Reduced dynamics with Poincaré symmetry in an open quantum system

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(Received 10 January 2023; accepted 5 October 2023; published 26 October 2023)

We consider how the reduced dynamics of an open quantum system coupled to an environment is realized in the Poincaré symmetry. The reduced dynamics is described by a dynamical map, which is a quantum channel (a completely positive and trace-preserving linear map) given by tracing out the environment from the total unitary evolution without initial correlations. We investigate the dynamical map invariant under the Poincaré transformations and discuss how the invariance constrains the form of the map. Based on the unitary representation theory of the Poincaré group, we develop a systematic way to construct the dynamical map with the Poincaré invariance. Using this method, we derive such a dynamical map for a spinless massive particle, and the conservation of the Poincaré generators is discussed. We then find the map with the Poincaré invariance and the four-momentum conservation. Further, we show that the conservation of the angular momentum and the boost operator makes the map of a spinless massive particle unitary.

DOI: 10.1103/PhysRevA.108.042217

I. INTRODUCTION

It is difficult to isolate a quantum system perfectly, which is affected by the inevitable influence of a surrounding environment. Such a quantum system is called an open quantum system. Since we encounter open quantum systems in a wide range of fields such as quantum information science [1,2], condensed-matter physics [3,4], and high-energy physics [5], it is important to understand their dynamics. In general, the dynamics of an open quantum system, the so-called reduced dynamics, is very complicated. This is because the environment may have infinitely many uncontrollable degrees of freedom. One needs the effective theory with relevant degrees of freedom to describe the reduced dynamics of an open quantum system [2].

As is well known, symmetry is a powerful tool for capturing relevant degrees of freedom in the dynamics of interest. For example, let us focus on the symmetry in the Minkowski space-time, which is called the Poincaré symmetry. Imposing the Poincaré symmetry on a quantum theory, one finds that quantum dynamics in the theory is described by the fundamental degrees of freedom such as a massive particle and a massless particle [6]. The approach based on symmetries may provide a way to get the effective theory of open quantum systems.

In this paper we discuss the consequences of the Poincaré symmetry on the reduced dynamics of an open quantum system. This is motivated by the desire to understand relativistic theories of open quantum systems (see, for example, [7–15]) and the theory of quantum gravity. At the present time, quantum mechanics and gravity have not been unified yet. This situation has prompted the proposal of many models of gravitating quantum systems. In Ref. [16] the model of a classical gravitational interaction between quantum systems was proposed, which is called the Kafri-Taylor-Milburn model. In addition, the Diósi-Penrose model [17–19] and the Tilloy-Diósi model [20] were advocated, for which gravitating quantum system intrinsically decoheres. The above models are formulated in nonrelativistic theories of open quantum systems. One concern is how they are incorporated in relativistic theories. This paper would help to obtain a relativistic extension of the above models.

For our analysis, we describe the reduced dynamics of an open quantum system by a dynamical map. The dynamical map is a quantum channel obtained by tracing out the environment from the total unitary evolution with an initial product state. It is known that the dynamical map (the quantum channel) is a completely positive and trace-preserving linear map and has an operator-sum representation given by Kraus operators [2,21–23]. We consider the condition of a dynamical map invariant under the Poincaré transformations. It is first shown that this condition is satisfied for unitary evolution in quantum theory with the Poincaré symmetry. We then consider how the condition restricts the form of Kraus operators associated with the dynamical map. With the help of the representation theory of the Poincaré group, we obtain a systematic way to deduce the Kraus operators.

Applying this method, we get a model of the dynamical map of a spinless massive particle. Discussing the conservation of the Poincaré generators, we obtain the following consequences. (i) There is the nonunitary dynamical map (i.e., nonunitary channel) with the Poincaré invariance and the four-momentum conservation. (ii) If we impose the conservation of the Poincaré generators, then the map of a spinless massive particle is reduced to the unitary map (i.e., unitary channel) generated by the time-translation operator. These imply that the Poincaré symmetry can strongly constrain the reduced dynamics of an open quantum system. We further discuss a covariant formulation of the dynamical map with the Poincare invariance.

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The structure of this paper is as follows. In Sec. II we discuss the dynamical map describing the reduced dynamics of an open quantum system and introduce the invariance of the dynamical map. In Sec. III we derive the condition that the dynamical map is invariant under the Poincaré group. In Sec. IV, focusing on the dynamics of a spinless massive particle, we present a model of the dynamical map with the Poincaré invariance. We then investigate the model in terms of the conservation law of the Poincaré generators. Section V discusses a covariant formulation of our theory. Section VI provides a summary. The natural unit $\hbar = c = 1$ is used in this paper.

II. QUANTUM DYNAMICAL MAP AND ITS SYMMETRY

In this section we consider the reduced dynamics of an open quantum system and discuss the symmetry of the dynamics. The reduced dynamics is given as the time evolution of system density operator. The evolution from a time slice $\tau = t_0$ to $\tau = t$ is assumed to be given by

$$\rho(t) = \Phi_{t,t_0}[\rho(t_0)] = \mathrm{Tr}_{\mathrm{E}}[\hat{U}(t,t_0)\rho(t_0) \otimes \rho_{\mathrm{E}}\hat{U}^\dagger(t,t_0)], \quad (1)$$

where $\rho(\tau)$ is the system density operator, ρ_{E} is the density operator of an environment, and $\hat{U}(t,t_0)$ is the unitary evolution operator of the total system. In this paper the map Φ_{t,t_0} is called a dynamical map, which has the property of being completely positive and trace preserving [2,21–23]. The dynamical map Φ_{t,t_0} is rewritten in the operator-sum representation

$$\Phi_{t,t_0}[\rho(t_0)] = \sum_\lambda \hat{F}^{t,t_0}_\lambda \rho(t_0) \hat{F}^{t,t_0\dagger}_\lambda, \quad (2)$$

where $\hat{F}^{t,t_0}_\lambda$ are the Kraus operators. Note that this generally follows from Eq. (1) (see Appendix A). The Kraus operators satisfy the completeness condition

$$\sum_\lambda \hat{F}^{t,t_0\dagger}_\lambda \hat{F}^{t,t_0}_\lambda = \hat{\mathbb{I}}, \quad (3)$$

which guarantees the trace-preserving property $\mathrm{Tr}\{\Phi_{t,t_0}[\rho(t_0)]\} = \mathrm{Tr}[\rho(t_0)]$. In the operator-sum representation, λ takes a discrete value. When λ is a continuous value, we should replace the summation $\sum_\lambda$ with the integration $\int d\mu(\lambda)$ with an appropriate measure $\mu(\lambda)$. It is known that two dynamical maps Φ and Φ', with

$$\Phi[\rho] = \sum_\lambda \hat{F}_\lambda \rho \hat{F}^\dagger_\lambda, \quad \Phi'[\rho] = \sum_\lambda \hat{F}'_\lambda \rho \hat{F}'^\dagger_\lambda, \quad (4)$$

are equivalent to each other (i.e., $\Phi[\rho] = \Phi'[\rho]$ for any density operator ρ) if and only if there is a unitary matrix $\mathcal{U}_{\lambda\lambda'}$ satisfying $\sum_\lambda \mathcal{U}_{\lambda_1\lambda}\mathcal{U}^*_{\lambda_2\lambda} = \delta_{\lambda_1\lambda_2} = \sum_\lambda \mathcal{U}_{\lambda\lambda_1}\mathcal{U}^*_{\lambda\lambda_2}$ and

$$\hat{F}'_\lambda = \sum_{\lambda'} \mathcal{U}_{\lambda\lambda'}\hat{F}_{\lambda'}. \quad (5)$$

This is the uniqueness of a dynamical map [2,21–23].

To introduce symmetry in the above formulation, we schematically consider the differential equation of the density operator

$$d\rho(\tau) = d\mathcal{L}_\tau[\rho(\tau)], \quad (6)$$

whose solution is $\rho(t) = \Phi_{t,t_0}[\rho(t_0)]$ from a time slice $\tau = t_0$ to $\tau = t$. For example, if $d\mathcal{L}_\tau = d\tau\mathcal{L}$ with a Lindbladian $\mathcal{L}$, then $\Phi_{t,t_0} = e^{\mathcal{L}(t-t_0)}$, which is nothing but the superoperator of the quantum dynamical semigroup [2,21,22,24,25]. The differential equation (6) is called covariant if the equation

$$d\rho'(\tau) = d\mathcal{L}'_\tau[\rho'(\tau)] \quad (7)$$

holds under a transformation with $\rho(\tau) \to \rho'(\tau)$ and $d\mathcal{L}_\tau \to d\mathcal{L}'_\tau$. The map $d\mathcal{L}_\tau$ is invariant under the transformation when $d\mathcal{L}'_\tau = d\mathcal{L}_\tau$. This leads to the equation

$$d\rho'(\tau) = d\mathcal{L}_\tau[\rho'(\tau)]. \quad (8)$$

We adopt the transformation rule given by $\rho'(\tau) = \hat{U}_\tau(g)\rho(\tau)\hat{U}^\dagger_\tau(g)$, where $\hat{U}_\tau(g)$ with $g \in G$ is the unitary representation of a group G. Substituting $\rho'(\tau) = \hat{U}_\tau(g)\rho(\tau)\hat{U}^\dagger_\tau(g)$ into $d\rho'(\tau) = d\mathcal{L}_\tau[\rho'(\tau)]$ and solving the differential equation, we get

$$\hat{U}_t(g)\Phi_{t,t_0}[\rho(t_0)]\hat{U}^\dagger_t(g) = \Phi_{t,t_0}\big[\hat{U}_{t_0}(g)\rho(t_0)\hat{U}^\dagger_{t_0}(g)\big]. \quad (9)$$

This defines the dynamical map Φ_{t,t_0} as being invariant under the group G, which was introduced in Refs. [23,26,27]. In the next section we will see that Eq. (9) holds for unitary evolution in quantum theory with the Poincaré symmetry. Our aim is to extend this to a general dynamical map and to construct the map which is invariant under the Poincaré group.

III. DYNAMICAL MAP WITH POINCARÉ INVARIANCE

In this section we consider a quantum theory with the Poincaré symmetry and introduce the dynamical map with the Poincaré invariance. The generators of the unitary representation of the Poincaré group [6] are given by

$$\hat{P}^\mu = \int d^3x\, \hat{T}^{0\mu}, \quad \hat{J}^{\mu\nu} = \int d^3x\, \hat{M}^{\mu\nu 0}, \quad (10)$$

where $\hat{T}^{\mu\nu}$ is the energy-momentum tensor of a system and $\hat{M}^{\mu\nu\rho}$ is defined as

$$\hat{M}^{\mu\nu\rho} = x^\mu \hat{T}^{\nu\rho} - x^\nu \hat{T}^{\mu\rho}. \quad (11)$$

In the Schrödinger picture, the components of the generators are

$$\hat{H} = \hat{P}^0 = \int d^3x\, \hat{T}^{00}(\boldsymbol{x}, 0), \quad (12)$$

$$\hat{P}^i = \int d^3x\, \hat{T}^{0i}(\boldsymbol{x}, 0), \quad (13)$$

$$\hat{J}^i = \tfrac{1}{2}\epsilon^{jki}\hat{J}_{jk} = \int d^3x\, \epsilon^{jki}x_j\hat{T}^0_k(\boldsymbol{x}, 0), \quad (14)$$

$$\hat{K}^i(t) = \hat{J}^{i0} = \int d^3x[x^i\hat{T}^{00}(\boldsymbol{x}, 0) - t\hat{T}^{0i}(\boldsymbol{x}, 0)], \quad (15)$$

where we note that the boost generator $\hat{K}^i(t)$ explicitly depends on a time t. The operators satisfy the Poincaré algebra,

$$[\hat{P}_i, \hat{P}_j] = 0, \quad (16)$$

$$[\hat{P}_i, \hat{H}] = 0, \quad (17)$$

$$[\hat{J}_i, \hat{H}] = 0, \quad (18)$$

$$[\hat{J}_i, \hat{J}_j] = i\epsilon_{ijk}\hat{J}^k, \tag{19}$$

$$[\hat{J}_i, \hat{P}_j] = i\epsilon_{ijk}\hat{P}^k, \tag{20}$$

$$[\hat{J}_i, \hat{K}_j] = i\epsilon_{ijk}\hat{K}^k, \tag{21}$$

$$[\hat{K}_i, \hat{P}_j] = i\delta_{ij}\hat{H}, \tag{22}$$

$$[\hat{K}_i, \hat{H}] = i\hat{P}_i, \tag{23}$$

$$[\hat{K}_i, \hat{K}_j] = -i\epsilon_{ijk}\hat{J}^k. \tag{24}$$

For later analysis, we clarify the explicit time dependence of the boost generator $\hat{\boldsymbol{K}}(t) = [\hat{K}^1(t), \hat{K}^2(t), \hat{K}^3(t)]$. Since the boost operator is conserved during the evolution generated by the system Hamiltonian $\hat{H}$, we have the conservation law $d\hat{\boldsymbol{K}}_{\mathrm{H}}(t)/dt = 0$, with $\hat{\boldsymbol{K}}_{\mathrm{H}}(t) = e^{i\hat{H}t}\hat{\boldsymbol{K}}(t)e^{-i\hat{H}t}$. The solution of $d\hat{\boldsymbol{K}}_{\mathrm{H}}(t)/dt = 0$ is $\hat{\boldsymbol{K}}_{\mathrm{H}}(t) = \hat{\boldsymbol{K}}_{\mathrm{H}}(0) = \hat{\boldsymbol{K}}(0)$ and hence we get

$$\hat{\boldsymbol{K}}(t) = e^{-i\hat{H}t}\hat{\boldsymbol{K}}(0)e^{i\hat{H}t}, \tag{25}$$

where

$$\hat{\boldsymbol{K}}(0) = \int d^3x\, \boldsymbol{x}\hat{T}^{00}(\boldsymbol{x}, 0). \tag{26}$$

Let us assume that the system dynamics is described by a dynamical map Φ_{t,t_0} from $\rho(t_0)$ to $\rho(t) = \Phi_{t,t_0}[\rho(t_0)]$, where $\rho(\tau)$ is the system's density operator. According to Eq. (9), the Poincaré invariance of the dynamical map is formulated as

$$\begin{aligned} &\hat{U}_t(\Lambda, a)\Phi_{t,t_0}[\rho(t_0)]\hat{U}_t^\dagger(\Lambda, a) \\ &\quad = \Phi_{t,t_0}\big[\hat{U}_{t_0}(\Lambda, a)\rho(t_0)\hat{U}_{t_0}^\dagger(\Lambda, a)\big], \end{aligned} \tag{27}$$

where the unitary operator $\hat{U}_t(\Lambda, a)$ depends on the proper ($\det\Lambda = 1$) orthochronous ($\Lambda^0{}_0 \geqslant 1$) Lorentz transformation matrix $\Lambda^\mu{}_\nu$ and on the real parameters a^μ of space-time translations. The unitary operator $\hat{U}_t(\Lambda, a)$ generated by $\hat{H}$, $\hat{\boldsymbol{P}} = [\hat{P}^1, \hat{P}^2, \hat{P}^3]$, $\hat{\boldsymbol{J}} = [\hat{J}^1, \hat{J}^2, \hat{J}^3]$, and $\hat{\boldsymbol{K}}(t)$ has the group multiplication rule

$$\hat{U}_t(\Lambda', a')\hat{U}_t(\Lambda, a) = \hat{U}_t(\Lambda'\Lambda, a' + \Lambda' a), \tag{28}$$

where we adopted the nonprojective unitary representation of the Poincaré group [6]. The time dependence of $\hat{U}_t$ comes from the boost generator $\hat{\boldsymbol{K}}(t)$.

Here it is worth discussing the present approach and emphasizing the scope of this paper. One may imagine that the reduced dynamics of a system is not Poincaré invariant even if the total dynamics of the system and its surrounding environment (1) are Poincaré invariant. In the present analysis, we just investigate the form of the dynamical map satisfying (27) and we do not care about how such a dynamical map is derived from the total dynamics of the system and the environment. In general, the system's reduced dynamics depends not only on the interaction Hamiltonian between the system and its surrounding environment but also on the initial state of the environment. Hence, it is possible to find equivalent reduced dynamics obtained from different models of environment. From this viewpoint, the approach based on a dynamical map (quantum channel) is independent of how the map is derived from possible models. Nevertheless, it is important for discussing what model of environment gives the dynamical map with the Poincaré invariance. This would help us to grasp a physical picture of the present approach. In this paper we do not investigate such a model.

Before we start analyzing the dynamical map consistent with (27), let us understand how the condition (27) holds for the unitary map,

$$\mathcal{U}_{t,t_0}[\rho(t_0)] = e^{-i\hat{H}(t-t_0)}\rho(t_0)e^{i\hat{H}(t-t_0)}. \tag{29}$$

According to the Poincaré algebra and Eq. (25), we have

$$\hat{U}_t(\Lambda, a) = e^{-i\hat{H}t}\hat{U}_0(\Lambda, a)e^{i\hat{H}t}, \tag{30}$$

where $\hat{U}_0(\Lambda, a)$ is the unitary representation of the Poincaré group with the generators $\hat{H}$, $\hat{\boldsymbol{P}}$, $\hat{\boldsymbol{J}}$, and $\hat{\boldsymbol{K}}(0)$. Using Eq. (30), we can check the invariance condition (27) of the unitary map as

$$\begin{aligned} &\mathcal{U}_{t,t_0}\big[\hat{U}_{t_0}(\Lambda, a)\rho(t_0)\hat{U}_{t_0}^\dagger(\Lambda, a)\big] \\ &\quad = e^{-i\hat{H}(t-t_0)}\hat{U}_{t_0}(\Lambda, a)\rho(t_0)\hat{U}_{t_0}^\dagger(\Lambda, a)e^{i\hat{H}(t-t_0)} \\ &\quad = e^{-i\hat{H}(t-t_0)}\hat{U}_{t_0}(\Lambda, a)e^{i\hat{H}(t-t_0)}\mathcal{U}_{t,t_0}[\rho(t_0)]e^{-i\hat{H}(t-t_0)} \\ &\quad \times \hat{U}_{t_0}^\dagger(\Lambda, a)e^{i\hat{H}(t-t_0)} \\ &\quad = \hat{U}_t(\Lambda, a)\mathcal{U}_{t,t_0}[\rho(t_0)]\hat{U}_t^\dagger(\Lambda, a). \end{aligned}$$

Let us extend the invariant property of the unitary map to a general dynamical map. In the operator-sum representation, Eq. (27) is written as

$$\begin{aligned} &\hat{U}_t(\Lambda, a)\sum_\lambda \hat{F}_\lambda^{t,t_0}\rho(t_0)\hat{F}_\lambda^{t,t_0\dagger}\hat{U}_t^\dagger(\Lambda, a) \\ &\quad = \sum_\lambda \hat{F}_\lambda^{t,t_0}\hat{U}_{t_0}(\Lambda, a)\rho(t_0)\hat{U}_{t_0}^\dagger(\Lambda, a)\hat{F}_\lambda^{t,t_0\dagger}. \end{aligned}$$

The uniqueness of the Kraus operators $\hat{F}_\lambda^{t,t_0}$ [see Eq. (5)] yields

$$\hat{U}_t^\dagger(\Lambda, a)\hat{F}_\lambda^{t,t_0}\hat{U}_{t_0}(\Lambda, a) = \sum_{\lambda'}\mathcal{U}_{\lambda\lambda'}(\Lambda, a)\hat{F}_{\lambda'}^{t,t_0}. \tag{31}$$

We can always choose $\hat{F}_\lambda^{t,t_0}$ so that $\{\hat{F}_\lambda^{t,t_0}\}_\lambda$ is the set of linearly independent operators. This linear independence and the group multiplication rule of $\hat{U}_t(\Lambda, a)$ given in (28) lead to the multiplication rule of $\mathcal{U}_{\lambda\lambda'}(\Lambda, a)$ as

$$\sum_{\lambda'}\mathcal{U}_{\lambda\lambda'}(\Lambda', a')\mathcal{U}_{\lambda'\lambda''}(\Lambda, a) = \mathcal{U}_{\lambda\lambda''}(\Lambda'\Lambda, \Lambda a + a'). \tag{32}$$

Hence, the unitary matrix $\mathcal{U}(\Lambda, a)$ with the components $\mathcal{U}_{\lambda\lambda'}(\Lambda, a)$ is a representation of the Poincaré group. Equation (30) helps us to simplify the invariance condition (31) on the Kraus operators. Defining the Kraus operators $\hat{E}_\lambda^{t,t_0}$ as

$$\hat{E}_\lambda^{t,t_0} = e^{i\hat{H}t}\hat{F}_\lambda^{t,t_0}e^{-i\hat{H}t_0}, \tag{33}$$

which have the completeness condition

$$\sum_\lambda \hat{E}_\lambda^{t,t_0\dagger}\hat{E}_\lambda^{t,t_0} = \hat{\mathbb{I}}, \tag{34}$$

we can rewrite Eq. (31) as

$$\hat{U}_0^\dagger(\Lambda, a)\hat{\boldsymbol{E}}\hat{U}_0(\Lambda, a) = \mathcal{U}(\Lambda, a)\hat{\boldsymbol{E}}. \tag{35}$$

Here we introduced the vector $\hat{\boldsymbol{E}}$ with the λ component $\hat{E}^{t,t_0}_\lambda$. Let the dynamical map $\mathcal{E}_{t,t_0}$ be given by

$$\mathcal{E}_{t,t_0}[\rho] = \sum_\lambda \hat{E}^{t,t_0}_\lambda \rho \hat{E}^{t,t_0\dagger}_\lambda. \tag{36}$$

The condition (35) implies that the map $\mathcal{E}_{t,t_0}$ is invariant under the Poincaré group in the sense that

$$\hat{U}_0(\Lambda, a)\mathcal{E}_{t,t_0}[\rho]\hat{U}^\dagger_0(\Lambda, a) = \mathcal{E}_{t,t_0}[\hat{U}_0(\Lambda, a)\rho\hat{U}^\dagger_0(\Lambda, a)]. \tag{37}$$

Then the dynamical map Φ_{t,t_0} is written with the unitary map $\mathcal{U}_{t,t_0}$ and the dynamical map $\mathcal{E}_{t,t_0}$ as

$$\begin{aligned}\Phi_{t,t_0}[\rho] &= \sum_\lambda \hat{F}^{t,t_0}_\lambda \rho \hat{F}^{t,t_0\dagger}_\lambda \\ &= e^{-i\hat{H}t}\sum_\lambda \hat{E}^{t,t_0}_\lambda e^{i\hat{H}t_0}\rho e^{-i\hat{H}t_0}\hat{E}^{t,t_0\dagger}_\lambda e^{i\hat{H}t} \\ &= e^{-i\hat{H}t}\mathcal{E}_{t,t_0}[e^{i\hat{H}t_0}\rho e^{-i\hat{H}t_0}]e^{i\hat{H}t} \\ &= e^{-i\hat{H}(t-t_0)}\mathcal{E}_{t,t_0}[\rho]e^{i\hat{H}(t-t_0)} \\ &= \mathcal{U}_{t,t_0}\circ\mathcal{E}_{t,t_0}[\rho],\end{aligned} \tag{38}$$

where in the fourth equality we used the condition (37) and the fact that $e^{i\hat{H}t_0}$ is the unitary transformation representing the time translation. Our task is to determine $\hat{\boldsymbol{E}}$ satisfying Eq. (35) [or $\mathcal{E}_{t,t_0}$ satisfying Eq. (37)]. The irreducible unitary representation of the Poincaré group is useful for our analysis because Eq. (35) is decomposed into equations for each irreducible representation subspace.

Let us present how to classify the unitary representations of the Poincaré group [6]. An arbitrary four-momentum q^μ is represented by a standard momentum ℓ^μ and the Lorentz transformation matrix $(S_q)^\mu{}_\nu$ with

$$q^\mu = (S_q)^\mu{}_\nu \ell^\nu. \tag{39}$$

The unitary matrix $\mathcal{U}(\Lambda, a)$ is written as

$$\mathcal{U}(\Lambda, a) = \mathcal{U}(I, a)\mathcal{U}(\Lambda, 0) = \mathcal{T}(a)\mathcal{V}(\Lambda), \tag{40}$$

where I is the identity matrix, $\mathcal{U}(I, a) = \mathcal{T}(a) = e^{-iP_\mu a^\mu}$, and $\mathcal{U}(\Lambda, 0) = \mathcal{V}(\Lambda)$. We define the vector $\boldsymbol{v}_{q,\xi}$ as

$$\boldsymbol{v}_{q,\xi} = N_q\mathcal{V}(S_q)\boldsymbol{v}_{\ell,\xi}, \tag{41}$$

where $P_\mu \boldsymbol{v}_{\ell,\xi} = \ell_\mu \boldsymbol{v}_{\ell,\xi}$, N_q is the normalization, and the label ξ describes the degrees of freedom other than those determined by ℓ^μ. The vector $\boldsymbol{v}_{q,\xi}$ follows the transformation rules

$$\begin{aligned}\mathcal{T}(a)\boldsymbol{v}_{q,\xi} &= N_q e^{-iP^\mu a_\mu}\mathcal{V}(S_q)\boldsymbol{v}_{\ell,\xi} \\ &= N_q\mathcal{V}(S_q)e^{-i(S_q)^\mu{}_\nu P^\nu a_\mu}\boldsymbol{v}_{\ell,\xi} \\ &= N_q\mathcal{V}(S_q)e^{-i(S_q)^\mu{}_\nu \ell^\nu a_\mu}\boldsymbol{v}_{\ell,\xi} \\ &= N_q\mathcal{V}(S_q)e^{-iq^\mu a_\mu}\boldsymbol{v}_{\ell,\xi} \\ &= e^{-iq^\mu a_\mu}\boldsymbol{v}_{q,\xi}\end{aligned} \tag{42}$$

and

$$\begin{aligned}\mathcal{V}(\Lambda)\boldsymbol{v}_{q,\xi} &= N_q\mathcal{V}(\Lambda)\mathcal{V}(S_q)\boldsymbol{v}_{\ell,\xi} \\ &= N_q\mathcal{V}(\Lambda S_q)\boldsymbol{v}_{\ell,\xi} \\ &= N_q\mathcal{V}(S_{\Lambda q})\mathcal{V}\big(S^{-1}_{\Lambda q}\Lambda S_q\big)\boldsymbol{v}_{\ell,\xi} \\ &= N_q\mathcal{V}(S_{\Lambda q})\sum_{\xi'}\mathcal{D}_{\xi'\xi}(Q(\Lambda, q))\boldsymbol{v}_{\ell,\xi'} \\ &= \frac{N_q}{N_{\Lambda q}}\sum_{\xi'}\mathcal{D}_{\xi'\xi}(Q(\Lambda, q))\boldsymbol{v}_{\Lambda q,\xi'},\end{aligned} \tag{43}$$

TABLE I. Classification of the standard momentum ℓ^μ and the little group associated with ℓ^μ, composed of $Q^\mu{}_\nu$ with $Q^\mu{}_\nu\ell^\nu = \ell^\mu$.

Standard momentum ℓ^μ	Little group
$\ell^\mu = [M, 0, 0, 0],\ M > 0$	SO(3)
$\ell^\mu = [-M, 0, 0, 0],\ M > 0$	SO(3)
$\ell^\mu = [\kappa, 0, 0, \kappa],\ \kappa > 0$	ISO(2)
$\ell^\mu = [-\kappa, 0, 0, \kappa],\ \kappa > 0$	ISO(2)
$\ell^\mu = [0, 0, 0, N],\ N^2 > 0$	SO(2,1)
$\ell^\mu = [0, 0, 0, 0]$	SO(3,1)

where $Q(\Lambda, q) = S^{-1}_{\Lambda q}\Lambda S_q$. The matrix $Q(\Lambda, q)$ satisfies $Q^\mu{}_\nu\ell^\nu = \ell^\mu$ and the set of such matrices forms a group called the little group. In Eq. (43), $\mathcal{D}_{\xi\xi'}(Q)$ forms a unitary matrix $\mathcal{D}(Q)$ and gives a unitary representation of the little group. The irreducible unitary representations of the Poincaré group are classified by the standard momentum ℓ^μ and the irreducible unitary representations of the little group. Table I lists the standard momenta ℓ^μ and the little groups. For simplicity, ξ is regarded as the label of basis vectors of the irreducible representation subspaces of the little group.

We investigate Eq. (35) restricted on each irreducible representation. For convenience, we separately focus on the Lorentz transformation and the space-time translation in Eq. (35). The unitary operator $\hat{U}_0(\Lambda, a)$ is written as

$$\hat{U}_0(\Lambda, a) = \hat{U}_0(I, a)\hat{U}_0(\Lambda, 0) = \hat{T}(a)\hat{V}(\Lambda), \tag{44}$$

where $\hat{U}_0(I, a) = \hat{T}(a) = e^{-i\hat{P}_\mu a^\mu}$ with the four-momentum operator $\hat{P}^\mu$ and $\hat{U}_0(\Lambda, 0) = \hat{V}(\Lambda)$ with the generators $\hat{\boldsymbol{J}}$ and $\hat{\boldsymbol{K}}(0)$. From Eq. (35) for $\Lambda = I$ we have

$$\hat{T}^\dagger(a)\hat{\boldsymbol{E}}\hat{T}(a) = \mathcal{T}(a)\hat{\boldsymbol{E}}. \tag{45}$$

Equation (35) for $a^\mu = 0$ gives

$$\hat{V}^\dagger(\Lambda)\hat{\boldsymbol{E}}\hat{V}(\Lambda) = \mathcal{V}(\Lambda)\hat{\boldsymbol{E}}. \tag{46}$$

Introducing $\hat{E}_{q,\xi} = \boldsymbol{v}^\dagger_{q,\xi}\hat{\boldsymbol{E}}$, we obtain the following equations from Eqs. (45) and (46):

$$\hat{T}^\dagger(a)\hat{E}_{q,\xi}\hat{T}(a) = e^{-iq_\mu a^\mu}\hat{E}_{q,\xi} \tag{47}$$

and

$$\hat{V}^\dagger(\Lambda)\hat{E}_{q,\xi}\hat{V}(\Lambda) = \frac{N^*_q}{N^*_{\Lambda^{-1}q}}\sum_{\xi'}\mathcal{D}^*_{\xi'\xi}(Q(\Lambda^{-1}, q))\hat{E}_{\Lambda^{-1}q,\xi'}. \tag{48}$$

Here we used Eqs. (42) and (43) and $Q(\Lambda, q) = S^{-1}_{\Lambda q}\Lambda S_q$. The label ξ can take discrete or continuous values. For the continuous case, the summation $\sum_\xi$ is replaced with the integration $\int d\mu(\xi)$ with a measure $\mu(\xi)$. Focusing on Eq. (48) for $\Lambda = S_q$, we get

$$\hat{V}^\dagger(S_q)\hat{E}_{q,\xi}\hat{V}(S_q) = N^*_q\hat{E}_{\ell,\xi}, \tag{49}$$

where note that $N_\ell = 1$ and $Q(S_q^{-1}, q) = S^{-1}_{S_q^{-1}q} S_q^{-1} S_q = S_\ell^{-1} = I$ hold by the definition of $\boldsymbol{v}_{q,\xi}$. Equation (49) tells us that the Kraus operators $\hat{E}_{q,\xi}$ are determined from the Kraus operators $\hat{E}_{\ell,\xi}$ with the standard momentum ℓ^μ. All we have to do is to give the form of the Kraus operators $\hat{E}_{\ell,\xi}$. To this end, we present the equations given by Eq. (47) for $q^\mu = \ell^\mu$ and by Eq. (48) for $q^\mu = \ell^\mu$ and $\Lambda = W$ with $W^\mu{}_\nu \ell^\nu = \ell^\mu$, respectively,

$$\hat{T}^\dagger(a)\hat{E}_{\ell,\xi}\hat{T}(a) = e^{-i\ell_\mu a^\mu}\hat{E}_{\ell,\xi}, \tag{50}$$

$$\hat{V}^\dagger(W)\hat{E}_{\ell,\xi}\hat{V}(W) = \sum_{\xi'} \mathcal{D}^*_{\xi'\xi}(W^{-1})\hat{E}_{\ell,\xi'}, \tag{51}$$

where $Q(\Lambda^{-1}, q) = Q(W^{-1}, \ell) = S^{-1}_{W^{-1}\ell} W^{-1} S_\ell = W^{-1}$. In the next section we construct a model of the dynamical map with the Poincaré invariance to describe the reduced dynamics of a spinless massive particle.

IV. MODEL OF THE DYNAMICAL MAP FOR A SPINLESS MASSIVE PARTICLE

In this section, based on Eqs. (50) and (51), we give a model of the dynamical map with the Poincaré invariance. To simplify the analysis, we consider a spinless particle with a mass m and its Hilbert space $\mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_0$ is the one-dimensional Hilbert space with the vacuum state $|0\rangle$ and $\mathcal{H}_1$ is the irreducible subspace with one-particle states. A state vector $|\Psi\rangle$ in $\mathcal{H}_1$ ($|\Psi\rangle \in \mathcal{H}_1$) is

$$|\Psi\rangle = \int d^3p\, \Psi(\boldsymbol{p})\hat{a}^\dagger(\boldsymbol{p})|0\rangle, \tag{52}$$

where the vacuum state $|0\rangle$ satisfies $\hat{a}(\boldsymbol{p})|0\rangle = 0$, $\Psi(\boldsymbol{p})$ with the momentum $\boldsymbol{p}$ is the wave function, and $\hat{a}(\boldsymbol{p})$ and $\hat{a}^\dagger(\boldsymbol{p})$ are the annihilation and creation operators, respectively, with

$$[\hat{a}(\boldsymbol{p}), \hat{a}(\boldsymbol{p}')] = 0 = [\hat{a}^\dagger(\boldsymbol{p}), \hat{a}^\dagger(\boldsymbol{p}')],$$
$$[\hat{a}(\boldsymbol{p}), \hat{a}^\dagger(\boldsymbol{p}')] = \delta^3(\boldsymbol{p} - \boldsymbol{p}'). \tag{53}$$

Here $[\hat{A}, \hat{B}]$ is the commutator, $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. In Refs. [6,28], the transformation rules of $\hat{a}^\dagger(\boldsymbol{p})$ are given by

$$\hat{T}(a)\hat{a}^\dagger(\boldsymbol{p})\hat{T}^\dagger(a) = e^{-ip^\mu a_\mu}\hat{a}^\dagger(\boldsymbol{p}), \tag{54}$$

$$\hat{V}(\Lambda)\hat{a}^\dagger(\boldsymbol{p})\hat{V}^\dagger(\Lambda) = \sqrt{\frac{E_{\boldsymbol{p}_\Lambda}}{E_{\boldsymbol{p}}}}\hat{a}^\dagger(\boldsymbol{p}_\Lambda), \tag{55}$$

where $E_{\boldsymbol{p}} = p^0 = \sqrt{\boldsymbol{p}^2 + m^2}$, $E_{\boldsymbol{p}_\Lambda} = (\Lambda p)^0$, and $\boldsymbol{p}_\Lambda$ is the vector with the components $(\boldsymbol{p}_\Lambda)^i = (\Lambda p)^i$.

We consider the Kraus operators $\hat{E}_{\ell,\xi}$ acting on the Hilbert space $\mathcal{H}_0 \oplus \mathcal{H}_1$, that is, $\hat{E}_{\ell,\xi} : \mathcal{H}_0 \oplus \mathcal{H}_1 \to \mathcal{H}_0 \oplus \mathcal{H}_1$, which have the form

$$\hat{E}_{\ell,\xi} = A_{\ell,\xi}\hat{\mathbb{I}} + \int d^3p\, B_{\ell,\xi}(\boldsymbol{p})\hat{a}(\boldsymbol{p}) + \int d^3p' d^3p\, C_{\ell,\xi}(\boldsymbol{p}', \boldsymbol{p})\hat{a}^\dagger(\boldsymbol{p}')\hat{a}(\boldsymbol{p}). \tag{56}$$

The dynamical map given by these operators describes the reduced dynamics of the particle, which can decay into the vacuum state. Substituting Eq. (56) into Eqs. (50) and (51), we obtain

$$A_{\ell,\xi} = e^{-i\ell^\mu a_\mu}A_{\ell,\xi}, \tag{57}$$

$$B_{\ell,\xi}(\boldsymbol{p})e^{-ip^\mu a_\mu} = B_{\ell,\xi}(\boldsymbol{p})e^{-i\ell^\mu a_\mu}, \tag{58}$$

$$C_{\ell,\xi}(\boldsymbol{p}', \boldsymbol{p})e^{i(p'^\mu - p^\mu)a_\mu} = C_{\ell,\xi}(\boldsymbol{p}', \boldsymbol{p})e^{-i\ell^\mu a_\mu} \tag{59}$$

and

$$A_{\ell,\xi} = \sum_{\xi'} \mathcal{D}^*_{\xi'\xi}(W^{-1})A_{\ell,\xi'}, \tag{60}$$

$$\sqrt{\frac{E_{\boldsymbol{p}_W}}{E_{\boldsymbol{p}}}}\sum_\sigma B_{\ell,\xi}(\boldsymbol{p}_W) = \sum_{\xi'} \mathcal{D}^*_{\xi'\xi}(W^{-1})B_{\ell,\xi'}(\boldsymbol{p}), \tag{61}$$

$$\sqrt{\frac{E_{\boldsymbol{p}'_W}E_{\boldsymbol{p}_W}}{E_{\boldsymbol{p}'}E_{\boldsymbol{p}}}}C_{\ell,\xi}(\boldsymbol{p}'_W, \boldsymbol{p}_W) = \sum_{\xi'} \mathcal{D}^*_{\xi'\xi}(W^{-1})C_{\ell,\xi'}(\boldsymbol{p}', \boldsymbol{p}). \tag{62}$$

The derivation of these equations is in Appendix B.

We can analyze the form of $A_{\ell,\xi}$, $B_{\ell,\xi}(\boldsymbol{p})$, and $C_{\ell,\xi}(\boldsymbol{p}', \boldsymbol{p})$ for a spinless massive particle. The long computations presented in Appendix C give the dynamical map with the Poincaré invariance $\Phi_{t,t_0}[\rho(t_0)] = \mathcal{U}_{t,t_0} \circ \mathcal{E}_{t,t_0}[\rho(t_0)]$, with the unitary map $\mathcal{U}_{t,t_0}$ given in (29) and $\mathcal{E}_{t,t_0}$ as

$$\begin{aligned}\mathcal{E}_{t,t_0}[\rho(t_0)] = {} & \beta_{t,t_0}\int d^3p\, \hat{a}(\boldsymbol{p})\rho(t_0)\hat{a}^\dagger(\boldsymbol{p}) \\ & + \big(\hat{\mathbb{I}} + \gamma_{t,t_0}\hat{N}\big)\rho(t_0)\big(\hat{\mathbb{I}} + \gamma_{t,t_0}\hat{N}\big)^\dagger \\ & + \int d^3p \int d^3q\, \delta_{t,t_0}(p, q)\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p}) \\ & \times \rho(t_0)\hat{a}^\dagger(\boldsymbol{q})\hat{a}(\boldsymbol{q}),\end{aligned} \tag{63}$$

where $\hat{N}$ is the number operator defined by

$$\hat{N} = \int d^3p\, \hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p}). \tag{64}$$

The function $\delta_{t,t_0}(p, q)$ is non-negative and Lorentz invariant in the sense that

$$\int d^3p d^3q\, f^*(p)\delta_{t,t_0}(p, q)f(q) \geqslant 0,$$

and

$$\delta_{t,t_0}(\Lambda p, \Lambda q) = \delta_{t,t_0}(p, q). \tag{65}$$

Hence, $\delta_{t,t_0}(p, p)$ does not depend on the three-momentum of the particle. In the following, $\delta_{t,t_0}(p, p)$ is simply denoted by δ_{t,t_0}. The parameters β_{t,t_0}, γ_{t,t_0}, and δ_{t,t_0} satisfy

$$\beta_{t,t_0} \geqslant 0, \quad \beta_{t,t_0} + \gamma^*_{t,t_0} + \gamma_{t,t_0} + |\gamma_{t,t_0}|^2 + \delta_{t,t_0} = 0, \tag{66}$$

where the inequality is required from the fact that the density operator $\mathcal{E}_{t,t_0}[\rho(t_0)]$ should be positive and the latter condition in (66) is yielded from the completeness of the Kraus operators (3).

From the transformation rules of the creation and annihilation operators (54) and (55), we can check that the map $\mathcal{E}_{t,t_0}$ satisfies the invariance condition (37). Since the unitary map $\mathcal{U}_{t,t_0}$ is invariant under the Poincaré group, which is discussed around Eq. (29), we can confirm that Φ_{t,t_0} is also invariant.

We discuss the conservation laws of the Poincaré generators $\hat{H}$, $\hat{\boldsymbol{P}}$, $\hat{\boldsymbol{J}}$, and $\hat{\boldsymbol{K}}(t)$ in quantum mechanics, which means that the all-order moments of each operator are conserved during time evolution. For this purpose, it is useful to consider the characteristic function

$$\chi_t(\theta, a) = \mathrm{Tr}[e^{-ia_\mu \hat{P}^\mu + (i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t)}\rho(t)], \tag{67}$$

where $\hat{P}^\mu = [\hat{H}, \hat{\boldsymbol{P}}]$ is the four-momentum and the antisymmetric tensor $\hat{J}^{\mu\nu}(t)$ is given by $\hat{J}^{jk}(t) = \epsilon^{ijk}\hat{J}_i$ and $J^{i0}(t) = \hat{K}^i(t)$, which depends on time in the Schrödinger picture. For example, the nth-order moment of the energy is given as $\mathrm{Tr}[\hat{H}^n \rho(t)] = (-i)^n \partial^n_{a^0}\chi_t(\theta, a)|_{\theta=0=a}$. The time evolution of the density operator is $\rho(t) = \Phi_{t,t_0}[\rho(t_0)] = \mathcal{U}_{t,t_0} \circ \mathcal{E}_{t,t_0}[\rho(t_0)]$.

Let us investigate the conservation of the four-momentum $\hat{P}^\mu$ for the above model of the massive particle. The characteristic function $\chi_t(0, a)$ is computed as

$$\begin{aligned}\chi_t(0, a) &= \mathrm{Tr}[e^{-ia_\mu \hat{P}^\mu}\rho(t)]\\ &= \chi_s(0, a) + \beta_{t,t_0}\mathrm{Tr}[\hat{N}(\hat{\mathbb{I}} - e^{-ia_\mu\hat{P}^\mu})\rho(t_0)].\end{aligned} \tag{68}$$

We thus find that the energy of the particle is not conserved, $\chi_t(0, a) \neq \chi_s(0, a)$, even when the map is invariant under the Poincaré group. Such a deviation between symmetry and conservation law was discussed in, for example, Refs. [26,27]. If the parameter β_{t,t_0} vanishes, then $\chi_t(0, a) = \chi_s(0, a)$ and hence the energy is conserved. For such a case, we find the dynamical map Φ_{t,t_0} with the Poincaré invariance, which guarantees the four-momentum conservation.

Under the four-momentum conservation (the condition $\beta_{t,t_0} = 0$), we further examine the conservation of $\hat{J}^{\mu\nu}(t)$. The characteristic function $\chi_t(\theta, 0)$ is

$$\begin{aligned}&\chi_t(\theta, 0)\\ &= \mathrm{Tr}[e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t)}\rho(t)]\\ &= \chi_s(\theta, 0) - \delta_{t,t_0}\mathrm{Tr}[\hat{N}e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(s)}\rho(t_0)]\\ &\quad + \mathrm{Tr}\left(\int d^3p\, \delta_{t,t_0}(p, \Lambda p)\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(s)}\rho(t_0)\right),\end{aligned} \tag{69}$$

where $\Lambda = \Lambda(\theta)$ is the Lorentz transformation matrix determined by $\theta_{\mu\nu}$. To conserve $\hat{J}^{\mu\nu}(t)$, $\delta_{t,t_0} = \delta_{t,t_0}(p, \Lambda p)$ should hold for all Λ, and hence $\delta_{t,t_0}(p, q) = \delta_{t,t_0}$. The dynamical map $\mathcal{E}_{t,t_0}$ for a spinless massive particle becomes

$$\mathcal{E}_{t,t_0}[\rho(t_0)] = (\hat{\mathbb{I}} + \gamma_{t,t_0}\hat{N})\rho(t_0)(\hat{\mathbb{I}} + \gamma_{t,t_0}\hat{N})^\dagger + \delta_{t,t_0}\hat{N}\rho(t_0)\hat{N}. \tag{70}$$

When the density operator $\rho(t_0)$ is given by one-particle states, we have $\hat{N}\rho(t_0) = \rho(t_0) = \rho(t_0)\hat{N}$, and then the dynamical map Φ_{t,t_0} with the Poincaré invariance is

$$\begin{aligned}\Phi_{t,t_0}[\rho(t_0)] &= \mathcal{U}_{t,t_0} \circ \mathcal{E}_{t,t_0}[\rho(t_0)]\\ &= (|1 + \gamma_{t,t_0}|^2 + \delta_{t,t_0})\mathcal{U}_{t,t_0}[\rho(t_0)]\\ &= \mathcal{U}_{t,t_0}[\rho(t_0)],\end{aligned} \tag{71}$$

where we used the condition $\delta_{t,t_0} = -\gamma_{t,t_0} - \gamma^*_{t,t_0} - |\gamma_{t,t_0}|^2$ given by setting $\beta_{t,t_0} = 0$ for the second equation in (66). Hence, the dynamical map with the Poincaré invariance for a spinless massive particle is reduced to the unitary map when the conservation of the Poincaré generators holds. The result corresponds to an extension of the analysis in [29].

V. COVARIANT FORMULATION

In the preceding section we obtained the form of the dynamical map with the Poincaré invariance in Minkowski time in a special coordinate system. In the following, we rewrite the previous formulation in a covariant way. We consider the foliation of Cauchy surfaces $\{\Sigma_\tau\}_\tau$ and the unit timelike vector n^μ normal to Σ_τ, where the parameter τ is generated by n^μ. For example, the parameter τ is defined by $dx^\mu = n^\mu d\tau$ with a constant n^μ in the Minkowski space. Letting x_0^μ be a constant of integration, the solution of the equation yields $\tau = -n_\mu(x^\mu - x_0^\mu)$, which is invariant in any inertial coordinate system. In the present formulation, the density operator $\rho(\tau)$ of a quantum system is defined on the Cauchy surface Σ_τ. The change of the density operator from a Cauchy surface Σ_{τ_0} to another Σ_τ is regarded as the time evolution of the density operator from τ_0 to τ ($>\tau_0$).

Since the parameter τ is coordinate invariant, the Cauchy surface Σ_τ given as a $\tau =$ constant hypersurface does not depend on any choice of an inertial coordinate system. On the other hand, the density operator depends on it. This is because the density operator of a quantum system is specified by the statistical outcome of observables such as a momentum and a spin. When the density operator in one inertial coordinate system is $\rho(\tau)$ on Σ_τ, the density operator of the same quantum system in another inertial coordinate system is given as $\hat{U}_\tau\rho(\tau)\hat{U}_\tau^\dagger$ on the same Cauchy surface Σ_τ. The two coordinate systems are connected by a Poincaré transformation, which is represented as the unitary operator $\hat{U}_\tau$ acting on the density operator. In the Schrödinger picture, the Poincaré generators of $\hat{U}_\tau$ are (see also Ref. [30])

$$\hat{\Theta} = -n_\mu\hat{P}^\mu, \quad \hat{\Pi}^\mu = \hat{P}^\mu - n^\mu\hat{\Theta}, \quad \hat{L}^\mu = \tfrac{1}{2}\epsilon^{\mu\alpha\beta\gamma}\hat{J}_{\alpha\beta}n_\gamma,$$
$$\hat{N}^\mu = \hat{J}^{\mu\nu}n_\nu, \tag{72}$$

which satisfy the commutation relations

$$[\hat{\Pi}_\mu, \hat{\Pi}_\nu] = 0, \tag{73}$$

$$[\hat{\Pi}_\mu, \hat{\Theta}] = 0, \tag{74}$$

$$[\hat{L}_\mu, \hat{\Theta}] = 0, \tag{75}$$

$$[\hat{L}_\mu, \hat{L}_\nu] = i\epsilon_{\mu\nu\alpha\beta}n^\alpha\hat{L}^\beta, \tag{76}$$

$$[\hat{L}_\mu, \hat{\Pi}_\nu] = i\epsilon_{\mu\nu\alpha\beta}n^\alpha\hat{\Pi}^\beta, \tag{77}$$

$$[\hat{L}_\mu, \hat{N}_\nu] = i\epsilon_{\mu\nu\alpha\beta}n^\alpha\hat{N}^\beta, \tag{78}$$

$$[\hat{N}_\mu, \hat{\Pi}_\nu] = i(\eta_{\mu\nu} + n_\mu n_\nu)\hat{\Theta}, \tag{79}$$

$$[\hat{N}_\mu, \hat{\Theta}] = i\hat{\Pi}_\mu, \tag{80}$$

$$[\hat{N}_\mu, \hat{N}_\nu] = -i\epsilon_{\mu\nu\alpha\beta}n^\alpha\hat{L}^\beta. \tag{81}$$

In the foliation with $n^\mu = [1, 0, 0, 0]$ in a coordinate, the generators $\hat{\Theta}$, $\hat{\Pi}^\mu$, $\hat{L}^\mu$, and $\hat{N}^\mu$ are with $\hat{\Theta} = \hat{H}$, $\hat{\Pi}^\mu = [0, \hat{\boldsymbol{P}}]$, $\hat{L}^\mu = [0, \hat{\boldsymbol{J}}]$, and $\hat{N}^\mu = [0, \hat{\boldsymbol{K}}]$, respectively. These are nothing but the Poincaré generators for the Minkowski time considered in the previous sections. Since each generator defined with the Cauchy surface Σ_τ is conserved in the flow by the system Hamiltonian $\hat{\Theta}$, the equation $d\hat{N}^\mu_{\rm H}/d\tau = 0$, with $\hat{N}^\mu_{\rm H}(\tau) = e^{i\hat{\Theta}\tau}\hat{N}^\mu(\tau)e^{-i\hat{\Theta}\tau}$, holds. This suggests that the operator $\hat{N}^\mu$ depends on τ as

$$\hat{N}^\mu(\tau) = e^{-i\hat{\Theta}\tau}\hat{N}^\mu(0)e^{i\hat{\Theta}\tau} \tag{82}$$

in the Schrödinger picture. The time dependence of $\hat{\boldsymbol{K}}(t)$ was similarly discussed around Eq. (25) in Sec. III. Then the τ dependence of the unitary operator $\hat{U}_\tau(\Lambda, a)$ generated by $\hat{\Theta}$, $\hat{\Pi}^\mu$, $\hat{L}^\mu$, and $\hat{N}^\mu(\tau)$ is given by

$$\hat{U}_\tau(\Lambda, a) = e^{-i\hat{\Theta}\tau}\hat{U}_0(\Lambda, a)e^{i\hat{\Theta}\tau}, \tag{83}$$

where the commutation relations (74) and (75) were used and the unitary operator $\hat{U}_0(\Lambda, a)$ is generated by $\hat{\Theta}$, $\hat{\Pi}^\mu$, $\hat{L}^\mu$, and $\hat{N}^\mu(0)$.

To manifest covariance, we use the Lorentz invariant measure $d\mu(p)$, with $d\mu(p) = d^4p\delta(p^2 + m^2)\theta(p^0)$, for a massive particle. Also, we introduce the annihilation and creation operators $\hat{A}(p)$ and $\hat{A}^\dagger(p)$ satisfying

$$[\hat{A}(f), \hat{A}(g)] = 0 = [\hat{A}^\dagger(f), \hat{A}^\dagger(g)], \quad [\hat{A}(f), \hat{A}^\dagger(g)] = \int d\mu(p)f(p)g^*(p), \tag{84}$$

where $f(p)$ and $g(p)$ are complex functions and

$$\hat{A}(f) = \int d\mu(p)f(p)\hat{A}(p). \tag{85}$$

The Poincaré transformation rule of $\hat{A}^\dagger(p)$ is

$$\hat{U}_0(\Lambda, a)\hat{A}^\dagger(p)\hat{U}_0^\dagger(\Lambda, a) = e^{-i(\Lambda p)^\mu a_\mu}\hat{A}^\dagger(\Lambda p). \tag{86}$$

In the previous formulation, the annihilation operator $\hat{A}(p)$ corresponds to $\sqrt{E_{\boldsymbol{p}}}\hat{a}(\boldsymbol{p})$. We can use the same procedure as in the previous sections and Appendix C to write the dynamical map with the Poincaré invariance in a covariant way. The dynamical map $\Phi_{\tau,\tau_0} = \mathcal{U}_{\tau,\tau_0} \circ \mathcal{E}_{\tau,\tau_0}$ for a massive spinless particle is given by

$$\mathcal{U}_{\tau,\tau_0}[\rho] = e^{-i\hat{\Theta}(\tau-\tau_0)}\rho e^{i\hat{\Theta}(\tau-\tau_0)} \tag{87}$$

and

$$\begin{aligned}\mathcal{E}_{\tau,\tau_0}[\rho] = {}&\beta_{\tau,\tau_0}\int d\mu(p)\hat{A}(p)\rho\hat{A}^\dagger(p)\\ &+ (\hat{\mathbb{I}} + \gamma_{\tau,\tau_0}\hat{N})\rho(\hat{\mathbb{I}} + \gamma_{\tau,\tau_0}\hat{N})^\dagger\\ &+\int d\mu(p)d\mu(q)\delta_{\tau,\tau_0}(p, q)\hat{A}^\dagger(p)\hat{A}(p)\rho\hat{A}^\dagger(q)\hat{A}^\dagger(q),\end{aligned} \tag{88}$$

where the number operator $\hat{N}$ is

$$\hat{N} = \int d\mu(p)\hat{A}(p)\hat{A}^\dagger(p). \tag{89}$$

For the Minkowski time in a special coordinate, in which $n^\mu = [1, 0, 0, 0]$, $\tau = t$, and $\tau_0 = t_0$, the above dynamical map is reduced to that derived in the preceding section.

We briefly comment on what may happen if the obtained dynamics is described in another foliation of Cauchy surfaces $\{\tilde{\Sigma}_\lambda\}_\lambda$ with a normal vector $m^\mu = \Lambda^\mu{}_\nu n^\nu$, where $\Lambda^\mu{}_\nu$ is a Lorentz transformation matrix. Here note that m^μ is not given by a Lorentz (coordinate) transformation but is a different objective vector from n^μ. In the another foliation, the dynamics may give the action at distance, that is, induce the violation of causality. To investigate this conceptual problem clearly, a local description of the dynamics would be required. This is because we should carefully specify each description of the dynamics in different foliations by a local quantity at a spacetime point in the intersection of two Cauchy surfaces Σ_τ and $\tilde{\Sigma}_\lambda$. Elucidating what constraints to the present framework are derived from causality and locality remains a challenge for the future.

VI. CONCLUSION

We discussed how a dynamical map describing the reduced dynamics of an open quantum system is realized under the Poincaré symmetry. The unitary representation theory of the Poincaré group refines the condition for the dynamical map with the Poincaré invariance. We derived the model of the dynamical map for a spinless massive particle. In the model, the particle can decay into the vacuum state, and we found a model of the dynamical map with four-momentum conservation. Further, we showed that the map is unitary under the conservation of the Poincaré generators if the map is restricted to density operators of one-particle states of the spinless massive particle. In this way, it was exemplified that the Poincaré symmetry strongly constrains the possible dynamics of an open quantum system. We also formulated the dynamical map with the Poincaré invariance in a covariant way. In the present analysis, we did not clarify how the total dynamics of the system and the environment lead to the dynamical map obtained here. This is left for future work.

In this paper we assumed an open system with a single particle. It is possible to extend our analysis to the case with many particles. Considering interactions among many particles, we can understand more general effective theories of open quantum systems in terms of the Poincaré symmetry. For the particles interacting via gravity, we can also discuss the models with intrinsic gravitational decoherence, which have been proposed in [16–20]. These models are written in the theory of open quantum systems. In the weak-field regime of gravity, the Poincaré symmetry may provide guidance for establishing the theory of gravitating particles.

This paper has the potential to lead to the development of a relativistic theory of open quantum systems. To describe the reduced dynamics of an open quantum system, a Markovian quantum master equation is often adopted. How such a master equation is consistent with relativity has been discussed [15,31,32]. Applying the present approach, it will be possible to discuss the quantum Markov dynamics with the Poincaré invariance. In doing so, it is also worth considering the description of the dynamics in quantum field theory and examining relativistic causality. The previous works [33–36] discussed the consistency between the measurements of local observables and the relativistic causality in quantum field theory. It may be interesting to understand the dynamics of an open quantum system with the Poincaré symmetry as a causal measurement process by an environment.

ACKNOWLEDGMENTS

We thank Y. Kuramochi for useful discussions and comments related to this paper. A.M. was supported by 2022 Research Start Program No. 202203 and Japan Society for the Promotion of Science (JSPS) KAKENHI, Grant No. JP23K13103.

APPENDIX A: DERIVATION OF EQ. (2)

We here give the derivation of Eq. (2) starting from Eq. (1). For simplicity, we assume that the initial state of the environment is pure, $\rho_{\rm E} = |0\rangle_{\rm E}\langle 0|$. We then find

$$\begin{aligned}\rho(t) &= \mathrm{Tr}_{\rm E}[\hat{U}(t,t_0)\rho(t_0)\otimes|0\rangle_{\rm E}\langle 0|\hat{U}^\dagger(t,t_0)]\\ &= \sum_n {}_{\rm E}\langle n|\hat{U}(t,t_0)\rho(t_0)\otimes|0\rangle_{\rm E}\langle 0|\hat{U}^\dagger(t,t_0)|n\rangle_{\rm E}\\ &= \sum_n [{}_{\rm E}\langle n|\hat{U}(t,t_0)|0\rangle_{\rm E}]\rho(t_0)[{}_{\rm E}\langle 0|\hat{U}^\dagger(t,t_0)|n\rangle_{\rm E}]\\ &= \sum_n \hat{F}_n^{t,t_0}\rho(t_0)\hat{F}_n^{t,t_0\dagger},\end{aligned} \tag{A1}$$

where $\hat{F}_n^{t,t_0} = {}_{\rm E}\langle n|\hat{U}(t,t_0)|0\rangle_{\rm E}$ is the Kraus operator acting on the system density operator $\rho(t_0)$. Replacing n with λ, we get Eq. (2).

Even when the environment is initially in a mixed state $\rho_{\rm E}$, for example, a thermal state, we can derive the operator-sum representation (2). Equation (1) is rewritten as

$$\begin{aligned}\rho(t) &= \mathrm{Tr}_{\rm E}[\hat{U}(t,t_0)\rho(t_0)\otimes\rho_{\rm E}\hat{U}^\dagger(t,t_0)]\\ &= \mathrm{Tr}_{\rm E}[\hat{U}(t,t_0)\rho(t_0)\otimes\sqrt{\rho_{\rm E}}\sqrt{\rho_{\rm E}}\hat{U}^\dagger(t,t_0)]\\ &= \mathrm{Tr}_{\rm E}\left(\hat{U}(t,t_0)\rho(t_0)\otimes\sqrt{\rho_{\rm E}}\sum_m|m\rangle_{\rm E}\langle m|\sqrt{\rho_{\rm E}}\hat{U}^\dagger(t,t_0)\right)\\ &= \sum_n {}_{\rm E}\langle n|\hat{U}(t,t_0)\rho(t_0)\otimes\sqrt{\rho_{\rm E}}\sum_m|m\rangle_{\rm E}\\ &\quad\times\langle m|\sqrt{\rho_{\rm E}}\hat{U}^\dagger(t,t_0)|n\rangle_{\rm E}\\ &= \sum_{n,m}[{}_{\rm E}\langle n|\hat{U}(t,t_0)\sqrt{\rho_{\rm E}}|m\rangle_{\rm E}]\rho(t_0)\\ &\quad\times[{}_{\rm E}\langle m|\sqrt{\rho_{\rm E}}\hat{U}^\dagger(t,t_0)|n\rangle_{\rm E}]\\ &= \sum_{n,m}\hat{F}_{n,m}^{t,t_0}\rho(t_0)\hat{F}_{n,m}^{t,t_0\dagger},\end{aligned} \tag{A2}$$

where in the second line we used the fact that $\rho_{\rm E}$ is a non-negative operator and uniquely has its square root $\sqrt{\rho_{\rm E}}$ and in the third line we inserted the completeness relation $\sum_m |m\rangle_{\rm E}\langle m| = \hat{\mathbb{I}}_{\rm E}$. Replacing the label (n,m) of the Kraus operator $\hat{F}_{n,m}^{t,t_0} = {}_{\rm E}\langle n|\hat{U}(t,t_0)\sqrt{\rho_{\rm E}}|m\rangle_{\rm E}$ acting on $\rho(t_0)$ with λ, we get Eq. (2) again. Here we assumed the initial uncorrelated state of the system and the environment. In the theory of an open quantum system, we may consider an initial state with correlation, which may lead to a more general reduced dynamics of the system (see, for example, the review [1]). In this paper we do not discuss such dynamics.

APPENDIX B: DERIVATION OF EQS. (57)–(62)

We present the transformation rules of $A_{\ell,\xi}$, $B_{\ell,\xi}$, and $C_{\ell,\xi}$ given in Eqs. (57)–(62). Using the assumed form of the Kraus operators $\hat{E}_{\ell,\xi}$ defined by (56), we can compute the right-hand side of Eq. (50) as

$$\begin{aligned}\hat{T}^\dagger(a)\hat{E}_{\ell,\xi}\hat{T}(a) &= A_{\ell,\xi}\hat{\mathbb{I}} + \int d^3p\,B_{\ell,\xi}(\boldsymbol{p})e^{-ip^\mu a_\mu}\hat{a}(\boldsymbol{p})\\ &\quad+\int d^3p'd^3p\,C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{i(p'^\mu-p^\mu)a_\mu}\hat{a}^\dagger(\boldsymbol{p}')\hat{a}(\boldsymbol{p}).\end{aligned}$$

From Eq. (50) we have

$$A_{\ell,\xi} = e^{-i\ell^\mu a_\mu}A_{\ell,\xi}, \tag{B1}$$

$$B_{\ell,\xi}(\boldsymbol{p})e^{-ip^\mu a_\mu} = B_{\ell,\xi}(\boldsymbol{p})e^{-i\ell^\mu a_\mu}, \tag{B2}$$

$$C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{i(p'^\mu-p^\mu)a_\mu} = C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{-i\ell^\mu a_\mu}. \tag{B3}$$

The right-hand side of Eq. (51) is evaluated as

$$\begin{aligned}&\hat{V}^\dagger(W)\hat{E}_{\ell,\xi}\hat{V}(W)\\ &= A_{\ell,\xi}\hat{\mathbb{I}} + \int d^3p\,B_{\ell,\xi}(\boldsymbol{p})\sqrt{\frac{E_{\boldsymbol{p}_{W^{-1}}}}{E_{\boldsymbol{p}}}}\hat{a}(\boldsymbol{p}_{W^{-1}})\\ &\quad+\int d^3p'd^3p\,C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})\sqrt{\frac{E_{\boldsymbol{p}'_{W^{-1}}}}{E_{\boldsymbol{p}'}}}\sqrt{\frac{E_{\boldsymbol{p}_{W^{-1}}}}{E_{\boldsymbol{p}}}}\hat{a}^\dagger(\boldsymbol{p}'_{W^{-1}})\hat{a}(\boldsymbol{p}_{W^{-1}})\\ &= A_{\ell,\xi}\hat{\mathbb{I}} + \int d^3p\,B_{\ell,\xi}(\boldsymbol{p}_W)\sqrt{\frac{E_{\boldsymbol{p}_W}}{E_{\boldsymbol{p}}}}\hat{a}(\boldsymbol{p})\\ &\quad+\int d^3p'd^3p\,C_{\ell,\xi}(\boldsymbol{p}'_W,\boldsymbol{p}_W)\sqrt{\frac{E_{\boldsymbol{p}'_W}}{E_{\boldsymbol{p}'}}}\sqrt{\frac{E_{\boldsymbol{p}_W}}{E_{\boldsymbol{p}}}}\hat{a}^\dagger(\boldsymbol{p}')\hat{a}(\boldsymbol{p}),\end{aligned}$$

where note that the Lorentz invariant measure is $d^3p/E_{\boldsymbol{p}}$ and hence $f(\boldsymbol{p})d^3p = E_{\boldsymbol{p}}f(\boldsymbol{p})d^3p/E_{\boldsymbol{p}} = E_{\boldsymbol{p}_\Lambda}f(\boldsymbol{p}_\Lambda)d^3p/E_{\boldsymbol{p}}$. From Eq. (51) we have

$$A_{\ell,\xi} = \sum_{\xi'}\mathcal{D}^*_{\xi'\xi}(W^{-1})A_{\ell,\xi'}, \tag{B4}$$

$$\sqrt{\frac{E_{\boldsymbol{p}_W}}{E_{\boldsymbol{p}}}}B_{\ell,\xi}(\boldsymbol{p}_W) = \sum_{\xi'}\mathcal{D}^*_{\xi'\xi}(W^{-1})B_{\ell,\xi'}(\boldsymbol{p}), \tag{B5}$$

$$\sqrt{\frac{E_{\boldsymbol{p}'_W}E_{\boldsymbol{p}_W}}{E_{\boldsymbol{p}'}E_{\boldsymbol{p}}}}C_{\ell,\xi}(\boldsymbol{p}'_W,\boldsymbol{p}_W) = \sum_{\xi'}\mathcal{D}^*_{\xi'\xi}(W^{-1})C_{\ell,\xi'}(\boldsymbol{p}',\boldsymbol{p}). \tag{B6}$$

APPENDIX C: ANALYSIS OF A SPINLESS MASSIVE PARTICLE

We assume that the spectrum of $\hat{P}^\mu$ on any state $|\Psi\rangle$ in the Hilbert space of one-particle states, $\mathcal{H}_1$, satisfies

$$\hat{P}^\mu\hat{P}_\mu|\Psi\rangle = -m^2|\Psi\rangle, \quad \langle\Psi|\hat{P}^0|\Psi\rangle > 0. \tag{C1}$$

These equations are equivalent to the fact that the Hamiltonian $\hat{H} = \hat{P}^0$ has the form $\hat{H} = \sqrt{\hat{P}_k\hat{P}^k + m^2}$, which implies that $|\Psi\rangle$ is the state of a massive particle. In this Appendix we derive the form of the dynamical map $\mathcal{E}_{t,t_0}$ of a spinless massive particle.

Case I: $\ell^\mu = [\pm M, 0, 0, 0]$, $M > 0$. We focus on the spectrum $\ell^\mu = [\pm M, 0, 0, 0]$, $M > 0$. From Eq. (B1) for all

$a^\mu=[a,0,0,0]$, $A_{\ell,\xi}$ must vanish,

$$A_{\ell,\xi}=e^{\pm iMa}A_{\ell,\xi}\ \therefore\ A_{\ell,\xi}=0.$$

Equation (B2) for all $a^\mu=[0,\boldsymbol{a}]$ leads to

$$B_{\ell,\xi}(\boldsymbol{p})e^{-i\boldsymbol{p}\cdot\boldsymbol{a}}=B_{\ell,\xi}(\boldsymbol{p})\ \therefore\ B_{\ell,\xi}(\boldsymbol{p})=B_{\ell,\xi}\delta^3(\boldsymbol{p}).$$

From Eq. (B2) for all $a^\mu=[a,0,0,0]$, we get

$$B_{\ell,\xi}(\boldsymbol{p})e^{iE_{\boldsymbol{p}}a}=B_{\ell,\xi}(\boldsymbol{p})e^{\pm iMa},$$

and combined with $B_{\ell,\xi}(\boldsymbol{p})=B_{\ell,\xi}\delta^3(\boldsymbol{p})$ we obtain

$$B_{\ell,\xi}e^{ima}=B_{\ell,\xi}e^{\pm iMa}.$$

Since the mass m is positive, to get a nontrivial result, we should choose $+M$ with $M=m$. Using Eq. (B5) for $Q=R\in$ SO(3) and adopting the result $B_{\ell,\xi}(\boldsymbol{p})=B_{\ell,\xi}\delta^3(\boldsymbol{p})$, we find

$$B_{\ell,\xi}=\sum_{\xi'}\mathcal{D}^*_{\xi'\xi}(R^{-1})B_{\ell,\xi'}.$$

Since $\mathcal{D}_{\xi'\xi}$ is an irreducible (unitary) representation, to get a nontrivial $B_{\ell,\xi}$, we should choose the spinless representation. Hence, $B_{\ell,\xi}$ is reduced to B_ℓ, which has no leg labeled by ξ. Therefore, $B_\ell(\boldsymbol{p})$, which is given by removing the label ξ from $B_{\ell,\xi}(\boldsymbol{p})$, is

$$B_\ell(\boldsymbol{p})=B_\ell\delta^3(\boldsymbol{p}).$$

From Eq. (B3) for all $a^\mu=[0,\boldsymbol{a}]$, we deduce

$$C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{a}}=C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})$$
$$\therefore\ C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})=C_{\ell,\xi}(\boldsymbol{p})\delta^3(\boldsymbol{p}'-\boldsymbol{p}).$$

Equation (B3) for all $a^\mu=[a,0,0,0]$ leads to

$$C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{-i(E_{\boldsymbol{p}'}-E_{\boldsymbol{p}})a}=C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{\pm iMa},$$

and substituting $C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})=C_{\ell,\xi}(\boldsymbol{p})\delta^3(\boldsymbol{p}'-\boldsymbol{p})$ into the above equation, we find that $C_{\ell,\xi}(\boldsymbol{p})$ vanishes and hence $C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})$ is zero,

$$C_{\ell,\xi}(\boldsymbol{p})=C_{\ell,\xi}(\boldsymbol{p})e^{\pm iMa}\ \therefore\ C_{\ell,\xi}(\boldsymbol{p})=0\ \therefore\ C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})=0.$$

The above results of $A_{\ell,\xi}$, $B_{\ell,\xi}(\boldsymbol{p})$, and $C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})$ give the Kraus operator $\hat{E}_{\ell,\xi}$ with $\ell^\mu=[m,0,0,0]$ as

$$\hat{E}_{\ell,\xi}=B_\ell\hat{a}(\boldsymbol{0}).$$

Equation (49) tells us that

$$\hat{E}_{q,\xi}=N_q^*\hat{V}(S_q)\hat{E}_{\ell,\xi}\hat{V}^\dagger(S_q)=N_q^*B_\ell\sqrt{\frac{E_{\boldsymbol{q}}}{m}}\hat{a}(\boldsymbol{q}),$$

where $E_q=(S_q\ell)^0$ and $q^i=(S_q\ell)^i$. Choosing the normalization of the inner product $\boldsymbol{v}^\dagger_{q',\xi'}\boldsymbol{v}_{q,\xi}$ as

$$\boldsymbol{v}^\dagger_{q',t_0'}\boldsymbol{v}_{q,\xi}=\delta^3(\boldsymbol{q}'-\boldsymbol{q})\delta_{\xi'\xi},$$

we have $N_q=\sqrt{m/E_{\boldsymbol{q}}}$ up to a phase factor and the completeness condition,

$$\int d^3q\sum_s\boldsymbol{v}_{q,\xi}\boldsymbol{v}^\dagger_{q,\xi}=\boldsymbol{I}.$$

We then derive a part of the dynamical map $\mathcal{E}_{t,t_0}$ as

$$\mathcal{E}_{t,t_0}[\rho(t_0)]\supset|B_\ell|^2\int d^3q\,\hat{a}(\boldsymbol{q})\rho(t_0)\hat{a}^\dagger(\boldsymbol{q}).\tag{C2}$$

Case II: $\ell^\mu=[\pm\kappa,0,0,\kappa]$, $\kappa>0$. We consider the spectrum $\ell^\mu=[\pm\kappa,0,0,\kappa]$, $\kappa>0$. From Eq. (B1) for all $a^\mu=[a,0,0,0]$, $A_{\ell,\xi}$ turns out to be zero,

$$A_{\ell,\xi}=e^{\pm i\kappa a}A_{\ell,\xi}\ \therefore\ A_{\ell,\xi}=0.$$

From Eq. (B2) for all $a^\mu=[0,\boldsymbol{a}]$, we get

$$B_{\ell,\xi}(\boldsymbol{p})e^{-i\boldsymbol{p}\cdot\boldsymbol{a}}=B_{\ell,\xi}(\boldsymbol{p})e^{-i\boldsymbol{\ell}\cdot\boldsymbol{a}}\ \therefore\ B_{\ell,\xi}(\boldsymbol{p})=B_{\ell,\xi}\delta^3(\boldsymbol{p}-\boldsymbol{\ell}),$$

where $\boldsymbol{\ell}=[0,0,\kappa]^{\mathrm{T}}$. Equation (B2) for all $a^\mu=[a,0,0,0]$ leads to

$$B_{\ell,\xi}(\boldsymbol{p})e^{iE_{\boldsymbol{p}}a}=B_{\ell,\xi}(\boldsymbol{p})e^{\pm i\kappa a},$$

and substituting $B_{\ell,\xi}(\boldsymbol{p})=B_{\ell,\xi}\delta^3(\boldsymbol{p}-\boldsymbol{\ell})$ and $E_{\boldsymbol{\ell}}=\sqrt{\boldsymbol{\ell}^2+m^2}=\sqrt{\kappa^2+m^2}$ into the above equation, we find that $B_{\ell,\xi}$ and $B_{\ell,\xi}(\boldsymbol{p})$ are trivial,

$$B_{\ell,\xi}e^{i\sqrt{\kappa^2+m^2}a}=B_{\ell,\xi}e^{\pm i\kappa a}\ \therefore\ B_{\ell,\xi}=0\ \therefore\ B_{\ell,\xi}(\boldsymbol{p})=0.$$

Equation (B3) for all $a^\mu=[0,\boldsymbol{a}]$ gives

$$C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{a}}=C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{-i\boldsymbol{\ell}\cdot\boldsymbol{a}}$$
$$\therefore\ C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})=C_{\ell,\xi}(\boldsymbol{p})\delta^3(\boldsymbol{p}'-\boldsymbol{p}+\boldsymbol{\ell}).$$

Using Eq. (B3) for all $a^\mu=[a,0,0,0]$, we get

$$C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{-i(E_{\boldsymbol{p}'}-E_{\boldsymbol{p}})a}=C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{\pm i\kappa a},$$

and substituting $C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})=C_{\ell,\xi}(\boldsymbol{p})\delta^3(\boldsymbol{p}'-\boldsymbol{p}+\boldsymbol{\ell})$ into the above equation, we have

$$C_{\ell,\xi}(\boldsymbol{p})e^{-i(E_{\boldsymbol{p}-\boldsymbol{\ell}}-E_{\boldsymbol{p}})a}=C_{\ell,\xi}(\boldsymbol{p})e^{\pm i\kappa a}.$$

Noting the fact that $E_{\boldsymbol{p}-\boldsymbol{\ell}}-E_{\boldsymbol{p}}\pm\kappa\neq0$, we get $C_{\ell,\xi}(\boldsymbol{p})=0$ and

$$C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})=0.$$

Combined with the above analyses on $A_{\ell,\xi}$, $B_{\ell,\xi}(\boldsymbol{p})$, and $C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})$, the Kraus operator $\hat{E}_{\ell,\xi}$ vanishes:

$$\hat{E}_{\ell,\xi}=0\ \therefore\ \hat{E}_{q,\xi}=N_q\hat{V}(S_q)\hat{E}_{\ell,\xi}\hat{V}^\dagger(S_q)=0.\tag{C3}$$

Case III: $\ell^\mu=[0,0,0,w]$, $N^2>0$. We focus on the spectrum $\ell^\mu=[0,0,0,w]$, $N^2>0$. From Eq. (B1) for all $a^\mu=[0,\boldsymbol{a}]$, we have

$$A_{\ell,\xi}=e^{-i\boldsymbol{\ell}\cdot\boldsymbol{a}}A_{\ell,\xi}\ \therefore\ A_{\ell,\xi}=0.$$

Equation (B2) for all $a^\mu=[a,0,0,0]$ leads to

$$B_{\ell,\xi}(\boldsymbol{p})e^{iE_{\boldsymbol{p}}a}=B_{\ell,\xi}(\boldsymbol{p})\ \therefore\ B_{\ell,\xi}(\boldsymbol{p})=0,$$

where note that $E_{\boldsymbol{q}}=\sqrt{\boldsymbol{q}^2+m^2}\neq0$. From Eq. (B3) for all $a^\mu=[0,\boldsymbol{a}]$, we deduce

$$C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{i(\boldsymbol{p}'-\boldsymbol{p})\cdot\boldsymbol{a}}=C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{-i\boldsymbol{\ell}\cdot\boldsymbol{a}}$$
$$\therefore\ C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})=C_{\ell,\xi}(\boldsymbol{p})\delta^3(\boldsymbol{p}'-\boldsymbol{p}+\boldsymbol{\ell}),$$

where $\boldsymbol{\ell}=[0,0,w]^{\mathrm{T}}$. From Eq. (B3) for all $a^\mu=[a,0,0,0]$, we get

$$C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})e^{-i(E_{\boldsymbol{p}'}-E_{\boldsymbol{p}})a}=C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p}),$$

and substituting $C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p}) = C_{\ell,\xi}(\boldsymbol{p})\delta^3(\boldsymbol{p}'-\boldsymbol{p}+\boldsymbol{\ell})$ into the above equation, we have

$$C_{\ell,\xi}(\boldsymbol{p})e^{-i(E_{\boldsymbol{p}-\boldsymbol{\ell}}-E_{\boldsymbol{p}})a} = C_{\ell,\xi}(\boldsymbol{p})$$
$$\therefore\ C_{\ell,\xi}(\boldsymbol{p}) = C_{\ell,\xi}(\boldsymbol{p}_\perp)\delta(p^z - N/2).$$

Substituting this into $C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p}) = C_{\ell,\xi}(\boldsymbol{p})\delta^3(\boldsymbol{p}'-\boldsymbol{p}+\boldsymbol{\ell})$, we obtain

$$C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p}) = C_{\ell,\xi}(\boldsymbol{p}_\perp)\delta^2(\boldsymbol{p}'_\perp-\boldsymbol{p}_\perp)\delta(p'^z+N/2)\delta(p^z-N/2).$$

With the above results of $A_{\ell,\xi}$, $B_{\ell,\xi}(\boldsymbol{p})$ and $C_{\ell,\xi}(\boldsymbol{p}',\boldsymbol{p})$, the Kraus operator $\hat{E}_{\ell,\xi}$ is written as

$$\hat{E}_{\ell,\xi} = \int d^2p_\perp C_{\ell,\xi}(\boldsymbol{p}_\perp)\hat{a}^\dagger(\boldsymbol{p}_\perp,-N/2)\hat{a}(\boldsymbol{p}_\perp,N/2).$$

By the completeness condition of the Kraus operators [Eq. (34)], the above Kraus operator $\hat{E}_{\ell,\xi}$ should satisfy $\hat{E}^\dagger_{\ell,\xi}\hat{E}_{\ell,\xi} \leqslant \hat{\mathbb{I}}$. Concretely, $\hat{E}^\dagger_{\ell,\xi}\hat{E}_{\ell,\xi}$ is evaluated as

$$\begin{aligned}\hat{E}^\dagger_{\ell,\xi}\hat{E}_{\ell,\xi} &= \int d^2p'_\perp C^*_{\ell,\xi}(\boldsymbol{p}'_\perp)\hat{a}^\dagger(\boldsymbol{p}'_\perp,N/2)\hat{a}(\boldsymbol{p}'_\perp,-N/2)\\ &\quad\times\int d^2p_\perp C_{\ell,\xi}(\boldsymbol{p}_\perp)\hat{a}^\dagger(\boldsymbol{p}_\perp,-N/2)\hat{a}(\boldsymbol{p}_\perp,N/2)\\ &= \delta(0)\int d^2p_\perp[C_{\ell,\xi}(\boldsymbol{p}_\perp)\hat{a}(\boldsymbol{p}_\perp,N/2)]^\dagger\\ &\quad\times[C_{\ell,\xi}(\boldsymbol{p}_\perp)\hat{a}(\boldsymbol{p}_\perp,N/2)],\end{aligned}$$

where the term given by the linear combination of $\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}$ vanishes on $\mathcal{H}_0\oplus\mathcal{H}_1$. To satisfy $\hat{E}^\dagger_{\ell,\xi}\hat{E}_{\ell,\xi} \leqslant \hat{\mathbb{I}}$, we find that

$$\int d^2p_\perp[C_{\ell,\xi}(\boldsymbol{p}_\perp)\hat{a}(\boldsymbol{p}_\perp,N/2)]^\dagger[C_{\ell,\xi}(\boldsymbol{p}_\perp)\hat{a}(\boldsymbol{p}_\perp,N/2)] = 0,$$

which leads to $C_{\ell,\xi}(\boldsymbol{p}_\perp) = 0$ and $\langle\Phi|\hat{E}_{\ell,\xi}|\Psi\rangle = 0$ for all $|\Psi\rangle, |\Phi\rangle \in \mathcal{H}_0\oplus\mathcal{H}_1$. Hence, the Kraus operator $\hat{E}_{q,\xi}$ vanishes,

$$\hat{E}_{q,\xi} = N^*_q\hat{V}(S_q)\hat{E}_{\ell,\xi}\hat{V}^\dagger(S_q) = 0, \tag{C4}$$

on the Hilbert space $\mathcal{H}_0\oplus\mathcal{H}_1$.

Case IV: $\ell^\mu = [0,0,0,0]$. We consider the case where $\ell^\mu = [0,0,0,0]$. Before computations, let us discuss the unitary irreducible representations $\mathcal{D}(\Lambda)$ of the Lorentz group [37,38]. Let $\mathcal{J}$ and $\mathcal{K}$ be the generators of $\mathcal{D}(\Lambda)$, which give rotations and boosts, respectively. The unitary irreducible representation $\mathcal{D}(\Lambda)$ is classified by two parameters j_0 and ν, by which the eigenvalues of the two Casimir operators $I_1 = \mathcal{J}^2 - \mathcal{K}^2$ and $I_2 = \mathcal{J}\cdot\mathcal{K}$ are determined. The value $j_0(j_0+1)$ gives the minimum eigenvalue of $\mathcal{J}^2$. The parameters j_0 and ν are decomposed into two parts called (i) the principal series $-i\nu\in\mathbb{R}$ and $j_0 = 0, \frac{1}{2}, 1, \dots$ and (ii) the complementary series $0\leqslant\nu^2\leqslant 1$ and $j_0 = 0$. We denote by $\mathcal{D}^{j_0,\nu}(\Lambda)$ the unitary irreducible representation with j_0 and ν. In particular, $\mathcal{D}^{0,\pm1}(\Lambda)$ is the trivial one-dimensional representation $\mathcal{D}^{0,1}(\Lambda) = \mathcal{D}^{0,-1}(\Lambda) = 1$ and the others are infinite-dimensional representations.

With the above knowledge on the unitary irreducible representation of the Lorentz group, we proceed with computations. In the following, we drop the label ℓ. Equation (B1) is identical for all a^μ. Since the little group associated with ℓ^μ is SO(3, 1), Eq. (B4) for $W = \Lambda \in$ SO(3, 1) is given as

$$A_\xi = \sum_{\xi'}\mathcal{D}^*_{\xi'\xi}(\Lambda^{-1})A_{\xi'}. \tag{C5}$$

For the one-dimensional representation, since the representation is trivial $\mathcal{D}^{0,1}(\Lambda) = \mathcal{D}^{0,-1}(\Lambda) = 1$, A_ξ is a scalar, $A_\xi = A$. Then Eq. (C5) trivially holds. For the infinite-dimensional representation, $A_\xi = 0$ from Eq. (C5).

Equation (B2) for all $a^\mu = [a,0,0,0]$ gives the condition

$$B_\xi(\boldsymbol{p})e^{iE_{\boldsymbol{p}}a} = B_\xi(\boldsymbol{p})\ \therefore\ B_\xi(\boldsymbol{p}) = 0,$$

where note that $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2+m^2} \neq 0$. From Eq. (B3) for all a^μ, we obtain

$$C_\xi(\boldsymbol{p}',\boldsymbol{p})e^{i(p'^\mu-p^\mu)a_\mu} = C_\xi(\boldsymbol{p}',\boldsymbol{p})$$
$$\therefore\ C_\xi(\boldsymbol{p}',\boldsymbol{p}) = C_\xi(\boldsymbol{p})\delta^3(\boldsymbol{p}'-\boldsymbol{p}).$$

Equation (B6) for $W = \Lambda \in$ SO(3, 1) is written as

$$\sqrt{\frac{E_{\boldsymbol{p}'_\Lambda}E_{\boldsymbol{p}_\Lambda}}{E_{\boldsymbol{p}'}E_{\boldsymbol{p}}}}C_\xi(\boldsymbol{p}'_\Lambda,\boldsymbol{p}_\Lambda) = \sum_{\xi'}\mathcal{D}^*_{\xi'\xi}(\Lambda^{-1})C_{\xi'}(\boldsymbol{p}',\boldsymbol{p}).$$

The equation $C_\xi(\boldsymbol{p}',\boldsymbol{p}) = C_\xi(\boldsymbol{p})\delta^3(\boldsymbol{p}'-\boldsymbol{p})$ and the fact that the invariant delta function is $E_{\boldsymbol{p}}\delta^3(\boldsymbol{p}-\boldsymbol{p}')$ yield

$$C_\xi(\boldsymbol{p}_\Lambda) = \sum_{\xi'}\mathcal{D}^*_{\xi'\xi}(\Lambda^{-1})C_{\xi'}(\boldsymbol{p}). \tag{C6}$$

Choosing $\boldsymbol{p} = \boldsymbol{0}$ and $\Lambda = S_q$ with $(S_q)^\mu{}_\nu k^\nu = q^\mu$ for $k^\mu = [m,0,0,0]$, we have

$$C_\xi(\boldsymbol{q}) = \sum_{\xi'}\mathcal{D}^*_{\xi'\xi}(S_q^{-1})C_{\xi'}(\boldsymbol{0}). \tag{C7}$$

Using Eq. (C7), Eq. (C6) is written as

$$C_\xi(\boldsymbol{0}) = \sum_{\xi'}\mathcal{D}^*_{\xi'\xi}(Q)C_{\xi'}(\boldsymbol{0}), \tag{C8}$$

where $\sum_{\xi''}\mathcal{D}_{\xi\xi''}(\Lambda)\mathcal{D}_{\xi''\xi'}(\Lambda') = \mathcal{D}_{\xi\xi'}(\Lambda\Lambda')$ and $\mathcal{D}^*_{\xi'\xi}(\Lambda) = \mathcal{D}_{\xi\xi'}(\Lambda^{-1})$ were used and $Q = Q(\Lambda,p) = S^{-1}_{\Lambda p}\Lambda S_p \in$ SO(3). For the trivial representation $\mathcal{D}(\Lambda) = \mathcal{D}^{0,\pm1}(\Lambda) = 1$, the label ξ is removed and we should use $C(\boldsymbol{p})$ instead of $C_\xi(\boldsymbol{p})$. Equation (C8) is then automatically satisfied. From Eq. (C7) we find that $C(\boldsymbol{q})$ is just a constant C:

$$C(\boldsymbol{q}) = C(\boldsymbol{0}) = C.$$

For the infinite-dimensional representation $\mathcal{D}(\Lambda) = \mathcal{D}^{j_0,\nu}(\Lambda)$ with $(j_0,\nu)\neq(0,\pm1)$, we have $C_\xi(\boldsymbol{p}',\boldsymbol{p}) = C_{\xi;j_0,\nu}(\boldsymbol{p}',\boldsymbol{p})$ with

$$C_{\xi;j_0,\nu}(\boldsymbol{p}',\boldsymbol{p}) = C_{\xi;j_0,\nu}(\boldsymbol{p})\delta^3(\boldsymbol{p}'-\boldsymbol{p}),$$

where the dependence of j_0 and ν was explicitly indicated. For a spinless massive particle, the equation

$$C_{\xi;j_0,\nu}(\boldsymbol{0}) = \sum_{\xi'}\mathcal{D}^{j_0,\nu*}_{\xi'\xi}(Q)C_{\xi';j_0,\nu}(\boldsymbol{0})$$

is satisfied from Eq. (C8), where $Q\in$ SO(3). To get a nontrivial solution, j_0 must be zero and $C_{\xi;0,\nu}(\boldsymbol{0})$ belongs to the representation space with $j_0 = 0$. Hence, $C_{\xi;0,\nu}(\mathbf{0}) = C_\nu \delta_{\xi\xi_0}$ with $\mathcal{D}^{0,\nu}_{\xi\xi_0}(Q) = \delta_{\xi\xi_0}$ for $Q \in \mathrm{SO}(3)$. Equation (C7) gives

$$C_{\xi;0,\nu}(\boldsymbol{q}) = C_\nu \mathcal{D}^{0,\nu*}_{\xi_0\xi}\big(S_q^{-1}\big), \tag{C9}$$

where note that S_q is not an element of SO(3).

The above analysis for the case $\ell^\mu = [0, 0, 0, 0]$ tells us the Kraus operators

$$\hat{E} = A\hat{\mathbb{I}} + C\hat{N}, \quad \hat{E}_{\xi;0,\nu} = \int d^3p\, C_\nu \mathcal{D}^{0,\nu*}_{\xi_0\xi}\big(S_p^{-1}\big)\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p}),$$

where A and C are complex numbers and $\hat{N}$ is the number operator defined in (64). A part of the dynamical map $\mathcal{E}_{t,t_0}$ is given as

$$\begin{aligned}\mathcal{E}_{t,t_0}[\rho(t_0)] &\supset (A\hat{\mathbb{I}} + C\hat{N})\rho(t_0)(A\hat{\mathbb{I}} + C\hat{N})^\dagger \\ &\quad + \sum_{\xi,\nu} \hat{E}_{\xi;0,\nu}\rho(t_0)\hat{E}^\dagger_{\xi;0,\nu} \\ &= (A\hat{\mathbb{I}} + C\hat{N})\rho(t_0)(A\hat{\mathbb{I}} + C\hat{N})^\dagger \\ &\quad + \int d^3p\, d^3q\, D(p, q)\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})\rho(t_0)\hat{a}^\dagger(\boldsymbol{q})\hat{a}(\boldsymbol{q}),\end{aligned} \tag{C10}$$

where $\sum_\nu$ is defined by

$$\sum_\nu F_\nu = \int_{-1<\nu<1} d\nu\, F_\nu + \int_{-i\nu\in\mathbb{R}} d(-i\nu)F_\nu$$

and $D(p, q)$ is

$$D(p, q) = \sum_{\nu,\xi} |C_\nu|^2 \mathcal{D}^{0,\nu*}_{\xi_0\xi}\big(S_p^{-1}\big)\mathcal{D}^{0,\nu}_{\xi_0\xi}\big(S_q^{-1}\big). \tag{C11}$$

This function $D(p, q)$ is non-negative and Lorentz invariant in the sense that

$$\int d^3p\, d^3q\, f^*(p)D(p, q)f(q) \geqslant 0, \quad D(\Lambda p, \Lambda q) = D(p, q), \tag{C12}$$

where $f(p)$ is a complex function of the particle's four-momentum and Λ is the Lorentz transformation matrix. It is easy to show the former condition, and the latter equation is shown as

$$\begin{aligned}D(\Lambda p, \Lambda q) &= \sum_{\nu,\xi} |C_\nu|^2 \mathcal{D}^{0,\nu*}_{\xi_0\xi}\big(S^{-1}_{\Lambda p}\big)\mathcal{D}^{0,\nu}_{\xi_0\xi}\big(S^{-1}_{\Lambda q}\big) \\ &= \sum_{\nu,\xi} |C_\nu|^2 \mathcal{D}^{0,\nu*}_{\xi_0\xi}\big(Q(\Lambda, p)S_p^{-1}\Lambda^{-1}\big) \\ &\quad \times \mathcal{D}^{0,\nu}_{\xi_0\xi}\big(Q(\Lambda, q)S_q^{-1}\Lambda^{-1}\big) \\ &= \sum_{\nu,\xi} |C_\nu|^2 \sum_{\xi'} \mathcal{D}^{0,\nu*}_{\xi_0\xi'}(Q(\Lambda, p))\mathcal{D}^{0,\nu*}_{\xi'\xi}\big(S_p^{-1}\Lambda^{-1}\big) \\ &\quad \times \sum_{\xi''} \mathcal{D}^{0,\nu}_{\xi_0\xi''}(Q(\Lambda, q))\mathcal{D}^{0,\nu}_{\xi''\xi}\big(S_q^{-1}\Lambda^{-1}\big) \\ &= \sum_{\nu,\xi} |C_\nu|^2 \mathcal{D}^{0,\nu*}_{\xi_0\xi}\big(S_p^{-1}\Lambda^{-1}\big)\mathcal{D}^{0,\nu}_{\xi_0\xi}\big(S_q^{-1}\Lambda^{-1}\big) \\ &= \sum_{\nu,\xi} |C_\nu|^2 \sum_{\xi'} \mathcal{D}^{0,\nu*}_{\xi_0\xi'}\big(S_p^{-1}\big)\mathcal{D}^{0,\nu*}_{\xi'\xi}(\Lambda^{-1}) \\ &\quad \times \sum_{\xi''} \mathcal{D}^{0,\nu}_{\xi_0\xi''}\big(S_q^{-1}\big)\mathcal{D}^{0,\nu}_{\xi''\xi}(\Lambda^{-1}) \\ &= \sum_\nu |C_\nu|^2 \sum_{\xi',\xi''} \mathcal{D}^{0,\nu*}_{\xi_0\xi'}\big(S_p^{-1}\big)\delta_{\xi'\xi''}\mathcal{D}^{0,\nu}_{\xi_0\xi''}\big(S_q^{-1}\big) \\ &= \sum_{\nu,\xi} |C_\nu|^2 \mathcal{D}^{0,\nu*}_{\xi_0\xi}\big(S_p^{-1}\big)\mathcal{D}^{0,\nu}_{\xi_0\xi}\big(S_q^{-1}\big) \\ &= D(p, q),\end{aligned}$$

where we used the equations $Q(\Lambda, p) = S^{-1}_{\Lambda p}\Lambda p$, $\sum_{\xi''} \mathcal{D}^{0,\nu}_{\xi\xi''}(\Lambda)\mathcal{D}^{0,\nu}_{\xi''\xi'}(\Lambda') = \mathcal{D}^{0,\nu}_{\xi\xi'}(\Lambda\Lambda')$, $\mathcal{D}^{0,\nu*}_{\xi'\xi}(\Lambda) = \mathcal{D}^{0,\nu}_{\xi\xi'}(\Lambda^{-1})$, and $\mathcal{D}^{0,\nu}_{\xi_0\xi}(Q) = \delta_{\xi_0\xi}$ for $Q \in \mathrm{SO}(3)$.

Summary from cases I–IV. Let us summarize the analysis of a spinless massive particle. The above results given in Eqs. (C2)–(C4) and (C10) provide the following form of $\mathcal{E}_{t,t_0}$:

$$\begin{aligned}\mathcal{E}_{t,t_0}[\rho(t_0)] &= |B|^2 \int d^3p\, \hat{a}(\boldsymbol{p})\rho(t_0)\hat{a}^\dagger(\boldsymbol{p}) \\ &\quad + (A\hat{\mathbb{I}} + C\hat{N})\rho(t_0)(A\hat{\mathbb{I}} + C\hat{N})^\dagger \\ &\quad + \int d^3p\, d^3q\, D(p, q)\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})\rho(t_0)\hat{a}^\dagger(\boldsymbol{q})\hat{a}(\boldsymbol{q}).\end{aligned}$$

Here B_ℓ is denoted by B for simplicity. For the Hilbert space $\mathcal{H}_0 \oplus \mathcal{H}_1$ of the vacuum state and one-particle states of the massive particle, the completeness condition of the Kraus operators gives

$$\begin{aligned}\hat{\mathbb{I}} &= |B|^2 \int d^3p\, \hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p}) + (\hat{A\mathbb{I}} + C\hat{N})^\dagger(\hat{A\mathbb{I}} + C\hat{N}) \\ &\quad + \int d^3p\, d^3q\, D(p, q)\hat{a}^\dagger(\boldsymbol{q})\hat{a}(\boldsymbol{q})\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p}) \\ &= |A|^2\hat{\mathbb{I}} + [|B|^2 + AC^* + A^*C + |C|^2 + D]\hat{N}.\end{aligned} \tag{C13}$$

where we note that the equations

$$\hat{N}^2 = \hat{N}, \quad \hat{a}^\dagger(\boldsymbol{q})\hat{a}(\boldsymbol{q})\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p}) = \hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})\delta^3(\boldsymbol{p} - \boldsymbol{q}) \tag{C14}$$

hold on $\mathcal{H}_0 \oplus \mathcal{H}_1$ and that $D = D(p, p)$ does not depend on the three-momentum $\boldsymbol{p}$ due to the Lorentz invariance [the second of Eqs. (C12)]. Equation (C13) gives

$$|A|^2 = 1, \quad |B|^2 + AC^* + A^*C + |C|^2 + D = 0. \tag{C15}$$

The parameters A, B, and C and the function $D(p, q)$ may depend on t and t_0. Redefining $|B|^2$, C/A, and $D(p, q)$ as β_{t,t_0}, γ_{t,t_0}, and $\delta_{t,t_0}(p, q)$, respectively, we get the following dynamical map $\mathcal{E}_{t,t_0}$,

$$\begin{aligned}\mathcal{E}_{t,t_0}[\rho(t_0)] &= \beta_{t,t_0} \int d^3p\, \hat{a}(\boldsymbol{p})\rho(t_0)\hat{a}^\dagger(\boldsymbol{p}) \\ &\quad + (\hat{\mathbb{I}} + \gamma_{t,t_0}\hat{N})\rho(t_0)(\hat{\mathbb{I}} + \gamma_{t,t_0}\hat{N})^\dagger \\ &\quad + \int d^3p\, d^3q\, \delta_{t,t_0}(p, q)\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})\rho(t_0)\hat{a}^\dagger(\boldsymbol{q})\hat{a}(\boldsymbol{q}).\end{aligned} \tag{C16}$$

This is nothing but (63). According to the definitions β_{t,t_0}, γ_{t,t_0}, and $\delta_{t,t_0}(p, q)$, we find that the function $\delta_{t,t_0}(p, q)$ is non-negative and Lorentz invariant, that is

$$\int d^3p d^3q\, f^*(p)\delta_{t,t_0}(p,q) f(q) \geqslant 0,$$

and

$$\delta_{t,t_0}(\Lambda p, \Lambda q) = \delta_{t,t_0}(p,q). \tag{C17}$$

and that the parameters β_{t,t_0}, γ_{t,t_0}, and $\delta_{t,t_0} = \delta_{t,t_0}(p,p)$ satisfy

$$\beta_{t,t_0} \geqslant 0, \quad \beta_{t,t_0} + \gamma^*_{t,t_0} + \gamma_{t,t_0} + |\gamma_{t,t_0}|^2 + \delta_{t,t_0} = 0. \tag{C18}$$

APPENDIX D: COMPUTATION OF THE CHARACTERISTIC FUNCTION

In this Appendix we derive Eqs. (68) and (69). Since $\rho(t) = \Phi_{t,t_0}[\rho(t_0)] = \mathcal{U}_{t,t_0} \circ \mathcal{E}_{t,t_0}[\rho(t_0)]$ and $\mathcal{U}_{t,t_0}[\rho] = e^{-i\hat{H}(t-t_0)} \rho e^{i\hat{H}(t-t_0)}$, we have

$$\chi_t(0,a) = \mathrm{Tr}[e^{-ia_\mu \hat{P}^\mu}\rho(t)] = \mathrm{Tr}\{e^{-ia_\mu \hat{P}^\mu}\mathcal{U}_{t,t_0} \circ \mathcal{E}_{t,t_0}[\rho(t_0)]\} = \mathrm{Tr}\{e^{-ia_\mu \hat{P}^\mu}\mathcal{E}_{t,t_0}[\rho(t_0)]\},$$

where $[\hat{H}, \hat{P}^\mu] = 0$ was used. Substituting the form of $\mathcal{E}_{t,t_0}$ given in (63) into the above equation, we get

$$\begin{aligned}\chi_t(0,a) = {}& \beta_{t,t_0}\mathrm{Tr}\left(e^{-ia_\mu \hat{P}^\mu}\int d^3p\, \hat{a}(\boldsymbol{p})\rho(t_0)\hat{a}^\dagger(\boldsymbol{p})\right) + \mathrm{Tr}\big[e^{-ia_\mu \hat{P}^\mu}(\hat{\mathbb{I}} + \gamma_{t,t_0}\hat{N})\rho(t_0)(\hat{\mathbb{I}} + \gamma^*_{t,t_0}\hat{N})\big] \\ & + \mathrm{Tr}\left(e^{-ia_\mu \hat{P}^\mu}\int d^3p \int d^3q\, \delta_{t,t_0}(p,q)\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})\rho(t_0)\hat{a}^\dagger(\boldsymbol{q})\hat{a}(\boldsymbol{q})\right).\end{aligned}$$

Since $e^{-i\hat{P}^\mu a_\mu}\hat{N}e^{i\hat{P}^\mu a_\mu} = \hat{N}$ and $e^{-i\hat{P}^\mu a_\mu}\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})e^{i\hat{P}^\mu a_\mu} = \hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})$, the characteristic function is written as

$$\begin{aligned}\chi_t(0,a) = {}& \beta_{t,t_0}\mathrm{Tr}\left(e^{-ia_\mu \hat{P}^\mu}\int d^3p\, \hat{a}(\boldsymbol{p})\rho(t_0)\hat{a}^\dagger(\boldsymbol{p})\right) + \mathrm{Tr}\big[(\hat{\mathbb{I}} + \gamma^*_{t,t_0}\hat{N})(\hat{\mathbb{I}} + \gamma_{t,t_0}\hat{N})e^{-ia_\mu \hat{P}^\mu}\rho(t_0)\big] \\ & + \mathrm{Tr}\left(\int d^3p \int d^3q\, \delta_{t,t_0}(p,q)\hat{a}^\dagger(\boldsymbol{q})\hat{a}(\boldsymbol{q})\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})e^{-ia_\mu \hat{P}^\mu}\rho(t_0)\right).\end{aligned}$$

On the Hilbert space $\mathcal{H}_0 \oplus \mathcal{H}_1$ of the vacuum state and one-particle states of the massive particle, the following equations hold:

$$e^{-i\hat{P}_\mu a^\mu}\hat{a}(\boldsymbol{p}) = \hat{a}(\boldsymbol{p}), \quad \hat{N}^2 = \hat{N}, \quad \hat{a}^\dagger(\boldsymbol{q})\hat{a}(\boldsymbol{q})\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p}) = \hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})\delta^3(\boldsymbol{p}-\boldsymbol{q}). \tag{D1}$$

Using them, we then find

$$\begin{aligned}\chi_t(0,a) &= \beta_{t,t_0}\mathrm{Tr}\left(\int d^3p\, \hat{a}(\boldsymbol{p})\rho(t_0)\hat{a}^\dagger(\boldsymbol{p})\right) + \mathrm{Tr}\big\{\big[\hat{\mathbb{I}} + \big(\gamma^*_{t,t_0} + \gamma_{t,t_0} + |\gamma_{t,t_0}|^2\big)\hat{N}\big]e^{-ia_\mu \hat{P}^\mu}\rho(t_0)\big\} \\ &\quad + \mathrm{Tr}\left(\int d^3p\, \delta_{t,t_0}(p,p)\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})e^{-ia_\mu \hat{P}^\mu}\rho(t_0)\right) \\ &= \beta_{t,t_0}\mathrm{Tr}[\hat{N}\rho(t_0)] + \mathrm{Tr}\big\{\big[\hat{\mathbb{I}} + \big(\gamma^*_{t,t_0} + \gamma_{t,t_0} + |\gamma_{t,t_0}|^2\big)\hat{N}\big]e^{-ia_\mu \hat{P}^\mu}\rho(t_0)\big\} + \delta_{t,t_0}\mathrm{Tr}[\hat{N}e^{-ia_\mu \hat{P}^\mu}\rho(t_0)] \\ &= \beta_{t,t_0}\mathrm{Tr}[\hat{N}\rho(t_0)] + \mathrm{Tr}\big\{\big[\hat{\mathbb{I}} - \big(\beta_{t,t_0} + \delta_{t,t_0}\big)\hat{N}\big]e^{-ia_\mu \hat{P}^\mu}\rho(t_0)\big\} + \delta_{t,t_0}\mathrm{Tr}[\hat{N}e^{-ia_\mu \hat{P}^\mu}\rho(t_0)] \\ &= \chi_s(0,a) + \beta_{t,t_0}\mathrm{Tr}[\hat{N}(\hat{\mathbb{I}} - e^{-ia_\mu \hat{P}^\mu})\rho(t_0)],\end{aligned}$$

where in the second equality note that $\delta_{t,t_0}(p,p) = \delta_{t,t_0}$ is independent of the three-momentum $\boldsymbol{p}$ due to the Lorentz invariance, $\delta_{t,t_0}(p,p) = \delta_{t,t_0}(\Lambda p, \Lambda p)$, and in the third equality we used $\gamma^*_{t,t_0} + \gamma_{t,t_0} + |\gamma_{t,t_0}|^2 = -\beta_{t,t_0} - \delta_{t,t_0}$ satisfied by the second of Eqs. (66). Hence, we get Eq. (68).

Let us compute the characteristic function $\chi_t(\theta, 0)$. Using $e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t)} = e^{-i\hat{H}(t-t_0)}e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}e^{i\hat{H}(t-t_0)}$, which follows by $\hat{\boldsymbol{K}}(t) = e^{-i\hat{H}t}\hat{\boldsymbol{K}}(0)e^{i\hat{H}t}$ and $\hat{\boldsymbol{J}} = e^{-i\hat{H}t}\hat{\boldsymbol{J}}e^{i\hat{H}t}$ [see also the discussion around Eq. (25)], we have

$$\chi_t(\theta,0) = \mathrm{Tr}[e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t)}\rho(t)] = \mathrm{Tr}\{e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t)}\mathcal{U}_{t,t_0} \circ \mathcal{E}_{t,t_0}[\rho(t_0)]\} = \mathrm{Tr}\{e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\mathcal{E}_{t,t_0}[\rho(t_0)]\}.$$

Substituting $\mathcal{E}_{t,t_0}$ into this equation and assuming $\beta_{t,t_0} = 0$, we obtain

$$\begin{aligned}\chi_t(\theta,0) = {}& \mathrm{Tr}\big[e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}(\hat{\mathbb{I}} + \gamma_{t,t_0}\hat{N})\rho(t_0)(\hat{\mathbb{I}} + \gamma^*_{t,t_0}\hat{N})\big] \\ & + \mathrm{Tr}\left(e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\int d^3p \int d^3q\, \delta_{t,t_0}(p,q)\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})\rho(t_0)\hat{a}^\dagger(\boldsymbol{q})\hat{a}(\boldsymbol{q})\right).\end{aligned}$$

With the help of the invariance of the number operator $e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\hat{N}e^{-(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)} = \hat{N}$ and the transformation rule $e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})e^{-(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)} = \frac{E_{\boldsymbol{p}_\Lambda}}{E_{\boldsymbol{p}}}\hat{a}^\dagger(\boldsymbol{p}_\Lambda)\hat{a}(\boldsymbol{p}_\Lambda)$, the characteristic function $\chi_t(\theta, 0)$ is computed as

$$\begin{aligned}
\chi_t(\theta, 0) = {}& \mathrm{Tr}\big[\big(\hat{\mathbb{I}} + \gamma_{t,t_0}\hat{N}\big)e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\rho(t_0)\big(\hat{\mathbb{I}} + \gamma^*_{t,t_0}\hat{N}\big)\big] \\
&+ \mathrm{Tr}\bigg(\int d^3p \int d^3q\, \delta_{t,t_0}(p,q)\frac{E_{\boldsymbol{p}_\Lambda}}{E_{\boldsymbol{p}}}\hat{a}^\dagger(\boldsymbol{p}_\Lambda)\hat{a}(\boldsymbol{p}_\Lambda)e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\rho(t_0)\hat{a}^\dagger(\boldsymbol{q})\hat{a}(\boldsymbol{q})\bigg) \\
= {}& \mathrm{Tr}\big[\big(\hat{\mathbb{I}} + \gamma^*_{t,t_0}\hat{N}\big)\big(\hat{\mathbb{I}} + \gamma_{t,t_0}\hat{N}\big)e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\rho(t_0)\big] \\
&+ \mathrm{Tr}\bigg(\int d^3p \int d^3q\, \delta_{t,t_0}(p,q)\frac{E_{\boldsymbol{p}_\Lambda}}{E_{\boldsymbol{p}}}\hat{a}^\dagger(\boldsymbol{q})\hat{a}(\boldsymbol{q})\hat{a}^\dagger(\boldsymbol{p}_\Lambda)\hat{a}(\boldsymbol{p}_\Lambda)e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\rho(t_0)\bigg) \\
= {}& \mathrm{Tr}\big\{\big[\hat{\mathbb{I}} + \big(\gamma^*_{t,t_0} + \gamma_{t,t_0} + |\gamma_{t,t_0}|^2\big)\hat{N}\big]e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\rho(t_0)\big\} \\
&+ \mathrm{Tr}\bigg(\int d^3p\, \delta_{t,t_0}(p,\Lambda p)\frac{E_{\boldsymbol{p}_\Lambda}}{E_{\boldsymbol{p}}}\hat{a}^\dagger(\boldsymbol{p}_\Lambda)\hat{a}(\boldsymbol{p}_\Lambda)e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\rho(t_0)\bigg) \\
= {}& \mathrm{Tr}\big[\big(\hat{\mathbb{I}} - \delta_{t,t_0}\hat{N}\big)e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\rho(t_0)\big] + \mathrm{Tr}\bigg(\int d^3p\, \delta_{t,t_0}(p,\Lambda p)\frac{E_{\boldsymbol{p}_\Lambda}}{E_{\boldsymbol{p}}}\hat{a}^\dagger(\boldsymbol{p}_\Lambda)\hat{a}(\boldsymbol{p}_\Lambda)e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\rho(t_0)\bigg) \\
= {}& \chi_{t_0}(\theta, 0) - \delta_{t,t_0}\mathrm{Tr}[\hat{N}e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\rho(t_0)] + \mathrm{Tr}\bigg(\int d^3p\, \delta_{t,t_0}(p,\Lambda p)\hat{a}^\dagger(\boldsymbol{p})\hat{a}(\boldsymbol{p})e^{(i/2)\theta_{\mu\nu}\hat{J}^{\mu\nu}(t_0)}\rho(t_0)\bigg),
\end{aligned}$$

where the second and third equations of (D1) were used in the third equality, and in the fourth equality the equation $\gamma^*_{t,t_0} + \gamma_{t,t_0} + |\gamma_{t,t_0}|^2 = -\delta_{t,t_0}$ was substituted. The Lorentz invariance of $d^3p/E_{\boldsymbol{p}}$ and $\delta_{t,t_0}(p,q)$ leads to the last equality, which is nothing but Eq. (69).

[1] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, *Colloquium*: Non-Markovian dynamics in open quantum systems, Rev. Mod. Phys. **88**, 021002 (2016).

[2] H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, New York, 2002).

[3] R. P. Feynman and F. L. Vernon, Jr., The theory of a general quantum system interacting with a linear dissipative system, Ann. Phys. (NY) **24**, 118 (1963).

[4] A. O. Caldeira and A. J. Leggett, Path integral approach to quantum Brownian motion, Physica A **121**, 587 (1983).

[5] E. A. Calzetta and B. L. Hu, *Nonequilibrium Quantum Field Theory* (Cambridge University Press, Cambridge, 2008).

[6] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1995), Vol. I.

[7] C. Jones, T. Guaita, and A. Bassi, Impossibility of extending the Ghirardi-Rimini-Weber model to relativistic particles, Phys. Rev. A **103**, 042216 (2021).

[8] C. Jones, G. Gasbarri, and A. Bassi, Mass-coupled relativistic spontaneous collapse models, J. Phys. A: Math. Theor. **54**, 295306 (2021).

[9] D. Bedingham, D. Dürr, G. Ghirardi, S. Goldstein, R. Tumulka, and N. Zanghí, Matter density and relativistic models of wave function collapse, J. Stat. Phys. **154**, 623 (2014).

[10] D. Bedingham and P. Pearle, Continuous-spontaneous-localization scalar-field relativistic collapse model, Phys. Rev. Res. **1**, 033040 (2019).

[11] P. Pearle, Relativistic dynamical collapse model, Phys. Rev. D **91**, 105012 (2015).

[12] M. A. Kurkov and V. A. Franke, Local fields without restrictions on the spectrum of 4-momentum operator and relativistic Lindblad equation, Found. Phys. **41**, 820 (2011).

[13] P. Wang, Relativistic quantum field theory of stochastic dynamics in the Hilbert space, Phys. Rev. D **105**, 115037 (2022).

[14] D. Ahn, H. J. Lee, and S. W. Hwang, Lorentz-covariant reduced-density-operator theory for relativistic-quantum-information processing, Phys. Rev. A **67**, 032309 (2003).

[15] X. Meng, Double-trace deformation in open quantum field theory, Phys. Rev. D **104**, 016016 (2021).

[16] D. Kafri, J. M. Taylor, and G. J. Milburn, A classical channel model for gravitational decoherence, New J. Phys. **16**, 065020 (2014).

[17] L. Diósi, A universal master equation for the gravitational violation of quantum mechanics, Phys. Lett. A **120**, 377 (1987).

[18] L. Diósi, Models for universal reduction of macroscopic quantum fluctuations, Phys. Rev. A **40**, 1165 (1989).

[19] R. Penrose, On gravity's role in quantum state reduction, Gen. Relativ. Gravit. **28**, 581 (1996).

[20] A. Tilloy and L. Diósi, Sourcing semiclassical gravity from spontaneously localized quantum matter, Phys. Rev. D **93**, 024026 (2016).

[21] E. B. Davies, *Quantum Theory of Open Systems* (Academic Press, New York, 1976).

[22] A. S. Holevo, *Statistical Structure of Quantum Theory*, Lecture Notes in Physics Monographs Vol. 67 (Springer, Berlin, 2001).

[23] M. Keyl, Fundamentals of quantum information theory, Phys. Rep. **369**, 431 (2002).

[24] G. Lindblad, On the generators of quantum dynamical semigroups, Commun. Math. Phys. **48**, 119 (1976).

[25] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, Completely positive dynamical semigroup of N-level system, J. Math. Phys. **17**, 821 (1976).

[26] C. Cîrstoiu, K. Korzekwa, and D. Jennings, Robustness of Noether's principle: Maximal disconnects between conservation laws and symmetries in quantum theory, Phys. Rev. X **10**, 041035 (2020).

[27] I. Marvian and R. W. Spekkens, Extending Noether's theorem by quantifying the asymmetry of quantum states, Nat. Commun. **5**, 3821 (2014).

[28] A. Peres and D. R. Terno, Quantum information and relativity theory, Rev. Mod. Phys. **76**, 93 (2004).

[29] M. Toroš, Constraints on the spontaneous collapse mechanism: Theory and experiments, Ph.D. thesis, Universitá degli Studi di Trieste, 2017.

[30] G. N. Fleming, A manifestly covariant Description of arbitrary dynamical variables in relativistic quantum mechanics, J. Math. Phys. **7**, 1959 (1966).

[31] E. Alicki, M. Fannes, and A. Verbeure, Unstable particles and the Poincare semigroup in quantum field theory, J. Phys. A: Math. Gen. **19**, 919 (1986)

[32] L. Diósi, Is there a relativistic Gorini-Kossakowski-Lindblad-Sudarshan master equation? Phys. Rev. D **106**, L051901 (2022).

[33] R. D. Sorkin, in *Directions in General Relativity*, Proceedings of the 1993 International Symposium, Maryland, edited by B. L. Hu and T. A. Jacobson (Cambridge University Press, Cambridge, 1993), Vol. 2, pp. 293–305.

[34] C. J. Fewster and R. Verch, Quantum fields and local measurements, Commun. Math. Phys. **378**, 851 (2020).

[35] H. Bostelmann, C. J. Fewster, and M. H. Ruep, Impossible measurements require impossible apparatus, Phys. Rev. D **103**, 025017 (2021).

[36] C. J. Fewster, I. Jubb, and M. H. Ruep, Asymptotic measurement schemes for every observable of a quantum field theory, Ann. Henri Poincaré **24**, 1137 (2023).

[37] V. Bargmann, Irreducible unitary representations of the Lorentz group, Ann. Math. **48**, 568 (1947).

[38] W. Tung, *Group Theory in Physics* (World Scientific, Singapore, 1985).