Universe as a nonlinear quantum simulation: Large-*n* limit of the central-spin model

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We investigate models of nonlinear qubit evolution based on mappings to an n-qubit central-spin model (CSM) in the large-n limit, where mean-field theory is exact. Extending a theorem of Erdös and Schlein [J. Stat. Phys. **134**, 859 (2009)], we establish that the CSM is rigorously dual to a nonlinear qubit when $n \rightarrow \infty$. The duality supports a type of nonlinear quantum computation in systems, such as a condensate, where a large number of ancillas couple symmetrically to a central qubit. It also enables a gate-model implementation of nonlinear quantum simulation with a rigorous error bound. Two variants of the model, with and without coupling between ancillas, map to effective models with different nonlinearity and symmetry. Without coupling the CSM simulates initial-condition nonlinearity, where the Hamiltonian is a linear combination of $\mathrm{tr}(\rho_0\sigma^x)\sigma^x$, $\mathrm{tr}(\rho_0\sigma^y)\sigma^y$, and $\mathrm{tr}(\rho_0\sigma^z)\sigma^z$, where σ^x, σ^y, and σ^z are Pauli matrices and ρ_0 is the initial density matrix. With symmetric ancillas coupling it simulates linear combinations of $\mathrm{tr}(\rho\sigma^x)\sigma^x$, $\mathrm{tr}(\rho\sigma^y)\sigma^y$, and $\mathrm{tr}(\rho\sigma^z)\sigma^z$, where ρ is the current state. This case can simulate qubit torsion, which has been shown by Abrams and Lloyd [Phys. Rev. Lett. **81**, 3992 (1998)] to enable an exponential speedup for state discrimination in an idealized setting. The duality discussed here might also be interesting from a quantum foundation perspective. There has long been interest in whether quantum mechanics might possess some type of small unobserved nonlinearity. If not, what is the principle prohibiting it? The duality implies that there is not a sharp distinction between universes evolving according to linear and nonlinear quantum mechanics: A one-qubit universe prepared in a pure state $|\varphi\rangle$ at the time of the big bang and symmetrically coupled to ancillas prepared in the same state would appear to evolve nonlinearly for any finite time $t > 0$ as long as there are exponentially many ancillas $n \gg \exp[O(t)]$.

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I. INTRODUCTION

There is a growing interest in exploring, as a purely theoretical question, the computational power of hypothetical forms of quantum nonlinearity [1–17]. One motivation is the intriguing paper by Abrams and Lloyd [2] arguing that evolution by certain nonlinear Schrödinger equations, in an idealized setting, would allow NP-complete problems to be solved efficiently. Meanwhile, there is a growing body of algorithms developed to simulate nonlinear problems, such as dissipative fluid flow, with a linear quantum computer [18–33]. Such algorithms provide a link between linear and nonlinear representations of the same problem and might teach us something about quantum nonlinearity itself. Here we explore this question in the context of a recent algorithm proposal by Lloyd *et al.* [23] for the quantum simulation of nonlinear differential equations. In their mean-field approach, nonlinear evolution of a quantum state $|\varphi\rangle$ is generated through coupling to many identical, weakly interacting copies of $|\varphi\rangle$, as in a Bose-Einstein condensate. In quantum many-body models for n indistinguishable atoms satisfying Bose statistics and prepared in a product state, it has been rigorously established that the nonlinear Gross-Pitaevskii equation for the one-particle density matrix becomes exact in the large-n or thermodynamic limit, i.e., the one-particle nonlinear Gross-Pitaevskii equation is dual to the n-particle linear Schrödinger equation when $n \rightarrow \infty$ [34–48]. As with bosons and some spin models [34,49], the mean-field approach of Ref. [23] is also expected to become exact in the large-n limit, but the precise form of this convergence has not been determined.

Here we extend the linear-nonlinear duality to n qubits subjected to arbitrary one-qubit and SWAP-symmetric two-qubit unitaries, a generalized central-spin model (CSM) [50–59]. The objectives are as follows. (i) Use mean-field theory to construct a rigorous duality between nonlinear qubits and a many-body CSM evolving under standard linear quantum mechanics. (ii) Provide an upper bound for the model error associated with the use of mean-field theory and investigate its breakdown at large times. (iii) Highlight the origin of qubit torsion (twisting of the Bloch ball) which leads to expansive dynamics, where the trace distance between a pair of close qubit states increases with time [1–6]. Section II defines the CSM. Section III employs the proof techniques of [40,60] to establish the duality. Section IV explains the origin of qubit torsion within this framework and contains the conclusions. Simulated examples are provided in Ref. [61].

II. CENTRAL-SPIN MODEL

A. Model definition

Let $\{1, 2, \ldots, n\}$ denote the vertices of a star graph of n qubits. Qubit 1 is the central qubit and the remaining ancilla qubits $\{2, \ldots, n\}$ are used to simulate a certain type of environment for the central qubit. However, this simulated environment is far from that of a random noisy bath. Instead, the ancilla qubits are initialized in the same pure state $|\varphi\rangle$ and they couple symmetrically to the central qubit. We consider a generalized homogeneous CSM with the Hamiltonian

$$H = \sum_{i=1}^{n} H_i^0 + \frac{1}{n-1}\left(\sum_{j>1}^{n} V_{1j} + \lambda \sum_{i>1}^{n-1}\sum_{j>i}^{n} V_{ij}\right),$$
$$[V_{ij}, \chi_{ij}] = 0, \ -1 \leqslant \lambda \leqslant 1. \tag{1}$$

The Hamiltonian H_i^0 acts as $H^0 \in \mathrm{su}(2)$ on qubit i and as the identity otherwise. Each qubit $i \in \{1, 2, \ldots, n\}$ sees the same single-qubit Hamiltonian H^0. This can be further expanded in a basis of Pauli matrices as $H_i^0 = \sum_{\mu=1}^{3} B_\mu \sigma_i^\mu$, where the field $\vec{B} = (B_1, B_2, B_3) \in \mathbb{R}^3$ has no dependence on the qubit index i. The interaction V_{ij} acts as $V \in \mathrm{su}(4)$ on the edge (i, j) and as the identity otherwise. In addition, we require V_{ij} to be SWAP symmetric, where SWAP is a two-qubit operator that acts on a product state as $\chi_{ij}|\alpha\rangle_i \otimes |\beta\rangle_j = |\beta\rangle_i \otimes |\alpha\rangle_j$. Note that the interaction in (1) has infinite range, favoring a mean-field description. A factor $O(1/n)$ is needed to control the large-n limit and is typical in large-n problems.

The parameter λ controls the ancilla-ancilla coupling and therefore affects the permutation symmetry of the Hamiltonian. We are mainly interested in $\lambda = 0$ but also consider cases with $|\lambda| \leqslant 1$. A CSM with $\lambda \neq 0$ might apply to two species of atomic qubits with inhomogeneous interactions. The case $\lambda = 1$ applies when all qubits are symmetrically coupled and the interaction graph is complete. Call this the complete graph (CG) model:

$$H_{\mathrm{CG}} = \sum_{i=1}^{n} H_i^0 + \frac{1}{n-1}\sum_{i=1}^{n-1}\sum_{j>i}^{n} V_{ij}. \tag{2}$$

The CG model (2) is a qubit analog of a weakly interacting monatomic Bose gas. Although we treat it as a special case of the CSM, they are distinct models with different symmetries.

A general SWAP-symmetric interaction can be obtained from the Cartan decomposition of su(4) [62], with which any $U \in \mathrm{SU}(4)$ can be written as an element of $\mathrm{SU}(2)_i \otimes \mathrm{SU}(2)_j$, followed by a symmetric entangling gate $\exp(-\mathrm{i}\sum_\mu J_\mu \sigma_i^\mu \otimes \sigma_j^\mu)$ and then a second $\mathrm{SU}(2)_i \otimes \mathrm{SU}(2)_j$. SWAP symmetry requires that the SU(2) unitaries in V_{ij} are the same on every qubit. They can therefore be generated by a single-qubit Hamiltonian H^0 and are not explicitly included in the interaction, which then takes the form

$$V_{ij} = \sum_{\mu=1}^{3} J_\mu \sigma_i^\mu \otimes \sigma_j^\mu, \quad \vec{J} = (J_1, J_2, J_3) \in \mathbb{R}^3, \tag{3}$$

where the couplings J_μ have no dependence on the edge label (i, j). The qubits interact via a vector coupling and have three coupling constants J_1, J_2, and J_3 instead of one as in the monatomic Bose gas case.

The operators H_i^0 and V_{ij} are time dependent and subject to the conditions that the quantities

$$\nu_0 := \sup_t \left\|H_i^0(t)\right\|_\infty, \quad J_0 := \sup_{\mu,t} |J_\mu(t)| \tag{4}$$

exist and are finite. Here $\|\cdot\|_\infty$ is the operator norm (relevant norm properties are collected in Appendix B). The quantity J_0 bounds the coupling, and hence the buildup of multiqubit correlation and corresponding breakdown of mean-field theory.

The time-evolution operator for the CSM is

$$U_t = T\exp\left(-\mathrm{i}\int_0^t H(\tau)d\tau\right),$$
$$\frac{dU_t}{dt} = -\mathrm{i}H(t)U_t, \quad U_0 = I, \tag{5}$$

where T is the time-ordering operator, I is the identity, $\mathrm{i} = \sqrt{-1}$, and factors of $\hbar$ are suppressed throughout this paper. We will also need the time-evolution operator for any single uncoupled qubit, which is

$$u_t = T\exp\left(-\mathrm{i}\int_0^t H^0(\tau)d\tau\right),$$
$$\frac{du_t}{dt} = -\mathrm{i}H^0(t)u_t, \quad u_0 = I. \tag{6}$$

The CSM with $\lambda = 0$ has a long history and many variants have been investigated [50–59]. Models with XXX symmetry [by which we mean $\vec{J} = (J_1, J_1, J_1)$] and some with XXZ symmetry [$\vec{J} = (J_1, J_1, J_3)$] are integrable and exactly solvable by the Bethe ansatz [50–54]. The $\lambda = 0$ CSM with Heisenberg interaction, XXX, has been studied extensively [50–57]. Time-dependent mean-field solutions in the XXX case have been obtained in terms of hyperelliptic functions [51]. Phase transitions have also been studied [58,59]. In this paper we study solutions of the CSM with XYZ interaction [arbitrary bounded $\vec{J} = (J_1, J_2, J_3)$], general λ, and high degrees of permutation symmetry. Specifically, we consider two levels of permutation symmetry.

Level S_{n-1}. This is the symmetry of the $\lambda \neq 1$ model, which includes the set of all permutations among ancillas $\{2, \ldots, n\}$. The symmetry group of the model then contains a subgroup of the symmetric group S_n (permutations on n qubits) that we simply call S_{n-1}.

Level S_n. The higher-symmetry case has full permutation symmetry, including the central qubit. This is the symmetry of the $\lambda = 1$ model. Now the symmetry group contains S_n.

We note that the initial condition $\rho(0)$ will respect both symmetries.

B. Linear picture: Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy

At time $t = 0$ the central qubit and ancilla are prepared in a product state

$$\rho(0) = |\varphi\rangle\langle\varphi|^{\otimes n}, \quad |\varphi\rangle = \varphi_0|0\rangle + \varphi_1|1\rangle,$$
$$\varphi_{0,1} \in \mathbb{C}, \quad |\varphi_0|^2 + |\varphi_1|^2 = 1. \tag{7}$$

This initial condition has complete permutation symmetry S_n. At later times $t > 0$ the state is $\rho(t) = U_t(|\varphi\rangle\langle\varphi|^{\otimes n})U_t^\dagger$ and the evolution equation is

$$\frac{d\rho}{dt} = -\mathrm{i}\left[\sum_{i=1}^{n} H_i^0, \rho\right] - \mathrm{i}\left[\sum_{j>1}^{n}\frac{V_{1j}}{n-1} + \lambda\sum_{i>1}^{n-1}\sum_{j>i}^{n}\frac{V_{ij}}{n-1}, \rho\right],$$
$$-1 \leqslant \lambda \leqslant 1. \tag{8}$$

Let $\mathrm{tr}_i(\cdot) = \sum_{x=0,1}\langle x|\cdot|x\rangle_i$ denote the partial trace over the Hilbert space of qubit i. The density matrix for the central qubit is $\rho_1(t) = \mathrm{tr}_{>1}[\rho(t)]$, where $\mathrm{tr}_{>i}(\cdot) := \mathrm{tr}_{i+1}\mathrm{tr}_{i+2}\cdots\mathrm{tr}_n(\cdot)$. Similarly, $\rho_2(t) = \mathrm{tr}_1[\rho_{12}(t)]$, where $\rho_{12} = \mathrm{tr}_{>2}[\rho(t)]$. Then we have

$$\frac{d\rho_1}{dt} = -\mathrm{i}[H^0, \rho_1] - \mathrm{i}\,\mathrm{tr}_{>1}\left[\sum_{j>1}^{n}\frac{V_{1j}}{n-1} + \lambda\sum_{i>1}^{n-1}\sum_{j>i}^{n}\frac{V_{ij}}{n-1}, \rho\right] \tag{9}$$

$$= -\mathrm{i}[H^0, \rho_1] - \mathrm{i}\,\mathrm{tr}_{>1}\left[\sum_{j>1}^{n}\frac{V_{1j}}{n-1}, \rho\right], \tag{10}$$

$$\frac{d\rho_2}{dt} = -\mathrm{i}[H^0, \rho_2] - \mathrm{i}\,\mathrm{tr}_1\mathrm{tr}_3\cdots\mathrm{tr}_n \times\left[\sum_{j>1}^{n}\frac{V_{1j}}{n-1} + \lambda\sum_{i>1}^{n-1}\sum_{j>i}^{n}\frac{V_{ij}}{n-1}, \rho\right] \tag{11}$$

$$= -\mathrm{i}[H^0, \rho_2] - \mathrm{i}\,\mathrm{tr}_1\mathrm{tr}_3\cdots\mathrm{tr}_n \times\left[\frac{V_{12}}{n-1} + \lambda\sum_{j>2}^{n}\frac{V_{2j}}{n-1}, \rho\right], \tag{12}$$

using (A1) and (A5). Next we assume S_{n-1} ancilla permutation symmetry to obtain

$$\frac{d\rho_1}{dt} = -\mathrm{i}[H^0, \rho_1] - \mathrm{i}\,\mathrm{tr}_2([V_{12}, \rho_{12}]). = -\mathrm{i}\sum_{\mu=1}^{3}B_\mu[\sigma_1^\mu, \rho_1] - \mathrm{i}\sum_{\mu=1}^{3}J_\mu[\sigma_1^\mu, \mathrm{tr}_2(\rho_{12}\sigma_2^\mu)], \tag{13}$$

$$\frac{d\rho_2}{dt} = -\mathrm{i}[H^0, \rho_2] - \mathrm{i}\frac{\mathrm{tr}_1[V_{12}, \rho_{12}] + \lambda(n-2)\mathrm{tr}_3[V_{23}, \rho_{23}]}{n-1} = -\mathrm{i}\sum_{\mu=1}^{3}B_\mu[\sigma_2^\mu, \rho_2] - \mathrm{i}\sum_{\mu=1}^{3}\frac{J_\mu}{n-1}[\sigma_2^\mu, \mathrm{tr}_1(\rho_{12}\sigma_1^\mu) + \lambda(n-2)\mathrm{tr}_3(\rho_{23}\sigma_3^\mu)], \tag{14}$$

where $\vec{B}$ and $\vec{J}$ are possibly time dependent. From these we obtain

$$\rho_1(t) = u_t\left(|\varphi\rangle\langle\varphi| - \mathrm{i}\sum_\mu\int_0^t d\tau\, J_\mu u_\tau^\dagger[\sigma_1^\mu, \mathrm{tr}_2(\rho_{12}\sigma_2^\mu)]u_\tau\right)u_t^\dagger,$$

$$\rho_2(t) = u_t\left(|\varphi\rangle\langle\varphi| - \mathrm{i}\sum_\mu\int_0^t d\tau\frac{J_\mu}{n-1}u_\tau^\dagger[\sigma_2^\mu, \mathrm{tr}_1(\rho_{12}\sigma_1^\mu) + \lambda(n-2)\mathrm{tr}_3(\rho_{23}\sigma_3^\mu)]u_\tau\right)u_t^\dagger, \tag{15}$$

where $\rho_{23} = \mathrm{tr}_1(\rho_{123}) = \mathrm{tr}_1(\mathrm{tr}_{>3}\rho)$. Here u_t is the time-evolution operator (6) for a single uncoupled qubit. The equations for $\rho_{1,2}$ are quantum Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy equations [63] for the generalized CSM.

C. Nonlinear picture: Mean-field theory

Theorem 1 in Sec. III relates the solutions of (13) and (14) to that of a mean-field-theory model. To construct that model, assume that the order parameter

$$\vec{m}_i := \langle\vec{\sigma}_i\rangle = \mathrm{tr}(\omega\vec{\sigma}_i), \quad i\in\{1, 2, \ldots, n\} \tag{16}$$

is nonvanishing, where the expectation is with respect to some (possibly time-dependent) state ω. To find equilibrium properties, ω is assumed to be a thermal state $e^{-\beta H}/\mathrm{tr}e^{-\beta H}$ at temperature $1/\beta$. Here we assume that ω is arbitrary (to be specified) and time dependent. Expanding the Hamiltonian (1) in powers of fluctuations $\delta\sigma_i^\mu = \sigma_i^\mu - m_i^\mu$ to first order leads to

$$H = \sum_{i=1}^{n}H_i^0 + \sum_\mu\frac{J_\mu}{n-1}\sum_{j>1}^{n}(m_1^\mu\sigma_j^\mu + \sigma_1^\mu m_j^\mu) + \lambda\sum_\mu\frac{J_\mu}{n-1}\sum_{i>1}^{n-1}\sum_{j>i}^{n}(m_i^\mu\sigma_j^\mu + \sigma_i^\mu m_j^\mu) + \Delta E, \tag{17}$$

where

$$\Delta E = -\sum_\mu\sum_{i>1}^{n}\frac{J_\mu m_1^\mu m_i^\mu}{n-1} - \lambda\sum_\mu\sum_{i>1}^{n-1}\sum_{j>i}^{n}\frac{J_\mu m_i^\mu m_j^\mu}{n-1}. \tag{18}$$

The background energy ΔE has no affect on the dynamics but contributes to thermodynamic properties such as the free energy.

In the following section we construct a mean-field theory for CSM solutions with S_{n-1} symmetry. The result is a pair of coupled equations of motion for the mean-field state X of the central qubit and the mean-field state Y of an ancilla (qubit 2). Because the equations of motion are coupled, they must be solved together. Hence, the dual mean-field model is a two-qubit model in a separable state $X\otimes Y$. This is the primary mean-field theory for the CSM. An exception occurs if $\lambda = 1$: In this case, assuming $X(0) = Y(0) = |\varphi\rangle\langle\varphi|$, the coupled equations of motion yield $X(t) = Y(t)$ for all time, leading to a solution with S_n symmetry. The mean-field theory for this case is also discussed below. The CSM with $\lambda = 1$ preserves the S_n symmetry of the initial condition, leading to a single-qubit dual model with self-interaction.

1. *Symmetry S_{n-1}*

If the CSM exhibits S_{n-1} symmetry, the order parameter satisfies $\vec{m}_2 = \vec{m}_3 = \cdots = \vec{m}_n$. Then from (17) we obtain

$$H = \sum_{i=1}^{n}H_i^0 + \sum_\mu J_\mu m_2^\mu\sigma_1^\mu + \sum_\mu\frac{J_\mu m_1^\mu}{n-1}\sum_{i>1}^{n}\sigma_i^\mu + \lambda\sum_\mu\frac{J_\mu m_2^\mu}{n-1}\sum_{i>1}^{n-1}\sum_{j>i}^{n}(\sigma_i^\mu + \sigma_j^\mu) + \Delta E$$

$$= \sum_{i=1}^{n}H_i^0 + \sum_\mu J_\mu m_2^\mu\sigma_1^\mu + \sum_\mu\frac{J_\mu m_1^\mu + \lambda(n-2)J_\mu m_2^\mu}{n-1}\sum_{i>1}^{n}\sigma_i^\mu + \Delta E, \tag{19}$$

where

$$\Delta E = -\sum_\mu J_\mu m_1^\mu m_2^\mu - \frac{\lambda}{2}\sum_\mu(n-2)J_\mu m_2^\mu m_2^\mu. \tag{20}$$

In the mean-field approximation (neglecting quadratic fluctuations) the qubits are decoupled and the mean-field Hamiltonians for qubits 1 and 2 are

$$H_1^{\rm eff} = H^0 + \sum_\mu J_\mu {\rm tr}(Y\sigma^\mu)\sigma_1^\mu, \tag{21}$$

$$H_2^{\rm eff} = H^0 + \sum_\mu J_\mu \frac{{\rm tr}(X\sigma^\mu) + \lambda(n-2){\rm tr}(Y\sigma^\mu)}{n-1}\sigma_2^\mu, \tag{22}$$

where X and Y are the mean-field density matrices for qubits 1 and 2, respectively. Here we have set $\omega = X \otimes Y$, the current mean-field state of qubits 1 and 2. The evolution equations for X and Y are

$$\frac{dX}{dt} = -{\rm i}[H^0, X] - {\rm i}\sum_{\mu=1}^3 J_\mu {\rm tr}(Y\sigma^\mu)[\sigma^\mu, X], \tag{23}$$

$$\frac{dY}{dt} = -{\rm i}[H^0, Y] - {\rm i} \times \sum_{\mu=1}^3 J_\mu \frac{{\rm tr}(X\sigma^\mu) + \lambda(n-2){\rm tr}(Y\sigma^\mu)}{n-1}[\sigma^\mu, Y] \tag{24}$$

$$\approx -{\rm i}[H^0, Y] - {\rm i}\lambda \sum_\mu J_\mu {\rm tr}(Y\sigma^\mu)[\sigma^\mu, Y], \tag{25}$$

where (25) applies in the large-n limit. The initial conditions are

$$X(0) = Y(0) = |\varphi\rangle\langle\varphi|. \tag{26}$$

Next, using (6), we obtain

$$X(t) = u_t\left(|\varphi\rangle\langle\varphi| - {\rm i}\sum_\mu \int_0^t d\tau\, J_\mu {\rm tr}(Y\sigma^\mu) u_\tau^\dagger([\sigma^\mu, X])u_\tau\right)u_t^\dagger,$$

$$Y(t) = u_t\left(|\varphi\rangle\langle\varphi| - {\rm i}\sum_\mu \int_0^t d\tau\, J_\mu \frac{{\rm tr}(X\sigma^\mu) + \lambda(n-2){\rm tr}(Y\sigma^\mu)}{n-1} u_\tau^\dagger([\sigma^\mu, Y])u_\tau\right)u_t^\dagger. \tag{27}$$

The nonlinear evolution equations (23) and (24) are dual to the linear BBGKY equations (13) and (14) in the large-n limit in the sense that $X = \rho_1$ and $Y = \rho_2$ in this limit. This is because Theorem 1 implies $\lim_{n\to\infty} \|X - \rho_1\| \to 0$ and $\lim_{n\to\infty} \|Y - \rho_2\| \to 0$.

2. Symmetry S_n

If the CSM exhibits S_n symmetry, the order parameter satisfies $\vec{m}_1 = \vec{m}_2 = \cdots = \vec{m}_n$. For $\vec{m}_1$ and $\vec{m}_2$ to be equal, we must have $X = Y$,[1] indicating symmetry between the central and ancilla qubits. Here we use the mean-field equations (23) and (24) to investigate S_n symmetry as a special case of S_{n-1} symmetry. First transform to

$$\rho_{\rm ave} := \frac{X+Y}{2}, \quad \rho_\Delta := \frac{X-Y}{2}. \tag{28}$$

While $\rho_{\rm ave}$ is a state (positive-semidefinite matrix with unit trace), ρ_Δ is not. For large n,

$$\frac{d\rho_{\rm ave}}{dt} = -{\rm i}[H^0, \rho_{\rm ave}] - {\rm i}\sum_\mu J_\mu {\rm tr}(\rho_{\rm ave}\sigma^\mu - \rho_\Delta\sigma^\mu) \times \left[\sigma^\mu, \rho_{\rm ave} + (\lambda - 1)\frac{\rho_{\rm ave} - \rho_\Delta}{2}\right], \tag{29}$$

$$\frac{d\rho_\Delta}{dt} = -{\rm i}[H^0, \rho_\Delta] - {\rm i}\sum_\mu J_\mu {\rm tr}(\rho_{\rm ave}\sigma^\mu - \rho_\Delta\sigma^\mu) \times \left[\sigma^\mu, \rho_\Delta + (1 - \lambda)\frac{\rho_{\rm ave} - \rho_\Delta}{2}\right], \tag{30}$$

with initial conditions $\rho_{\rm ave}(0) = |\varphi\rangle\langle\varphi|$ and $\rho_\Delta(0) = 0$. At time zero, $\rho_\Delta = 0$, so the system initially possesses S_n symmetry. If $\lambda \neq 1$, the initial rate of change $(d\rho_\Delta/dt)_0 = -{\rm i}(\frac{1-\lambda}{2})\sum_\mu J_\mu {\rm tr}(\rho_{\rm ave}\sigma^\mu)[\sigma^\mu, \rho_{\rm ave}]$ is nonzero, breaking the symmetry between X and Y. However ρ_Δ remains zero if $\lambda = 1$, preserving the S_n symmetry and leading to a single-qubit mean-field theory for X with self-interaction:

$$\frac{dX}{dt} = -{\rm i}[H^0, X] - {\rm i}\sum_{\mu=1}^3 J_\mu {\rm tr}(X\sigma^\mu)[\sigma^\mu, X]. \tag{31}$$

[1]This is because, for a qubit, the order parameter $\vec{m} = {\rm tr}(\rho\vec{\sigma})$ uniquely specifies the state $\rho = (I + \vec{m}\cdot\vec{\sigma})/2$.

III. LARGE-n LIMIT

In this section we establish the duality between the linear BBGKY equations and the nonlinear mean-field theory in the large-n limit of the generalized CSM, following the proof techniques of [40,60]. Our work also builds on recent papers by Fernengel and Drossel [64] and Kłobus *et al.* [65], who studied nonlinear mean-field dynamics of related spin models. The following are some features of our analysis. (1) In contrast to particle models, we do not assume indistinguishable particles with Bose or Fermi statistics. (2) The $\lambda = 0$ model has reduced permutation symmetry and no interaction between ancilla. Full permutational symmetry is broken, but the ancilla qubits $\{2, \ldots, n\}$ remain identical. (3) Qubits interact via an arbitrary $V \in {\rm su}(4)$. (4) The interaction is long ranged and does not decay with distance. (5) All terms in the Hamiltonian are assumed to be time dependent.

Theorem 1 (extended Erdős-Schlein theorem [40]). Let $X(t)$ and $Y(t)$ be solutions of the coupled nonlinear evolution equations (23) and (24) [or (25)] for the n-qubit generalized CSM (1), with initial conditions $X(0) = Y(0) = |\varphi\rangle\langle\varphi|$, where $|\varphi\rangle = \varphi_0|0\rangle + \varphi_1|1\rangle$, $\varphi_{0,1} \in \mathbb{C}$, and $|\varphi_0|^2 + |\varphi_1|^2 = 1$. Also let $\rho_1 = {\rm tr}_{>1}(\rho)$ and $\rho_2 = {\rm tr}_1(\rho_{12})$ be the exact reduced density matrices on qubits 1 and 2, respectively (partial trace notation is defined in Sec. II B). Then the distance in trace norm between the mean field and exact state satisfies

$$\|X(t)-\rho_1(t)\|_1 \leqslant 4\frac{e^{12(1+|\lambda|)J_0 t}-1}{n(1+|\lambda|)}, \quad t\geqslant 0 \tag{32}$$

and

$$\|Y(t)-\rho_2(t)\|_1 \leqslant 4\frac{e^{12(1+|\lambda|)J_0 t}-1}{n(1+|\lambda|)}, \quad t\geqslant 0, \tag{33}$$

where J_0 is an interaction strength bound defined in (4). The same upper bound applies to both X and Y. The inequalities imply that, for any fixed $t \geqslant 0$,

$$\lim_{n\to\infty} \|X(t)-\rho_1(t)\|_1 = 0, \tag{34}$$

$$\lim_{n\to\infty} \|Y(t)-\rho_2(t)\|_1 = 0, \tag{35}$$

establishing the duality.

The proof of Theorem 1 uses the following lemmas.

Lemma 1 (Lieb-Robinson bound [40,66]). For any $k \in \{1,\dots,n-1\}$, let $A_{1,\dots,k} \in \mathbb{C}^{2^n\times 2^n}$ and $B_{k+1} \in \mathbb{C}^{2^n\times 2^n}$ be Hermitian bounded linear operators (observables) with support exclusively in subsets $\{1,2,\dots,k\}$ and $\{k+1\}$, respectively, of the n-qubit generalized CSM (1). Here $A_{1,\dots,k}$ acts nontrivially on the first k qubits $\{1,2,\dots,k\}$ (including the central qubit) and as the identity elsewhere. Similarly, B_{k+1} acts nontrivially on qubit $k+1$ only. Let

$$\Gamma_{kt} := \sup_{A\neq 0, B\neq 0} \frac{\|[U_t^\dagger A_{1,\dots,k}U_t, B_{k+1}]\|_\infty}{\|A_{1,\dots,k}\|_\infty \|B_{k+1}\|_\infty}, \tag{36}$$

where the supremum is over the set of all bounded linear operators $A_{1,\dots,k}$ with support on qubits $\{1,\dots,k\}$ such that $\|A_{1,\dots,k}\|_\infty \neq 0$ and over all bounded linear operators B_{k+1} with support on qubit $k+1$ such that $\|B_{k+1}\|_\infty \neq 0$. Then

$$\Gamma_{kt} \leqslant 2 \tag{37}$$

holds for any $k = 1,2,\dots,n-1$. Furthermore, for $k = 1, 2$,

$$\Gamma_{kt} \leqslant 2\frac{e^{6(1+|\lambda|)J_0 t}-1}{n-1}, \tag{38}$$

where J_0 is defined in (4).

The quantity Γ_{kt} is a measure of the largest possible correlation between a cluster containing the first k qubits (including the central) and qubit $k+1$, due to their interaction. Only cases $k = 1, 2$ are required below. The bound (37) shows that correlation measured this way does not blow up at long times, in contrast with (38). Therefore, the interesting regime occurs when the bound in (38) is small, namely, $n \gg e^{6(1+|\lambda|)J_0 t}$.

Proof. The bound (37) follows from unitary invariance and submultiplicativity of the Schatten p-norm (see Appendix B). To obtain (38), transform to a representation where time evolution is generated exclusively by the cross interactions

$$W^{(k)} := \frac{1}{n-1}\left(\sum_{j=k+1}^{n} V_{1j} + \lambda \sum_{i=2}^{k}\sum_{j=k+1}^{n} V_{ij}\right) \tag{39}$$

between the k-qubit cluster on which $A_{1,\dots,k}$ acts and its environment. In particular,

$$W^{(k=1)} = \frac{V_{12}}{n-1} + \frac{V_{13}+\cdots+V_{1n}}{n-1}, \tag{40}$$

independent of λ, and

$$W^{(k=2)} = \frac{V_{13}+\lambda V_{23}}{n-1} + \frac{V_{14}+\cdots+V_{1n}+\lambda(V_{24}+\cdots+V_{2n})}{n-1}. \tag{41}$$

In these expressions, terms that do not commute with B_{k+1} have been isolated. The first step of the proof is to note that

$$\frac{d}{dt}(U_t^\dagger S_{kt} A_{1,\dots,k} S_{kt}^\dagger U_t) = \mathrm{i}[U_t^\dagger W^{(k)} U_t, U_t^\dagger S_{kt} A_{1,\dots,k} S_{kt}^\dagger U_t] = \mathrm{i}[\mathbb{W}^{(k)}, U_t^\dagger S_{kt} A_{1,\dots,k} S_{kt}^\dagger U_t], \tag{42}$$

where, for any $k \in \{1,2,\dots,n-1\}$,

$$H^{(k)} = H - W^{(k)}, \quad S_{kt} = T\exp\left(-\mathrm{i}\int_0^t H^{(k)}(\tau)d\tau\right), \quad \frac{dS_{kt}}{dt} = -\mathrm{i}H^{(k)}(t)S_{kt}, \quad S_{k0} = I, \tag{43}$$

$$\mathbb{W}^{(k)} = U_t^\dagger W^{(k)} U_t, \quad \mathbb{S}_{kt} = T\exp\left(\mathrm{i}\int_0^t \mathbb{W}^{(k)}(\tau)d\tau\right), \quad \frac{d\mathbb{S}_{kt}}{dt} = \mathrm{i}\mathbb{W}^{(k)}(t)\mathbb{S}_{kt}, \quad \mathbb{S}_{k0} = I. \tag{44}$$

The time-evolution operators S_{kt} and $\mathbb{S}_{kt}$ are generated by $-\mathrm{i}H^{(k)}$ and $\mathrm{i}\,\mathbb{W}^{(k)}$, respectively. The Hamiltonian $H^{(k)}$ has the cross interactions $W^{(k)}$ between the k-qubit cluster and its surroundings removed. Next let $f_{kt} := [U_t^\dagger S_{kt} A_{1,\dots,k} S_{kt}^\dagger U_t, B_{k+1}]$. Then

$$\frac{df_{kt}}{dt} = \mathrm{i}[[\mathbb{W}^{(k)}, U_t^\dagger S_{kt} A_{1,\dots,k} S_{kt}^\dagger U_t], B_{k+1}] = \mathrm{i}[\mathbb{W}^{(k)}, f_{kt}] + c_{kt}, \tag{45}$$

where $c_{kt} = \mathrm{i}[[\mathbb{W}^{(k)}, B_{k+1}], U_t^\dagger S_{kt} A_{1,\dots,k} S_{kt}^\dagger U_t]$. We then have $\frac{d}{dt}(\mathbb{S}_{kt}^\dagger f_{kt} \mathbb{S}_{kt}) = \mathbb{S}_{kt}^\dagger c_{kt} \mathbb{S}_{kt}$ and $\mathbb{S}_{kt}^\dagger f_{kt}\mathbb{S}_{kt} = \int_0^t \mathbb{S}_{k\tau}^\dagger c_{k\tau}\mathbb{S}_{k\tau} d\tau$, because $f_{k0} = [A_{1,\dots,k}, B_{k+1}] = 0$. Therefore,

$$\|[U_t^\dagger S_{kt} A_{1,\dots,k} S_{kt}^\dagger U_t, B_{k+1}]\|_\infty \leqslant \int_0^t \|c_{k\tau}\|_\infty d\tau \leqslant 2\|A_{1,\dots,k}\|_\infty \int_0^t \|[\mathbb{W}^{(k)}(\tau), B_{k+1}]\|_\infty d\tau. \tag{46}$$

Separating out terms in $\mathbb{W}^{(k)}$ that might become large at short times due to noncommutativity with B_{k+1} and using $\|\vec{\sigma}_i \cdot \vec{\sigma}_j\|_\infty = 3$ leads to

$$\Gamma_{1t} \leqslant \frac{12 J_0 t}{n-1} + 6 J_0 \int_0^t dt_1 \Gamma_{2t_1}, \tag{47}$$

$$\Gamma_{2t} \leqslant \frac{12(1+|\lambda|)J_0 t}{n-1} + 6(1+|\lambda|)J_0 \int_0^t dt_1 \Gamma_{2t_1}. \tag{48}$$

First we solve (48) iteratively, obtaining a bound for Γ_{2t}. Then we use (47) to bound Γ_{1t}. After q iterations we have

$$\Gamma_{2t} \leqslant \frac{2}{n-1} \sum_{\ell=1}^{q} \frac{[6(1+|\lambda|)J_0 t]^\ell}{\ell!} + [6(1+|\lambda|)J_0]^q \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{q-1}} dt_q \Gamma_{2t_q} \tag{49}$$

or

$$\Gamma_{2t} \leqslant \frac{2}{n-1} \sum_{\ell=1}^{q} \frac{[6(1+|\lambda|)J_0 t]^\ell}{\ell!} + 2\frac{[6(1+|\lambda|)J_0 t]^q}{q!}, \tag{50}$$

using (37). In the large-q limit,

$$\Gamma_{2t} \leqslant 2\frac{e^{6(1+|\lambda|)J_0 t} - 1}{n-1}. \tag{51}$$

Inserting this into (47) and integrating leads to

$$\Gamma_{1t} \leqslant \frac{2}{1+|\lambda|} \frac{e^{6(1+|\lambda|)J_0 t} - 1}{n-1} \leqslant 2\frac{e^{6(1+|\lambda|)J_0 t} - 1}{n-1}, \tag{52}$$

as required. ■

Lemma 2. Let A_1 and B_2 be Hermitian observables with support exclusively on qubits 1 and 2, respectively, of the n-qubit generalized CSM (1), and let

$$\langle A_1\rangle := \langle\varphi|^{\otimes n} U_t^\dagger A_1 U_t |\varphi\rangle^{\otimes n}, \quad \langle B_2\rangle := \langle\varphi|^{\otimes n} U_t^\dagger B_2 U_t |\varphi\rangle^{\otimes n}, \quad \langle A_1 B_2\rangle := \langle\varphi|^{\otimes n} U_t^\dagger A_1 B_2 U_t |\varphi\rangle^{\otimes n}$$

be their expectations in the exact many-body state $U_t|\varphi\rangle^{\otimes n}$. Here $|\varphi\rangle = \varphi_0|0\rangle + \varphi_1|1\rangle$ is a pure single-qubit state with $\varphi_{0,1} \in \mathbb{C}$ and $|\varphi_0|^2 + |\varphi_1|^2 = 1$, and U_t is the exact time-evolution operator (5) of the CSM. Then

$$C_t := \sup_{A\neq 0, B\neq 0} \frac{|\langle A_1 B_2\rangle - \langle A_1\rangle\langle B_2\rangle|}{\|A_1\|_\infty \|B_2\|_\infty} \leqslant 4\frac{e^{12(1+|\lambda|)J_0 t} - 1}{n-1}, \tag{53}$$

where the supremum is over the set of all bounded linear operators A_1 and B_2 with support on qubits 1 and 2, respectively, such that $\|A_1\|_\infty$ and $\|B_2\|_\infty$ are nonzero, and J_0 is defined in (4).

Proof. The proof works by rewriting the correlation function on the left-hand side of (53) in terms of commutators and using Lemma 1. First note the equality

$$I^{\otimes n} = |\varphi\rangle\langle\varphi|^{\otimes n} + \sum_{j=1}^{n} |\varphi\rangle\langle\varphi|_1 \otimes |\varphi\rangle\langle\varphi|_2 \otimes \cdots \otimes |\varphi\rangle\langle\varphi|_{j-1} \otimes (I - |\varphi\rangle\langle\varphi|)_j \otimes I_{j+1} \otimes \cdots \otimes I_n,$$

where I is the two-dimensional identity. Then insert $I^{\otimes n}$ in

$$\langle\varphi|^{\otimes n} U_t^\dagger (A_1 \otimes B_2) U_t |\varphi\rangle^{\otimes n} = \langle\varphi|^{\otimes n} (U_t^\dagger A_1 U_t)(U_t^\dagger B_2 U_t)|\varphi\rangle^{\otimes n} \tag{54}$$

to obtain

$$\begin{aligned}
&\langle\varphi|^{\otimes n} U_t^\dagger A_1 B_2 U_t |\varphi\rangle^{\otimes n} - \langle\varphi|^{\otimes n} U_t^\dagger A_1 U_t |\varphi\rangle^{\otimes n} \langle\varphi|^{\otimes n} U_t^\dagger B_2 U_t |\varphi\rangle^{\otimes n} \\
&= \sum_{j=1}^{n} \langle\varphi|^{\otimes n} U_t^\dagger A_1 U_t (|\varphi\rangle\langle\varphi|_1 \otimes \cdots \otimes |\varphi\rangle\langle\varphi|_{j-1}) \otimes (I - |\varphi\rangle\langle\varphi|)_j U_t^\dagger B_2 U_t |\varphi\rangle^{\otimes n} \\
&= \sum_{j=1}^{n} \mathrm{tr}[|\varphi\rangle\langle\varphi|^{\otimes n} U_t^\dagger A_1 U_t (|\varphi\rangle\langle\varphi|_1 \otimes \cdots \otimes |\varphi\rangle\langle\varphi|_{j-1}) \otimes (I - |\varphi\rangle\langle\varphi|)_j U_t^\dagger B_2 U_t]
\end{aligned} \tag{55}$$

and

$$\begin{aligned}
&|\langle\varphi|^{\otimes n} U_t^\dagger A_1 B_2 U_t |\varphi\rangle^{\otimes n} - \langle\varphi|^{\otimes n} U_t^\dagger A_1 U_t |\varphi\rangle^{\otimes n} \langle\varphi|^{\otimes n} U_t^\dagger B_2 U_t |\varphi\rangle^{\otimes n}| \\
&\leqslant \sum_{j=1}^{n} |\mathrm{tr}[|\varphi\rangle\langle\varphi|^{\otimes n} U_t^\dagger A_1 U_t (|\varphi\rangle\langle\varphi|_1 \otimes \cdots \otimes |\varphi\rangle\langle\varphi|_{j-1}) \otimes (I - |\varphi\rangle\langle\varphi|)_j U_t^\dagger B_2 U_t]|.
\end{aligned} \tag{56}$$

Next isolate the first two terms in the summation and rewrite in terms of commutators,

$$\begin{aligned}
&|\langle\varphi|^{\otimes n}U_t^\dagger A_1 B_2 U_t|\varphi\rangle^{\otimes n} - \langle\varphi|^{\otimes n}U_t^\dagger A_1 U_t|\varphi\rangle^{\otimes n}\langle\varphi|^{\otimes n}U_t^\dagger B_2 U_t|\varphi\rangle^{\otimes n}|\\
&\quad\leqslant |\mathrm{tr}(|\varphi\rangle\langle\varphi|^{\otimes n}U_t^\dagger A_1 U_t[I-|\varphi\rangle\langle\varphi|_1, U_t^\dagger B_2 U_t])|\\
&\qquad + |\mathrm{tr}(|\varphi\rangle\langle\varphi|^{\otimes n}[U_t^\dagger A_1 U_t, I-|\varphi\rangle\langle\varphi|_2]|\varphi\rangle\langle\varphi|_1 U_t^\dagger B_2 U_t)|\\
&\qquad + \sum_{j>2}^{n}|\mathrm{tr}(|\varphi\rangle\langle\varphi|^{\otimes n}[U_t^\dagger A_1 U_t, I-|\varphi\rangle\langle\varphi|_j]|\varphi\rangle\langle\varphi|_1\otimes\cdots\otimes|\varphi\rangle\langle\varphi|_{j-1}[I-|\varphi\rangle\langle\varphi|_j, U_t^\dagger B_2 U_t])|,
\end{aligned}\tag{57}$$

using the property that $I-|\varphi\rangle\langle\varphi|_i = (I-|\varphi\rangle\langle\varphi|_i)^2$ annihilates the initial state $|\varphi\rangle^{\otimes n}$. This leads to

$$\begin{aligned}
&|\langle\varphi|^{\otimes n}U_t^\dagger A_1 B_2 U_t|\varphi\rangle^{\otimes n} - \langle\varphi|^{\otimes n}U_t^\dagger A_1 U_t|\varphi\rangle^{\otimes n}\langle\varphi|^{\otimes n}U_t^\dagger B_2 U_t|\varphi\rangle^{\otimes n}|\\
&\quad\leqslant \||\varphi\rangle\langle\varphi|^{\otimes n}U_t^\dagger A_1 U_t[I-|\varphi\rangle\langle\varphi|_1, U_t^\dagger B_2 U_t]\|_1\\
&\qquad + \||\varphi\rangle\langle\varphi|^{\otimes n}[U_t^\dagger A_1 U_t, I-|\varphi\rangle\langle\varphi|_2]|\varphi\rangle\langle\varphi|_1 U_t^\dagger B_2 U_t\|_1\\
&\qquad + \sum_{j>2}^{n}\||\varphi\rangle\langle\varphi|^{\otimes n}[U_t^\dagger A_1 U_t, I-|\varphi\rangle\langle\varphi|_j]|\varphi\rangle\langle\varphi|_1\otimes\cdots\otimes|\varphi\rangle\langle\varphi|_{j-1}[I-|\varphi\rangle\langle\varphi|_j, U_t^\dagger B_2 U_t]\|_1\\
&\quad\leqslant \||\varphi\rangle\langle\varphi|^{\otimes n}U_t^\dagger A_1 U_t\|_1\|[I-|\varphi\rangle\langle\varphi|_1, U_t^\dagger B_2 U_t]\|_\infty\\
&\qquad + \||\varphi\rangle\langle\varphi|^{\otimes n}[U_t^\dagger A_1 U_t, I-|\varphi\rangle\langle\varphi|_2]\|_\infty\||\varphi\rangle\langle\varphi|_1 U_t^\dagger B_2 U_t\|_1\\
&\qquad + \sum_{j>2}^{n}\||\varphi\rangle\langle\varphi|^{\otimes n}[U_t^\dagger A_1 U_t, I-|\varphi\rangle\langle\varphi|_j]\|_1\||\varphi\rangle\langle\varphi|_1\otimes\cdots\otimes|\varphi\rangle\langle\varphi|_{j-1}[I-|\varphi\rangle\langle\varphi|_j, U_t^\dagger B_2 U_t]\|_\infty\\
&\quad\leqslant 2\|A_1\|_\infty\|B_2\|_\infty\Gamma_{1t} + (n-2)\|A_1\|_\infty\|B_2\|_\infty\Gamma_{1t}^2.
\end{aligned}\tag{58}$$

Here we have used the fact that both the operator and trace norms of a state (positive-semidefinite matrix with unit trace) are equal to 1. Then

$$\begin{aligned}
&|\langle\varphi|^{\otimes n}U_t^\dagger A_1 B_2 U_t|\varphi\rangle^{\otimes n} - \langle\varphi|^{\otimes n}U_t^\dagger A_1 U_t|\varphi\rangle^{\otimes n}\langle\varphi|^{\otimes n}U_t^\dagger B_2 U_t|\varphi\rangle^{\otimes n}|\\
&\quad\leqslant \|A_1\|_\infty\|B_2\|_\infty[2\Gamma_{1t} + (n-1)\Gamma_{1t}^2] \leqslant 4\|A_1\|_\infty\|B_2\|_\infty\frac{e^{12(1+|\lambda|)J_0 t}-1}{n-1}.
\end{aligned}\tag{59}$$

Hence, for any pair of observables A_1 and B_2 with nonvanishing operator norms, it follows that $\frac{|\langle A_1 B_2\rangle - \langle A_1\rangle\langle B_2\rangle|}{\|A_1\|_\infty\|B_2\|_\infty} \leqslant 4\frac{e^{12(1+|\lambda|)J_0 t}-1}{n-1}$, leading to (53), as required. ■

Next we turn to the proof of Theorem 1.

Proof. Let A_1 and B_2 be observables for qubits 1 and 2, respectively. Use (15) and (27) to obtain

$$\begin{aligned}
|\mathrm{tr}_1[A_1 X_1(t) - A_1\rho_1(t)]| &= \left|\sum_{\mu=1}^{3}\int_0^t d\tau\, J_\mu \mathrm{tr}_1\big\{(u_\tau u_t^\dagger A_1 u_t u_\tau^\dagger)\big[\sigma_1^\mu, \mathrm{tr}_2\big[(X_1\otimes Y_2 - \rho_{12})\sigma_2^\mu\big]\big]\big\}\right|\\
&= \left|\sum_{\mu}\int_0^t d\tau\, J_\mu \mathrm{tr}_1\mathrm{tr}_2\big\{(u_\tau u_t^\dagger A_1 u_t u_\tau^\dagger)\big[\sigma_1^\mu, (X_1\otimes Y_2 - \rho_{12})\sigma_2^\mu\big]\big\}\right| \qquad (60)\\
&= \left|\sum_{\mu}\int_0^t d\tau\, J_\mu \mathrm{tr}_1\mathrm{tr}_2\big\{(X_1\otimes Y_2 - \rho_{12})\sigma_2^\mu\big[u_\tau u_t^\dagger A_1 u_t u_\tau^\dagger, \sigma_1^\mu\big]\big\}\right| \qquad (61)\\
&\leqslant J_0\sum_{\mu}\int_0^t d\tau\big|\mathrm{tr}_1\mathrm{tr}_2\big\{(X_1\otimes Y_2 - \rho_{12})\sigma_2^\mu\big[u_\tau u_t^\dagger A_1 u_t u_\tau^\dagger, \sigma_1^\mu\big]\big\}\big| \qquad (62)
\end{aligned}$$

and

$$\begin{aligned}
|\mathrm{tr}_2[B_2 Y_2(t) - B_2\rho_2(t)]| &= \Bigg|\sum_{\mu=1}^{3}\int_0^t d\tau\frac{J_\mu}{n-1}\mathrm{tr}_1\mathrm{tr}_2\big\{(u_\tau u_t^\dagger B_2 u_t u_\tau^\dagger)\big[\sigma_2^\mu, (X_1\otimes Y_2 - \rho_{12})\sigma_1^\mu\big]\big\}\\
&\quad + \lambda(n-2)\sum_{\mu}\int_0^t d\tau\frac{J_\mu}{n-1}\mathrm{tr}_2\mathrm{tr}_3\big\{(u_\tau u_t^\dagger B_2 u_t u_\tau^\dagger)\big[\sigma_2^\mu, (Y_2\otimes Y_3 - \rho_{23})\sigma_3^\mu\big]\big\}\Bigg|
\end{aligned}\tag{63}$$

$$= \left| \sum_\mu \int_0^t d\tau \frac{J_\mu}{n-1} \mathrm{tr}_1\mathrm{tr}_2\big\{(X_1 \otimes Y_2 - \rho_{12})\sigma_1^\mu\big[u_\tau u_t^\dagger B_2 u_t u_\tau^\dagger, \sigma_2^\mu\big]\big\} \right.$$

$$\left. + \lambda(n-2) \sum_\mu \int_0^t d\tau \frac{J_\mu}{n-1} \mathrm{tr}_2\mathrm{tr}_3\big\{(Y_2 \otimes Y_3 - \rho_{23})\sigma_3^\mu\big[u_\tau u_t^\dagger B_2 u_t u_\tau^\dagger, \sigma_2^\mu\big]\big\} \right| \tag{64}$$

$$\leqslant J_0 \frac{1}{n-1} \sum_\mu \int_0^t d\tau \big|\mathrm{tr}_1\mathrm{tr}_2\big\{(X_1 \otimes Y_2 - \rho_{12})\sigma_1^\mu\big[u_\tau u_t^\dagger B_2 u_t u_\tau^\dagger, \sigma_2^\mu\big]\big\}\big|$$

$$+ |\lambda| J_0 \frac{n-2}{n-1} \sum_\mu \int_0^t d\tau \big|\mathrm{tr}_2\mathrm{tr}_3\big\{(Y_2 \otimes Y_3 - \rho_{23})\sigma_3^\mu\big[u_\tau u_t^\dagger B_2 u_t u_\tau^\dagger, \sigma_2^\mu\big]\big\}\big|. \tag{65}$$

Using the identities

$$X_1 \otimes Y_2 = (X_1 - \rho_1) \otimes Y_2 + \rho_1 \otimes (Y_2 - \rho_2) + \rho_1 \otimes \rho_2, \tag{66}$$

$$Y_2 \otimes Y_3 = (Y_2 - \rho_2) \otimes Y_3 + \rho_2 \otimes (Y_3 - \rho_3) + \rho_2 \otimes \rho_3 \tag{67}$$

leads to

$$|\mathrm{tr}[A_1(X - \rho_1)]| \leqslant J_0 \sum_\mu \int_0^t d\tau \|[u_\tau u_t^\dagger A_1 u_t u_\tau^\dagger, \sigma_1^\mu]\|_\infty \|\sigma_2^\mu\|_\infty \bigg(\|X - \rho_1\|_1 \|Y\|_1 + \|Y - \rho_2\|_1 \|\rho_1\|_1$$

$$+ \frac{|\langle [u_\tau u_t^\dagger A_1 u_t u_\tau^\dagger, \sigma_1^\mu]\sigma_2^\mu \rangle - \langle [u_\tau u_t^\dagger A_1 u_t u_\tau^\dagger, \sigma_1^\mu]\rangle\langle \sigma_2^\mu\rangle|}{\|[u_\tau u_t^\dagger A_1 u_t u_\tau^\dagger, \sigma_1^\mu]\|_\infty \|\sigma_2^\mu\|_\infty} \bigg) \tag{68}$$

$$\leqslant 6J_0 \|A_1\|_\infty \int_0^t d\tau \left(\|X - \rho_1\|_1 + \|Y - \rho_2\|_1 + 4\frac{e^{12(1+|\lambda|)J_0\tau} - 1}{n-1} \right), \tag{69}$$

where $\langle \cdot \rangle = \mathrm{tr}(\rho\, \cdot)$ denotes expectation in the state $\rho = U_t(|\varphi\rangle\langle\varphi|^{\otimes n})U_t^\dagger$. Similarly,

$$|\mathrm{tr}(B_2(Y - \rho_2))| \leqslant \frac{6J_0\|B_2\|_\infty}{n-1} \int_0^t d\tau \bigg[\|X - \rho_1\|_1 + \|Y - \rho_2\|_1 + 4\frac{e^{12(1+|\lambda|)J_0\tau} - 1}{n-1}$$

$$+ |\lambda|(n-2)\left(2\|Y - \rho_2\|_1 + 4\frac{e^{12(1+|\lambda|)J_0\tau} - 1}{n-1} \right) \bigg]. \tag{70}$$

Assuming $\|A_1\|_\infty \neq 0$ and $\|B_2\|_\infty \neq 0$,

$$\frac{|\mathrm{tr}[A_1(X - \rho_1)]|}{\|A_1\|_\infty} \leqslant 6J_0 \int_0^t d\tau \left(\|X - \rho_1\|_1 + \|Y - \rho_2\|_1 + 4\frac{e^{12(1+|\lambda|)J_0\tau} - 1}{n-1} \right), \tag{71}$$

$$\frac{|\mathrm{tr}[B_2(Y - \rho_2)]|}{\|B_2\|_\infty} \leqslant \frac{6J_0}{n-1} \int_0^t d\tau \bigg[\|X - \rho_1\|_1 + \|Y - \rho_2\|_1 + 4\frac{e^{12(1+|\lambda|)J_0\tau} - 1}{n-1}$$

$$+ |\lambda|(n-2)\left(2\|Y - \rho_2\|_1 + 4\frac{e^{12(1+|\lambda|)J_0\tau} - 1}{n-1} \right) \bigg]. \tag{72}$$

These hold for any A_1 and B_2 such that $\|A_1\|_\infty \neq 0$ and $\|B_2\|_\infty \neq 0$. Therefore,

$$\sup_{A\neq 0} \frac{|\mathrm{tr}[A_1(X - \rho_1)]|}{\|A_1\|_\infty} \leqslant 6J_0 \int_0^t d\tau \left(\|X - \rho_1\|_1 + \|Y - \rho_2\|_1 + 4\frac{e^{12(1+|\lambda|)J_0\tau} - 1}{n-1} \right), \tag{73}$$

$$\sup_{B\neq 0} \frac{|\mathrm{tr}[B_2(Y - \rho_2)]|}{\|B_2\|_\infty} \leqslant \frac{6J_0}{n-1} \int_0^t d\tau \bigg[\|X - \rho_1\|_1 + \|Y - \rho_2\|_1 + 4\frac{e^{12(1+|\lambda|)J_0\tau} - 1}{n-1}$$

$$+ |\lambda|(n-2)\left(2\|Y - \rho_2\|_1 + 4\frac{e^{12(1+|\lambda|)J_0\tau} - 1}{n-1} \right) \bigg]. \tag{74}$$

Then, after using (B6),

$$\|X - \rho_1\|_1 \leqslant 6J_0 \int_0^t d\tau \left(\|X - \rho_1\|_1 + \|Y - \rho_2\|_1 + 4\frac{e^{12(1+|\lambda|)J_0\tau} - 1}{n-1} \right), \tag{75}$$

$$\|Y-\rho_2\|_1 \leqslant \frac{6J_0}{n-1}\int_0^t d\tau\left[\|X-\rho_1\|_1+\|Y-\rho_2\|_1+4\frac{e^{12(1+|\lambda|)J_0\tau}-1}{n-1}+|\lambda|(n-2)\left(2\|Y-\rho_2\|_1+4\frac{e^{12(1+|\lambda|)J_0\tau}-1}{n-1}\right)\right]. \tag{76}$$

Up to this point in the proof we have assumed that $n \geqslant 2$. If $n \gg 1$,

$$\|X-\rho_1\|_1 \leqslant 2\frac{e^{12(1+|\lambda|)J_0t}-1}{n(1+|\lambda|)}+6J_0\int_0^t dt_1(\|X-\rho_1\|_1+\|Y-\rho_2\|_1)+O(1/n^2), \tag{77}$$

$$\|Y-\rho_2\|_1 \leqslant 2|\lambda|\frac{e^{12(1+|\lambda|)J_0t}-1}{n(1+|\lambda|)}+6J_0\int_0^t dt_1\left[\frac{\|X-\rho_1\|_1}{n}+\left(\frac{1}{n}+2|\lambda|\right)\|Y-\rho_2\|_1\right]+O(1/n^2). \tag{78}$$

We solve these iteratively. After q iterations we have

$$\begin{aligned}\|X-\rho_1\|_1 \leqslant{}& 2\frac{e^{12(1+|\lambda|)J_0t}-1}{n(1+|\lambda|)}\left[1+(a_1+|\lambda|b_1)\left(\frac{1}{2(1+|\lambda|)}\right)+\cdots+(a_{q-1}+|\lambda|b_{q-1})\left(\frac{1}{2(1+|\lambda|)}\right)^{q-1}\right]\\ &+(6J_0)^q\int_0^t dt_1\cdots\int_0^{t_{q-1}}dt_q(a_q\|X-\rho_1\|_1+b_q\|Y-\rho_2\|_1)+O(1/n^2)\end{aligned} \tag{79}$$

and

$$\begin{aligned}\|Y-\rho_2\|_1 \leqslant{}& 2\frac{e^{12(1+|\lambda|)J_0t}-1}{n(1+|\lambda|)}\left[|\lambda|+(a'_1+|\lambda|b'_1)\left(\frac{1}{2(1+|\lambda|)}\right)+\cdots+(a'_{q-1}+|\lambda|b'_{q-1})\left(\frac{1}{2(1+|\lambda|)}\right)^{q-1}\right]\\ &+(6J_0)^q\int_0^t dt_1\cdots\int_0^{t_{q-1}}dt_q(a'_q\|X-\rho_1\|_1+b'_q\|Y-\rho_2\|_1)+O(1/n^2),\end{aligned} \tag{80}$$

where the positive real coefficients a_k and b_k satisfy

$$a_1=1,\quad b_1=1, \tag{81}$$

and

$$a_k=a_{k-1}+\frac{b_{k-1}}{n}, \tag{82}$$

$$b_k=a_{k-1}+mb_{k-1} \tag{83}$$

for $k>1$, where

$$m:=\frac{1}{n}+2|\lambda|. \tag{84}$$

The coefficients a'_k and b'_k in (80) satisfy the identical recurrence relation but start with

$$a'_1=\frac{1}{n},\quad b'_1=m \tag{85}$$

instead of (81). Equations (82) and (83) can be solved for arbitrary a_1 and b_1:

$$\begin{aligned}a_k ={}& \left(1+\frac{1+(1+m)+(1+m+m^2)+\cdots+(1+m+m^2+m^3+\cdots m^{k-3})}{n}\right)a_1\\ &+\left(\frac{1+m+m^2+m^3+\cdots+m^{k-2}}{n}\right)b_1+O(1/n^2)\end{aligned} \tag{86}$$

$$=\left(1+\frac{1-2m+(k-3)(1-m)+m^{k-1}}{n(1-m)^2}\right)a_1+\frac{1-m^{k-1}}{n(1-m)}b_1+O(1/n^2), \tag{87}$$

$$\begin{aligned}b_k ={}& \left(\frac{1-m^{k-1}}{1-m}+\frac{(k-3)m^{k+1}+(1-k)m^k+2m^3-m^2+(k-1)m+2-k}{nm^2(m-1)^3}\right)a_1\\ &+\left(m^{k-1}+\frac{1-m^{k-1}+(k-1)(m-1)m^{k-2}}{n(1-m)^2}\right)b_1+O(1/n^2).\end{aligned} \tag{88}$$

Anticipating the large-n limit, we have dropped terms $1/n^2$ and smaller. The second forms of the above expressions are obtained by assuming $m \neq 1$ and summing geometric series and their derivatives. Note that for $(a_1, b_1) = (1, 1)$ we have

$$a_k + |\lambda| b_k = 1 + |\lambda| \frac{1 - m^k}{1 - m} + O(1/n), \tag{89}$$

whereas for $(a'_1, b'_1) = (\frac{1}{n}, m)$ we have

$$a'_k + |\lambda| b'_k = |\lambda| m^k + O(1/n). \tag{90}$$

Using (89) and (90),

$$\lim_{n\to\infty} \sum_{k=1}^{q-1} \frac{a_k + |\lambda| b_k}{(2 + 2|\lambda|)^k} = \frac{1}{1 - 2|\lambda|} \sum_{k=1}^{q-1} \frac{(1 - |\lambda|) - |\lambda|(2|\lambda|)^k}{(2 + 2|\lambda|)^k} + O(1/n), \tag{91}$$

$$\lim_{n\to\infty} \sum_{k=1}^{q-1} \frac{a'_k + |\lambda| b'_k}{(2 + 2|\lambda|)^k} = |\lambda| \sum_{k=1}^{q-1} \frac{|2\lambda|^k}{(2 + 2|\lambda|)^k} + O(1/n). \tag{92}$$

Then we obtain, for $|\lambda| \leqslant 1$,

$$\lim_{q\to\infty} \lim_{n\to\infty} \left(1 + \sum_{k=1}^{q-1} \frac{a_k + |\lambda| b_k}{(2 + 2|\lambda|)^k} \right) \leqslant 1 + \frac{1 - |\lambda| - |\lambda|^2 - 2|\lambda|^3}{(1 + 2|\lambda|)(1 - 2|\lambda|)} \leqslant 2 \tag{93}$$

and

$$\lim_{q\to\infty} \lim_{n\to\infty} \left(|\lambda| + \sum_{k=1}^{q-1} \frac{a'_k + |\lambda| b'_k}{(2 + 2|\lambda|)^k} \right) \leqslant |\lambda| + \lambda^2 \leqslant 2. \tag{94}$$

Finally, note that

$$(6J_0)^q \int_0^t dt_1 \cdots \int_0^{t_{q-1}} dt_q (a_q \|X - \rho_1\|_1 + b_q \|Y - \rho_2\|_1) \leqslant 2(a_q + b_q) \frac{(6J_0 t)^q}{q!}, \tag{95}$$

$$(6J_0)^q \int_0^t dt_1 \cdots \int_0^{t_{q-1}} dt_q (a'_q \|X - \rho_1\|_1 + b'_q \|Y - \rho_2\|_1) \leqslant 2(a'_q + b'_q) \frac{(6J_0 t)^q}{q!} \tag{96}$$

both vanish in the large-q limit. Then we obtain (32), as required. ■

IV. DISCUSSION

Mean-field errors are bounded by a competition between an exponential growth in time and a $1/n$ suppression in system size, so the bounds are mainly interesting when $n \gg \exp[O(t)]$. Thus, it is tempting to conclude that the CSM requires exponentially many qubits to simulate nonlinearity, but this is not the case for a finite-time simulation. This can be understood by assuming $12(1 + |\lambda|)J_0 t \ll 1$, which defines a particular short-time limit, and linearizing the exponential in (32). This leads to

$$\|X(t) - \rho_1(t)\|_1 \leqslant \frac{48J_0 t}{n} = \epsilon, \tag{97}$$

where ϵ is the desired model error. Then duality within ϵ holds for a time

$$t_{\max} = \frac{n\epsilon}{48J_0} = n\Delta t, \quad \Delta t := \frac{\epsilon}{48J_0}. \tag{98}$$

In the short-time regime, increasing n merely increases the simulation interval $t_{\max}$, each ancilla qubit contributing a unit of propagation time Δt.

If $\lambda = 1$ and complete permutation symmetry is respected, the CSM is described by mean-field theory (31), which has self-interaction. This nonlinearity generates qubit torsion and other nonrigid distortions of the Bloch ball determined by the couplings J_μ [64,65]. To see this, write the Hamiltonian in (31) as

$$H^{\text{eff}} = H^0 + \sum_\mu J_\mu \text{tr}(X\sigma^\mu)\sigma^\mu, \tag{99}$$

where X is the current state of the central (or any other) qubit. Suppose $J_\mu = (J_1, 0, 0)$. The nonlinear term in (99) generates an x rotation with frequency $2J_1 x$, where x is the projection of the Bloch vector on the x axis. States with larger x components rotate faster and states with negative projections rotate in the opposite direction, twisting the Bloch ball. Couplings $(0, J_2, 0)$ and $(0, 0, J_3)$ similarly generate pure torsion about the y and z axes of the Bloch ball, respectively. Single-axis torsions have been investigated previously [2,5,6]. More general couplings $J_\mu = (J_1, J_2, J_3)$ with two or three nonzero components generate higher-order distortions beyond pure torsion,[2] which have not been studied.

The CSM with $\lambda \neq 1$ is described by the coupled nonlinear equations (23) and (24). The CSM with $\lambda = 0$ is particularly interesting: In this case the Hamiltonian for the central qubit

[2]Unlike rigid rotations, simultaneous twisting about two perpendicular axes is not equivalent to a twist about an intermediate axis.

is

$$H^{\text{eff}} = H^0 + \sum_{\mu} J_\mu \text{tr}(Y\sigma^\mu)\sigma^\mu, \tag{100}$$

where, in the large-n limit, Y is governed by H^0 only. Thus, the central qubit interacts with a bath of synchronized ancillas but produces a vanishing reaction on any individual ancilla qubit. To use this for information processing, set $H^0 = 0$. Then $\frac{dY}{dt} = 0$ and the resulting Hamiltonian

$$H^{\text{eff}} = \sum_{\mu} J_\mu \langle\varphi|\sigma^\mu|\varphi\rangle\sigma^\mu \tag{101}$$

implements initial-condition nonlinearity ($\langle\sigma^\mu\rangle$ is static and fixed by the initial condition). Different initial states $|\varphi\rangle$ are subjected to different Hamiltonians. If J_μ is time independent, these are static Hamiltonians, whereas (99) is typically time dependent (because X is).

Finally, we speculate on the relevance of the duality to the question of whether quantum mechanics is fundamentally nonlinear. While there is no experimental evidence for such nonlinearity [67–73], it would be more illuminating to have a theoretical argument or no-go theorem showing that its presence would violate a stronger property, such as relativistic invariance [74–78]. However, no such argument is currently available. Dualities like that discussed here suggest that there might not be a sharp distinction between universes evolving according to linear and nonlinear quantum mechanics. This observation is consistent both with the absence of a nonlinear no-go theorem and with other dualities based on nonlinear gauge transformations [79]. If quantum nonlinearity is indeed allowed, how can we experimentally test for it? Beyond laboratory experiments [67–73], one possibility is to consider the cosmological implications of potential quantum nonlinearity [80–84]. Lloyd [82] has argued that the universe itself might be regarded as a giant quantum information processor and that this perspective explains how the complexity observed today could arise from a homogeneous, isotropic initial state evolving according to simple laws. In the future it would be interesting to reexamine the question of cosmological complexity generation with the hypothesis of real or simulated quantum nonlinearity.

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APPENDIX A: PARTIAL TRACES OF COMMUTATORS

Here we explain some properties of partial traces used in the proofs.

(1) Let $\rho \in B(\mathcal{H}, \mathbb{C})$ be any bounded linear operator and let B_i be an operator acting on qubit i exclusively. Then the partial trace of their commutator vanishes:

$$\text{tr}_i([B_i, \rho]) = 0. \tag{A1}$$

To see this, evaluate $\text{tr}_i([B_i, \rho])$ in the $\{|0\rangle, |1\rangle\}$ basis of qubit i,

$$\text{tr}_i([B_i, \rho]) = \sum_{x,x'=0,1} (\langle x|B_i|x'\rangle_i \langle x'|\rho|x\rangle_i - \langle x|\rho|x'\rangle_i \langle x'|B_i|x\rangle_i) \tag{A2}$$

$$= \sum_{x,x'=0,1} (\langle x|B_i|x'\rangle_i \langle x'|\rho|x\rangle_i - \langle x'|\rho|x\rangle_i \langle x|B_i|x'\rangle_i) \tag{A3}$$

$$= \sum_{x,x'=0,1} \langle x|B_i|x'\rangle_i (\langle x'|\rho|x\rangle_i - \langle x'|\rho|x\rangle_i) = 0, \tag{A4}$$

because $\langle x|B_i|x'\rangle_i \in \mathbb{C}$ commutes with the operator $\langle x'|\rho|x\rangle_i$.

(2) Let $\rho \in B(\mathcal{H}, \mathbb{C})$ be any bounded linear operator and let B_i be an operator acting on qubit i exclusively. Then

$$\text{tr}_{>j}([B_i, \rho]) = \text{tr}_{j+1}\text{tr}_{j+2} \cdots \text{tr}_n([B_i, \rho]) = \begin{cases} [B_i, \text{tr}_{>j}(\rho)] & \text{for } i \leqslant j \\ 0 & \text{for } i > j. \end{cases} \tag{A5}$$

If $i \leqslant j$ then $\text{tr}_{j+1} \cdots \text{tr}_n(B_i\rho - \rho B_i) = [B_i, \text{tr}_{>j}(\rho)]$. If $i > j$ the required result follows from (A1).

APPENDIX B: SCHATTEN *P*-NORMS

Here we collect a few properties of the matrix norms used in this paper. Let $X \in \mathbb{C}^{2^n \times 2^n}$ be a complex matrix on n qubits. The norms $\|X\|_1$ and $\|X\|_\infty$ used in Theorem 1 (Sec. III) are special cases of Schatten p-norms

$$\|X\|_p := [\text{tr}(|X|^p)]^{1/p}, \quad p \geqslant 1, \tag{B1}$$

where $|X| := \sqrt{X^\dagger X}$ is the absolute value of a matrix. Because $A = X^\dagger X = UDU^\dagger$ is Hermitian and positive semidefinite, we can define $\sqrt{A} = U\sqrt{D}U^\dagger$ through its spectral decomposition, leading to $|X| = U\sqrt{D}U^\dagger = U\Sigma U^\dagger$, where Σ is a diagonal matrix containing the singular values $\sqrt{\text{spec}(X^\dagger X)}$ of X. Here $\text{spec}(Y)$ denotes the set of eigenvalues of $Y \in B(\mathcal{H}, \mathbb{C})$ and $\sqrt{\text{spec}(Y)}$ are their square roots. Then $\|X\|_p = [\text{tr}(\Sigma^p)]^{1/p} = [\sum_{i=1}^{2^n} (\Sigma_{ii})^p]^{1/p}$.

We use the following properties.

(1) The Schatten p-norm is unitarily invariant. Let $U, V \in \mathbb{C}^{2^n \times 2^n}$ be unitary. Then $\|UXV^\dagger\|_p = \|X\|_p$.

(2) The Schatten p-norm is submultiplicative:

$$\|XY\|_p \leqslant \|X\|_p \|Y\|_p. \tag{B2}$$

(3) The Schatten 1-norm $\|X\|_1$ is equal to the trace norm (sum of singular values).

(4) The Schatten 1-norm satisfies

$$|\text{tr}(X)| \leqslant \|X\|_1. \tag{B3}$$

(5) The Schatten 1-norm is not normalized: $\|I^{\otimes n}\|_1 = 2^n$. Here I is the two-dimensional identity.

(6) The limit $\|X\|_\infty := \lim_{p\to\infty} \|X\|_p$ exists and is equal to the operator norm (maximum singular value).

(7) The operator norm is normalized: $\|I^{\otimes n}\|_\infty = \|I\|_\infty = 1$.

(8) The trace and operator norms satisfy the inequality

$$\|X\|_\infty \leqslant \|X\|_1. \tag{B4}$$

(9) The trace and operator norms also satisfy a Holder inequality

$$\|XY\|_1 \leqslant \|X\|_1 \|Y\|_\infty, \tag{B5}$$

which is tighter than that provided by (B2).

(10) Let $A \in B(\mathcal{H}, \mathbb{C})$ be a bounded linear operator. Then

$$\sup_{B\neq 0} \frac{|\mathrm{tr}(AB)|}{\|B\|_\infty} = \|A\|_1, \tag{B6}$$

where the supremum is over the set of all $B \in B(\mathcal{H}, \mathbb{C})$ with $\|B\|_\infty \neq 0$.

(11) Let X_α and X_β be arbitrary states (positive-semidefinite operators with unit trace). Then

$$\|X_\alpha - X_\beta\|_1 \leqslant 2. \tag{B7}$$

(12) Let $A, B \in \mathbb{C}^{N\times N}$ and $C \in \mathbb{C}^{N^2\times N^2}$. Then

$$\int_0^t d\tau |\mathrm{tr}(CA\otimes B)| \leqslant \int_0^t d\tau \|C(\tau)\|_1 \|A(\tau)\|_\infty \|B(\tau)\|_\infty, \tag{B8}$$

$$\int_0^t d\tau |\mathrm{tr}(CA\otimes B)| \leqslant \int_0^t d\tau \|C(\tau)\|_\infty \|A(\tau)\|_1 \|B(\tau)\|_1. \tag{B9}$$

(13) Let $\vec{\sigma}_i \cdot \vec{\sigma}_j = \sigma_i^1 \otimes \sigma_j^1 + \sigma_i^2 \otimes \sigma_j^2 + \sigma_i^3 \otimes \sigma_j^3$. Then

$$\|\vec{\sigma}_i \cdot \vec{\sigma}_j\|_\infty = 3, \quad \|\vec{\sigma}_i \cdot \vec{\sigma}_j\|_1 = 6. \tag{B10}$$

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