circuit execution.

Let us first explain the way to virtually project a quantum state  $\rho$  into the code space by using the circuit shown in Fig. 2(a). We can derive the expectation value obtained through this circuit as

$$\frac{1}{2}(\text{tr}[S_i \rho S_j O] + \text{tr}[S_j \rho S_i O]). \quad (7)$$

Thus, by uniformly sampling  $i, j \in \{1, \dots, 2^{n-k}\}$  and taking the average of the distribution, we can obtain the expectation value of the projected state as

$$\text{tr}[P \rho P O] \quad (8)$$

since the average of  $S_i$  can be written as  $\langle S_i \rangle = 2^{-(n-k)} \sum_i S_i = P$ .

We can further simplify the circuit as in Fig. 2(b) for stabilizer codes. The expectation value obtained through this circuit is

$$\frac{1}{2}(\text{tr}[S_j S_i \rho S_i O] + \text{tr}[S_i \rho S_i S_j O]). \quad (9)$$

Thus, by uniformly sampling  $i, j \in \{1, \dots, 2^{n-k}\}$  and taking the average of the distribution, we can also obtain Eq. (8) since  $P S_i = S_i P = P$  holds.

These methods can be used to virtually detect errors in noisy quantum circuits. We consider a logical quantum circuit composed of a state preparation of a logical initial state  $\rho_0$  followed by the  $L$  logical unitary gate  $\mathcal{U}_l(\cdot) = U_l \cdot U_l^\dagger$  ( $l = 1, \dots, L$ ) and a measurement of an observable  $O$  in the hope of estimating the expectation value of  $O$  for the state  $\rho_{\text{id}} = \mathcal{U}_L \circ \dots \circ \mathcal{U}_1(\rho_0)$ . However, we assume that these logical quantum gates are affected by Markovian noise and that the actual gates are represented as  $\mathcal{U}'_l = \mathcal{E}_l \circ \mathcal{U}_l$ . For simplicity, we ignore state preparation and measurement errors, but these effects can easily be reflected. When we can perform

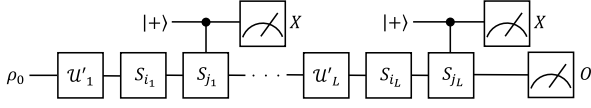


FIG. 3. Quantum circuit for virtual quantum error detection.

quantum error detection after each gate, we will have

$$\rho_{\text{det}} = \frac{\rho'_{\text{det}}}{\text{tr}[\rho'_{\text{det}}]}, \quad (10)$$

where

$$\rho'_{\text{det}} = \mathcal{P} \circ \mathcal{E}_L \circ \mathcal{U}_L \circ \cdots \circ \mathcal{P} \circ \mathcal{E}_1 \circ \mathcal{U}_1(\rho_0). \quad (11)$$

Here, we define  $\mathcal{P}(\cdot) = P \cdot P$ .

In order to obtain the expectation value for  $\rho_{\text{det}}$  through VQED, we construct a quantum circuit represented in Fig. 3. This circuit allows for computing the expectation values corresponding to the error-detection circuits by performing the  $S_{i_l}$  gate on the noisy circuit, preparing a single qubit ancilla initialized to  $|+\rangle$ , coupling the ancilla qubit with the noisy circuit through the controlled- $S_{j_l}$  gate, and measuring the ancilla in the  $X$  bases. Note that the frequency of applying these operations of VQED can be reduced according to the noise level, although we discuss gatewise VQED for generality. The state immediately before the measurement of this circuit  $\rho_{\text{bf}}$  reads

$$\rho_{\text{bf}} = \frac{1}{2^L} \sum_{pq} |\mathbf{p}\rangle \langle \mathbf{q}| \otimes \rho_{ij}^{pq},$$

$$\rho_{ij}^{pq} = \mathcal{P}_{i_l j_l}^{p_l q_l} \circ \mathcal{E}_L \circ \mathcal{U}_L \circ \cdots \circ \mathcal{P}_{i_1 j_1}^{p_1 q_1} \circ \mathcal{E}_1 \circ \mathcal{U}_1(\rho_0), \quad (12)$$

where  $\mathbf{p}$  and  $\mathbf{q}$  are bit strings of length  $L$  and

$$\mathcal{P}_{i_l j_l}^{p_l q_l}(\cdot) = S_{j_l}^{p_l} S_{i_l} \cdot S_{i_l} S_{j_l}^{q_l}. \quad (13)$$

Then, the expectation value of the observable  $X^{\otimes L} \otimes O$  in this state is

$$\langle X^{\otimes L} \otimes O \rangle = \text{tr}[\rho_{\text{bf}} X^{\otimes L} \otimes O]$$

$$= \frac{1}{2^L} \sum_{\mathbf{p}} \text{tr}[\rho_{ij}^{pp+1} O] \quad (14)$$

where  $\mathbf{1}$  is a bit string of length  $L$  whose elements are all 1. When we uniformly sample  $i_l, j_l \in \{1, \dots, 2^{n-k}\}$  ( $1 \leq l \leq L$ ) and denote the expectation value under the probability distribution as  $\langle \cdot \rangle_{ij}$ , we can project the noisy state into the code space after each noisy gate as

$$\langle \rho_{ij}^{pp+1} \rangle_{ij} = \mathcal{P} \circ \mathcal{E}_L \circ \mathcal{U}_L \circ \cdots \circ \mathcal{P} \circ \mathcal{E}_1 \circ \mathcal{U}_1(\rho_0)$$

$$= \rho'_{\text{det}}, \quad (15)$$

where we use

$$\langle \mathcal{P}_{i_l j_l}^{p_l q_l+1}(\cdot) \rangle_{i_l j_l} = \langle S_{j_l}^{p_l} S_{i_l} \cdot S_{i_l} S_{j_l}^{1-p_l} \rangle_{i_l j_l}$$

$$= \begin{cases} \langle S_{i_l} \cdot S_{i_l} S_{j_l} \rangle_{i_l j_l} & (p_l = 0) \\ \langle S_{j_l} S_{i_l} \cdot S_{i_l} \rangle_{i_l j_l} & (p_l = 1) \end{cases}$$

$$= P \cdot P. \quad (16)$$

Thus, the expectation value of the observable  $O$  for the postselected state  $\rho_{\text{det}}$  can be represented as

$$\text{tr}[\rho_{\text{det}} O] = \frac{\langle \text{tr}[(\frac{1}{2^L} \sum_{pq} |\mathbf{p}\rangle \langle \mathbf{q}| \otimes \rho_{ij}^{pq}) X^{\otimes L} \otimes O] \rangle_{ij}}{\langle \text{tr}[(\frac{1}{2^L} \sum_{pq} |\mathbf{p}\rangle \langle \mathbf{q}| \otimes \rho_{ij}^{pq}) X^{\otimes L} \otimes I] \rangle_{ij}}. \quad (17)$$

Therefore, we can perform our VQED in the noisy quantum circuit with the following procedure.

- (1) For  $s = 1, \dots, N$ , repeat the following operation.
  - (a) Uniformly sample  $i_l, j_l \in \{1, \dots, 2^{n-k}\}$  ( $1 \leq l \leq L$ ).
  - (b) Run the circuit illustrated in Fig. 3.
  - (c) Record the product of the  $X$  measurement as  $a_s$  and the product of  $a_s$  and the  $O$  measurement as  $b_s$ .
- (2) Calculate  $a = \frac{1}{N} \sum_s a_s$  and  $b = \frac{1}{N} \sum_s b_s$ .
- (3) Output  $b/a$ .

In this way, with the sampling overhead of  $N = O(\varepsilon^{-2} \text{tr}[\rho'_{\text{det}}]^{-2})$ , we can perform VQED to virtually detect errors that occurred during the computation with some fixed accuracy (standard deviation)  $\varepsilon$ . Note that while we focus on the stabilizer QEC and QED codes our method can be straightforwardly applied to the stabilizer-based QEM method for the spin and electron number preservation [23,29] for more near-term quantum hardware.

Note that our VQED protocol circumvents the syndrome measurements of stabilizer generators that need high-fidelity single-shot measurement of ancilla qubits; our method only measures expectation values of the observable. Moreover, while quantum error detection requires measurements of  $n - k$  stabilizer generators via Hadamard test circuits shown in Fig. 1, our method only necessitates a single controlled operation irrespective of the number of stabilizer generators.

Furthermore, for certain simple error models, the obtained expectation value is robust against noise that occurs in the ancilla qubit. Since we calculate the expectation value of  $X$  for each ancilla, the only terms of the ancilla that affect Eq. (17) are  $|0\rangle\langle 1|$  and  $|1\rangle\langle 0|$ . Thus, even if the single qubit depolarizing noise  $\mathcal{E}_p : \rho \mapsto (1-p)\rho + pI/2$  affects each ancilla during the execution of the controlled- $S_{j_l}$  gate, the numerator and the denominator of Eq. (17) are only multiplied by  $(1-p)^L$ . Therefore, the value obtained through VQED remains unchanged. Note that the circuit level noise model, where each CNOT gate and CZ gate to implement the controlled- $S_{j_l}$  gate is affected by a single qubit depolarizing noise, also does not affect the expectation value. This is because the noisy term represented as  $|0\rangle\langle 0| \otimes \rho + |1\rangle\langle 1| \otimes \rho'$ , where  $\rho$  and  $\rho'$  are the states of the system qubits where noise may be propagated, cancels out when we take the expectation value of  $X$ . The same principle applies to other noise models which are not biased by Pauli  $X$  or  $Y$ , such as local dephasing and amplitude damping noise. We further discuss these points in Appendix A. We also want to mention that, by combining the readout error mitigation [27,28] method with our method for the ancilla qubits, we can perform high-fidelity virtual projection onto the code space even under the existence of measurement errors.

The disadvantages of VQED are that we can only obtain the error-mitigated expectation values, not the quantum state itself, as well as quadratically worse sampling cost for the projection probability  $\text{tr}[\rho'_{\text{det}}]$ . While sampling costs can only

be overcome by increased parallelization, lightweight quantum phase estimation algorithms only employing expectation values are proposed [32–35] in addition to the fact that most NISQ algorithms use expectation values. Our VQED methods can be used in such algorithms.

#### IV. VIRTUAL IMPLEMENTATION OF QUANTUM ERROR CORRECTION

We also discuss how to perform quantum error correction virtually without any syndrome measurements and feedback operations. The main idea is that the error-corrected state as in Eq. (4) can be also written as

$$\rho_{\text{cor}} = \sum_{s \in \{-1, 1\}^{n-k}} \mathcal{P}(R_s \rho R_s). \quad (18)$$

Thus, we can virtually correct errors by uniformly sampling  $s_1, \dots, s_{n-k} \in \{+1, -1\}$ , applying  $R_s$  to the noisy state, virtually projecting the state into the code space using the way mentioned above, and multiplying the result by  $2^{n-k}$ . However, the sampling cost of this method scales as  $2^{2(n-k)}$ , which grows exponentially with the number of redundant qubits. We may decrease the cost by limiting the scope of the sum. Let  $B \subset \{-1, 1\}^n$  be a subset of highly probable measurement results such as the measurement results when an error did not occur or occurred only once. Then, we may approximate the error-corrected state as

$$\rho_{\text{cor}'} = \frac{1}{\sum_{s \in B} p_s} \sum_{s \in B} \mathcal{P}(R_s \rho R_s) \quad (19)$$

where  $p_s = \text{tr}[PR_s \rho R_s]$  represents the probability of obtaining  $s$  at the syndrome measurement. However, the sampling cost of virtually calculating this state is  $|B|^2 (1/\sum_{s \in B} p_s)^2$ , which is still significantly higher than just performing VQED with  $B = \{1\}^n$ . Furthermore, while error detection can detect errors of at most  $d$  qubits, error correction can only correct errors of at most  $\lfloor (d-1)/2 \rfloor$  qubits. This means that even the accuracy of this virtual implementation of QEC is generally worse than VQED. Even though these methods may be more effective than VQED in the case where the noise maps the state in the code space outside of it with high probability, finding practical scenarios to utilize these methods is left as our future work.

#### V. NUMERICAL SIMULATION

In this section, we numerically evaluate the performance of our method for  $[[4,1,2]]$ ,  $[[5,1,3]]$ , and  $[[7,1,3]]$  stabilizer codes [11,12,14]. The generators and logical operators of these codes are shown in Table I. Similar to the numerical calculation presented in the previous study on SE [24], we initialize the state  $\rho_0$  as the logical state  $|0\rangle_L$ , and the unitary gate  $U_l$  is randomly chosen from a set of transversal single-qubit gates [31,36]. We specify the set of transversal single-qubit gates we use in Appendix B. We assume that the local depolarizing noise  $\mathcal{E} = \mathcal{E}_p^{\otimes n}$  ( $\mathcal{E}_p(\rho) = (1-p)\rho + pI/2$ ) disturbs the circuit with noise strength  $p = 0.01$  each after the gate. We numerically calculate the depth  $L$  dependence of the infidelity  $1 - \langle \tilde{\Psi} | \rho_{\text{det}} | \tilde{\Psi} \rangle$  between the output state of the noisy circuit  $\rho_{\text{det}}$  and the noiseless circuit  $|\tilde{\Psi}\rangle$  and the scaling of the sampling cost  $\text{tr}[\rho'_{\text{det}}]^{-2}$  by using QUTIP [37].

TABLE I. Generators and logical operators for  $[[4,1,2]]$ ,  $[[5,1,3]]$ , and  $[[7,1,3]]$  stabilizer codes.

Stabilizer code	Name	Operator
$[[4, 1, 2]]$	$G_1$	$XXXX$
	$G_2$	$ZZZZ$
	$G_3$	$IZZI$
	$Z_L$	$ZZII$
	$X_L$	$IXXI$
$[[5,1,3]]$	$G_1$	$XZZXI$
	$G_2$	$IXZZX$
	$G_3$	$XIXZZ$
	$G_4$	$ZXIXZ$
	$Z_L$	$ZZZZZ$
$[[7,1,3]]$	$X_L$	$XXXXXX$
	$G_1$	$IIIZZZZ$
	$G_2$	$IZZIIZZ$
	$G_3$	$ZIZIZIZ$
	$G_1$	$IIIXXXX$
	$G_2$	$IXXIIXX$
	$G_3$	$XIXIXIX$
	$Z_L$	$ZZZZZZZ$
	$X_L$	$XXXXXXXX$

Our results are shown in Figs. 4 and 5. As shown in Fig. 4, we can reduce the infidelity using VQED compared to a single physical qubit affected by the error  $\mathcal{E}_p$  without encoding. Furthermore, frequent application of VQED during the circuit execution prevents the noisy state from being highly mixed in the code space. This allows us to suppress logical errors that cannot be mitigated by the conventional SE performed only on the state immediately before the measurement [24,25]. We also find that we can reduce infidelity without performing error detection after every gate: we can sufficiently mitigate errors simply by performing VQED after every fixed number of gates. This fact can be useful when the measurement time is much longer than the gate execution time. By comparing infidelity among different codes, we can say that infidelity becomes smaller as the code distance gets larger.

Figure 5 shows that the sampling cost increases exponentially with the circuit depth  $L$  when we perform VQED frequently. This scaling can be roughly considered to be given by the square inverse of the probability that a state in the code space remains in the code space when the noise is applied; thus we can write  $N \sim \text{tr}[\rho'_{\text{det}}]^{-2} = O[(1 - \frac{3}{4}p)^{-2nL}]$ . It is noteworthy that the sampling cost is not significantly influenced by the frequency of VQED, even when comparing the cases of the conventional SE and VQED each applied after the gate (see the numerical results up to the depth  $L = 40$ , for example). Meanwhile, it may appear that the sampling cost of SE approaches a constant value for large  $L$ . However, this is because the accumulated errors increase and the noisy state approaches a completely mixed state, and thus the sampling cost converges to  $N \sim \text{tr}[I/2^n P] = \frac{1}{2^{n-k}}$ .

In Figs. 4(d) and 5(d), we also present the performance of VQED for the  $[[4, 1, 2]]$  stabilizer code when the VQED gadget  $S_{ii}$  gate and controlled- $S_{ji}$  gate in Fig. 3 are each

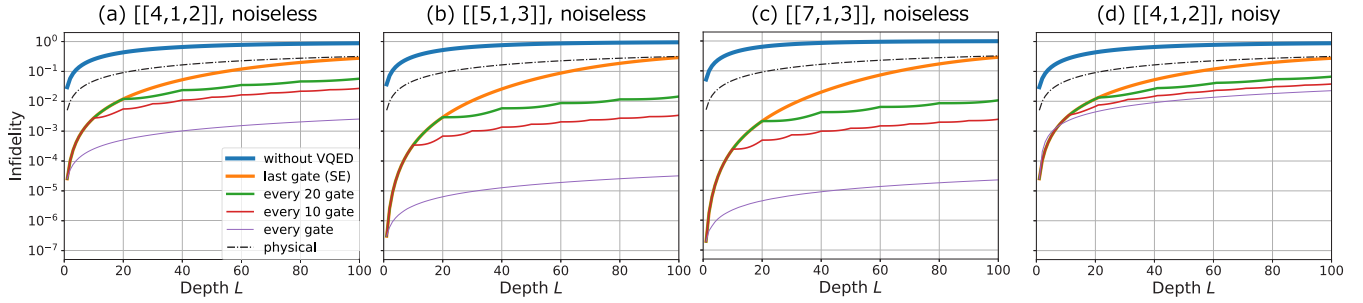


FIG. 4. Depth  $L$  dependence of infidelity  $1 - \text{tr}[\rho_{\text{det}}|\bar{\Psi}\rangle\langle\bar{\Psi}|] = 1 - \langle\bar{\Psi}|\rho_{\text{det}}|\bar{\Psi}\rangle$  between the output state of the noisy circuit with VQED  $\rho_{\text{det}}$  and the noiseless circuit  $|\bar{\Psi}\rangle$  for [(a), (d)]  $[[4,1,2]]$ , (b)  $[[5,1,3]]$ , and (c)  $[[7,1,3]]$  stabilizer codes. [(a)–(c)] Results when the controlled-stabilizer gates are noiseless. (d) Results when the VQED gadgets are affected by local depolarizing noise. The “without VQED” line represents infidelity when we did not perform VQED. The “last gate (SE)” line represents infidelity when we perform VQED only before the measurement, which is just a normal SE, as in Refs. [24,25]. The “every 20 gates” and the “every 10 gates” lines represent infidelity when we perform VQED after every 20 and 10 gates. The “every gate” line represents infidelity when we perform VQED after every gate. The “physical” line represents the infidelity of a single physical qubit without encoding.

affected by the local depolarizing noise  $\mathcal{E}_p^{\otimes n}$  and  $\mathcal{E}_p^{\otimes(n+1)}$  with the same error rate  $p = 0.01$ . See Appendix C for the results for  $[[5, 1, 3]]$  and  $[[7, 1, 3]]$  stabilizer codes. In the previous section, we have mentioned that a wide class of realistic noise that occurred in the ancilla does not affect the accuracy of VQED. By comparing Figs. 4(a) and 4(d), we can also say that even in the presence of additional noise acting on the system qubits VQED still significantly outperforms the unmitigated results and the conventional SE. Meanwhile, we can see from Figs. 5(a) and 5(d) that the sampling cost increases when we assume that the noise affects the VQED gadget. This is mainly because the probability that a state in the code space remains in the space decreases due to the additional noise acting on the system qubits, and the denominator in Eq. (17) is multiplied by the factor of  $(1-p)^L$ . Thus, the sampling cost scales as  $N \sim O[(1 - \frac{3}{4}p)^{-6nL}(1-p)^{-2L}]$ . However, these effects can be circumvented by reducing the frequency of applying the operation of VQED.

From the above, we can say that our method can be used effectively by adjusting the code distance or frequency of VQED according to the hardware constraints, the desired accuracy, or the allowable sampling cost.

## VI. DISCUSSION

We propose VQED so that the computation errors during the circuit execution can be flexibly suppressed by using additional two-qubit operations and measurements in the  $X$  basis. We verify in the numerical simulations that our virtual quantum error detection protocol allows for the realization of significantly higher-fidelity calculation of expectation values, compared with the conventional symmetry expansion method, at the cost of sampling costs. We also discuss the virtual implementation of quantum error correction; however, even though the fidelity of the quantum state after quantum error correction is generally lower than that for quantum error detection, the sampling cost of virtual implementation of quantum error correction becomes larger than VQED.

Although we mainly discuss the stabilizer codes based on the Pauli group, we can apply our method to other types of codes such as RSBCs [38] as well. In Ref. [26], symmetry expansion in RSBCs is proposed, but it is restricted to state preparation and immediately before measurement. By considering the rotation symmetry operators rather than

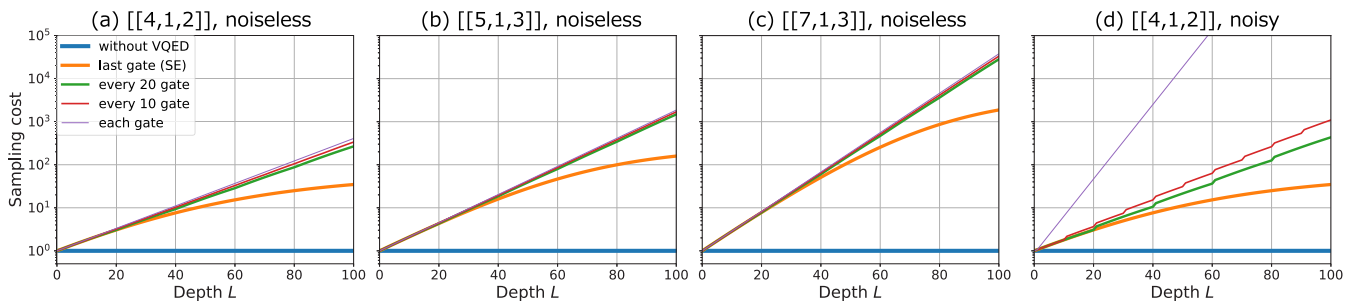


FIG. 5. Scaling of the sampling cost  $\text{tr}[\rho'_{\text{det}}]^{-2}$  with respect to the depth  $L$  of the quantum circuit for [(a), (d)]  $[[4,1,2]]$ , (b)  $[[5,1,3]]$ , and (c)  $[[7,1,3]]$  stabilizer codes. [(a)–(c)] Results when the controlled-stabilizer gates are noiseless. (d) Results when the VQED gadgets are affected by local depolarizing noise. The “without VQED” line represents sampling cost when we did not perform VQED. The “last gate (SE)” line represents sampling cost when we perform VQED only before the measurement, which is just a normal SE, as in Refs. [24,25]. The “every 20 gates” and the “every 10 gates” lines represent the sampling cost when we perform VQED after every 20 and 10 gates. The “every gate” line represents the sampling cost when we perform VQED after every gate.

Pauli symmetries, we can also perform virtual quantum error detection for RSBCs, which is a significant generalization of Ref. [26]. In this case, we need controlled-rotation gates, which are implemented by the dispersive interactions [39] between the resonator and the ancilla qubit.

Even after the application of VQED, there remains a finite logical error, e.g., due to the limitations of code distances. Therefore, the efficient combination of VQED with other QEM methods, e.g., purification-based QEM [40–45], will also be an important research direction for the realization of even more accurate quantum computing. Also, because VQED can be regarded as a QEM method implemented on the code space, the relationship between VQED and other hybrid QEM and QEC methods is worth exploring [32,46–48].

Experimental implementation of VQED is also an important direction for future work. The VQED circuit we propose in Fig. 3 requires connectivity between the ancilla qubit and all the qubits constructing logical qubits, and we generally need SWAP operations to implement controlled-stabilizer operations in the case of restricted connectivity, e.g., current superconducting hardware. However, we can relax the requirement by assigning different ancilla qubits for each logical qubit and flexibly choosing the arrangement of the ancilla qubits according to the constraints of the experiment. For example, for trapped-ion systems, all-to-all connectivity is a relatively reasonable assumption [49]; therefore, one ancilla qubit may be iteratively used for all logical qubits. Meanwhile, superconducting qubit devices have restricted connectivity [6], and it may be better to assign different ancilla qubits to each logical qubit.

Finally, information-theoretic analysis of QEM is one of the intensively studied topics [50–55]. As far as we know, symmetries of the system are not explicitly considered in these works while our paper shows that they can play a crucial role for QEM. Therefore, the construction of an information-theoretic analysis of QEM incorporating the symmetries may shed light on, e.g., the characterization cost of the noise model for performing QEM.

**ACKNOWLEDGMENTS**

This work is supported by Japan Science and Technology Agency (JST) PRESTO Grants No. JPMJPR1916, No. JPMJPR2114, and No. JPMJPR2119; JST CREST Grants No. JPMJCR1771 and No. JPMJCR23I4; Ministry of Education, Culture, Sports, Science, and Technology Q-LEAP Grants No. JPMXS0120319794 and No. JPMXS0118068682; JST Moonshot Research and Development Grant No. JPMJMS2061; COI-NEXT Grant No. JPMJPF2221; and JST ERATO Grant No. JPMJER2302. K.T. is supported by the Worldleading Innovative Graduate Study Program for Materials Research, Industry, and Technology of the University of Tokyo.

**APPENDIX A: ROBUSTNESS AGAINST NOISE IN ANCILLA QUBITS**

In this section, we discuss the robustness of the VQED gadget against the noise in the ancilla qubits. In order to perform  $S_i$  or controlled- $S_j$  gates in the VQED gadget, we

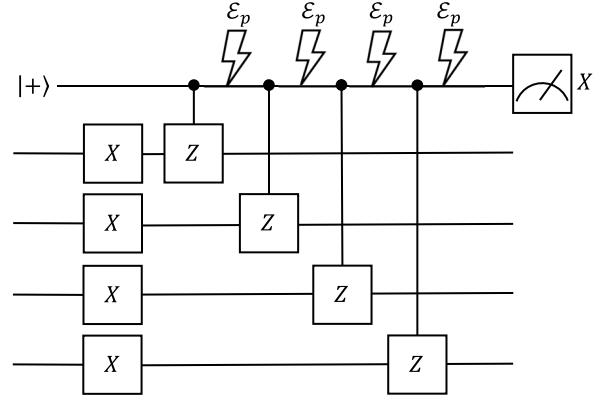


FIG. 6. Gate-based decomposition of the VQED gadget in Fig. 2(b) for the  $[[4,1,2]]$  stabilizer code with  $S_i = XXXX$  and  $S_j = ZZZZ$ . We assume that the single-qubit depolarizing noise  $\mathcal{E}_p : \rho \mapsto (1 - p)\rho + pI/2$  affects the ancilla qubit every time we perform controlled-Pauli gates.

need to perform Pauli or controlled-Pauli gates as in Fig. 6. Here, let us assume that the ancilla qubit is noisy and affected by the single-qubit depolarizing noise  $\mathcal{E}_p : \rho \mapsto (1 - p)\rho + pI/2$  every time we perform controlled-Pauli gates. If the error occurred on the controlled-Pauli gates, the error propagates to the system qubits and may cause an undetectable error. This may seem to ruin the performance of VQED. However, our VQED protocol is not based on single-shot stabilizer measurements that are highly sensitive to such noise; our method only measures the expectation value of observables, and thus the effect of the error on the ancilla qubits can be removed.

When we apply the noisy VQED gadget as in Fig. 6 to the state  $\rho$ , the state before the measurement will be

$$\begin{aligned} & \frac{(1 - p)^{w(S_j)}}{2} (|0\rangle\langle 0| \otimes S_i \rho S_i + |1\rangle\langle 1| \otimes S_j S_i \rho S_i S_j \\ & + |1\rangle\langle 0| \otimes S_j S_i \rho S_i + |0\rangle\langle 1| \otimes S_i \rho S_i S_j) \\ & + |0\rangle\langle 0| \otimes \rho' + |1\rangle\langle 1| \otimes \rho'', \end{aligned} \tag{A1}$$

where  $w(S_j)$  is the Pauli weight of  $S_j$  (the number of Pauli operators in  $S_j$ ) and  $\rho'$  and  $\rho''$  are un-normalized noisy quantum states in the system qubits. By inserting additional noise in the ancilla qubits before the measurement, we can further convert this state to

$$\begin{aligned} & \frac{(1 - p)^n}{2} (|0\rangle\langle 0| \otimes S_i \rho S_i + |1\rangle\langle 1| \otimes S_j S_i \rho S_i S_j \\ & + |1\rangle\langle 0| \otimes S_j S_i \rho S_i + |0\rangle\langle 1| \otimes S_i \rho S_i S_j) \\ & + |0\rangle\langle 0| \otimes \rho''' + |1\rangle\langle 1| \otimes \rho''', \end{aligned} \tag{A2}$$

where  $\rho'''$  and  $\rho''''$  are also un-normalized noisy quantum states in the system qubits. When we take the expectation value of the operator  $X \otimes O$  for this state, we obtain

$$\frac{(1 - p)^n}{2} (\text{tr}[S_j S_i \rho S_i O] + \text{tr}[S_i \rho S_i S_j O]), \tag{A3}$$

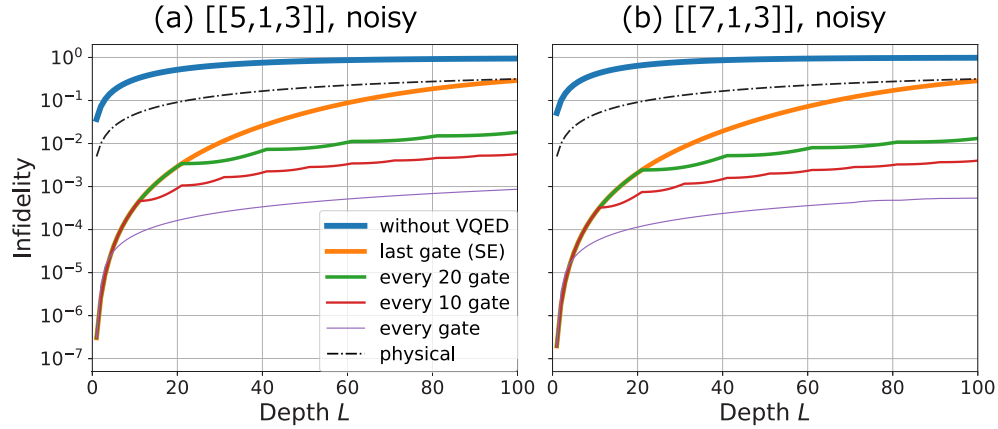


FIG. 7. Depth  $L$  dependence of infidelity  $1 - \text{tr}[\rho_{\text{det}}|\bar{\Psi}\rangle\langle\bar{\Psi}|] = 1 - \langle\bar{\Psi}|\rho_{\text{det}}|\bar{\Psi}\rangle$  between the output state of the noisy circuit with VQED  $\rho_{\text{det}}$  and the noiseless circuit  $|\bar{\Psi}\rangle$  for (a)  $[[5,1,3]]$  and (b)  $[[7,1,3]]$  stabilizer codes. All of the panels denote the results when the VQED gadgets are affected by local depolarizing noise. The “without VQED” line represents the infidelity when we did not perform VQED. The “last gate (SE)” line represents infidelity when we perform VQED only before the measurement, which is just a normal SE, as in Refs. [24,25]. The “every 20 gates” and the “every 10 gates” lines represent infidelity when we perform VQED after every 20 and 10 gates. The “every gate” line represents infidelity when we perform VQED after every gate. The “physical” line represents the infidelity of a single physical qubit without encoding.

and by uniformly sampling  $i, j \in \{1, \dots, 2^{n-k}\}$  and taking the average of the distribution we obtain

$$(1-p)^n \text{tr}[P\rho PO]. \quad (\text{A4})$$

Thus, even under the existence of noise on the ancilla qubit, we can still calculate the expectation value corresponding to the postselected state  $\rho_{\text{det}}$  as

$$\text{tr}[\rho_{\text{det}}O] = \frac{(1-p)^n \text{tr}[P\rho PO]}{(1-p)^n \text{tr}[P\rho P]}. \quad (\text{A5})$$

In the same way, we can say that the value obtained through VQED in Eq. (17) remains unchanged even under the existence of such noise, since the numerator and the

denominator of Eq. (17) are only multiplied by  $(1-p)^{nL}$ . The only change to the performance of VQED is the slight increase in the sampling overhead from  $N = O(\varepsilon^{-2} \text{tr}[\rho'_{\text{det}}]^{-2})$  to  $N = O\{\varepsilon^{-2}(1-p)^{-2nL} \text{tr}[\rho'_{\text{det}}]^{-2}\}$ .

The essential point of this robustness is that the noisy term  $|0\rangle\langle 0| \otimes \rho''' + |1\rangle\langle 1| \otimes \rho''''$  in Eq. (A2) is removed when we take the expectation value of  $X$  for the ancilla qubit. We note that the same principle applies to other noise models that are not biased by Pauli  $X$  or  $Y$ , such as local dephasing and amplitude damping noise. We also note that the noise model we used for the VQED gadget in our numerical simulation is different from what we consider in this section: instead of assuming that the noise affects the ancilla qubits after every

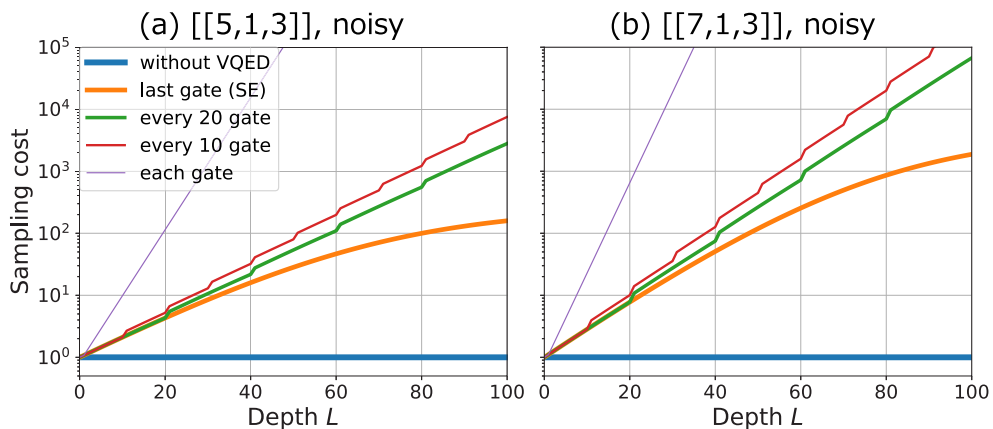


FIG. 8. Scaling of the sampling cost  $\text{tr}[\rho'_{\text{det}}]^{-2}$  with respect to the depth  $L$  of the quantum circuit for (a)  $[[5,1,3]]$  and (b)  $[[7,1,3]]$  stabilizer codes. All of the panels denote the results when the VQED gadgets are affected by local depolarizing noise. The “without VQED” line represents sampling cost when we did not perform VQED. The “last gate (SE)” line represents sampling cost when we perform VQED only before the measurement, which is just a normal SE, as in Refs. [24,25]. The “every 20 gates” and the “every 10 gates” lines represent the sampling cost when we perform VQED after every 20 and 10 gates. The “every gate” line represents the sampling cost when we perform VQED after every gate.



execution of the controlled-Pauli gates, we assumed that the noise affects both the system and ancilla qubits after every execution of  $S_i$  and controlled- $S_j$  gates.

### APPENDIX B: TRANSVERSAL SINGLE-QUBIT GATES IN STABILIZER CODES

In this section, we clarify the sets of transversal single-qubit gates we use in the numerical simulation. For the  $[[4, 1, 2]]$  stabilizer code, we use a set of single-qubit Pauli gates as a set of transversal single-qubit gates [36]. For the  $[[5, 1, 3]]$  stabilizer code, we use  $\{X, Y, Z, SH\}$  as a set of transversal single-qubit gates [31]. For the  $[[7, 1, 3]]$  stabilizer

code, we use a set of single-qubit Clifford gates as a set of transversal single-qubit gates [31].

### APPENDIX C: NUMERICAL SIMULATION OF VQED WHEN THE VQED GADGETS ARE NOISY

In this section, we present the performance of VQED for  $[[5, 1, 3]]$  and  $[[7, 1, 3]]$  stabilizer codes when the VQED gadget  $S_{ij}$  gate and controlled- $S_{ji}$  gate in Fig. 3 are each affected by the local depolarizing noise  $\mathcal{E}_p^{\otimes n}$  and  $\mathcal{E}_p^{\otimes(n+1)}$  with the same error rate  $p = 0.01$ . Our results are shown in Figs. 7 and 8. The results for  $[[5, 1, 3]]$  and  $[[7, 1, 3]]$  stabilizer codes are qualitatively similar to those for the  $[[4, 1, 2]]$  stabilizer codes shown in Figs. 4(d) and 5(d).

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