Measuring nonstabilizerness via multifractal flatness

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Universal quantum computing requires nonstabilizer (magic) quantum states. Quantifying the nonstabilizerness and relating it to other quantum resources is vital for characterizing the complexity of quantum many-body systems. In this work, we prove that a quantum state is a stabilizer if and only if all states belonging to its Clifford orbit have a flat probability distribution on the computational basis. This implies, in particular, that multifractal states are nonstabilizers. We introduce multifractal flatness, a measure based on the participation entropy that quantifies the wave-function distribution flatness. We demonstrate that this quantity is analytically related to the stabilizer entropy of the state and present several examples elucidating the relationship between multifractality and nonstabilizerness. In particular, we show that the multifractal flatness provides an experimentally and computationally viable nonstabilizerness certification. Our work unravels the direct relation between the nonstabilizerness of a quantum state and its wave-function structure.

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I. INTRODUCTION

Originally developed in the context of fault-tolerant quantum computation and error correction [1,2], stabilizer states and operations play an essential role in quantum information theory [3], in our understanding of thermalization [4,5] and entanglement propagation in many-body systems [6–8], and in providing insights on holography [9]. However, their classical simulability [10] implies that additional ingredients are required to attain computational quantum speedup [11–13]. Quantifying the nonstabilizerness (or "magic") of a quantum state is therefore a key task in the resource theory of quantum computation [14–21], related to the classical computational cost of simulation quantum physics [18,22–24] and to the onset of quantum chaos [25,26]. While various nonstabilizerness monotones have been proposed [17,27–33], their geometrical nature requires minimization over large spaces, resulting in prohibitive computational costs already for few-qubit systems. In fact, quantifying nonstabilizerness in many qubit systems remains a major challenge. Stabilizer entropy provides a (generally nonmonotone [34]) measure of nonstabilizerness that is calculable for few-qubit states [35–38] and that is related to the flatness of the entanglement spectrum [39]. Recent works demonstrated that while being measurable in present quantum devices via randomized measurement protocols or Bell measurements [40,41], the stabilizer entropy is efficiently computable for many-body matrix product states [34,42,43]. These results allow for studying the static and dynamical properties of nonstabilizerness in one-dimensional many-body systems.

A seemingly unrelated quantity is the inverse participation ratio and the participation entropy of many-body wave functions. Originally developed in the context of Anderson localization [44–50], these quantities measure the spread of the many-body states on a certain basis of the Hilbert space. Studies on the ground states of many-body systems [51–61], the problem of many-body localization [47,62–74], and monitored unitary dynamics [75,76] demonstrated that most quantum states have multifractal features [77]; that is, they are described by different fractal properties in different geometrical regions.

This work demonstrates that nonstabilizerness is directly encoded in the structure of wave functions. Employing methods inspired by recent works [39,78], we show that a state has a flat participation distribution along its Clifford orbit if and only if it is a stabilizer state. Conversely, a state with nonflat participation distribution, for instance, a multifractal state, has nonstabilizerness. After reviewing the key concepts of interest, we introduce a *multifractal flatness*, a measure of the flatness of participation distributions that we prove is a simple function of the stabilizer entropy. This allows us to demonstrate that the participation entropy in the computational basis provides a useful magic witness that is amenable to computational methods. We illustrate the relation between nonstabilizerness and participation entropy using the examples of a single qubit, many-body product states, and Haar random states. Finally, we demonstrate that the nonstabilizerness quantifier introduced in this paper is measurable in noisy intermediate-scale quantum devices [79–81].

II. STABILIZER AND PARTICIPATION ENTROPIES

We consider a system of N qubits with Hilbert space dimension $d = 2^N$ and denote by $\{\sigma^\alpha\}_{\alpha=0,1,2,3}$ the Pauli matrices ($\sigma^0 = \mathbb{1}$), by $|0\rangle$ and $|1\rangle$ the local computational basis of σ^3, and by $\mathcal{P}_N$ the set of all N-qubit Pauli strings. The Clifford group $\mathcal{C}_N$ is the subset of unitary operations that map a Pauli string into a *single* Pauli string. The stabilizer entropy is defined for a pure normalized state $|\Psi\rangle$ as [35]

$$M_q(|\Psi\rangle) = \frac{1}{1-q}\log_2 \sum_{P\in\mathcal{P}_N} \frac{(\langle\Psi|P|\Psi\rangle)^{2q}}{d}. \quad (1)$$

It is a measure of nonstabilizerness because (i) $M_q(|\Psi\rangle) \geqslant 0$, with the equality holding if and only if the state is a stabilizer $|\Psi\rangle \in \mathrm{STAB}$, (ii) it is invariant under Clifford conjugation $M_q(C|\Psi\rangle) = M_q(|\Psi\rangle)$ for any $C \in \mathcal{C}_N$, and (iii) it is additive $M_q(|\Psi\rangle \otimes |\Phi\rangle) = M_q(|\Psi\rangle) + M_q(|\Phi\rangle)$ [35]. Despite the fact that the stabilizer entropy is not a magic monotone for the generic Rényi index q [34], it is computationally amenable compared to the magic robustness and stabilizer fidelity [22]. We aim to show that the stabilizer entropy $M_2(|\Psi\rangle)$ is related to the structure of the wave function quantified by the participation entropy.

Given a pure state $|\Psi\rangle$, we introduce the participation distribution in the computational basis $\mathcal{B} \equiv \{|\vec{\sigma}\rangle,\ \sigma_i = 0, 1,\ i = 1,\dots,N\}$ as the probability distribution $p(\vec{\sigma}) \equiv |\langle\vec{\sigma}|\Psi\rangle|^2$. Then, the participation entropy is

$$S_q(|\Psi\rangle) = (1-q)^{-1}\log_2 I_q(|\Psi\rangle), \quad (2)$$

where $I_q(|\Psi\rangle) = \sum_{\vec{\sigma}\in\mathcal{B}} p(\vec{\sigma})^q$ is the inverse participation ratio. The participation entropy (2) quantifies the spreading of the state $|\Psi\rangle$ over the basis $\mathcal{B}$. Conventionally, in condensed-matter settings, the system-size dependence of the participation entropy is parametrized as [50,63,82]

$$S_q = D_q N + c_q, \quad (3)$$

where D_q is the multifractal dimension and c_q is a subleading term. In the field of Anderson and many-body localization transition [46], it is customary to denote the state $|\Psi\rangle$ localized (fully extended) when $D_q = 0$ ($D_q = 1$). The intermediate regimes, for which $0 < D_q < 1$ and D_q depend nontrivially on the Rényi index q, are said to be multifractal [83]. Here, we are concerned with the *participation flatness*, occurring when $S_{q_1}(|\Psi\rangle) = S_{q_2}(|\Psi\rangle)$ for all $q_1, q_2 > 0$. [This condition is equivalent to $p(\vec{\sigma})$ being uniform in its domain, hence justifying the name.] Participation flatness implies the absence of wave-function multifractality. However, we remark that the two notions are not equivalent: Participation flatness is defined for a given state at a fixed system size N. Instead, the multifractality is intrinsically related to the N dependence of the participation entropy (3). In particular, as we will discuss in the following, a state can be fully extended ($D_q = 1$) or localized ($D_q = 0$) and still not have a flat participation distribution due to the nontrivial q dependence of c_q.

Finally, let us note that the stabilizer entropy (1) has an intrinsic basis dependence in the choice of Pauli strings as generators of the operator space. This basis dependence is a feature of the nonstabilizerness, and different frames yield different results [28]. Similarly, the participation entropy (2), and hence also the participation flatness, depends explicitly on the choice of the many-body basis $\mathcal{B}$. Therefore, we choose the Pauli strings to quantify the degree of nonstabilizerness and consider the computational basis to probe the many-body wave-function structure.

III. NONSTABILIZERNESS AND MULTIFRACTALITY

Stabilizer states are not multifractal and always possess a flat participation distribution, as shown in [75], which also provides an efficient way to compute the participation entropy of stabilizer state using its tableau representation [10,84]. As we argue in the following, the converse statement is also true, leading to a characterization of nonstabilizerness in terms of the participation entropies of the many-body wave function.

Before stating the main results of this work, we introduce the *multifractal flatness* $\mathcal{F}(|\Psi\rangle) = I_3(|\Psi\rangle) - I_2^2(|\Psi\rangle)$, which is a measure of the participation flatness. By the concavity of the participation entropy and using Jensen's inequality, it is easy to see that $\mathcal{F}(|\Psi\rangle) \geqslant 0$, with the equality holding if and only if the participation distribution $p(\vec{\sigma})$ is flat or, equivalently, when $S_{q_1}(|\Psi\rangle) = S_{q_2}(|\Psi\rangle)$ for all $q_1, q_2 > 0$.

Theorem 1. The average of the multifractal flatness $\mathcal{F}$ over the Clifford orbit $\mathcal{C}_\Psi \equiv \{C|\Psi\rangle\ |C \in \mathcal{C}_N\}$ is a measure of nonstabilizerness, with

$$\overline{\mathcal{F}}(|\Psi\rangle) \equiv \mathbb{E}_{C\in\mathcal{C}_N}[\mathcal{F}(C|\Psi\rangle)] = \frac{2(1-2^{-M_2(|\Psi\rangle)})}{(d+1)(d+2)}. \quad (4)$$

Let us highlight an immediate consequence that conceptually bridges the participation flatness and that of nonstabilizerness.

Corollary 1. A state is a stabilizer if and only if every element of its Clifford orbit is participation flat. Conversely, a state is a nonstabilizer if and only if a $C \in \mathcal{C}_N$ exists for which $C|\Psi\rangle$ is not participation flat.

As already mentioned, a stabilizer state has a flat participation distribution [75]. In this case, Eq. (4) is trivially satisfied. Conversely, we must prove the flatness of the participation distribution for all the states in $\mathcal{C}_\Psi$, or, in other words, that there is no $C \in \mathcal{C}_N$ for which $S_q(C|\Psi\rangle)$ is q dependent. From the positive monotonicity of $\mathcal{F}$, this fact is equivalent to proving Eq. (4) holds for generic states. To this end, let us rephrase I_q^r in the replica formalism as

$$I_q^r = \mathrm{tr}\big[|\Phi^{(rq)}\rangle\langle\Phi^{(rq)}|\big(\Lambda_1^{(q)}\big)^{\otimes r}\cdots\big(\Lambda_N^{(q)}\big)^{\otimes r}\big], \quad (5)$$

where $|\Phi^{(rq)}\rangle = |\Psi\rangle^{\otimes rq}$ is the replica state and $\Lambda_k^{(q)} = (|0\rangle\langle 0|)^q + (|1\rangle\langle 1|)^q$ are the operators enforcing the book replica boundary condition [52,75] (see Fig. 1). From Eq. (5) we see that I_q^r has a permutation invariance $\mathcal{S}_q^{\otimes r}$ over the replica space. This fact will allow for simplifications in evaluating the orbit average in Eq. (4). First, let us compute the $q = 3, r = 1$ term. Using the three-design property of the Clifford group [85,86], we have

$$\mathbb{E}_{\mathcal{C}_N}[C^{\otimes 3}|\Phi^{(3)}\rangle\langle\Phi^{(3)}|(C^\dagger)^{\otimes 3}] = \frac{6}{d(d+1)(d+2)}\Pi_{[3]}, \quad (6)$$

with $\Pi_{[k]} = \sum_{\pi\in\mathcal{S}_k} U_\pi/k!$ being the projector onto the symmetric permutation of k elements. It follows that $\mathbb{E}_{C\in\mathcal{C}_N}[I_3(C|\Psi\rangle)] = 6/[(d+1)(d+2)]$. Instead, $q = 2$ and $r = 2$ require the average of $|\Phi^{(4)}\rangle$. This object is less trivial and requires insights into the commutant of the Clifford group (cf. Refs. [87,88]). We have

$$\mathbb{E}_{\mathcal{C}_N}[C^{\otimes 4}|\Phi^{(4)}\rangle\langle\Phi^{(\otimes 4)}|(C^\dagger)^4] = \beta_+(|\Psi\rangle)\Pi_+ + \beta_-(|\Psi\rangle)\Pi_-, \quad (7)$$

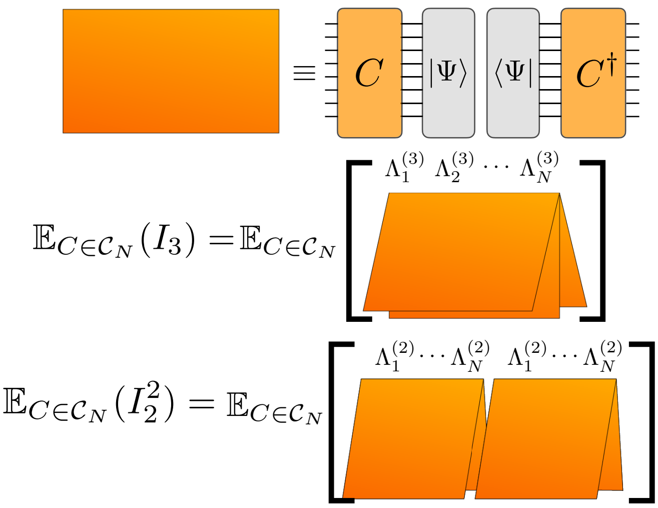

FIG. 1. Pictorial representation of the replica picture. Each *page* is a folding of $C|\Psi\rangle$ and its adjoint. Applying $\Lambda_k^{(q)}$ on each site implies a sum over the same physical index and results in *book* boundary conditions.

with $\Pi_+ = \Pi_{N,4}\Pi_{[4]}$ and $\Pi_- = (1-\Pi_{N,4})\Pi_{[4]}$ being two projectors and $\Pi_{N,4} = \sum_{P\in\mathcal{P}_N} P^{\otimes 4}/d^2$, while the coefficients are defined as $\beta_+ = 6||\Xi(|\Psi\rangle)||_2^2/[(d+1)(d+2)]$ and $\beta_- = 24[1-||\Xi(|\Psi\rangle)||_2^2]/[(d^2-1)(d+2)(d+4)]$, with $||\Xi(|\Psi\rangle)||_2^2 = \sum_{P\in\mathcal{P}_N} \langle\Psi|P|\Psi\rangle^4/d^2$. A simple computation imposing the book boundary condition in the replica space [75] gives

$$\mathbb{E}_{C\in\mathcal{C}_N}\left[I_2^2(C|\Psi\rangle)\right] = \frac{4-2d||\Xi(|\Psi\rangle)||_2^2}{(d+1)(d+2)}. \tag{8}$$

Combining these results, Eq. (4) and the proof's conclusion for $N \geqslant 3$ follow. The cases with $N = 1, 2$ have a rank-deficient Clifford group commutant [89,90] and must be evaluated separately. Nevertheless, one can show that the final result (4) holds, recalling $d = 2^N$. ■

A few remarks are in order here. At a practical level, the right-hand side of Eq. (4) can be evaluated with Monte Carlo sampling over the Clifford group $\mathcal{C}_N$, i.e., $\overline{\mathcal{F}}_{\text{mc}} \equiv \sum_C \mathcal{F}(C|\Psi\rangle)/\mathcal{N}_{\text{real}}$, with $\mathcal{N}_{\text{real}}$ being the number of random choices of Clifford unitaries C. We recall that randomly drawing from the Clifford group is efficiently implementable [91], and we present an explicit example of the Monte Carlo estimation in the following sections.

While our theorem (4) links the multifractal flatness averaged over the whole Clifford orbit $\mathcal{C}_\Psi$ to the stabilizer entropy M_2, we remark that $\mathcal{F}(C^*|\Psi\rangle)$ for a fixed $C^* \in \mathcal{C}_N$ is a convenient witness of nonstabilizerness, being computationally cheap compared to (1). Indeed, $\mathcal{F}(C|\Psi\rangle) \geqslant 0$ for any C and $|\Psi\rangle$ and is nonzero only if the state is not a stabilizer. Hence, if $\mathcal{F}(C^*|\Psi\rangle) > 0$, we certify that the state has some amount of nonstabilizerness.

Additionally, the multifractal flatness $\mathcal{F}$ can be extended to the class $\mathcal{F}(|\Psi\rangle; q, m) = I_q(|\Psi\rangle) - I^m_{(k-1+q)/m}(|\Psi\rangle)$, with $m, q > 0$. While we conjecture a result similar to (4) holds, the rapidly growing commutant dimension of the Clifford group already hinders analytical insights for $q, k = 5$ [92]. Nevertheless, these terms would scale as $O(d^{-3})$, therefore being more difficult to resolve than (4) for practical purposes. Last, we note an expression similar to (4) resembles analogous results for the entanglement spectrum [39], where an additional dependence on the entanglement bipartition is present.

In the following, we illustrate the relation between nonstabilizerness and the multifractal flatness $\mathcal{F}$ for several examples of quantum states. In particular, we show that resolving $\overline{\mathcal{F}}$ generally requires exponential resources in system size for generic systems, as shown using a numerical example below. However, we show that $\overline{\mathcal{F}}$ can be estimated in current noisy intermediate-scale quantum devices, provided the fidelity is preserved.

IV. MULTIFRACTALITY AND NONSTABILIZERNESS: EXAMPLES

A. Single qubit

Let us start with the intuitive example of a single qubit ($N = 1$) and highlight the relationship between the stabilizer and participation entropies of the quantum state

$$|\Psi_1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle, \tag{9}$$

where θ and ϕ are real parameters. The stabilizer entropy (1) of this state is given by

$$M_q(|\Psi_1\rangle) = \frac{1}{1-q}\log_2\left(\frac{1+\cos^{2q}(\theta)+\Omega(\theta,\phi)}{2}\right),$$
$$\Omega(\theta,\phi) = [\sin(\theta)\sin(\phi)]^{2q} + [\sin(\theta)\cos(\phi)]^{2q}, \tag{10}$$

which is nonvanishing for a generic choice of θ and ϕ. The calculation of the participation entropy (2) yields $S_q(|\Psi_1\rangle) = \log_2[\cos^{2q}(\theta/2) + \sin^{2q}(\theta/2)]/(1-q)$, which depends nontrivially on q for a generic parameter choice, providing an example of a typical scenario in which a nonstabilizer state does not have a flat participation distribution. However, for a fine-tuned choice $\theta = \pi/2$, the participation entropy is equal to unity, independent of the value of q; hence, $\mathcal{F}(|\Psi_1\rangle) = 0$. Still, the state $|\Psi_1\rangle$ has a nontrivial nonstabilizerness for $\theta = \pi/2$ and a generic value of ϕ. This shows that the flatness of the participation distribution for a selected point of the Clifford orbit $\mathcal{C}_{\Psi_1}$ does not imply that the state is a stabilizer state, illustrating the importance of the average over the whole Clifford orbit in (4). Indeed, for a single choice of $C \in \mathcal{C}_N$, $\mathcal{F}$ is only a magic witness: If it is nonzero, we know that the state has nonstabilizerness, but the converse is not true. If we act with a Hadamard gate $H \in \mathcal{C}_1$ on the state $|\Psi_1\rangle$, we discover a nontrivial q dependency $S_q(H|\Psi_1\rangle) = (\log_2\{[(1+\sin\theta\cos\phi)/2]^q + [(1-\sin\theta\cos\phi)/2]^q\})/(1-q)$ for generic values of ϕ even at $\theta = \pi/2$, consistent with the nonstabilizerness of the state $|\Psi_1\rangle$ revealed by the nonvanishing value of $M_q(\Psi)$ (10). Its multifractal flatness is $\mathcal{F}(H|\Psi\rangle) = [\sin^2(\theta)\cos^2(\phi) - \sin^4(\theta)\cos^4(\phi)]/4$, which must be compared with the Clifford orbit average in Eq. (4), given by $\overline{\mathcal{F}}(|\Psi\rangle) = \sin^2(\theta)[-2\sin^2(\theta)\cos(4\phi) + 7\cos(2\theta) + 9]/96$.

B. Many-qubit product states

We consider now a state $|\Psi_N\rangle$ of N qubits which is a product state of the single-qubit states $|\Psi_1\rangle$ (9), i.e., $|\Psi_N\rangle = |\Psi_1\rangle^{\otimes N}$. This state is not entangled but has an extensive stabilizer entropy $M_q(|\Psi_N\rangle) = NM_q(|\Psi_1\rangle)$, where $M_q(|\Psi_1\rangle)$ is given by (10), i.e., $|\Psi_N\rangle$ is a magic state. Moreover, it is easy to verify that $S_q(|\Psi_N\rangle) = NS_q(|\Psi_1\rangle)$; that is, for a generic choice of θ and ϕ, the state $M_q(|\Psi_N\rangle)$ is multifractal with the multifractal dimension $D_q = S_q(|\Psi_1\rangle)$ and the subleading term $c_q = 0$. This is an example of a situation in which the presence of multifractality at a given point of the stabilizer orbit of $|\Psi_N\rangle$ implies that the state is a nonstabilizer $|\Psi_N\rangle$. However, this is not the case for the fine-tuned choice $\theta = \pi/2$, for which the multifractality flatness vanishes, $\mathcal{F}(|\Psi_N\rangle) = 0$, and the state $|\Psi_N\rangle$ is fully extended, $D_q = 1$, while still being a nonstabilizer for a generic value of ϕ. Like for the single qubit, the action of the Hadamard gates on $|\Psi_N\rangle$ yields a state with a participation distribution that is not flat, with the multifractal flatness depending on the number of Hadamard gates considered, thus revealing the nonstabilizerness of the state.

C. Random Haar states

We consider a random state $|\Psi\rangle$ and quantify its nonstabilizerness by calculating the stabilizer entropy via (4). Any such state is obtainable for a (random) Haar unitary $U \in \mathcal{U}(d)$ acting on a reference state $|\Psi_0\rangle$. For convenience and without loss of generality, $|\Psi_0\rangle = |0\rangle^{\otimes N}$. We begin by computing the Haar average $\mathbb{E}_{U\in\mathcal{U}(d)}[\overline{\mathcal{F}}[U|\Psi\rangle]]$. As we shall argue, in the limit of large systems $N \gg 1$, $\overline{\mathcal{F}}$ is self-averaging for any fixed realization $|\Psi\rangle = U|\Psi_0\rangle$.

First, let us note that $\mathbb{E}_{U\in\mathcal{U}(d)}[\overline{\mathcal{F}}[U|\Psi_0\rangle]] = \mathbb{E}_{U\in\mathcal{U}(d)}\mathcal{F}[U|\Psi_0\rangle]$. This fact follows from the Clifford group being a subgroup of the unitary ensemble $\mathcal{C}_N \subset \mathcal{U}(d)$ and from the unitary invariance of the Haar measure. Recalling that $\mathbb{E}_{U\in\mathcal{U}(d)}[(U|\Psi_0\rangle\langle\Psi_0|U^\dagger)^{\otimes k}] = k!\Pi_{[k]}/[d(d+1)\cdots(d+k-1)]$ and using the book boundary conditions in replica space (see Fig. 1), it follows that

$$\mathbb{E}_{U\in\mathcal{U}(d)}\big[I_q^r(U|\Psi_0\rangle)\big] = \frac{\sum_{\lambda\vdash r} d(d-1)\cdots(d-K_\lambda+1)((\lambda_k q)!)^{n_k} a_\lambda}{d(d+1)\cdots(d+rq-1)}. \quad (11)$$

In Eq. (11), $\lambda = \{(\lambda_k, n_k)\}_{k=1,\dots,K_\lambda}$ runs over the partitions of $r = \sum_{k=1}^{K_\lambda} n_k\lambda_k$, and the remaining coefficients a_λ are given by

$$a_\lambda = \frac{r!}{\left[\prod_{k=1}^{K_\lambda}(\lambda_k!)^{n_k}\right]\left[\prod_{k=1}^{K_\lambda}(n_k!)\right]}. \quad (12)$$

Specializing to $q = 3$, $r = 1$, and $q = r = 2$, we have

$$\overline{\mathcal{F}}^U \equiv \mathbb{E}_{U\in\mathcal{U}(d)}[\overline{\mathcal{F}}[U|\Psi_0\rangle]] = \frac{2(d-1)}{(d+1)(d+2)(d+3)}. \quad (13)$$

In particular, we recover the stabilizer entropy computation in Ref. [35], with $\mathbb{E}_{U\in\mathcal{U}(d)}[2^{-M_2[U|\Psi_0\rangle]}] = 4/(d+3)$. For a realization $U \in \mathcal{U}(d)$, the state $|\Psi\rangle = U|\Psi_0\rangle$ has $\overline{\mathcal{F}} \simeq \overline{\mathcal{F}}^U$, with an exponentially small error in system size N as a consequence of quantum typicality [93,94]. Indeed, the standard deviation of $\overline{\mathcal{F}}$ over the Haar ensemble is, at leading order in $1/d$, given by $\mathrm{std}_{U\in\mathcal{U}(d)}(\overline{\mathcal{F}}[U|\Psi_0\rangle]) \simeq 2\sqrt{34}/d^{5/2} + O(d^{-6})$ [95].

The random Haar states constitute an ensemble of nonstabilizer states with stabilizer entropy $M_2[U|\Psi_0\rangle] = N - 2 + O(1/d)$, which is close to the maximal one, $M_2 = N$. Their wave function is fully extended over the many-body basis, as shown by the participation entropy $S_q(U|\Psi_0\rangle) = N + (1-q)^{-1}\log_2\Gamma(1+q)$; that is, the multifractal dimension is $D_q = 1$. Hence, according to our definition in this work, the random Haar states are not multifractal. At the same time, the wave function of a random Haar state does not have a flat participation distribution. The multifractality flatness is nonvanishing, $\mathcal{F}(U|\Psi_0\rangle) > 0$, and $S_q(U|\Psi_0\rangle)$ depends nontrivially on the index q via the subleading term c_q in its system-size dependence, consistent with their nonstabilizerness.

V. PROBING NONSTABILIZERNESS VIA MULTIFRACTAL FLATNESS

A. Numerical example

Here, we show that sampling over the Clifford group in Eq. (4) can be performed via the action of a circuit consisting of local Clifford gates acting on the state of interest $|\Psi\rangle$. We consider a system of $N \in [2, 10]$ qubits, and as the initial state $|\Psi\rangle$, we take either the random Haar state $U|\Psi_0\rangle$ or a product state $|\Psi_1\rangle^{\otimes N}$ of N single-qubit states (9) with $\phi = \pi/4$ and $\theta = \pi/2$. We assume that the qubits form a one-dimensional lattice, select random site i, and act with a random two-qubit Clifford gate U_2 [84,91] on sites i and $i+1$, imposing periodic boundary conditions. After each action of the Clifford gate, we compute the multifractal flatness $\mathcal{F}$ of the obtained state. By repeating this process, we obtain results $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{\mathcal{N}_{\text{real}}}$, which we average to calculate an estimator $\hat{\mathcal{F}} = \sum_i \mathcal{F}_1/\mathcal{N}_{\text{real}}$ of the mean multifractal flatness $\overline{\mathcal{F}}(|\Psi\rangle)$ over the Clifford orbit $\mathcal{C}_\Psi$.

Reverting Eq. (4), we find that

$$M_2(|\Psi\rangle) = -\log_2\left[1 - \frac{(d+1)(d+2)}{2}\hat{\mathcal{F}}\right]. \quad (14)$$

This allows us to calculate the stabilizer entropy $M_2(|\Psi\rangle)$ using $\hat{\mathcal{F}}$ and also shows that the statistical uncertainty $\sigma(\mathcal{F})$ of the multifractal flatness has to be multiplied by a factor exponential in the number of qubits N to yield the statistical uncertainty $\sigma(M_2)$ of M_2. For instance, the results from the preceding section for random Haar states imply that $\sigma(\mathcal{F}) \sim d^{5/2}/\sqrt{\mathcal{N}}$, which leads to $\sigma(M_2) \sim \sqrt{d/\mathcal{N}}$. This, in turn, indicates that in order to keep the same uncertainty in M_2 with increasing system size, we need to increase the number $\mathcal{N}$ of samples proportionally to the Hilbert space dimension d. Numerical results shown in the top panel of Fig. 2 confirm this expectation. Keeping the same uncertainty in the stabilizer entropy with $|\Psi_1\rangle^{\otimes N}$ as the initial state requires a number of samples that is smaller but still exponential in the number of qubits: $\mathcal{N} \sim 2^{cN}$ (since $c \approx 0.7$).

We also consider an alternative protocol in which each calculation of the multifractal flatness $\mathcal{F}$ is preceded by an action of a layer of $L/2$ random two-qubit Clifford gates. The

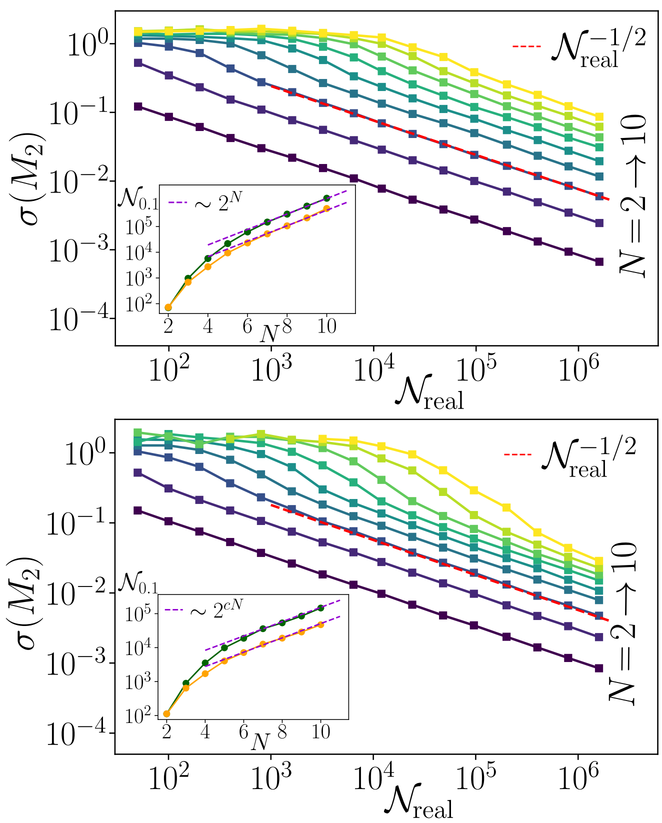

FIG. 2. Determining stabilizer entropy with multifractal flatness $\mathcal{F}$. Top: Statistical error $\sigma(M_2)$ of stabilizer entropy determined from $\mathcal{N}_{\rm real}$ measurements of $\mathcal{F}$ for a random Haar state; the asymptotic $\mathcal{N}_{\rm real}^{-1/2}$ scaling is denoted by the dashed line. Data are shown for number of qubits $N = 2, 3, \ldots, 10$; the green line in the inset shows the number of measurements $\mathcal{N}_{0.1}$ needed to achieve error $\sigma(M_2) = 0.1$ as a function of N, and the orange line shows $\mathcal{N}_{0.1}$ for a protocol in which a full layer of two-body Clifford gates acts between each measurement of $\mathcal{F}$. Bottom: The same as in the top panel, but for the product state $|\Psi_N\rangle$ with $\phi = \pi/2$ and $\theta = \pi/2$; the observed scaling $\mathcal{N}_{0.1} \sim 2^{cN}$ is slower than for the random Haar state (since $c \approx 0.7$) but still exponential in N.

Clifford gates from each layer act on pairs of neighboring sites covering the whole N-qubit chain, and the subsequent layers are shifted by one site with respect to each other. Intuitively, this protocol may allow for a more efficient exploration of the Clifford orbit $\mathcal{C}_\Psi$ and a faster estimation of the stabilizer entropy in (14). This intuition is confirmed by a comparison of the values of $\mathcal{N}_{0.1}$ in the insets in Fig. 2 for the two protocols. However, the asymptotic exponential scaling of the number of samples required to achieve a prescribed accuracy of $M_2(|\Psi\rangle)$ remains the same for the two protocols: $\mathcal{N} \sim 2^{cN}$.

The results of this numerical demonstration provide quantitative confirmation of our analytical calculations that relate the stabilizer entropy to the multifractal flatness of the many-body wave function. Moreover, the action of local two-body Clifford gates is sufficient to probe the Clifford orbit $\mathcal{C}_\Psi$ and obtain an accurate estimation of the multifractal flatness $\overline{\mathcal{F}}(|\Psi\rangle)$. Nevertheless, the sensitivity of the stabilizer entropy to the statistical uncertainty of $\overline{\mathcal{F}}(|\Psi\rangle)$ makes this method of calculation of $M_2(|\Psi\rangle)$ impractical—the computational

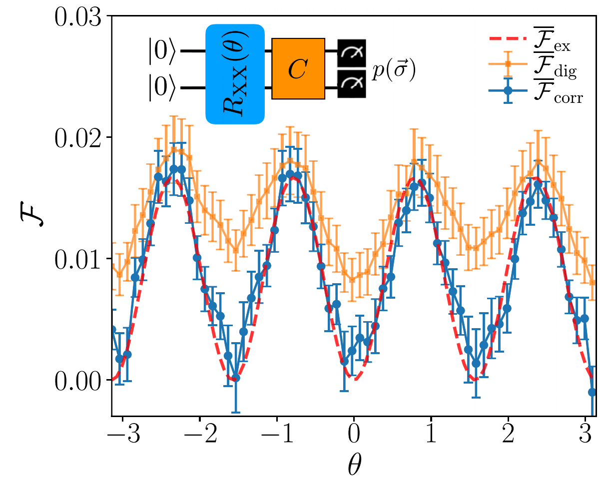

FIG. 3. Digital quantum simulation of the multifractal flatness $\overline{\mathcal{F}}$ using the IBM IBMQ Oslo transmon quantum device. The initial state is prepared by acting with the $R_{XX}(\theta)$ gate on the $|00\rangle$ state (see text). Application of a random Clifford gate C followed by a readout allows for a Monte Carlo estimate (14) of the multifractal flatness $\overline{\mathcal{F}}_{\rm dig}$. While the features in θ are qualitatively captured, we notice a systematic shift of the quantum demonstration result $\overline{\mathcal{F}}_{\rm dig}$ with respect to the exact analytical value $\overline{\mathcal{F}}_{\rm ex}$ due to imperfections of the IBM machine. Those errors can be mitigated, resulting in the corrected value of multifractal flatness $\overline{\mathcal{F}}_{\rm corr}$, which agrees quantitatively with $\overline{\mathcal{F}}_{\rm ex}$ for all values of θ considered. The error bars show the statistical uncertainty of the results associated with the number of circuit realizations, which was fixed as 60.

resources needed to compute $M_2(|\Psi\rangle)$ for larger numbers of qubits with a given accuracy are comparable to or larger than a direct calculation of $M_2(|\Psi\rangle)$ according to the definition (1) [96].

B. Two-qubit system on a quantum device

Last, we demonstrate that the multifractal flatness $\overline{\mathcal{F}}$ is a quantity observable in current quantum devices. As a simple proof of principle, we consider an $(N = 2)$-qubit system prepared in state $|\Psi_\theta\rangle = R_{XX}(\theta)|\Psi_0\rangle$, where $|\Psi_0\rangle = |00\rangle$ and $R_{XX}(\theta) = \exp[-i\theta/2\sigma^1 \otimes \sigma^1]$. We simulate the system in the IBM transmon quantum device [97]. For specific angles $\theta = k\pi/2$ (where k is an integer) the initial state $|\Psi_\theta\rangle$ is a stabilizer; otherwise, it contains nonstabilizerness. We estimate the multifractal flatness $\overline{\mathcal{F}}(|\Psi_\theta\rangle) = \sum_{C_i} \mathcal{F}(C_i|\Psi_\theta\rangle)/\mathcal{N}_{\rm real}$ in the IBMQ Oslo device using $\mathcal{N}_{\rm real} = 60$ realizations of random Clifford gates C_i. Our results are shown in Fig. 3.

The results of digital simulations $\overline{\mathcal{F}}_{\rm dig}$ (denoted with orange dots) quantitatively agree with the exact analytic prediction $\overline{\mathcal{F}}_{\rm ex}$ in the vicinity of the maxima of the multifractal flatness, where the stabilizer entropy of $|\Psi_\theta\rangle$ is close to maximal. However, in neighborhoods of minima of $\overline{\mathcal{F}}_{\rm ex}$, around $\theta = k\pi/2$ (for integer k), we observe a systematic deviation of $\overline{\mathcal{F}}_{\rm dig}$ from the exact value. Nevertheless, we would like to emphasize that even the bare results $\overline{\mathcal{F}}_{\rm dig}$ qualitatively describe the variation of the multifractal flatness with the value of θ.

The remaining discrepancies are due to a combination of decoherence and leakage errors, imperfect fidelity of gates, and readout errors. A closer inspection of the obtained vectors of quasiprobabilities $p_{\mathrm{noisy}}(\vec{\sigma})$ [which approximate the probabilities $p_{\mathrm{ideal}}(\vec{\sigma}) = |\langle\vec{\sigma}|C|\Psi_\theta\rangle\rangle|^2$] reveals that the errors in p_{noisy} affect the value of the multifractal most severely for θ around $k\pi/2$. In those instances, the final state $C|\Psi_\theta\rangle$ is likely to have $p_{\mathrm{ideal}}(\vec{\sigma}) = 0$ for two or three basis states $|\vec{\sigma}\rangle$. An error of order ϵ in p_{noisy} leads to an error of ϵ in the value of multifractal flatness. Hence, already, errors on the level of a few percent in the quasiprobability vector p_{noisy} lead to error in $\mathcal{F}$ on the level of the signal in Fig. 3. In contrast, around $\theta = (2k+1)\pi/4$, the probability $p_{\mathrm{ideal}}(\vec{\sigma})$ is distributed more uniformly over the basis $|\vec{\sigma}\rangle$ for the majority of the Clifford gates C. Hence, some of the errors in p_{noisy} propagate only at the second order in ϵ, while some errors cancel out, yielding $\overline{\mathcal{F}}_{\mathrm{dig}}$ close to the exact value.

Below, we show that a passive readout-error-mitigation scheme [98] is sufficient to correct the bare results $\overline{\mathcal{F}}_{\mathrm{dig}}$. We fix the Clifford gate C to be equal to unity and set $\theta = 0.01$. Considering all $4 = 2^N$ ($N = 2$) initial states, we obtain the quasiprobability vectors p_{noisy} and find matrix A, which links them with the ideal results p_{ideal}:

$$p_{\mathrm{noisy}}(\vec{\sigma}) = A\, p_{\mathrm{ideal}}(\vec{\sigma}). \tag{15}$$

We observe that the 4×4 matrix A can be parametrized in the basis $\{|\vec{\sigma}\rangle\} = \{|01\rangle, |00\rangle, |10\rangle, |11\rangle\}$ with good accuracy as

$$A=\begin{bmatrix} 1-2p-q & p & p & q \\ p & 1-2p-q & q & p \\ p & q & 1-2p-q & p \\ q & p & p & 1-2p-q \end{bmatrix}, \tag{16}$$

where p and q are characteristic parameters for a given device. For IBMQ Oslo we find that the results are broadly consistent with $p = 0.045$ and $q = 0.02$. Inverting formula (15), given the matrix A, allows us to find an estimate for $p_{\mathrm{ideal}}(\vec{\sigma})$. Performing this error mitigation for the data gathered for arbitrary θ and C, we obtain the corrected value of multifractal flatness $\overline{\mathcal{F}}_{\mathrm{corr}}$, which agrees, within the error bars associated with sampling over the Clifford orbit, with the exact result $\overline{\mathcal{F}}_{\mathrm{ex}}$.

The presented results illustrate that the multifractal flatness $\overline{\mathcal{F}}$ is, indeed, measurable on current quantum devices, albeit for a very small number of qubits, $N = 2$. Finding more controlled and scalable error-mitigation schemes is an interesting future challenge, which might involve the use of the zero-noise extrapolation [99] and active error-mitigation techniques [100–102].

VI. CONCLUSION

This work puts forward a correspondence between the concept of wave-function multifractality and nonstabilizerness. Specifically, we introduced multifractal flatness, a combination of inverse participation ratios that quantifies the flatness of the wave-function probability distribution. We showed that multifractal flatness is a witness of nonstabilizerness and translates to a measure of magic when the average over the Clifford orbit is considered. We illustrated the relationship between multifractality and nonstabilizerness using a few examples of quantum states, for instance, recovering known results [35] for random Haar states.

The connection put forward in this work has practical implications, as we showed by computing the stabilizer entropies by estimating the multifractal flatness with a repeated action of a local Clifford circuit. As a proof of principle, we demonstrated that nonstabilizerness, as estimated by the multifractal flatness $\overline{\mathcal{F}}$, is detectable in current quantum devices.

Recent works [26,103] revealed a tight relationship between quantum chaos and nonstabilizerness. It would be interesting to investigate this link through the lens of multifractality. For instance, in semiclassical quantum systems, the generalized Lyapunov exponents present multifractal features [104,105]. Furthermore, models of maximally chaotic quantum many-body systems have been shown to display rich multifractal behavior [106–108]. Additionally, it would be interesting to reveal the equilibrium and out-of-equilibrium nonstabilizerness of archetypal condensed-matter models [37,43,109].

Note added. Recently, a paper appeared [110] that implemented a scalable scheme for experimentally observing the multifractal flatness.

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[96] We note that the resources for the calculation of $M_2(|\Psi\rangle)$ directly according to the definition (1) scale as $O(2^{3N})$, while the cost of Monte Carlo sampling of the Clifford orbit $O(2^{2N})$ until a prescribed accuracy of $M_2(|\Psi\rangle)$ is achieved scales as $O(2^{2N})$. Nevertheless, the constant describing the scaling in the Monte Carlo sampling case is large, and direct numerical calculation of $M_2(|\Psi\rangle)$ according to (1) is more efficient at practically relevant system sizes $N \lesssim 14$.
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