Coherence manipulation with stochastic incoherent operation

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An important problem in the coherent resource theory is the convertibility of coherent states by free operations. In this paper we consider such a problem from a mixed coherent state into a pure one by using both stochastic incoherent operations and incoherent operations. We prove that stochastic incoherent operations can transform a mixed coherent state into a pure coherent state if and only if the density matrix of the initial state contains a singular principal submatrix. Then we provide two sufficient conditions with explicit expressions for the probabilistic transformation and the deterministic transformation. These two sufficient conditions facilitate the construction of concrete operators of stochastic incoherent operations and incoherent operations. These results demonstrate that incoherent operations are strictly more powerful than strictly incoherent operations within the probabilistic transformation as well as the deterministic transformation.

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I. INTRODUCTION

Quantum coherence, or superposition, is an intrinsic feature of quantum mechanics and offers many advantages over the classical world. The fact that quantum coherence is the central component of many quantum information processing protocols [1] gives rise to broad interest in areas such as quantum communication, quantum computing, and quantum cryptography.

Quantum coherence can be regarded precisely as resources, which provide the advantages enabled by the quantum information tasks, within the so-called resource-theoretic setting [2–7]. The resource-theoretic framework consists of two elements: a set of free states and a set of free operations which specifically act invariantly on the free states [8–10]. For the resource theory of coherence, the free states enjoy a simplified form, i.e., a diagonal density matrix on a fixed computational basis. However, several free operations are proposed via taking diverse physical and mathematical motivations into consideration [5,6], such as maximal incoherent operations (MIOs) [2], incoherent operations (IOs) [4], dephasing-covariant incoherent operations (DIOs) [11,12], strictly incoherent operations (SIOs) [13,14], and physically incoherent operations [8,15,16]. Here we focus on IOs, which can be seen as generalized measurements that are performed on a quantum system and coherence nongenerating for each measurement outcome.

It is necessary to study the different free operation abilities, which can help us make better use of them to manipulate coherence in practice. Thus, understanding their capabilities and limitations is one of the fundamental problems posed in coherent resource theory. This can be formulated as the question of resource convertibility: When can a coherent state be transformed into another under a given set of free operations? This problem only has been fully resolved for the pure state case. It was shown in Refs. [16–19] that determining the convertibility between pure coherent states only requires one to compare the majorization relationship of pure coherent states, i.e., by using SIOs, IOs, and DIOs, the deterministic transformation between pure coherent states must follow the majorization condition. These results tell us that SIOs, IOs, and DIOs possess the same operational capabilities in manipulating pure coherent states, despite SIOs forming a strictly smaller set than both IOs and DIOs. Moreover, SIOs are as powerful as IOs and even MIOs in qubit transformations [16,20–26]. These trivial results characterize the corresponding deterministic transformations and show that the larger sets of free operations IOs and DIOs do not give any advantage over SIOs. In practical quantum information processing protocols, the last step always refers to a quantum measurement, which itself reflects the probability of the whole quantum operation. The probabilistic transformation between pure coherent states via SIOs was studied in Refs. [18,27]. Notable progress on the above problem was presented in Ref. [28], where it was shown that IOs and stochastic IOs can increase the dimension of the maximal pure coherent subspace, which implies that there is indeed an operational gap between IOs and SIOs under coherent state transformations.

The purpose of the present article is to provide a comparative investigation between SIOs and IOs. This is an extension of Ref. [28] and covers its main results. The main contributions of this paper are as follows.

(i) Two complete characterizations of extracting pure coherence under stochastic IOs are provided for cases of the target pure coherent state of coherence rank 2 and greater coherence rank.

(ii) It is shown that a mixed state of rank $m-1$ cannot be converted to a pure coherent state under IOs, where m is the number of the nonzero diagonal elements in the considered mixed state.

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(iii) Based on maximal linearly independent sets, a method is introduced to transform a mixed state into a pure state under stochastic IOs.

(iv) A simple and operable sufficient condition is proposed for the deterministic IO transformation which transforms the initial state into a pure coherent state. Moreover, the specific IO Kraus operators, which output the target pure state, can be immediately constructed.

This paper is organized as follows. In Sec. II we recall several notions of the quantum resource theory of coherence, including IOs and stochastic IOs. In Sec. III we present our main results, i.e., we not only tackle and solve the condition to transform a mixed coherent state into a pure one under stochastic IOs but also provide a sufficient condition for such a transformation under IOs. In Sec. IV we summarize and discuss our results.

II. PRELIMINARIES

Let H be the Hilbert space of a d-dimensional quantum system. A particular basis of H is denoted by $\{|i\rangle\}_{i=1}^{d}$. Specifically, a state δ is said to be incoherent if it is diagonal in the basis, i.e., $\delta = \Sigma_{i=1}^{d}\delta_i|i\rangle\langle i|$, where the coefficients $\delta_i \geqslant 0$ form a probability distribution. We use I to represent the set of incoherent states. Any state that cannot be called a diagonal matrix is defined as a coherent state. For a pure coherent state $|\varphi_{\mathbf{m}}\rangle = \sum_{i=1}^{d}\varphi_i|i\rangle$, we will denote $|\varphi_{\mathbf{m}}\rangle\langle\varphi_{\mathbf{m}}|$ by $\varphi_{\mathbf{m}}$, where the bold subscript represents the number of nonzero diagonal terms ($\varphi_i \neq 0$), i.e., the coherence rank of the pure coherent state $\varphi_{\mathbf{m}}$. In particular, the maximally coherent state of dimension d is defined as $|\Psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|i\rangle$. For a mixed coherent state ρ, we express $\sqrt{\rho}$ as the column-vector form $(|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle)$, which plays a central role in the following transformations.

We mainly investigate the transformation from a mixed coherent state to a pure one (in short, a mixed to pure transformation) by using both incoherent operations and stochastic incoherent operations. An incoherent operation is a completely positive and trace-preserving (CPTP) map, expressed as

$$\Lambda(\rho) = \sum_{\alpha=1}^{N} K_\alpha\rho K_\alpha^\dagger = \sum_{\alpha=1}^{N}(\sqrt{\rho}K_\alpha^\dagger)^\dagger\sqrt{\rho}K_\alpha^\dagger, \tag{1}$$

where the Kraus operators K_α satisfy not only $\sum_{\alpha=1}^{N}K_\alpha^\dagger K_\alpha = \mathbb{I}$ but also $K_\alpha I K_\alpha^\dagger \subseteq I$ for all K_α, i.e., each K_α transforms an incoherent state into an incoherent state, and such a K_α is called an incoherent Kraus operator (incoherent operator). With the notion of incoherent operation, a stochastic incoherent operation is constructed by a subset of these incoherent Kraus operators. Without loss of generality, we use the subscript to represent the subset containing the incoherent operators. Then a stochastic incoherent operation, denoted by $\Lambda_S(\rho)$, is defined as [28,29]

$$\Lambda_S(\rho) = \frac{\sum_{\alpha=1}^{L}K_\alpha\rho K_\alpha^\dagger}{\mathrm{Tr}\big(\sum_{\alpha=1}^{L}K_\alpha\rho K_\alpha^\dagger\big)} = \frac{\sum_{\alpha=1}^{L}(\sqrt{\rho}K_\alpha^\dagger)^\dagger\sqrt{\rho}K_\alpha^\dagger}{\mathrm{Tr}\big(\sum_{\alpha=1}^{L}(\sqrt{\rho}K_\alpha^\dagger)^\dagger\sqrt{\rho}K_\alpha^\dagger\big)}, \tag{2}$$

FIG. 1. Ladder operator.

where the subset $\{K_1, K_2, \ldots, K_L\}$ satisfies $\sum_{\alpha=1}^{L}K_\alpha^\dagger K_\alpha \leqslant \mathbb{I}$. Clearly, the resultant state $\Lambda_S(\rho)$ is obtained with probability $p = \mathrm{Tr}(\sum_{\alpha=1}^{L}K_\alpha\rho K_\alpha^\dagger)$ under a stochastic incoherent operation Λ_S, while the resultant state $\Lambda(\rho)$ is fully deterministic under an incoherent operation Λ.

The following lemma characterizes the form of Kraus operators belonging to an incoherent operation.

Lemma 1 (from [16,28–31]). (a) For an incoherent operation $\Lambda = \sum_\alpha K_\alpha(\cdot)K_\alpha^\dagger$, the form of K_α is $K_\alpha = \sum_i c_{\alpha i}|f_\alpha(i)\rangle\langle i|$, where $f_\alpha : \{1, \ldots, d\} \to \{1, \ldots, d\}$ and the coefficients $c_{\alpha i}$ satisfy

$$\sum_{\substack{\alpha \\ f_\alpha(i)=f_\alpha(j)}} c_{\alpha i}^* c_{\alpha j} = \delta_{ij}.$$

(b) For an incoherent Kraus operator K_α, there is at most one nonzero element in each column of K_α. In other words, an incoherent Kraus operator can be represented in the form

$$K_\alpha = \left(\sum_i c_{\alpha i}|g_\alpha(i)\rangle\langle i|\right)P_\alpha, \tag{3}$$

where $g_\alpha : \{1, \ldots, d\} \to \{1, \ldots, d\}$ is a nondecreasing function and P_α is a permutation operator.

We note that the coefficients $c_{\alpha i}$ of the incoherent operator $K = \sum_i c_{\alpha i}|g_\alpha(i)\rangle\langle i|$ mentioned in Lemma 1(b), where $g_\alpha : \{1, \ldots, d\} \to \{1, \ldots, d\}$ is a nondecreasing function, are arranged in a ladder form, as shown in Fig. 1. This form will be used repeatedly in the article.

Let Δ denote the dephasing map in the basis $\{|i\rangle\}_{i=1}^{d}$, i.e., $\Delta(\cdot) := \sum_{i=1}^{d}|i\rangle\langle i|(\cdot)|i\rangle\langle i|$, and let Π_S denote incoherent projectors with the form $\Pi_S := \sum_{i\in S}|i\rangle\langle i|$ for some subset of indices $S \subseteq \{1, \ldots, d\}$. Moreover, we refer to SIOs, which are all IOs with $K_\alpha^\dagger I K_\alpha \subseteq I$ for all incoherent operators K_α, and DIOs, which are all CPTP maps satisfying $\Lambda \circ \Delta = \Delta \circ \Lambda$.

Here we need to provide further details about the characteristics of $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle)$. It is worth noting that $\sqrt{\cdot}$ is an operator function, which acts on the eigenvalues of the operator. As mentioned above, we express the matrix $\sqrt{\rho}$ by its column vectors. Besides $\rho = \sqrt{\rho}^\dagger\sqrt{\rho} = (\langle\rho_i|\rho_j\rangle)_{i,j\in\{1,\ldots,d\}}$, the following lemma gives general properties of these column vectors $|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle$.

Lemma 2 (from [32]). Let $|v_1\rangle, \ldots, |v_d\rangle$ be vectors in an inner product space V with inner product $\langle\cdot|\cdot\rangle$ and let $G := (\langle v_i|v_j\rangle)_{i,j=1,\ldots,d}$, which is a $d \times d$ matrix. Then (a) G is

Hermitian and positive semidefinite and (b) G is positive-definite (nonsingular) if and only if the vectors $|v_1\rangle, \dots, |v_d\rangle$ are linearly independent. Here the matrix G is called a Gram matrix.

The mixed to pure convertibility criterion under stochastic SIOs is simply presented as the existence of a singular principal submatrix of order 2 on the incoherent basis representation of the initial state [33–35]. By leveraging Lemma 2, we update the complete characterization of the mixed to pure transformation under stochastic SIOs.

Given a $d \times d$ matrix M, for an index subset $S \subseteq \{1, \dots, d\}$, we denote by $M[S]$ the principal submatrix of entries that lie in the rows and columns of M indexed by S. For example, for a three-dimensional quantum state $\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}$, we have

$$\rho[\{13\}] = \begin{pmatrix} \rho_{11} & \rho_{13} \\ \rho_{31} & \rho_{33} \end{pmatrix},$$

$$\Delta(\rho)[\{13\}] = \begin{pmatrix} \rho_{11} & 0 & 0 \\ 0 & \rho_{22} & 0 \\ 0 & 0 & \rho_{33} \end{pmatrix}[\{13\}] = \begin{pmatrix} \rho_{11} & 0 \\ 0 & \rho_{33} \end{pmatrix}.$$

In particular, for a Gram matrix G, we have

$$G[S] = (\langle v_i|v_j\rangle)_{i,j\in S}$$

for any $S \subseteq \{1, \dots, d\}$.

Lemma 3 (from [33–35]). Given a d-dimensional coherent state ρ, write $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \dots, |\rho_d\rangle)$. Then the following statements are equivalent.

(a) There exists a stochastic strictly incoherent operation $\Lambda_S(\cdot)$ such that the resultant state $\Lambda_S(\rho)$ is a pure coherent state $\varphi_{\mathbf{m}}$ $(m \geqslant 2)$.

(b) There is a subset $S \subseteq \{1, \dots, d\}$ with $|S| = m$ such that

$$\Delta(\rho)[S] = \sum_{i\in S} \langle\rho_i|\rho_i\rangle |l(i)\rangle\langle l(i)| \tag{4}$$

is positive and, for any $i, j \in S$ $(i \neq j)$, the equation

$$\det(\rho[\{i, j\}]) = \langle\rho_i|\rho_i\rangle\langle\rho_j|\rho_j\rangle - \langle\rho_i|\rho_j\rangle\langle\rho_j|\rho_i\rangle = 0 \tag{5}$$

holds, where $l: S \to \{1, \dots, m\}$ is a strictly increasing bijection, i.e., $l(i) > l(j)$ whenever $i > j$.

(c) There is a subset $S \in \{1, \dots, d\}$ with $|S| = m$ such that, for any $i, j \in S$ $(i \neq j)$, the column vectors $|\rho_i\rangle$ and $|\rho_j\rangle$ are nonzero vectors and linearly dependent.

Note that Lemma 3(b) is equivalent to Theorem 1 in Ref. [33], which provides the necessary and sufficient conditions for the probabilistic mixed to pure transformation via stochastic SIOs. Specifically, if there is an incoherent projector $\Pi_{\{i,j\}}$ $(i, j \in \{1, \dots, d\}$ and $i \neq j)$ such that $\frac{\Pi_{\{i,j\}}\rho\Pi_{\{i,j\}}}{\mathrm{Tr}(\Pi_{\{i,j\}}\rho\Pi_{\{i,j\}})} = \varphi_{\mathbf{2}}$, then we have

$$\Pi_{\{i,j\}}\rho\Pi_{\{i,j\}}[\{i, j\}] = \begin{pmatrix} \langle\rho_i|\rho_i\rangle & \langle\rho_i|\rho_j\rangle \\ \langle\rho_j|\rho_i\rangle & \langle\rho_j|\rho_j\rangle \end{pmatrix} = \begin{pmatrix} \langle\rho_i| \\ \langle\rho_j| \end{pmatrix}(|\rho_i\rangle, |\rho_j\rangle),$$

where $\langle\rho_i|\rho_i\rangle\langle\rho_j|\rho_j\rangle = \langle\rho_i|\rho_j\rangle\langle\rho_j|\rho_i\rangle$. Due to the Cauchy-Schwarz inequality, we note that $\langle\rho_i|\rho_i\rangle\langle\rho_j|\rho_j\rangle \geqslant \langle\rho_i|\rho_j\rangle\langle\rho_j|\rho_i\rangle$ and the equality holds if and only if $|\rho_i\rangle$ and $|\rho_j\rangle$ are linearly dependent. For more general cases, combine Lemma 2 and Theorem 3 in Ref. [35], sustaining the establishment of Lemma 3.

III. COHERENCE STATE TRANSFORMATION VIA STOCHASTIC INCOHERENT OPERATIONS

We begin our study by observing the important role played by individual incoherent operators of stochastic IOs under the mixed to pure transformation task. This leads to the following lemma.

Lemma 4 (from [28]). For a coherent state ρ there exists a stochastic incoherent operation Λ_S such that $\Lambda_S(\rho) = \varphi$. There must be an incoherent operator K_α belonging to Λ_S such that $\frac{K_\alpha\rho K_\alpha^\dagger}{\mathrm{Tr}(K_\alpha\rho K_\alpha^\dagger)}$ is the pure coherent state φ.

It is known that IOs, SIOs, and DIOs have the same power in pure to pure transformations, which can be completely characterized by majorization relations [17,19]. This yields $P_{\mathrm{(S)IO}}(\Psi_m \to \varphi_{\mathbf{n}}) = 1$ $(\mathbf{n} \leqslant m)$ and [18]

$$P_{\mathrm{(S)IO}}(\varphi_{\mathbf{n}} \to \Psi_m) = \begin{cases} 0, & \mathbf{n} < m \\ \min_{k\in[1,m]} \frac{m}{k} \sum_{i=m-k+1}^{d} \varphi_i^2, & \mathbf{n} \geqslant m, \end{cases}$$

where we have assumed without loss of generality that the coefficients of $|\varphi_{\mathbf{n}}\rangle = \sum_{i=1}^{d} \varphi_i|i\rangle$ are arranged in nonincreasing order. Therefore, $\varphi_{\mathbf{m}}$ can be probabilistically transformed into Ψ_m and vice versa. Apparently, it suffices to consider the maximally coherent state Ψ_m as the target state of the mixed to pure transformation under stochastic IOs. Equipped with the above knowledge, we can present the following theorem.

Theorem 1. Given a d-dimensional coherent state ρ, write $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \dots, |\rho_d\rangle)$. Then the following statements are equivalent.

(a) There exists a stochastic incoherent operation $\Lambda_S(\cdot)$ such that the resultant state $\Lambda_S(\rho)$ is a pure coherent state $\varphi_{\mathbf{2}}$.

(b) There is a subset $S \subseteq \{1, \dots, d\}$ $(S \neq \emptyset)$ such that

$$\Delta(\rho)[S] = \sum_{i\in S} \langle\rho_i|\rho_i\rangle |l(i)\rangle\langle l(i)| \tag{6}$$

is positive-definite and the equation

$$\det(\rho[S]) = \det[(\langle\rho_i|\rho_j\rangle)_{i,j\in S}] = 0 \tag{7}$$

holds, where $l: S \to \{1, \dots, |S|\}$ is a strictly increasing bijection.

(c) There is a subset $S \subseteq \{1, \dots, d\}$ $(S \neq \emptyset)$ such that the column vectors $|\rho_i\rangle$ $(i \in S)$ are nonzero vectors and linearly dependent.

Proof. (a) $\Rightarrow$ (c). First, we show if the maximally coherent state Ψ_2 can be obtained from the initial state ρ via a stochastic IO Λ_s, then there is a subset $S \subseteq \{1, \dots, d\}$ $(S \neq \emptyset)$ such that the linear system $\sqrt{\rho}\Pi_S|x\rangle = 0$ has nonzero solutions, which means that the column vectors $|\rho_i\rangle$ $(i \in S)$, which make up the coefficient matrix of $\sqrt{\rho}\Pi_S|x\rangle = 0$, are linearly dependent, while the diagonal entries of $\rho[S] = (\langle\rho_i|\rho_j\rangle)_{i,j\in S}$ are nonzero.

According to Lemma 4, let us assume that we can obtain the pure coherent state Ψ_2 from the coherent state ρ by using an incoherent operator K_α, which belongs to Λ_S. We have

$$\frac{K_\alpha\rho K_\alpha^\dagger}{\mathrm{Tr}(K_\alpha\rho K_\alpha^\dagger)} = \frac{(\sqrt{\rho}K_\alpha^\dagger)^\dagger\sqrt{\rho}K_\alpha^\dagger}{\mathrm{Tr}[(\sqrt{\rho}K_\alpha^\dagger)^\dagger\sqrt{\rho}K_\alpha^\dagger]} = \Psi_2, \tag{8}$$

where $|\Psi_2\rangle = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \Theta)^T$ and Θ represents the all-zero matrix of appropriate size. According to Lemma 1(b), the incoherent operator K_α has the form

$$K_\alpha = \begin{pmatrix} a_1^* & a_2^* & \cdots & a_k^* & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & b_{k+1}^* & \cdots & b_d^* \\ \Theta & \Theta & \cdots & \Theta & \Theta & \cdots & \Theta \end{pmatrix} P_\alpha, \tag{9}$$

where P_α is a permutation, Θ represents the all-zero matrix of appropriate size, and $\mathrm{rank}(K_\alpha) = 2$. Substituting Eq. (9) into Eq. (8), we can deduce the identity

$$\sqrt{\rho}K_\alpha^\dagger = |\gamma\rangle\langle\Psi_2| \quad (|\gamma\rangle \neq 0),$$

where $|\gamma\rangle = \sqrt{2}a_1|\rho_{1'}\rangle + \cdots + \sqrt{2}a_k|\rho_{k'}\rangle = \sqrt{2}b_{k+1}|\rho_{(k+1)'}\rangle + \cdots + \sqrt{2}b_d|\rho_{d'}\rangle$ [$i = P_\alpha^\dagger(i')$ and $i, i' = 1, \ldots, d$]. Thus, we obtain that $(a_1, \ldots, a_k, -b_{k+1}, \ldots, -b_d)^T$ is a nonzero solution of the linear system $\sqrt{\rho}P_\alpha^\dagger|x\rangle = 0$. Considering Lemma 2, we obtain that, as a positive-semidefinite matrix ρ, its elements of the rows and columns corresponding to the zero-diagonal elements, i.e., $\langle\rho_i|\rho_i\rangle = 0$, are all zero. This means that the linear system $\sqrt{\rho}\Pi_S|x\rangle = 0$ also has nonzero solutions. Here $S = \{i \mid |\rho_i\rangle$ is a nonzero vector, $i = 1, \ldots, d\}$.

(c) $\Rightarrow$ (a). First, we assume that $S = \{1, \ldots, d'\}$ $(d' \leqslant d)$ and that the column vectors $|\rho_i\rangle$ $(i \in S)$ are nonzero vectors as well as linearly dependent. It follows that $d' > r = \mathrm{rank}(\rho[S])$. Without loss of generality, we assume that the vectors $|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_r\rangle$ are linearly independent. Second, suppose that a nonzero solution of the linear system $x_1|\rho_1\rangle + x_2|\rho_2\rangle + \cdots + x_{r+1}|\rho_{r+1}\rangle = 0$ $(r + 1 \leqslant d')$ is $(a_1, a_2, \ldots, a_r, a_{r+1})^T$. Then we get the equation

$$a_1|\rho_1\rangle + a_2|\rho_2\rangle + \cdots + a_{r+1}|\rho_{r+1}\rangle = 0.$$

Then it is easy to get the following equation in terms of complex conjugation:

$$a_1^*\langle\rho_1| + a_2^*\langle\rho_2| + \cdots + a_{r+1}^*\langle\rho_{r+1}| = 0.$$

Since $|\rho_{r+1}\rangle$ is a nonzero vector, it means that a_{r+1} is also a nonzero number. Using these two equations, we construct an incoherent operator K_α,

$$K_\alpha = \begin{pmatrix} a_1^* & a_2^* & \cdots & a_r^* & 0 & \Theta \\ 0 & 0 & \cdots & 0 & -a_{r+1}^* & \Theta \\ \Theta & \Theta & \cdots & \Theta & \Theta & \Theta \end{pmatrix} \tag{10}$$

for which Θ represents the all-zero matrix of appropriate size. We can check that the equations

$$K_\alpha\rho K_\alpha^\dagger = (\sqrt{\rho}K_\alpha^\dagger)^\dagger\sqrt{\rho}K_\alpha^\dagger \tag{11}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}\langle\gamma| \\ \frac{1}{\sqrt{2}}\langle\gamma| \\ \Theta \end{pmatrix}\left(\frac{1}{\sqrt{2}}|\gamma\rangle, \frac{1}{\sqrt{2}}|\gamma\rangle, \Theta\right) \tag{12}$$

$$= \langle\gamma|\gamma\rangle\Psi_2 \tag{13}$$

hold, where $|\gamma\rangle = \sqrt{2}a_1|\rho_1\rangle + \cdots + \sqrt{2}a_r|\rho_r\rangle = -\sqrt{2}a_{r+1}|\rho_{r+1}\rangle \neq 0$. Third, we carefully choose $\sum_{i=1}^r |a_i|^2 = 1$ $(0 < |a_{r+1}|^2 \leqslant 1)$ or $\sum_{i=1}^r |a_i|^2 \leqslant 1$ $(|a_{r+1}|^2 = 1)$ by the following method. Assume that $(x_1, x_2, \ldots, x_{r+1})^T$ is feasible for the linear system $(|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_{r+1}\rangle)|x\rangle = 0$. Then, if $\sum_{i=1}^r |x_i|^2 \geqslant |x_{r+1}|^2$, let

$$(a_1, a_2, \ldots, a_r, a_{r+1})^T = \frac{1}{\sqrt{\sum_{i=1}^r |x_i|^2}}(x_1, x_2, \ldots, x_{r+1})^T,$$

and if not, let

$$(a_1, a_2, \ldots, a_r, a_{r+1})^T = \frac{1}{\sqrt{|x_{r+1}|^2}}(x_1, x_2, \ldots, x_{r+1})^T.$$

Thus, clearly $K_\alpha^\dagger K_\alpha \leqslant \mathbb{I}$, as follows from $\|K_\alpha^\dagger K_\alpha\|_\infty = \max\{\|K_\alpha^\dagger K_\alpha u\| : u \in H, \ \|u\| \leqslant 1\} = 1$.

Finally, according to Lemma 2, we can easily get the implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b). This completes the proof.

Note that Theorem 1(b) is equivalent to Theorem 1 in Ref. [29], which provides the criterion for a state to be distillable under stochastic IOs. Furthermore, we note that the coherent state ρ with $\mathrm{rank}(\rho) = \mathrm{rank}[\Delta(\rho)] = r$ actually cannot be converted into any coherent pure state because there is no subset S, which makes $\Delta(\rho)[S] > 0$ and $\det(\rho[S]) = 0$ simultaneously hold. In a sense, this state is a full-rank coherent state for the corresponding r-dimensional subsystem. In fact, there is no incoherent operator that allows us to distill any coherent pure state from a full-rank state, even probabilistically [27]. Further, this situation cannot be improved by embedding a full-rank state in a larger quantum system.

Theorem 1 manifests that stochastic IOs are generally stronger than stochastic SIOs when we want to transform a mixed coherent state into the maximally coherent state Ψ_2, i.e., the probabilistic coherence distillation [27]. Theorem 1 shows that, under probabilistic coherence distillation, stochastic IOs suit more general conditions than stochastic SIOs. In addition, we note that each nonzero solution of the linear system $\sqrt{\rho}\Pi_S|x\rangle = 0$ always corresponds to an incoherent operator K_α such that $K_\alpha\rho K_\alpha^\dagger \propto \Psi_2$.

As a supplement, we remind the reader that, for any two incoherent operations Λ_1 with Kraus operators $\{K_\alpha^1\}$ and Λ_2 with Kraus operators $\{K_\beta^2\}$, the operation $\Lambda = \Lambda_1 \circ \Lambda_2$ is also an incoherent operation with Kraus operators $\{K_\gamma = K_\alpha^1 K_\beta^2\}$ because $K_\gamma I K_\gamma^\dagger \subseteq I$ and $\sum_\gamma K_\gamma^\dagger K_\gamma = \mathbb{I}$. In addition, we know that any permutation P and its inverse are SIOs. With this knowledge, it is easy to show that, for an IO or a stochastic IO, there is always an incoherent operator that can be given in the form

$$K_\alpha = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1t_1} & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a_{21} & \cdots & a_{2t_2} & 0 & \cdots & 0 \\ & & & & & \cdots & & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & a_{m1} & \cdots & a_{mt_m} \end{pmatrix}. \tag{14}$$

Theorem 1 establishes necessary and sufficient criteria for the existence of a stochastic IO mapping a given mixed state ρ to any pure coherent state $\varphi_{\mathbf{2}}$ of coherence rank 2. In particular, the explicit condition (b) is in terms of principal subdeterminants of the incoherent basis representation of ρ. In Theorem 2 we generalize analogous results to the target pure coherent state of higher coherence rank.

Theorem 2. Given a d-dimensional coherent state ρ, write $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \dots, |\rho_d\rangle)$. Then the following statements are equivalent.

(a) There exists a stochastic incoherent operation $\Lambda_S(\cdot)$ such that the resultant state $\Lambda_S(\rho)$ is a pure coherent state $\varphi_{\mathbf{m}}$ ($d \geqslant \mathbf{m} \geqslant 2$).

(b) There is a subset $S \subseteq \{1, \dots, d\}$ ($S \neq \emptyset$) which can be partitioned into m disjoint subsets S_s ($s = 1, \dots, m$) such that, for all $s = 1, \dots, m$ the column vectors $|\rho_i\rangle$ ($i \in S_s$) are nonzero vectors and the intersection of the subspaces $\mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})$ includes nonzero vectors, i.e., $\dim[\bigcap_{s=1}^m \mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})] > 0$.

(c) There is a subset $S \subseteq \{1, \dots, d\}$ ($S \neq \emptyset$) which can be partitioned into m disjoint subsets S_s ($s = 1, \dots, m$) such that, for all $s = 1, \dots, m$, the column vectors $|\rho_i\rangle$ ($i \in S_s$) are not only nonzero vectors but also linearly independent and the intersection of the subspaces $\mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})$ includes nonzero vectors, i.e., $\dim[\bigcap_{s=1}^m \mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})] > 0$.

Proof. (a) $\Rightarrow$ (b). By Lemma 4, it suffices to show that if an incoherent operator K_α transforms the initial state ρ into the maximally coherent state Ψ_m, then there is a subset $S \subseteq \{1, \dots, d\}$ ($S \neq \emptyset$) which can be partitioned into m disjoint subsets S_s ($s = 1, \dots, m$) such that the column vectors $|\rho_i\rangle$ ($i \in S$) are nonzero vectors and the intersection of the subspaces $\mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})$ includes nonzero vectors, i.e., $\dim[\bigcap_{s=1}^m \mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})] > 0$.

Suppose $K_\alpha \rho K_\alpha^\dagger \propto \Psi_m$; then we have $(\sqrt{\rho}K_\alpha^\dagger)^\dagger \sqrt{\rho} K_\alpha^\dagger \propto |\Psi_m\rangle\langle\Psi_m|$, i.e., $\sqrt{\rho}K_\alpha^\dagger = |\gamma\rangle\langle\Psi_m|$, where $\langle\Psi_m| = (\frac{1}{\sqrt{m}}, \frac{1}{\sqrt{m}}, \dots, \frac{1}{\sqrt{m}}, \Theta)$ and $|\gamma\rangle$ is a nonzero vector of d elements. Here Θ represents the all-zero matrix of appropriate size. Consider the form of K_α as

$$K_\alpha = \begin{pmatrix} a^*_{\mathbf{11}} & a^*_{\mathbf{12}} & \cdots & a^*_{\mathbf{1}t_1} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a^*_{\mathbf{21}} & \cdots & 0 & \cdots & 0 \\ \cdots & & & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & a^*_{\mathbf{m}1} & \cdots & a^*_{\mathbf{m}t_m} \\ \Theta & \Theta & \cdots & \Theta & \Theta & \cdots & \Theta & \cdots & \Theta \end{pmatrix}, \tag{15}$$

while $\mathrm{rank}(K_\alpha) = m$. There are nonzero vectors $(a_{\mathbf{11}}, a_{\mathbf{12}}, \dots, a_{\mathbf{1}t_1})^T$, $(a_{\mathbf{21}}, \dots, a_{\mathbf{2}t_2})^T, \dots, (a_{\mathbf{m}1}, \dots, a_{\mathbf{m}t_m})^T$ such that the following equations hold:

$$\begin{aligned} \frac{1}{\sqrt{m}}|\gamma\rangle &= (a_{\mathbf{11}}|\rho_{\mathbf{11}}\rangle + \cdots + a_{\mathbf{1}t_1}|\rho_{\mathbf{1}t_1}\rangle) \\ &= (a_{\mathbf{21}}|\rho_{\mathbf{21}}\rangle + \cdots + a_{\mathbf{2}t_2}|\rho_{\mathbf{2}t_2}\rangle) \\ &\ \ \vdots \\ &= (a_{\mathbf{m}1}|\rho_{\mathbf{m}1}\rangle + \cdots + a_{\mathbf{m}t_m}|\rho_{\mathbf{m}t_m}\rangle). \end{aligned}$$

Here there is a bijection $f: \{1, \dots, d\} \to \{\mathbf{1}1, \dots, \mathbf{1}t_1, \mathbf{2}1, \dots, \mathbf{2}t_2, \dots, \mathbf{m}1, \dots, \mathbf{m}t_m\}$ such that $f(i) = \mathbf{s}j$ ($j > 0$) for which the boldface type of $s = 1, \dots, m$ is used to mark the partition. Therefore, we can obtain the subset $S = \{\mathbf{s}j \mid |\rho_{\mathbf{s}j}\rangle$ is a nonzero vector; $s = 1, \dots, m;\ j > 0\}$, which can be partitioned into m disjoint subsets $S_s = \{\mathbf{s}j \mid |\rho_{\mathbf{s}j}\rangle$ is a nonzero vector, $j > 0\}$, and meet the conditions in (b) due to $|\gamma\rangle \in \bigcap_{s=1}^m \mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})$.

(b) $\Rightarrow$ (c). For all $s = 1, \dots, m$ there is always a subset S'_s ($S'_s \subseteq S_s$) such that the principal submatrix $\rho[S'_s]$ is nonsingular, i.e.,

$$\det(\rho[S'_s]) > 0,$$

and the principal submatrix $\rho[S'_s \cup \{j\}]$ ($j \in S_s - S'_s$) is singular, i.e.,

$$\det(\rho[S'_s \cup \{j\}]) = 0 \quad (j \in S_s - S'_s).$$

Thus, according to Lemma 2, the vectors $|\rho_{i'}\rangle$ ($i' \in S'_s$) are linearly independent, while the vectors $|\rho_{i'}\rangle$ ($i' \in S'_s \cup \{j\}$ and $j \in S_s - S'_s$) are linearly dependent for each $s = 1, \dots, m$. Therefore, for all $s = 1, \dots, m$ there are vectors $|\rho_{i'}\rangle$ ($i' \in S'_s$ and $S'_s \subseteq S_s$) such that the vectors $|\rho_{i'}\rangle$ ($i' \in S'_s$) are linearly independent and

$$\mathrm{span}(\{|\rho_{i'}\rangle\}_{i'\in S'_s}) = \mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s}).$$

(c) $\Rightarrow$ (a). For all $s = 1, \dots, m$, the subspace $\mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})$ is equal to the space of all vectors that may be written as a linear combination of elements of $\{|\rho_i\rangle\}_{i\in S_s}$. Suppose that there is a nonzero vector $|\gamma\rangle$ such that $|\gamma\rangle \in \bigcap_{s=1}^m \mathrm{span}(\{|\rho_i\rangle\}_{i\in S_s})$. Specifically, there are nonzero vectors $(a_{\mathbf{11}}, a_{\mathbf{12}}, \dots, a_{\mathbf{1}t_1})^T$, $(a_{\mathbf{21}}, \dots, a_{\mathbf{2}t_2})^T, \dots, (a_{\mathbf{m}1}, \dots, a_{\mathbf{m}t_m})^T$

such that the following equations hold:

$$\begin{aligned}\frac{1}{\sqrt{m}}|\gamma\rangle &= (a_{\mathbf{1}1}|\rho_{\mathbf{1}1}\rangle + \cdots + a_{\mathbf{1}t_1}|\rho_{\mathbf{1}t_1}\rangle)\\ &= (a_{\mathbf{2}1}|\rho_{\mathbf{2}1}\rangle + \cdots + a_{\mathbf{2}t_2}|\rho_{\mathbf{2}t_2}\rangle)\\ &\vdots\\ &= (a_{\mathbf{m}1}|\rho_{\mathbf{m}1}\rangle + \cdots + a_{\mathbf{m}t_m}|\rho_{\mathbf{m}t_m}\rangle).\end{aligned}$$

Then we can construct an incoherent operator such that a stochastic IO transformation $\rho \to \Psi_m$ can be achieved. We choose that $\sum_{\mathbf{l}1}^{\mathbf{l}t_l}|a_{\mathbf{l}i}|^2 = 1$ holds for some $l \in \{1, 2, \ldots, m\}$ and $\sum_{\mathbf{k}1}^{\mathbf{k}t_k}|a_{\mathbf{k}i}|^2 \leqslant 1$ holds for all $k \neq l$ and $k \in \{1, 2, \ldots, m\}$, similarly to Theorem 1. There exists an incoherent operator K_α such that the coherent state ρ is transformed into Ψ_m. Here is the form of K_α,

$$K_\alpha = \begin{pmatrix} a_{\mathbf{1}1}^* & a_{\mathbf{1}2}^* & \cdots & a_{\mathbf{1}t_1}^* & 0 & \cdots & 0 & \cdots & 0 & \Theta\\ 0 & 0 & \cdots & 0 & a_{\mathbf{2}1}^* & \cdots & 0 & \cdots & 0 & \Theta\\ \cdots & & & & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & \cdots & a_{\mathbf{m}1}^* & \cdots & a_{\mathbf{m}t_m}^* & \Theta\\ \Theta & \Theta & \cdots & \Theta & \Theta & \cdots & \Theta & \cdots & \Theta & \Theta \end{pmatrix}, \tag{16}$$

for which Θ represents the all-zero matrix of appropriate size. It is easy to check that

$$\frac{K_\alpha\rho K_\alpha^\dagger}{\mathrm{Tr}(K_\alpha\rho K_\alpha^\dagger)} = \Psi_m, \quad K_\alpha^\dagger K_\alpha \leqslant \mathbb{I},$$

because $\|K_\alpha^\dagger K_\alpha\|_\infty = \max\{\|K_\alpha^\dagger K_\alpha u\| : u \in H,\ \|u\| \leqslant 1\} = 1$. This completes the proof. ■

Compared with the necessary and sufficient conditions for the probabilistic mixed to pure transformation via stochastic SIOs in Lemma 3(c), the conditions of stochastic IOs in Theorem 2(c) cover these of stochastic SIOs. This is consistent with the fact that SIO $\subset$ IO, i.e., a strict inclusion relation.

We already know that a full-rank state cannot be converted to a pure coherent state by IOs with certainty. For non-full-rank states, we can make the following observation through Theorem 2. We show that if the rank of the mixed coherent state ρ is too large (rank$[\Delta(\rho)] - 1$), it cannot be converted to any pure coherent state by using IOs.

Proposition 1. Given a d-dimensional coherent state ρ, write $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_d\rangle)$. Then the implication (a) $\Rightarrow$ (b) holds.

(a) There is a subset $S \subseteq \{1, \ldots, d\}$ satisfying $\mathrm{Tr}(\rho[S]) = 1$ which contains a subset S_1 with $|S_1| = |S| - 1$ such that

$$\Delta(\rho)[S] = \sum_{i\in S}\langle\rho_i|\rho_i\rangle|l(i)\rangle\langle l(i)| \tag{17}$$

is positive-definite and the equations

$$\det(\rho[S]) = 0, \tag{18}$$

$$\det(\rho[S_1]) > 0 \tag{19}$$

hold, where $l: S \to \{1, \ldots, |S|\}$ is a strictly increasing bijection.

(b) There exists no incoherent operation $\Lambda(\cdot)$ such that the resultant state $\Lambda(\rho)$ is a maximally coherent state Ψ_m ($d \geqslant m \geqslant 2$).

Proof. Note first that the initial state ρ is both a mixed coherent state and a non-full-rank state, so we obtain $|S| > 2$, that is, the case for qubits is excluded. Without loss of generality, we assume directly that $\Delta(\rho) = \sum_{i=1}^d\langle\rho_i|\rho_i\rangle|i\rangle\langle i| > 0$. Here $S = \{1, \ldots, d\}$. Since $\det(\rho) = 0$ and there is a subset S_1 with $|S_1| = d-1$ such that $\det(\rho[S_1]) > 0$, the rank of ρ is $d-1$. We suppose that the vectors $\{|\rho_1\rangle, |\rho_2\rangle, \ldots, |\rho_{d-1}\rangle\}$ are linearly independent. The linear equation $x_1|\rho_1\rangle + x_2|\rho_2\rangle + \cdots + x_d|\rho_d\rangle = 0$ has a unique nonzero solution $(a_1, a_2, \ldots, a_d)^T$ which satisfies $\sum_{i=1}^{d-1}|a_i|^2 = 1$, $0 < |a_d|^2 \leqslant 1$, due to the one-dimensional solution space. Then we obtain two equations

$$a_1|\rho_1\rangle + a_2|\rho_2\rangle + \cdots + a_d|\rho_d\rangle = 0$$

and

$$a_1^*\langle\rho_1| + a_2^*\langle\rho_2| + \cdots + a_d^*\langle\rho_d| = 0.$$

Based on these two equations, we can construct an incoherent operator

$$K_\alpha = \begin{pmatrix} a_1^* & \cdots & a_{d-1}^* & 0\\ 0 & \cdots & 0 & -a_d^* \end{pmatrix}$$

and obtain $K_\alpha\rho K_\alpha^\dagger/\mathrm{Tr}(K_\alpha\rho K_\alpha^\dagger) = \Psi_2$. It is easy to check that $K_\alpha^\dagger K_\alpha \leqslant \mathbb{I}$ with $\|K_\alpha^\dagger K_\alpha\|_\infty = 1$. At the same time, we get

$$\mathrm{Tr}(K_\alpha\rho K_\alpha^\dagger) = \mathrm{Tr}(|a'\rangle\langle a'|\rho[S_1]) + |a_d|^2\langle\rho_d|\rho_d\rangle \ < \ 1,$$

due to

$$\begin{aligned}\mathrm{Tr}(|a'\rangle\langle a'|\rho[S_1]) &\leqslant \sum_j \lambda_j^\downarrow(|a'\rangle\langle a'|)\lambda_j^\downarrow(\rho[S_1])\\ &\leqslant \lambda_{\max}(\rho[S_1]) < \sum_{i=1}^{d-1}\langle\rho_i|\rho_i\rangle\end{aligned}$$

and

$$\sum_{i=1}^d\langle\rho_i|\rho_i\rangle = 1,$$

where $\{\lambda_j^\downarrow(M)\}_j$ are the eigenvalues of positive-semidefinite matrix M in nonincreasing order, $|a'\rangle = (a_1, \ldots, a_{d-1})^T$, and $S_1 = \{1, \ldots, d-1\}$.

We note that any two disjoint subsets S_1 and S_2 such that the intersection space $\mathrm{span}(\{|\rho_i\rangle\}_{i\in S_1}) \cap \mathrm{span}(\{|\rho_i\rangle\}_{i\in S_2})$ includes nonzero vectors is equivalent to the linear system $x_1|\rho_1\rangle + x_2|\rho_2\rangle + \cdots + x_d|\rho_d\rangle = 0$ with nonzero solutions. Consider that the incoherent operator K_α needs at least one different incoherent operator K_β to achieve completion identity. However, any vector that is linearly independent with

$(a_1, a_2, \dots, a_d)^T$ cannot construct an incoherent operator to distill Ψ_2, i.e., any vector that is linearly independent with $(a_1, a_2, \dots, a_d)^T$ is not a feasible solution to the linear system $x_1|\rho_1\rangle + x_2|\rho_2\rangle + \cdots + x_d|\rho_d\rangle = 0$. This means that any IO satisfying completion identity cannot be used to transform ρ into the maximally coherent state Ψ_2 with certainty.

For the case of the maximally coherent state Ψ_m ($d \geqslant m > 2$), $\sqrt{\rho} = \{|\rho_1\rangle, \dots, |\rho_d\rangle\}$ has at most two disjoint subsets which meet the conditions of Theorem 2(c). Thus, no incoherent operators can transform ρ into Ψ_m ($m > 2$). This completes the proof of Proposition 1. ■

This result establishes a no-go theorem for deterministic coherence distillation [33], showing that there is no IO that can distill any perfect coherence from a d-dimensional coherent state, which has nonzero diagonal entries and rank $d-1$ ($d > 2$). According to Theorem 2, the result of Proposition 1 is also suitable for a generic pure state $\varphi_{\mathbf{m}}$ and covers the result of Theorem 3 in Ref. [28], which presents the same viewpoint for three-dimensional quantum states.

However, the conditions in Theorem 2 are very complex and difficult to verify. It could thus be interesting to provide an uncomplicated condition that involves the principal subdeterminant of the incoherent basis representation of ρ. To do this, we now make an observation that shows the relation between the maximal linearly independent vectors among given vectors and the principal subdeterminant of the corresponding Gram matrix.

Lemma 5. Let $|v_1\rangle, \dots, |v_d\rangle$ be vectors in an inner product space V with inner product $\langle\cdot|\cdot\rangle$ and let $G = (\langle v_i|v_j\rangle)_{i,j=1,\dots,d}$. If vectors $|v_1\rangle, \dots, |v_d\rangle$ are linearly dependent, i.e., $\det(G) = 0$, then there is always a nonempty proper subset S such that $\det(G[S]) > 0$ and $\det(G[S \cup \{j\}]) = 0$ hold for all $j \in \bar{S}$. Here $\bar{S}$ is the complement of the subset S.

We define the above vectors $|v_i\rangle$ ($i \in S$) as a maximal linearly independent set of $|v_1\rangle, \dots, |v_d\rangle$, where $S \subset \{1, 2, \dots, d\}$. By Lemma 5, we readily obtain one sufficient condition with an explicit expression, which is a common way to find a partition of the incoherent basis associated with maximal linearly independent sets.

Theorem 3. Given a d-dimensional coherent state ρ, write $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \dots, |\rho_d\rangle)$. Then the implication (a) ⇒ (b) holds.

(a) There is a subset $S \subseteq \{1, \dots, d\}$ ($S \neq \emptyset$) which can be partitioned into m ($m \geqslant 2$) disjoint subsets S_s ($s = 1, \dots, m$) such that the equations

$$\det(\rho[S_s]) > 0, \tag{20}$$

$$\det(\rho[S_s \cup \{j\}]) = 0 \tag{21}$$

hold, where $s = 1, \dots, m$ and $j \in \bigcup_{k=s+1}^{m} S_k$.

(b) There exists a stochastic incoherent operation $\Lambda_S(\cdot)$ such that the resultant state $\Lambda_S(\rho)$ is a pure coherent state $\varphi_{\mathbf{m}}$ ($\mathbf{m} \geqslant 2$).

Proof. Without loss of generality, we suppose that, for $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \dots, |\rho_d\rangle)$, the column vectors $|\rho_i\rangle$ ($i = 1, \dots, d$) are all nonzero vectors and $S = \{1, \dots, d\}$.

By Lemma 5, we can find a maximal linearly independent set of $|\rho_1\rangle, |\rho_2\rangle, \dots, |\rho_d\rangle$ and represent it by $|\rho_{\mathbf{11}}\rangle, |\rho_{\mathbf{12}}\rangle, \dots, |\rho_{\mathbf{1}t_1}\rangle$. We repeat this process, finding a maximal linearly independent set of the remaining vectors. Suppose that the second maximal linearly independent set is denoted by $|\rho_{\mathbf{21}}\rangle, |\rho_{\mathbf{22}}\rangle, \dots, |\rho_{\mathbf{2}t_2}\rangle$. We continue to do the same thing and get the vector sets

$$\begin{aligned}
&\{|\rho_1\rangle, |\rho_2\rangle, \dots, |\rho_d\rangle\}, && S = \{1, \dots, d\},\\
&\{|\rho_{\mathbf{11}}\rangle, |\rho_{\mathbf{12}}\rangle, \dots, |\rho_{\mathbf{1}t_1}\rangle\}, && S_1 = \{\mathbf{11}, \mathbf{12}, \dots, \mathbf{1}t_1\},\\
&\{|\rho_{\mathbf{21}}\rangle, |\rho_{\mathbf{22}}\rangle, \dots, |\rho_{\mathbf{2}t_2}\rangle\}, && S_2 = \{\mathbf{21}, \mathbf{22}, \dots, \mathbf{2}t_2\},\\
&\qquad\vdots && \qquad\vdots\\
&\{|\rho_{\mathbf{r}1}\rangle, |\rho_{\mathbf{r}2}\rangle, \dots, |\rho_{\mathbf{r}t_r}\rangle\}, && S_r = \{\mathbf{r}1, \mathbf{r}2, \dots, \mathbf{r}t_r\}
\end{aligned}$$

and employ the following relations, respectively:

$$\begin{aligned}
&\det(\rho[S_1]) > 0, \quad \det(\rho[S_1 \cup \{j\}]) = 0\, \forall\, j \in S - S_1,\\
&\det(\rho[S_2]) > 0, \quad \det(\rho[S_2 \cup \{j\}]) = 0\, \forall\, j \in S - S_1 - S_2,\\
&\qquad\vdots\\
&\det(\rho[S_r]) > 0, \quad S_r = S - \cup_{s=1}^{r-1} S_s.
\end{aligned}$$

Distinctly, there is a bijection $f : \{1, \dots, d\} \to \{\mathbf{11}, \dots, \mathbf{1}t_1, \mathbf{2}1, \dots, \mathbf{2}t_2, \dots, \mathbf{r}1, \dots, \mathbf{r}t_r\}$ such that $f(i) = \mathbf{s}j$ ($i = 1, \dots, d$ and $j > 0$) for which the boldface type of $s = 1, \dots, r$ is used to mark the partition. According to statement (a), which shows there is an m partition associated with maximal linearly independent sets under the subset $S = \{1, \dots, d\}$, we thus have $r \geqslant m$. For the case of $r < m$ we need to do the previous process after applying a permutation operator to the subset S until we get the desired result. Altogether, an m partition of maximal linearly independent sets is obtained by the above method.

Subsequently, there are nonzero vectors $(a_{\mathbf{m}1}, \dots, a_{\mathbf{m}t_m})^T, \dots, (a_{\mathbf{2}1}, \dots, a_{\mathbf{2}t_2})^T, (a_{\mathbf{11}}, a_{\mathbf{12}}, \dots, a_{\mathbf{1}t_1})^T$ such that the following equations hold:

$$\frac{1}{\sqrt{m}}|\gamma\rangle = a_{\mathbf{m}1}|\rho_{\mathbf{m}1}\rangle + \cdots + a_{\mathbf{m}t_m}|\rho_{\mathbf{m}t_m}\rangle \neq 0$$

and

$$\begin{aligned}
&a_{\mathbf{m}1}|\rho_{\mathbf{m}1}\rangle + \cdots + a_{\mathbf{m}t_m}|\rho_{\mathbf{m}t_m}\rangle\\
&\quad = a_{\mathbf{m-1}1}|\rho_{\mathbf{m-1}1}\rangle + \cdots + a_{\mathbf{m-1}t_{m-1}}|\rho_{\mathbf{m-1}t_{m-1}}\rangle\\
&\quad\ \vdots\\
&\quad = a_{\mathbf{11}}|\rho_{\mathbf{11}}\rangle + \cdots + a_{\mathbf{1}t_1}|\rho_{\mathbf{1}t_1}\rangle.
\end{aligned} \tag{22}$$

Here we start from the last maximal linearly independent set. Because the latter maximal linearly independent set always can be linearly represented by the former, this construction method is feasible.

Then we can construct an incoherent operator in a similar way to Theorem 2 such that a stochastic IO transformation $\rho \to \Psi_m$ can be achieved. This completes the proof. ■

Theorem 3 provides a relatively simple and operable method for the probabilistic transformation via stochastic IOs, for the mixed to pure transformation task. We illustrate the strategy in Theorem 3 by the following example.

Example 1. Let $\rho = \frac{1}{2}|\varphi_1\rangle\langle\varphi_1| + \frac{1}{2}|\varphi_2\rangle\langle\varphi_2|$, where

$$|\varphi_1\rangle = \frac{1}{5\sqrt{2}}(4, 3, \sqrt{5}, 2\sqrt{5})^T,$$
$$|\varphi_2\rangle = \frac{1}{5\sqrt{2}}(-3, 4, -2\sqrt{5}, \sqrt{5})^T.$$

We can get

$$\rho = \frac{1}{4}\begin{pmatrix} 1 & 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ 0 & 1 & -\frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} & 1 & 0 \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} & 0 & 1 \end{pmatrix}$$

and

$$\sqrt{\rho} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ 0 & 1 & -\frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} & 1 & 0 \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} & 0 & 1 \end{pmatrix}.$$

We thus obtain the following vectors:

$$|\rho_1\rangle = \frac{1}{2\sqrt{2}}\left(1, 0, \frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right)^T,$$
$$|\rho_2\rangle = \frac{1}{2\sqrt{2}}\left(0, 1, -\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)^T,$$
$$|\rho_3\rangle = \frac{1}{2\sqrt{2}}\left(\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}, 1, 0\right)^T,$$
$$|\rho_4\rangle = \frac{1}{2\sqrt{2}}\left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}, 0, 1\right)^T.$$

The maximal linearly independent partition of the vectors $|\rho_1\rangle, |\rho_2\rangle, |\rho_3\rangle$, and $|\rho_3\rangle$ is provided,

$$S_1 = \{|\rho_1\rangle, |\rho_2\rangle\},$$
$$S_2 = \{|\rho_3\rangle, |\rho_4\rangle\},$$

by straightforwardly computing the corresponding principal subdeterminants of ρ.

Furthermore, we have $(\frac{4}{5}, \frac{3}{5})^T, (\frac{1}{\sqrt{5}} \frac{2}{\sqrt{5}})^T$ and $(-\frac{4}{5}, \frac{3}{5})^T, (-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})^T$, which are nonzero solutions of the linear system $x_1|\rho_1\rangle + x_2|\rho_2\rangle = x_3|\rho_3\rangle + x_4|\rho_4\rangle$. We thus construct the incoherent operators K_0 and K_1,

$$K_0 = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix},$$
$$K_1 = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} & 0 & 0 \\ 0 & 0 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}.$$

By directly calculating, we have $K_0^\dagger K_0 + K_1^\dagger K_1 = \mathbb{I}$ and $K_0\rho K_0^\dagger = K_1\rho K_1^\dagger = \frac{1}{2}\Psi_2$.

In fact, different selection orders of vectors may lead to different numbers of components concerning the corresponding partition of maximal linearly independent sets. That reflects different coherence ranks for the final pure state. The following is an explicit example.

Example 2. Let $\rho = \frac{1}{6}|\psi_1\rangle\langle\psi_1| + \frac{5}{6}|\psi_2\rangle\langle\psi_2|$, where

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(1, -1, 0, 0)^T,$$
$$|\psi_2\rangle = \frac{1}{\sqrt{10}}(1, 1, 2, 2)^T.$$

We can get a partition with maximal linearly independent sets $\{1, 2\}$, $\{3\}$, and $\{4\}$, while the density matrix

$$\rho = \frac{1}{6}\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \end{pmatrix},$$

with

$$\sqrt{\rho} = \frac{1}{2\sqrt{6}}\begin{pmatrix} \frac{1+\sqrt{5}}{\sqrt{5}} & \frac{1-\sqrt{5}}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{1-\sqrt{5}}{\sqrt{5}} & \frac{1+\sqrt{5}}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} & \frac{4}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \frac{4}{\sqrt{5}} & \frac{4}{\sqrt{5}} \end{pmatrix},$$

has another partition with maximal linearly independent sets $\{1, 3\}$ and $\{2, 4\}$.

We note that the deterministic conversion of a general quantum state into a coherent pure state under IOs has not been fully characterized. This leads us to investigate other relationships between the submatrices of the initial state ρ and the column vectors of $\sqrt{\rho}$ as shown in Lemma 3. For two index sets $S_1, S_2 \subseteq \{1, \ldots, d\}$, we denote by $M[S_1, S_2]$ the submatrix of entries that lie in the rows of M indexed by S_1 and the columns of M indexed by S_2. We observe that the unitary equivalence between column vectors is equivalent to the identity relationship between submatrices of the corresponding Gram matrix.

Lemma 6. Let $|v_1\rangle, \ldots, |v_d\rangle$ be vectors in an inner product space V with inner product $\langle\cdot|\cdot\rangle$ and let $G = (\langle v_i|v_j\rangle)_{i,j=1,\ldots,d}$. Then the following statements are equivalent.

(a) For two disjoint subsets $S_1, S_2 \subseteq \{1, \ldots, d\}$ with $|S_1| = |S_2|$, the vectors $|v_i\rangle$ $(i \in S_1)$ and $|v_j\rangle$ $(j \in S_2)$ are unitarily equivalent, i.e., there is a unitary matrix U, which is an $|S_1| \times |S_1|$ matrix, such that $(|v_i\rangle)_{i\in S_1}U = (|v_j\rangle)_{j\in S_2}$.

(b) The principal submatrices $G[S_1]$ and $G[S_2]$ satisfy one of the two equations

$$G[S_1] = \sqrt{G[S_1, S_2]G[S_2, S_1]}, \tag{23}$$

$$G[S_2] = \sqrt{G[S_2, S_1]G[S_1, S_2]}. \tag{24}$$

Proof. First, let us show our proof with the following example, i.e., if

$$G[\{a, b, c\}] = \begin{pmatrix} \langle v_a| \\ \langle v_b| \\ \langle v_c| \end{pmatrix}(|v_a\rangle, |v_b\rangle, |v_c\rangle) \tag{25}$$

$$= \sqrt{G[\{a, b, c\}, \{a', b', c'\}]G[\{a', b', c'\}, \{a, b, c\}]}, \tag{26}$$

then vectors $|v_a\rangle, |v_b\rangle, |v_c\rangle$ and $|v_{a'}\rangle, |v_{b'}\rangle, |v_{c'}\rangle$ are unitarily equivalent, where $\{a, b, c\}, \{a', b', c'\} \subseteq \{1, \dots, d\}$ and $\{a, b, c\} \bigcap \{a', b', c'\} = \emptyset$. According to the singular value theorem, there are unitary matrices U_L and V_R such that

$$G[\{a, b, c\}, \{a', b', c'\}] = \begin{pmatrix} \langle v_a| \\ \langle v_b| \\ \langle v_c| \end{pmatrix} (|v_{a'}\rangle\ |v_{b'}\rangle\ |v_{c'}\rangle) = U_L D V_R^\dagger$$

and

$$G[\{a', b', c'\}, \{a, b, c\}] = \begin{pmatrix} \langle v_{a'}| \\ \langle v_{b'}| \\ \langle v_{c'}| \end{pmatrix} (|v_a\rangle\ |v_b\rangle\ |v_c\rangle) = V_R D U_L^\dagger$$

due to $G[\{a, b, c\}, \{a', b', c'\}] = G[\{a', b', c'\}, \{a, b, c\}]^\dagger$ for the Hermitian matrix G following Lemma 2(a), where the matrix D is a diagonal matrix. Then, combined with Eq. (26), the following reasoning process is presented:

$$\begin{aligned} D &= U_L^\dagger \begin{pmatrix} \langle v_a| \\ \langle v_b| \\ \langle v_c| \end{pmatrix} (|v_{a'}\rangle\ |v_{b'}\rangle\ |v_{c'}\rangle) V_R \\ &= U_L^\dagger \begin{pmatrix} \langle v_a| \\ \langle v_b| \\ \langle v_c| \end{pmatrix} (|v_a\rangle\ |v_b\rangle\ |v_c\rangle) U_L. \end{aligned}$$

We get the desired result,

$$(|v_a\rangle\ |v_b\rangle\ |v_c\rangle) U_L = (|v_{a'}\rangle\ |v_{b'}\rangle\ |v_{c'}\rangle) V_R.$$

For more general cases, it can be proved in the same way. On the other hand, it is obvious that if vectors $|v_i\rangle$ $(i \in S_1)$ and $|v_j\rangle$ $(j \in S_2)$, with $S_1, S_2 \subset \{1, \dots, d\}$, $S_1 \cap S_2 = \emptyset$, and $|S_1| = |S_2|$, are unitarily equivalent, then the corresponding submatrices are easily checked to satisfy Eqs. (23) and (24).

Using Lemma 6, we construct a class of IO deterministic transformations that contain the important example presented by Theorem 4 in Ref. [28], which demonstrates a mixed to pure transformation beyond the capabilities of SIOs. In particular, the following theorem characterizes a sufficient condition of mixed to pure transformations by IOs. We will see later that this type of mixed to pure transformation can be achieved by DIOs too, because we construct concrete incoherent operators that also satisfy the properties required by DIOs.

Theorem 4. Given a d-dimensional coherent state ρ, write $\sqrt{\rho} = (|\rho_1\rangle, |\rho_2\rangle, \dots, |\rho_d\rangle)$. Then the implication (a) $\Rightarrow$ (b) holds.

(a) There is a subset $S \subseteq \{1, \dots, d\}$ satisfying $\mathrm{Tr}(\rho[S]) = 1$, which can be partitioned into m $(m \geqslant 2)$ disjoint subsets S_s $(s = 1, \dots, m)$ with $|S_1| = |S_2| = \cdots = |S_m|$ such that

$$\Delta(\rho)[S] = \sum_{i \in S} \langle \rho_i | \rho_i \rangle |l(i)\rangle\langle l(i)| \tag{27}$$

is positive-definite and, for $s = 1, \dots, m-1$, the equation

$$\rho[S_s] = \sqrt{\rho[S_s, S_{s+1}]\rho[S_s S_{s+1}]} \tag{28}$$

holds, where $l : S \rightarrow \{1, \dots, |S|\}$ is a strictly increasing bijection.

(b) There exists an incoherent operation $\Lambda(\cdot)$ such that the resultant state $\Lambda(\rho)$ is a pure coherent state $\varphi_{\mathbf{m}}$ $(\mathbf{m} \geqslant 2)$.

Proof. Without loss of generality, we assume directly that $\Delta(\rho) = \sum_{i=1}^{d} \langle \rho_i | \rho_i \rangle |i\rangle\langle i| > 0$. According to Lemma 6, there are m, V_s and $t \times t$ $(t = \frac{d}{m}$ and $s = 1, \dots, m)$ unitary matrices such that $(|\rho_i\rangle)_{i \in S_1} V_1 = (|\rho_i\rangle)_{i \in S_2} V_2 = \cdots = (|\rho_i\rangle)_{i \in S_m} V_m$, where the disjoint subsets satisfy $\bigcup_{s=1}^{m} S_s = \{1, \dots, d\}$. Then, without loss of generality, we suppose that

$$\rho = \sqrt{\rho}^\dagger \sqrt{\rho} = \begin{pmatrix} V_1 D V_1^\dagger & V_1 D V_2^\dagger & \cdots & V_1 D V_m^\dagger \\ V_2 D V_1^\dagger & V_2 D V_2^\dagger & \cdots & V_2 D V_m^\dagger \\ \vdots & \vdots & & \vdots \\ V_m D V_1^\dagger & V_m D V_2^\dagger & \cdots & V_m D V_m^\dagger \end{pmatrix}, \tag{29}$$

where D is a diagonal matrix. We construct the IO Λ, whose incoherent operators $\{K_\alpha\}$ have the form

$$K_\alpha = \begin{pmatrix} \langle V_{1\alpha}| & \Theta & \cdots & \Theta \\ \Theta & \langle V_{2\alpha}| & \cdots & \Theta \\ \vdots & \vdots & \ddots & \vdots \\ \Theta & \Theta & \cdots & \langle V_{m\alpha}| \end{pmatrix}, \tag{30}$$

where $|V_{s\alpha}\rangle$ is the αth column vector of V_s and Θ represents the all-zero matrix of appropriate size $(s = 1, \dots, m$ and $\alpha = 1, \dots, t)$. In other words, we express V_s as $(|V_{s1}\rangle, |V_{s2}\rangle, \dots, |V_{st}\rangle)$. According to the above incoherent operators' definition, we get

$$\begin{aligned} K_\alpha \rho K_\alpha^\dagger &= \begin{pmatrix} \langle \alpha|D|\alpha\rangle & \langle \alpha|D|\alpha\rangle & \cdots & \langle \alpha|D|\alpha\rangle \\ \langle \alpha|D|\alpha\rangle & \langle \alpha|D|\alpha\rangle & \cdots & \langle \alpha|D|\alpha\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \alpha|D|\alpha\rangle & \langle \alpha|D|\alpha\rangle & \cdots & \langle \alpha|D|\alpha\rangle \end{pmatrix} \\ &= \begin{pmatrix} d_\alpha & d_\alpha & \cdots & d_\alpha \\ d_\alpha & d_\alpha & \cdots & d_\alpha \\ \vdots & \vdots & \ddots & \vdots \\ d_\alpha & d_\alpha & \cdots & d_\alpha \end{pmatrix} \\ &= (m d_\alpha) \Psi_{\mathbf{m}}, \end{aligned}$$

where $d_\alpha = \langle \alpha|D|\alpha\rangle$, $\alpha = 1, \dots, t$. Thus, we get that $\Lambda(\rho) = \sum_\alpha K_\alpha \rho K_\alpha^\dagger = \Psi_{\mathbf{m}}$. In addition, we can verify the completion identity of IO Λ through the following steps. Noting that

$$K_\alpha^\dagger K_\alpha = \begin{pmatrix} |V_{1\alpha}\rangle\langle V_{1\alpha}| & \Theta & \cdots & \Theta \\ \Theta & |V_{2\alpha}\rangle\langle V_{2\alpha}| & \cdots & \Theta \\ \vdots & \vdots & \ddots & \vdots \\ \Theta & \Theta & \cdots & |V_{m\alpha}\rangle\langle V_{m\alpha}| \end{pmatrix},$$

we obtain that

$$\sum_{\alpha} K_\alpha^\dagger K_\alpha = \begin{pmatrix} \sum_\alpha |V_{1\alpha}\rangle\langle V_{1\alpha}| & \Theta & \cdots & \Theta \\ \Theta & \sum_\alpha |V_{2\alpha}\rangle\langle V_{2\alpha}| & \cdots & \Theta \\ \vdots & \vdots & \ddots & \vdots \\ \Theta & \Theta & \cdots & \sum_\alpha |V_{m\alpha}\rangle\langle V_{m\alpha}| \end{pmatrix}$$
$$= \mathbb{I}.$$

Finally, because the maximally coherent state Ψ_m can be transformed into any pure coherence $\varphi_{\mathbf{m}}$ and the concatenation of IOs is still an IO, we can always find an IO Λ' to make the equation $\Lambda'(\rho) = \varphi_{\mathbf{m}}$ hold. This completes the proof. ■

It was shown in Ref. [34] that almost all states, except for states whose density matrix contains a rank-1 submatrix, are bound coherent under SIOs. Theorem 4 shows that there exist bound states under SIOs such that the probability of the coherent distillation by IOs can be reached up to 1, as illustrated by Example 1. Here

$$\begin{aligned}\rho &= \tfrac{1}{2}|\varphi_1\rangle\langle\varphi_1| + \tfrac{1}{2}|\varphi_2\rangle\langle\varphi_2| \\ &= \frac{1}{4}\begin{pmatrix} 1 & 0 & \frac{2\sqrt{5}}{5} & \frac{\sqrt{5}}{5} \\ 0 & 1 & -\frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} & 1 & 0 \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{I} & \Theta \\ \Theta & V \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \mathbb{I} & \Theta \\ \Theta & V^\dagger \end{pmatrix},\end{aligned}$$

where

$$V = \begin{pmatrix} \frac{2\sqrt{5}}{5} & -\frac{\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} & \frac{2\sqrt{5}}{5} \end{pmatrix}$$

and Θ represents the all-zero matrix of appropriate size.

Next we remind the reader of the notion $|V_{j|i}\rangle$ denoted by [16]

$$|V_{j|i}\rangle := \begin{pmatrix} \langle j|K_1|i\rangle \\ \langle j|K_2|i\rangle \\ \vdots \\ \langle j|K_m|i\rangle \end{pmatrix} \tag{31}$$

and the following lemma.

Lemma 7 (from [16]). Using the notion of Eq. (31), a CPTP map Λ is a dephasing-covariant incoherent operation if and only if there exists a conditional probability distribution $\{r_{j|i}\}$ such that

$$\langle V_{j'|i}|V_{j|i}\rangle = r_{j|i}\delta_{jj'},$$
$$\langle V_{j|i}|V_{j|i'}\rangle = r_{j|i}\delta_{ii'}.$$

Corollary 1. The incoherent operation in Theorem 4 is a dephasing-covariant incoherent operation.

Proof. With the construction of incoherent operators as shown in Eq. (30), we can easily check that $\langle V_{j|i}|V_{j|i'}\rangle = \delta_{ii'}$ and $\langle V_{j'|i}|V_{j|i}\rangle = \delta_{jj'}$, when $|V_{j|i}\rangle, |V_{j'|i}\rangle \neq 0$, where $i, i' = 1, \ldots, m$ and $j, j' = 1, \ldots, d$. We note that if

$$|V_{j^*|i}\rangle = \begin{pmatrix} \star \\ \vdots \\ \star \end{pmatrix},$$

then

$$|V_{j|i}\rangle = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

when $j \neq j^*$, where $\star$ represents a nonzero number, $i \in \{1, \ldots, m\}$, and $j, j^* \in \{1, \ldots, d\}$, because nonzero elements in each row of K_α for all α are in the same positions. We obtain the equations

$$\langle V_{j'|i}|V_{j|i}\rangle = r_{j|i}\delta_{jj'},$$
$$\langle V_{j|i}|V_{j|i'}\rangle = r_{j|i}\delta_{ii'},$$

where if $|V_{j^*|i}\rangle \neq 0$, $r_{j^*|i} = 1$; otherwise $r_{j|i} = 0$, with $i, i' \in \{1, \ldots, m\}$ and $j, j', j^* \in \{1, \ldots, d\}$. ■

Corollary 1 shows that DIOs also possess an operational advantage compared with SIOs, which is an unexpected conclusion and was not suggested in any previous work.

IV. CONCLUSION

In summary, we have investigated the mixed to pure transformation by using both stochastic IOs and IOs. Under stochastic IOs, we first proved two equivalence conditions for the case of coherence rank 2 and greater coherence rank of the target pure state. Then we provided two sufficient conditions with explicit expressions for the mixed to pure transformation under stochastic IOs and IOs, respectively. We showed that such sufficient conditions are in favor of the construction of concrete operators of stochastic IOs and IOs. Our results show that stochastic IOs and IOs for extracting pure coherent states do not depend on the nonzero pure coherent subspace of the initial state, which is the sufficient and necessary condition for these transformations under stochastic SIOs and SIOs [28]. This means that IOs are generally stronger than SIOs when we want to transform a mixed coherent state into a pure coherent one. Our work enriches the study of the mixed to pure transformation under stochastic IOs and IOs. In addition to the problems of obtaining maximum probability under concrete incoherent operators, it would be of interest to analyze the equivalence condition of the mixed to pure transformation under IOs.

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