Identifying primes from entanglement dynamics

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The distribution of primes over the set of natural numbers is a fascinating subject closely related to topics that range from the fundamental problem of factorization to applications in cryptography. Despite numerous efforts, efficient methods to locate huge primes are still under investigation. Here, we present an alternative approach to identifying prime numbers that is based on the evolution of the linear entanglement entropy. Specifically, we show that a singular behavior in the amplitudes of the Fourier series of this entropy is associated with prime numbers. We also discuss how this intriguing connection between primes and entanglement could be experimentally implemented using existing optical devices, and examine a possible relationship between our results and the zeros of the Riemann zeta function.

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I. INTRODUCTION

Prime numbers have captured the attention of researchers for centuries. In pure mathematics, given the prominent role of primes in factorizing a positive integer n, much effort has been made to unravel patterns in their distribution over the set of natural numbers $\mathbb{N}$. Remarkable results in this direction involve the counting function $\pi(n)$ [1], which gives the number of primes less than or equal to n. The complete knowledge of $\pi(n)$ would imply being able to determine the position of each prime number p, since jumps of this function expose their presence. However, achieving the counting function with satisfactory accuracy for large values of n has been shown to be practically impossible. In applied mathematics, this lack of knowledge is exploited for cryptographic protocols [2], such as the RSA (Rivest-Shamir-Adleman) algorithm. To break the RSA protocol, one needs to find the prime factors of a huge n, which would require the implementation of Shor's algorithm in a quantum computer [3].

Another fascinating result involving primes is their connection with the zeros of the Riemann zeta function $\zeta(s)$, where s is a complex variable, defined as [4]

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1-p^{-s}} = \frac{\Gamma(1-s)}{2\pi i} \int_{\gamma} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}. \tag{1}$$

The summation and the product above converge when the real part of the variable s satisfies $\mathrm{Re}[s] > 1$. The first equality in Eq. (1) is due to Euler and makes evident the relation between p and $\zeta(s)$. In 1859 [5], Riemann achieved the expression shown on the right side of Eq. (1), which involves a line integral along a particular path γ [6] and the gamma function $\Gamma(1-s) \equiv \int_0^{\infty} x^{-s} e^{-x} dx$. The expression is analytic for all values of s, except for a simple pole at $s = 1$. In the region $\mathrm{Re}[s] > 1$, the term with the integral recovers the other two expressions in Eq. (1), so one considers it as their analytic continuation. Using these ideas, Riemann [5] demonstrated how the complex zeros s_0 of $\zeta(s)$ encode the distribution of p. Specifically, he showed that a certain convergent series, running over all s_0, recovers the function $\pi(n)$. But, for large values of n, the number of zeros required to accurately get the counting function is so large that the use of the series to identify p becomes intractable. This sophisticated connection between $\zeta(s)$ and primes is but one of the impressive results of Ref. [5]. Arguments in that work also gave origin to the Riemann hypothesis, which conjectures that $\mathrm{Re}[s_0] = \frac{1}{2}$ for all nontrivial zeros of $\zeta(s)$.

The almost mystical status of prime numbers and their connection with the zeta function have also reached the physics community, especially researchers working with quantum mechanics [7,8]. A particularly inspiring idea is the Hilbert-Pólya conjecture that could lead to the proof of the Riemann hypothesis. The conjecture proposes that the nontrivial zeros of the Riemann zeta function, which supposedly fall on the critical line $\mathrm{Re}[s_0] = \frac{1}{2}$, correspond to the eigenvalues of a Hermitian Hamiltonian operator. The first substantial evidence supporting this conjecture arose in the semiclassical studies carried out by Berry and Keating [9]. They compared the distribution of s_0 over the critical line with the equivalent function for the eigenvalues of a given Hamiltonian achieved through the Gutzwiller trace formula. In this way, they were able to identify a Hamiltonian operator that fulfills the conjecture. The operator is obtained by quantizing the classical Hamiltonian $H_{\mathrm{cl}} = xp$, where x and p are the particle position and momentum, respectively.

Inspired by the Hilbert-Pólya conjecture and the Berry and Keating xp model, several works [10–16] have aimed at interpreting, extending, and circumventing technical difficulties of Ref. [9]. Other contributions looked for alternative physical

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systems, where the properties of $\zeta(s)$ could be identified. This has been done using quantum graph theory [17], many-body systems [18], wave-packet dynamics [19], statistical mechanics of random energy landscapes [20], random matrix theory [21,22], and quantum entanglement [23], and there is also an experiment based on a periodically driven single qubit [24]. In addition to these works, where the Riemann zeta function is the protagonist to link prime numbers and quantum physics, other quantum approaches deal directly with primes, studying, for instance, number factorization [25,26] and the properties of prime states [27,28], which are superpositions of states corresponding to prime numbers.

The literature cited above describes a broad context in which prime numbers connect to the quantum theory. In the present paper, we add a related but different approach to this subject by showing that the dynamics of the linear entanglement entropy encodes prime and semiprime numbers. In a sense, we borrow this idea from the Hilbert-Pólya conjecture. While the original statement concerns the existence of a system that reveals the nontrivial zeros of $\zeta(s)$ through an energy measurement, here we indicate which physical observation exposes the location of a prime number, a mathematical element as undomesticated as s_0. Specifically, we demonstrate that the Fourier modes c_n of the entropy have amplitudes that present a singular behavior when n corresponds to a prime. In addition, we find curves $c_n^{(f)}$ for the location of families f of semiprimes. This result is shown for a system of two coupled harmonic oscillators and a system of two coupled spins. The route to achieve our goals consists of letting a specific initial bipartite state evolve according to an appropriate Hamiltonian and calculating its linear entanglement entropy as a function of time. A careful inspection of the Fourier series representation of this expression reveals the location of primes and semiprimes. Finally, at the end of the paper, we discuss a possible experimental realization of this idea and speculate on how to link c_n and the counting function $\pi(n)$.

Our work focuses exclusively on this new connection between entanglement and prime numbers, and thus differs from the available literature on methods for efficiently identifying primes. Furthermore, our proposal for the experimental realization of our ideas serves as a proof of concept. We provide initial estimates for the values of the parameters, but further development and eventual implementation call for collaborations with experimental groups that are experts in the proposed approach.

II. ENTANGLEMENT DYNAMICS

We consider a system consisting of two interacting parts, A and B, to which we assign distinct Hilbert spaces, $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. The system is isolated and prepared in a pure state, $\rho(0)=|\Psi(0)\rangle\langle\Psi(0)|$, which evolves according to the total Hamiltonian

$$H = H_A\otimes\mathbb{1}_B + \mathbb{1}_A\otimes H_B + \lambda H_A\otimes H_B, \tag{2}$$

where λ is the coupling strength between the two subsystems. The linear entanglement entropy,

$$S_L(t) = 1-\mathrm{Tr}\big[\rho_A^2(t)\big], \tag{3}$$

where $\rho_A(t)\equiv\mathrm{Tr}_B[\rho(t)]$ is the reduced density matrix of system A, measures the bipartite entanglement in time between A and B. When $\rho(t)$ is separable, $S_L(t)=0$, otherwise $0<S_L(t)<1$.

III. COUPLED OSCILLATORS

The first Hamiltonian that we consider describes two coupled harmonic oscillators, where H_A is the Hamiltonian for part A with eigenvalues $\hbar\omega_0(n_A+\frac{1}{2})$ and eigenstates $|n_A\rangle$, and equivalently for part B. The initial state is the product of two canonical coherent states [29], $|\Psi(0)\rangle=|\alpha_A\,\alpha_B\rangle$, with

$$|\alpha_A\,\alpha_B\rangle \equiv e^{-u/2}\sum_{n_A,n_B=0}^{\infty}\frac{\alpha_A^{n_A}}{\sqrt{n_A!}}\frac{\alpha_B^{n_B}}{\sqrt{n_B!}}|n_A\,n_B\rangle, \tag{4}$$

where we chose $|\alpha_A|^2=|\alpha_B|^2=\frac{u}{2}$. Defining $\omega\equiv\hbar\lambda\omega_0^2$,

$$S_L^{\mathrm{osc}}(t) = 1-e^{-2u}\sum_{j,k,l,m}\frac{\left(\frac{u}{2}\right)^{j+k+l+m}}{j!\,k!\,l!\,m!}e^{-i\omega t(j-k)(l-m)}. \tag{5}$$

The sum above runs from 0 to ∞ for all indexes. Since the entropy in Eq. (5) is periodic in time, with period $T=2\pi/\omega$, we use the Fourier series and arrive at

$$S_L^{\mathrm{osc}}(t) = c_0(u)-\sum_{n=1}^{\infty}c_n(u)\cos(n\omega t), \tag{6}$$

where the coefficients $c_n(u)$ are given by

$$c_n(u) = 4\,e^{-2u}\sum_{k,m}\sum_{j>k}\sum_{l>m}\frac{\left(\frac{u}{2}\right)^{j+k+l+m}}{j!\,k!\,l!\,m!}\delta^n_{(j-k)(l-m)}, \tag{7}$$

representing the amplitude of the mode with frequency $n\omega$. In Eq. (7), the Kronecker delta function, δ_a^b for integers a and b, prompts the analysis of how n decomposes as a product of two integers. We introduce the set Λ_n composed of all distinct positive divisors of n and note that the non-null terms in Eq. (7) are those for which $(j-k)$ and $(l-m)$ belong to Λ_n and their product equals n. Therefore,

$$c_n(u) = 4\,e^{-2u}\sum_{\mu\in\Lambda_n}I_\mu(u)\,I_{\frac{n}{\mu}}(u), \tag{8}$$

where $I_\chi(w)\equiv\sum_{k=0}^{\infty}[k!(\chi+k)!]^{-1}(\frac{w}{2})^{2k+\chi}$ is the modified Bessel function of the first kind. As we show next, $c_n(u)$ is very sensitive to the primality of n.

A. Identifying primes

To show that the entanglement dynamics quantified by $S_L^{\mathrm{osc}}(t)$ is a prime identifier, we first assume that n is prime, so that $\Lambda_n=\{1,n\}$ and Eq. (8) yields $c_n(u)=c_n^{(p)}(u)$, where

$$c_n^{(p)}(u)\equiv 8\,e^{-2u}I_1(u)\,I_n(u). \tag{9}$$

In contrast, when n is a composite number, Λ_n necessarily consists of 1, n, and, at least, one more integer. By defining the set $\Lambda_n'=\Lambda_n-\{1,n\}$, Eq. (8) becomes

$$c_n(u) = c_n^{(p)}(u)+4\,e^{-2u}\sum_{\mu\in\Lambda_n'}I_\mu(u)\,I_{\frac{n}{\mu}}(u)\geqslant c_n^{(p)}(u), \tag{10}$$

which holds only for $n>1$.

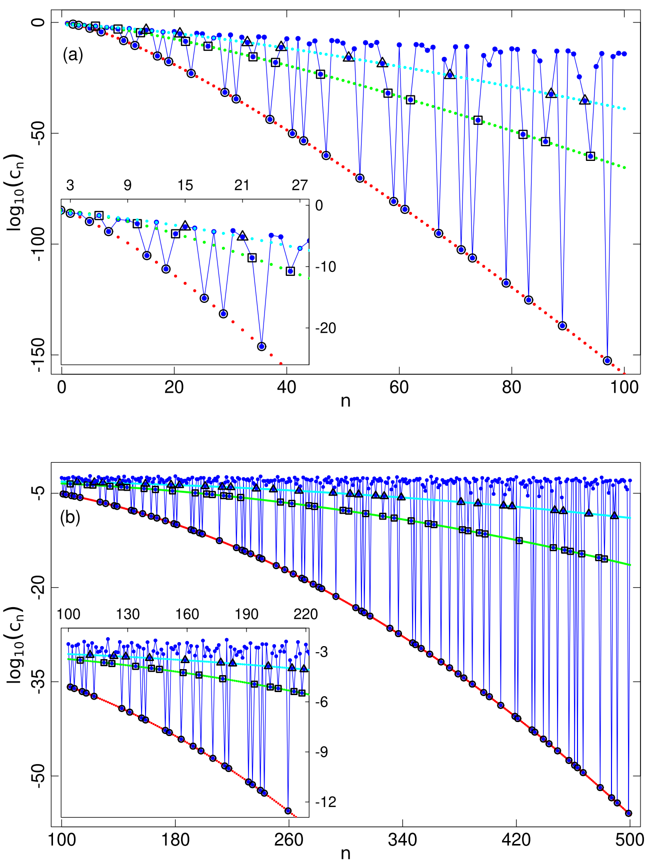

FIG. 1. Decimal logarithm of the coefficients $c_n(u)$ as a function of n for (a) $u = 1$ and (b) $u = 10^3$. The blue dots, connected with a blue solid line to guide the eye, represent $c_n(u)$, and the red points indicate the lower bound $c_n^{(p)}(u)$. For each prime n, the blue dots are encircled with black circles and touch the red dotted line. Green and cyan lines are for $c_n^{(f_2)}(u)$ and $c_n^{(f_3)}(u)$, respectively. Black squares [triangles] enclose the coefficients of the family f_2 [f_3]. Inset plots show a magnified region of the respective main graph.

Inequality (10) is a main result of this work. It shows that the coefficients $c_n(u)$ coincides with $c_n^{(p)}(u)$ in Eq. (9) if, and only if, n is prime, while for composite numbers the amplitudes are strictly lower bounded by $c_n^{(p)}(u)$. This means that if we have a way to determine the values of $c_n(u)$, other than through the construction of Λ_n, the application of Eq. (10) can reveal the primality of n. Indeed, $c_n(u)$ can be evaluated theoretically via Eq. (5) and possibly experimentally.

In Fig. 1, to illustrate the above analytical discussion, we mark the values of $c_n(u)$ with blue dots. For each n identified as a prime, the dots are encircled with a black circle. All values of $c_n(u)$ corresponding to primes coincide with the red dotted line that represents the lower bound $c_n^{(p)}(u)$ and can, therefore, be clearly distinguished. In Fig. 1(a) we show results for $u = 1$, but $c_n(1)$ becomes too small when n is large, so in Fig. 1(b) we use $u = 10^3$ and larger values of n.

B. Semiprimes

Figure 1 also reveals the square-free semiprimes, including the integer 2, which we denote by family f_2. For these numbers, $\Lambda_n = \{1, 2, P_n, n\}$, where $P_n = n/2 \neq 2$ is a prime, and $c_n(u) = c_n^{(f_2)}(u)$, where

$$c_n^{(f_2)}(u) \equiv 8\, e^{-2u}\big[I_1(u)\, I_n(u) + I_2(u)\, I_{\frac{n}{2}}(u)\big]. \tag{11}$$

For a composite n that presents the divisors 1, 2, $n/2$, n, and at least one more integer,

$$c_n^{(2)}(u) \equiv c_n^{(f_2)}(u) + 4\, e^{-2u} \sum_{\mu\in\Lambda_n'^{(2)}} I_\mu(u)\, I_{\frac{n}{\mu}}(u) > c_n^{(f_2)}(u),$$

where $\Lambda_n'^{(2)} = \Lambda_n - \{1, 2, n/2, n\}$. The semiprimes of family f_2 are identified with black squares in Fig. 1 and they all fall on the curve $c_n^{(f_2)}(u)$, indicated with a green line. This line consists of a lower bound for any integer composed by 2, except for 2, 2^2, and 2^3.

It is straightforward to extend the analysis above to any other family of semiprimes. The particular case of the square-free semiprimes containing the number 3, denoted by f_3, is shown in Fig. 1. The components of family f_3 are marked with black triangles and they are exactly located over the cyan curve $c_n^{(f_3)}(u)$ [30].

IV. INTERACTING SPINS

Analogously to the system of coupled harmonic oscillators, we now show that two interacting spins with a large quantum number S is another physical system that can be used to identify primes. In the Hamiltonian of Eq. (2), we assume that $H_A = \hbar\omega_0 S_A^z$, where S_A^z is the z component of the spin operator of part A and $S_A^z|m_A\rangle = m_A|m_A\rangle$, and equivalently for part B. The initial state is the product of spin coherent states [29], $|\Psi(0)\rangle = |s_A\, s_B\rangle$, where

$$|s_A\, s_B\rangle \equiv N_s \sum_{n_A,n_B=0}^{2S} s_A^{n_A} s_B^{n_B} \sqrt{\binom{2S}{n_A}\binom{2S}{n_B}}\,|n_A - S, n_B - S\rangle, \tag{12}$$

with $N_s = (1+u)^{-2S}$. Choosing $|s_A|^2 = |s_B|^2 = u$, the evolving linear entanglement entropy (3) becomes

$$S_L^{\rm spin}(t) = 1 - \sum_{j,k,l,m=0}^{2S} \xi_{j,k,l,m}\, u^{j+k+l+m} e^{-i\omega t(j-k)(l-m)}, \tag{13}$$

where $\xi_{j,k,l,m} \equiv \binom{2S}{j}\binom{2S}{k}\binom{2S}{l}\binom{2S}{m}(1+u)^{-8S}$. The Fourier series of Eq. (13) has the same structure as Eq. (6), but with the coefficients

$$\bar{c}_n(u) = 4 \sum_{k,m=0}^{2S-1} \sum_{j>k}^{2S} \sum_{l>m}^{2S} \xi_{j,k,l,m}\, u^{j+k+l+m} \delta^n_{(j-k)(l-m)}.$$

The nonvanishing terms in the equation above are again those for which $\mu \equiv (j-k)$ and $\nu \equiv (l-m)$ belong to the set Λ_n of divisors of n. However, this system is bounded, so the upper limits of the above summations add new constraints to μ and ν. Given a value for k and m, we have that $\mu \in \mathcal{I}_k \equiv [1, 2S-k]$ and $\nu \in \mathcal{I}_m \equiv [1, 2S-m]$. In addition, since $\delta^n_{\mu\nu}$ implies that $\nu = n/\mu$, we also get that $\mu \in \mathcal{I}_{m,n} \equiv [\frac{n}{2S-m}, n]$. Therefore,

$$\bar{c}_n(u) = 4 \sum_{k,m=0}^{2S-1} \sum_{\mu\in\bar{\Lambda}_n^{(k,m)}} \bar{\xi}\, u^{2k+2m+\mu+n/\mu}, \tag{14}$$

where $\bar{\xi} \equiv \xi_{k+\mu,k,m+\frac{n}{\mu},m}$ and $\bar{\Lambda}_n^{(k,m)}$ is the set of divisors of n that satisfies all constraints commented above:

$$\bar{\Lambda}_n^{(k,m)} \equiv \Lambda_n \cap \mathcal{I}_k \cap \mathcal{I}_{m,n}. \tag{15}$$

Contrary to $S_L^{\rm osc}(t)$ in Eq. (6), there is a finite number of Fourier modes in $S_L^{\rm spin}(t)$ due to the finite Hilbert space for the spin system. For $n > 4S^2$, we have that $\mathcal{I}_k \cap \mathcal{I}_{m,n} = \emptyset$ for any value of k and m, so $\bar{\Lambda}_n^{(k,m)} = \emptyset$ and $\bar{c}_n(u) = 0$. In what follows, to assess the primality of n, we conveniently divide the interval $1 < n \leqslant 4S^2$ in two regions of interest, region I ($1 < n \leqslant 2S$) and region II ($2S < n \leqslant 4S$). In the complementary range $4S < n \leqslant 4S^2$, one cannot find a behavior that distinguishes primes from composite numbers. There, even though all amplitudes of the Fourier modes for a prime n vanish, the same can also happen to some composite numbers.

Region I ($1 < n \leqslant 2S$): A careful inspection of Eq. (15) reveals that $1 \in \bar{\Lambda}_n^{(k,m)}$ for all values of k and $0 \leqslant m \leqslant 2S - n$, while $n \in \bar{\Lambda}_n^{(k,m)}$ for all values of m and $0 \leqslant k \leqslant 2S - n$. Applying these results to Eq. (14), we find that, for n prime, $\bar{c}_n(u) = \bar{c}_{n,I}^{(p)}(u)$, where

$$\bar{c}_{n,I}^{(p)}(u) \equiv 8\,(1+u)^{-8S} G_1(u) G_n(u), \tag{16}$$

with $G_\chi(w) \equiv \sum_{k=0}^{2S-\chi} \binom{2S}{k}\binom{2S}{k+\chi} w^{2k+\chi}$. On the other hand, for a composite number $n > 1$ inside region I, we have that $\bar{c}_n(u) = \bar{c}_n^{(p)}(u) + \bar{d}_{n,I}(u)$, where

$$\bar{d}_{n,I}(u) \equiv 4 \sum_{k,m=0}^{2S-1} \sum_{\mu \in \bar{\Lambda}_n^{\prime(k,m)}} \bar{\xi}\, u^{2k+2m+\mu+n/\mu} \tag{17}$$

and $\bar{\Lambda}_n^{\prime(k,m)} \equiv \Lambda_n^\prime \cap \mathcal{I}_k \cap \mathcal{I}_{m,n}$, where $\Lambda_n^\prime$ was defined above Eq. (10). One can show that, at least for $k = m = 0$, there is a positive non-null term in Eq. (17). Therefore, for a composite number, we necessarily have $\bar{c}_n(u) > \bar{c}_{n,I}^{(p)}(u)$ and this inequality can be employed in the search for prime numbers in region I.

Region II ($2S < n \leqslant 4S$): In this case, prime modes are not included in the Fourier series of Eq. (13). Indeed, for a prime number n, we now have that $1 \notin \mathcal{I}_{m,n}$ and $n \notin \mathcal{I}_k$, which implies that $\bar{\Lambda}_n^{(k,m)} = \emptyset$ and $\bar{c}_n(u) = 0$. For a composite n in the same interval, similarly to region I, there is at least one integer in Λ_n, in addition to 1 and n, so that $\bar{c}_n(u) > 0$. So we can distinguish primes from composite numbers by checking if $\bar{c}_n(u) = 0$ or $\bar{c}_n(u) \neq 0$.

V. EXPERIMENTAL PROPOSAL

The Hamiltonian for the two coupled harmonic oscillators studied above also describes the interaction between two optical fields via a Kerr nonlinear medium [31,32]. In Fig. 2, we show the sketch of an experimental setup for implementing this optical system. A laser beam of frequency ω_0 and linear polarization at 45° is sent to a polarizing beam splitter (PBS). The vertically polarized (↕) component goes directly to the homodyne detection scheme [33], shown in the lower right corner of the figure, to act as the local oscillator (LO). The horizontally polarized (↔) component passes through a half-wave plate (HWP), which rotates the polarization to 45° and enters an unbalanced Mach-Zehnder interferometer, identified in Fig. 2 with dashed lines. In the interferometer, after the PBS, the ↕ component beam propagates through the short path, and the ↔ one goes through the long course. The path difference is longer than the coherence length, so that the recombined ↔ and ↕ beams at the interferometer output PBS are separable and no longer result in a pure mode with linear diagonal polarization. Next, these two beams are injected in the nonlinear Kerr medium (KM) of length L and Kerr optical nonlinearity $\chi^{(3)}$. During the propagation time inside the KM, each beam will experience a modified index of refraction due to the action of the other beam. This is how the coupling between the oscillators is physically implemented. By varying the length L, one changes the interaction time t. After passing KM, the ↔ and ↕ beams are split again. To identify the prime numbers, we measure one of the beams and ignore the other, which is equivalent to performing the trace over one of the interacting systems. We can analyze any of the two beams because of the interaction symmetry. In Fig. 2, we choose the ↕ beam, which goes to the homodyne detector, a frequency-dependent measurement. By tuning the phase of LO, we can measure the fluctuations in the quadrature affected by the cross-Kerr modulation directly. These fluctuations are translated into electric current modulations, whose spectrum is analyzed. Therefore, it is possible to readily address fluctuation frequencies as high as $f_h \simeq 1$ GHz, the homodyne frequency, with current technology.

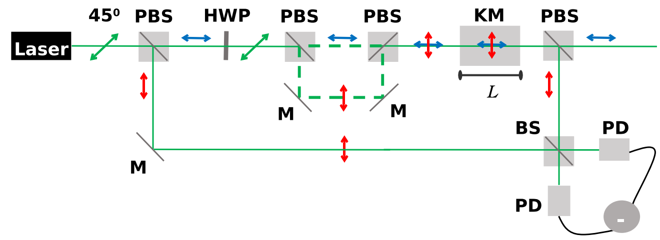

FIG. 2. Setup for the identification of prime numbers. The symbols BS, HWP, PBS, M, KM, and PD refer to beam splitter, half-wave plate, polarizing beam splitter, mirror, Kerr medium, and photodiode, respectively. See text for details.

We can estimate the range of prime numbers p that can be identified using this approach by noting that the homodyne frequency f_h must be equal to the mode frequency $p\omega/2\pi$, appearing in Eq. (5). This gives $p\omega/2\pi = p\hbar\lambda\omega_0^2/2\pi = f_h$. With angular optical frequency ω_0 in the range of 10^{14} Hz and interaction strength $\lambda \simeq 10^{-6}\,\mathrm{J}^{-1}$, we have $p \simeq 10^{21}$, which is a prime with 21 digits. Notice also that this is not an upper limit because the nonlinear interaction can be increased, reducing λ, and heterodyne detection can be used to increase the frequency analysis. Naturally, the actual implementation of this experiment will require further technical considerations and fine tuning.

VI. CONCLUSION

This paper shows that analyzing the bipartite entanglement in time works to identify prime and semiprime numbers in $\mathbb{N}$. The main ingredient is a Hamiltonian composed of two parts, A and B, with equidistant energy levels for H_A and H_B.

We discussed the implementation of this idea, emphasizing that it takes advantage of an *existing* experimental setup. The purpose of the approach is not to compete with standard numerical methods used to study the distribution of large primes. Instead, our focus is on the new fundamental relation between primes and entanglement, which offers a new perspective of the problem and may provide a new route of investigation.

Our work resonates with the Hilbert-Pólya conjecture in the sense of proposing a physically measurable quantity to determine prime numbers. We speculate that there may be a way to connect our results with $\zeta(s)$. In fact, by defining $A(n,u) \equiv c_n(u) - c_n^{(p)}(u)$ for a fixed u, the task of counting primes along $\mathbb{N}$ is equivalent to counting the zeros of $A(n,u)$. This alternative method to build $\pi(n)$ could be used to study the zeros of $\zeta(s)$ by means, for example, of a hypothetical inversion of the Riemann series that connects s_0 with $\pi(n)$, shedding new light on the Riemann hypothesis.

The last remark concerns the semiclassical approaches originated from the ideas of Berry and Keating [9]. The semiclassical version of the linear entanglement entropy was addressed in [34,35] for the same two physical systems treated here. In those papers, entanglement is reproduced by summing over sets of classical trajectories determined by the solutions of a given transcendental equation. The purely quantum formalism presented here encourages the reexamination of those works, aiming at connecting those solutions with the distribution of primes.

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