

Quantum nonlocality in the presence of strong measurement dependenceIvan Šupić ^{1,2,*}, Jean-Daniel Bancal ^{1,3} and Nicolas Brunner¹¹*Département de Physique Appliquée, Université de Genève, 1211 Genève, Switzerland*²*LIP6, CNRS, Sorbonne Université, 4 Place Jussieu, 75005 Paris, France*³*Institut de Physique Théorique, CNRS, CEA, Université Paris-Saclay 91191 Gif-sur-Yvette, France*

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It is well known that the effect of quantum nonlocality, as witnessed by violation of a Bell inequality, can be observed even when relaxing the assumption of measurement independence, i.e., allowing for the source to be partially correlated with the choices of measurement settings; however, what is the minimal amount of measurement independence needed to observe quantum nonlocality? Here we explore this question and consider models with strong measurement-dependent locality, where measurement choices can be perfectly determined in almost all rounds of the Bell test. Nevertheless, we show that quantum nonlocality can still be observed in this scenario, which we conjecture is minimal within the framework we use. We also discuss potential applications in randomness amplification.

DOI: [10.1103/PhysRevA.108.042207](https://doi.org/10.1103/PhysRevA.108.042207)**I. INTRODUCTION**

Quantum theory allows for strong nonlocal correlations as witnessed by the violation of Bell inequalities [1]. This phenomenon, known as quantum nonlocality, has been verified experimentally in a variety of physical platforms (see, e.g., [2–8]) and represents the basic resource for quantum information processing in the so-called device-independent setting.

A Bell test typically involves a number of distant observers (say, Alice and Bob) performing randomly chosen local measurements on a shared physical system. Bell inequalities are then usually derived under two assumptions (see, e.g., [9]): (i) The choices of local measurements made by each party are independent of the source distributing the shared physical resource and (ii) the measurement output of each party is determined solely by their input and a shared local (hidden) variable. Clearly, the first assumption is important, since if the source could know *a priori* the choice of measurement setting for each round, then all possible correlations can be reproduced by a classical model. Therefore, the observation of (quantum) nonlocality relies on the assumption that the choice of measurement settings cannot be perfectly determined at the source. In other words, there must be some level of measurement independence, but then how much measurement independence is required to observe quantum nonlocality?

This question has attracted broad attention in recent years and has been discussed following various approaches (see, e.g., [10–18]). A notable approach is that of Pütz *et al.* [19,20], who presented a general framework for addressing these questions. In particular, they derived Bell inequalities for testing local models with measurement-dependent locality (MDL), i.e., where the above assumption (i) is relaxed. Remarkably, violation of such inequalities is possible for any level of

measurement dependence, demonstrating that quantum nonlocality can be observed even if measurement settings can be almost perfectly determined by the source. Hence quantum theory allows for measurement-dependent nonlocality. These ideas have been tested experimentally [21] and shown to be relevant for the task of randomness amplification [22] and network quantum nonlocality [23].

A natural question at this point is whether the above works have identified the minimal requirements in terms of measurement independence, or if quantum nonlocality could in fact be demonstrated considering an even stronger form of measurement dependence. This is the main motivation behind the present work. Our main result is that quantum theory in fact allows for a much stronger form of measurement-dependent nonlocality. Specifically, we consider the approach of Pütz *et al.* [19,20], who showed that the measurement inputs can be almost perfectly determined in each round of the Bell test. This means that in every round, the inputs feature some (possibly arbitrarily small) level of randomness with respect to the source. In contrast, we consider models where, in almost all rounds, the source can fully determine the measurement inputs. Hence we only require the presence of rounds where the inputs cannot be perfectly determined and that these rounds have a nonvanishing probability to occur. We then show that quantum correlations are still incompatible with respect to these models. Our work thus shows that the requirement for demonstrating quantum nonlocality can still be considerably relaxed compared to previously known results. We conclude with a discussion of further questions and applications for randomness amplification.

II. GENERAL SETTING

We consider first bipartite Bell tests involving two noncommunicating agents, Alice and Bob. Alice can choose between several measurements, labeled with $x \in \{0, 1, \dots, m_A - 1\}$. Her outcomes are labeled with $a \in \{0, 1, \dots, n_A - 1\}$. For

*ivan.supic@lip6.fr

Bob the measurement choice label is $y \in \{0, 1, \dots, m_B - 1\}$ and the corresponding outcome $b \in \{0, 1, \dots, n_B - 1\}$. The experiment is characterized by the set of conditional probabilities of obtaining the pair of outcomes a, b when the chosen inputs are x and y ,

$$\mathcal{P} = \{p(a, b|x, y)\}_{a,b,x,y}, \quad (1)$$

usually called a behavior. Without loss of generality, the joint correlation probabilities can be written as

$$p(a, b, x, y) = \int_{\lambda} d\lambda p(\lambda) p(x, y|\lambda) p(a, b|x, y, \lambda), \quad (2)$$

where $p(\lambda)$, $p(x, y|\lambda)$, and $p(a, b|x, y, \lambda)$ are valid probability distributions. The classical variable λ thus encodes the correlations between various measurement outputs.

Bell [1] formalized the concept of a local hidden-variable (LHV) model via the following requirements: (i) measurement independence, wherein the variable λ is completely independent of the inputs x and y , i.e.,

$$p(x, y|\lambda) = p(x, y) \forall \lambda, \quad (3)$$

and (ii) Bell locality, wherein locally, each output is determined solely by the input and the shared variable λ , i.e.,

$$p(a, b|x, y, \lambda) = p(a|x, \lambda) p(b|y, \lambda) \forall x, y, a, b, \lambda. \quad (4)$$

A behavior is then termed (Bell) local if it admits a decomposition of the form

$$p(a, b|x, y) = \int_{\lambda} d\lambda p(\lambda) p(a|x, \lambda) p(b|y, \lambda), \quad (5)$$

which is obtained by taking into account the conditions (3) and (4) and dividing Eq. (2) by $p(x, y)$. We denote such a behavior by $\tilde{\mathcal{P}}_{\text{loc}}$ with corresponding probabilities $p_{\text{loc}}(a, b|x, y)$. All such behaviors satisfy Bell inequalities, which are based on the linear functional of the joint probabilities and take the form

$$\mathcal{I} \equiv \sum_{a,b,x,y} \omega_{a,b,x,y} p_{\text{loc}}(a, b|x, y) \leq \beta_{\text{loc}}, \quad (6)$$

where $\omega_{a,b,x,y}$ are real coefficients and β_{loc} is termed the local bound of the Bell inequality.

Violation of such a Bell inequality in the presence of measurement independence implies nonlocality. This is possible in quantum theory, by performing well-chosen sets of local measurements on a shared entangled state ρ^{AB} . In general, a quantum behavior takes the form

$$p(a, b|x, y) = \text{Tr}[\mathbf{M}_{a|x} \otimes \mathbf{M}_{b|y} \rho^{\text{AB}}], \quad (7)$$

where the sets of operators $\{\mathbf{M}_{a|x}\}$ and $\{\mathbf{M}_{b|y}\}$ represent the local measurements of Alice and Bob, respectively.

Before continuing, let us introduce two additional bounds for a Bell inequality, relevant for our work. First, the maximal value of a Bell expression [the functional defined in (6)] in quantum theory (optimized over all quantum behaviours) is called the quantum bound and denoted by β_{q} . Second, the maximal value of a Bell expression (considering any valid joint probability distribution) is called the algebraic bound and denoted by β_{alg} .

Let us now return to the assumptions behind local models. Our focus here is on the assumption of measurement

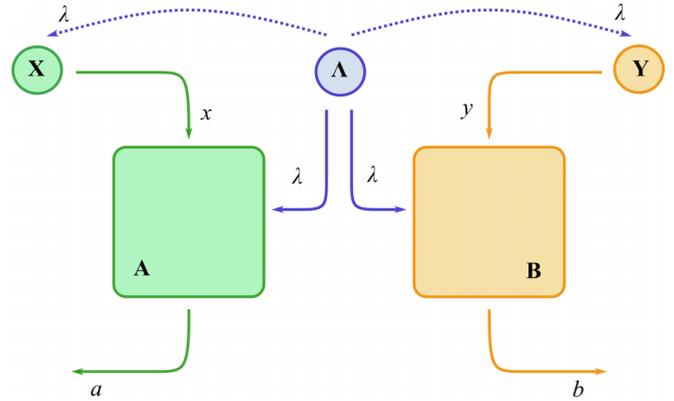


FIG. 1. We consider a Bell test with relaxed measurement independence. Specifically, the source, i.e., the classical variable Λ , can be correlated to the choice of measurement inputs (variables X and Y) of both parties. We show that quantum nonlocality can exhibit strong measurement-dependent nonlocality, i.e., the quantum predictions cannot be explained by a local model even if the source Λ completely determines both inputs in almost all rounds of the experiment.

independence. From a fundamental perspective, it is natural to ask whether this assumption could be partially relaxed. Specifically, could one still demonstrate quantum nonlocality when considering partial correlations between the source, i.e., the shared variable λ , and the measurement inputs x and y (see Fig. 1). References [15,19,20] introduced a general framework to tackle this question. In particular, they proposed to relax the usual measurement independence condition to

$$\xi \leq p(x, y|\lambda) \leq \eta \forall x, y, \lambda, \quad (8)$$

where $\xi > 0$ and consequently $\eta < 1$, as $\sum_{x,y} p(x, y|\lambda) = 1$. Hence the measurement inputs are no longer independent of the shared variable λ . Importantly, the inputs can still not be perfectly determined from λ , as ensured by the condition $\xi > 0$. More generally, one can set given values to the parameters ξ and η , which then quantify the level of measurement independence. One can then define classes of local models with relaxed measurement independence, satisfying the condition of Bell locality [Eq. (4)] and the partial measurement independence condition in Eq. (8) for given values of ξ and η . Reference [19] then showed how to construct Bell inequalities for these models. A key idea consists in considering joint distributions $p(a, b, x, y)$ instead of conditional ones. Starting from a standard Bell inequality, as in (6), one can then construct a novel Bell inequality of the form

$$\tilde{\mathcal{I}}_{\xi,\eta} \leq \xi \eta \beta_{\text{loc}} \equiv \beta_{\text{loc}}^{\text{MD}}, \quad (9)$$

where $\tilde{\mathcal{I}}_{\xi,\eta}$ is a linear functional of the joint probabilities $p(a, b, x, y)$ (with coefficients obtained from the original Bell functional).

Violation of such a Bell inequality under the condition (8) implies so-called measurement-dependent nonlocality. Remarkably, Pütz *et al.* [19] showed that quantum theory allows for measurement-dependent nonlocality even for an arbitrarily small level of measurement independence, i.e., for any $\xi > 0$. Their example is connected to the quantum correlations arising from the well-known Hardy paradox [24].

III. STRONG MEASUREMENT-DEPENDENT NONLOCALITY

We are now in position to present our main result, namely, that quantum correlations can in fact exhibit an even stronger form of measurement-dependent nonlocality. To be more precise, let us first note that in the approach of Pütz *et al.* [19] reviewed above, the relaxed measurement-independence condition (8) must hold in every round of the Bell experiment. In other words, in every round, the inputs cannot be perfectly determined from the shared variable λ .

Below we consider local models where this constraint is almost fully relaxed. Specifically, we consider two types of rounds. First, we have rounds where the shared variable perfectly determines the inputs, i.e., full measurement dependence. These rounds are denoted by the set of shared variables Λ'' . Second, we have rounds with relaxed measurement independence, i.e., the condition (8) applies for a fixed value of ξ and η . This set of rounds is denoted by Λ' . Let us characterize the relative probabilities of these sets:

$$\int_{\Lambda'} d\lambda p(\lambda) = q, \quad \int_{\Lambda''} d\lambda p(\lambda) = 1 - q. \quad (10)$$

Note that we must require that $q > 0$ in order to possibly observe nonlocality. Below we show how to construct a relevant Bell inequality, starting from any standard Bell inequality (6), in order to test the above model, for any values of $q > 0$ and $\xi > 0$. Moreover, we exhibit an example of such a Bell

inequality, where quantum correlations lead to a Bell violation for any values of $q > 0$ and $\xi > 0$. We say that such correlations feature strong measurement-dependent nonlocality.

Consider now an arbitrary Bell inequality (6). As in [19], we define $\omega_{a,b,x,y}^+ = \omega_{a,b,x,y}$ if $\omega_{a,b,x,y} > 0$ and $\omega_{a,b,x,y}^- = -\omega_{a,b,x,y}$ if $\omega_{a,b,x,y} < 0$. Then the Bell inequality has the form

$$\begin{aligned} \mathcal{I} &\equiv \sum_{a,b,x,y} [\omega_{a,b,x,y}^+ p_{\text{loc}}(a, b|x, y) - \omega_{a,b,x,y}^- p_{\text{loc}}(a, b|x, y)] \\ &\leq \beta_{\text{loc}}. \end{aligned} \quad (11)$$

The quantum bound of the inequality, equal to β_q , can be decomposed as $\beta_q = \beta_q^+ - \beta_q^-$, where we introduce notation

$$\beta_q^+ = \sum_{a,b,x,y} \omega_{a,b,x,y}^+ p_q(a, b|x, y), \quad (12)$$

$$\beta_q^- = \sum_{a,b,x,y} \omega_{a,b,x,y}^- p_q(a, b|x, y), \quad (13)$$

with $\{p_q(a, b|x, y)\}$ a behavior reaching the quantum bound. The measurement-dependent Bell inequality from [19] reads

$$\tilde{\mathcal{I}}_{\xi,\eta} = \sum_{a,b,x,y} [\xi \omega_{a,b,x,y}^+ p(a, b, x, y) - \eta \omega_{a,b,x,y}^- p(a, b, x, y)]. \quad (14)$$

By separating the sets Λ' and Λ'' and taking into account the decomposition of local joint probabilities we get

$$\begin{aligned} \tilde{\mathcal{I}}_{\xi,\eta} &= \sum_{a,b,x,y} \left(\xi \omega_{a,b,x,y}^+ q \int_{\Lambda'} d\lambda \frac{p(\lambda)}{q} p(x, y|\lambda) p(a|x, \lambda) p(b|y, \lambda) - \eta \omega_{a,b,x,y}^- q \int_{\Lambda'} d\lambda \frac{p(\lambda)}{q} p(x, y|\lambda) p(a|x, \lambda) p(b|y, \lambda) \right) \\ &\quad + \sum_{a,b,x,y} [\xi \omega_{a,b,x,y}^+ (1 - q) p(a, b, x, y|\Lambda'') - \eta \omega_{a,b,x,y}^- (1 - q) p(a, b, x, y|\Lambda'')], \end{aligned} \quad (15)$$

where $p(a, b, x, y|\Lambda'') = \int_{\Lambda''} d\lambda \frac{p(\lambda)}{1-q} p(a, b, x, y|\lambda')$ defines the average statistics in the case where λ belongs to the set Λ'' . The first line is simply the renormalized measurement-dependent inequality for the local variables satisfying the condition (8). Hence its local bound is $q\xi\eta\beta_{\text{loc}}$, where q comes from the normalization of the hidden-variable probability distribution. In the second line, the condition (8) is not satisfied; nevertheless, $p(a, b, x, y) \geq 0$ and therefore the second term is upper bounded by 0. The first term can also be bounded by noting that

$$\sum_{a,b,x,y} \xi \omega_{a,b,x,y}^+ (1 - q) p(a, b, x, y|\Lambda'') \leq \xi(1 - q)\omega_{\text{max}}^+, \quad (16)$$

where

$$\omega_{\text{max}}^+ = \max_{a,b,x,y} \omega_{a,b,x,y}^+. \quad (17)$$

This implies that the local bound for the strong measurement-dependent LHV models is

$$\beta_{\text{loc}}^{\text{SMD}} \equiv q\xi\eta\beta_{\text{loc}} + (1 - q)\xi\omega_{\text{max}}^+. \quad (18)$$

Thus, for hidden variables satisfying the condition (8) when $\lambda \in \Lambda'$ and Λ' satisfying condition (10), the bound of the

measurement-dependent Bell inequality becomes

$$\tilde{\mathcal{I}} \leq \beta_{\text{loc}}^{\text{SMD}}. \quad (19)$$

The next part consists in evaluating the quantum bound of the expression $\tilde{\mathcal{I}}_{\xi,\eta}$. For this, it is useful to know the distribution of measurement inputs, which can be observed in the Bell test. From now on, let us assume that all possible input pairs appear with equal probabilities, i.e., $p(x, y) = 1/m_{\text{A}}m_{\text{B}}$. From the definitions (12) and (13) we get the quantum bound

$$\beta_q^{\text{SMD}} = \frac{1}{m_{\text{A}}m_{\text{B}}} (\xi\beta_q^+ - \eta\beta_q^-). \quad (20)$$

Strong measurement-dependent nonlocality is then observed when $\beta_q^{\text{SMD}} > \beta_{\text{loc}}^{\text{SMD}}$. Below we discuss when such a Bell violation is possible.

For most Bell inequalities, we have that $\beta_q < \beta_q^+ < m_{\text{A}}m_{\text{B}}\omega_{\text{max}}^+$. It follows that one can detect strong measurement-dependent nonlocality for some limited range of parameters of the model. Notably, one will require that $q > q_0$, where q_0 is a strictly positive constant. This means that the rounds where the local variable cannot perfectly determine the inputs have a certain minimal probability

to occur. This is, for example, the case for the quantum correlations presented in Ref. [19].

It is however possible to construct quantum correlations that are incompatible with any local model with strong MDL, i.e., considering any parameter values of $q > 0$ and $\xi > 0$. We now present an example of such quantum correlations. Our starting point is quantum nonlocal correlation arising from the Peres-Mermin (PM) magic-square game [25,26], which was proposed in Ref. [27]. This work also constructs a Bell inequality to detect these nonlocal correlations. The key property of this Bell inequality is that the quantum bound coincides with the algebraic one, a feature sometimes referred to as quantum pseudotelepathy.

Following Ref. [27], we consider a bipartite Bell test with three inputs per party. For each input, each party outputs two bits, defined $a = (a_1, a_2)$ and for Bob with $b = (b_1, b_2)$, with $a_i, b_j \in \{-1, 1\}$. For our purpose, we need to construct a variant of the Bell inequality of Ref. [27]. Let us first introduce two types of correlators

$$\hat{C}_{x,y}^{f(a,b)} = \sum_{a,b} 2^{[f(a,b)+1]/2} (-16)^{-[f(a,b)-1]/2} p(a, b|x, y), \quad (21)$$

$$\bar{C}_{x,y}^{f(a,b)} = \sum_{a,b} 16^{[f(a,b)+1]/2} (-2)^{-[f(a,b)-1]/2} p(a, b|x, y). \quad (22)$$

While it is common to use correlators taking values in the interval $[-1, 1]$ for Bell inequalities, note that the above modified correlators take values in the following intervals: $-16 \leq \hat{C}_{x,y}^{f(a,b)} \leq 2$ and $-2 \leq \bar{C}_{x,y}^{f(a,b)} \leq 16$ (it will become clear below why we need this). We now construct the Bell expression

$$\mathcal{I} = \hat{C}_{1,1}^{a_1 b_1} + \hat{C}_{1,2}^{a_2 b_1} + \hat{C}_{2,1}^{a_1 b_2} + \hat{C}_{2,2}^{a_2 b_2} + \hat{C}_{1,3}^{a_1 a_2 b_1} + \hat{C}_{2,3}^{a_1 a_2 b_2} + \hat{C}_{3,3}^{a_1 a_2 b_1 b_2} + \hat{C}_{3,1}^{a_1 b_1 b_2} - \bar{C}_{3,2}^{a_2 b_1 b_2}, \quad (23)$$

with $\omega_{\max}^+ = 2$. Using the quantum strategy based on the PM magic-square game as in [27], we get a nonlocal distribution that reaches $\mathcal{I} = 18$; hence we have that $\beta_q = \beta_q^+ = m_A m_B \omega_{\max}^+$.

Now the important property for our purpose is that the local bound of \mathcal{I} is $\beta_{\text{loc}} = 0$ (hence the use of modified correlators), which simply follows from inspection over all deterministic strategies.

We can now apply the method outlined above to adapt this Bell inequality to the measurement-dependent scenario, namely,

$$\tilde{\mathcal{I}}_{\xi,\eta} = \hat{C}_{1,1,\xi,\eta}^{a_1 b_1} + \hat{C}_{1,2,\xi,\eta}^{a_2 b_1} + \hat{C}_{2,1,\xi,\eta}^{a_1 b_2} + \hat{C}_{2,2,\xi,\eta}^{a_2 b_2} + \hat{C}_{1,3,\xi,\eta}^{a_1 a_2 b_1} + \hat{C}_{2,3,\xi,\eta}^{a_1 a_2 b_2} + \hat{C}_{3,3,\xi,\eta}^{a_1 a_2 b_1 b_2} + \hat{C}_{3,1,\xi,\eta}^{a_1 b_1 b_2} - \bar{C}_{3,2,\xi,\eta}^{a_2 b_1 b_2}, \quad (24)$$

where $\hat{C}_{x,y,\xi,\eta}^{f(a,b)}$ and $\bar{C}_{x,y,\xi,\eta}^{f(a,b)}$ are now weighted correlators of joint distributions, defined as

$$\hat{C}_{x,y,\xi,\eta}^{f(a,b)} = \sum_{a,b} (2\xi)^{[f(a,b)+1]/2} (-16\eta)^{-[f(a,b)-1]/2} p(a, b, x, y), \quad (25)$$

$$\bar{C}_{x,y,\xi,\eta}^{f(a,b)} = \sum_{a,b} (16\eta)^{[f(a,b)+1]/2} (-2\xi)^{-[f(a,b)-1]/2} p(a, b, x, y). \quad (26)$$

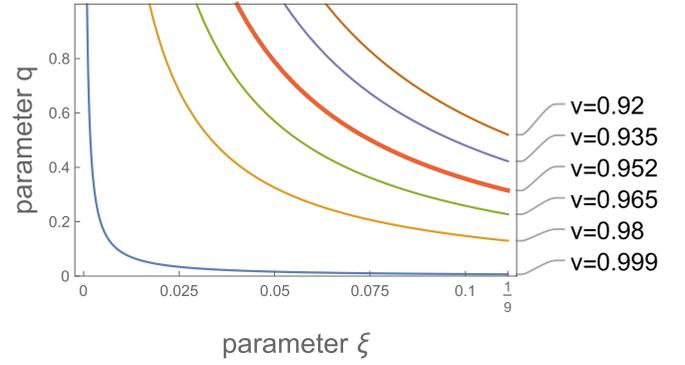


FIG. 2. Graph depicting nonlocality in the presence of a strong measurement dependence attained by the states $|\phi_v\rangle\langle\phi_v| = v|\phi_+\rangle\langle\phi_+| + (1-v)\mathbb{1}/16$, where $|\phi_+\rangle = \sum_{j=0}^3 |jj\rangle/2$. The measurements correspond to those used to obtain the maximal violation of the Mermin-Peres magic-square game. For given visibility v , nonlocality is observed if the values of the parameters ξ and q belong to the region above the corresponding curve. The parameter η is taken to be equal to $2/9 - \xi$. The red bold curve corresponds to a visibility comparable to that in the experiment of Ref. [27], based on a hyperentangled state.

Note that for $\tilde{\mathcal{I}}_{\xi,\eta}$ we have $\omega_{\max}^+ = 2$; hence from Eq. (18) we obtain the local bound $\beta_{\text{loc}}^{\text{SMD}} = 2(1-q)\xi$, while the quantum bound is $\beta_q^{\text{SMD}} = 2\xi$. Hence we obtain a quantum violation for any parameter values $q > 0$ and $\xi > 0$. In Fig. 2 we show for which values of parameters ξ and q nonlocality with strong measurement dependence can be observed if noisy states are used to violate the Mermin-Peres magic-square game.

In Appendix B we also present a similar construction based on the Mermin inequality, considering a tripartite Bell test.

IV. CONNECTION TO RANDOMNESS AMPLIFICATION

While the question of observing quantum nonlocality under conditions of relaxed measurement independence is of rather fundamental nature, these ideas also have implications from a more applied perspective. Indeed, one prominent application of quantum nonlocality is the task of generating randomness. This process is known as device-independent quantum random number generation (DIQRNG), which explores the possibility to generate certified random numbers under minimal assumptions on the devices (see, e.g., [28] for a recent review).

The most relevant aspect of DIQRNG here is the task of device-independent randomness amplification [29]. The idea is to perform a Bell test using imperfect sources of randomness to generate the inputs and being able to certify more randomness in the outputs, hence amplifying randomness via nonlocal correlations. Previous works in this direction have demonstrated randomness amplification considering so-called Santha-Vazirani sources for inputs [29–33]. Interestingly, randomness amplification can be directly connected to the measurement-dependent nonlocality [22], specifically to the approach of Ref. [19], which leads to protocols with improved performance.

Given this strong connection, we believe that our approach can have direct consequences in the context of randomness

amplification. In particular, our results should translate into stronger forms of randomness amplifications, going beyond Santha-Vazirani sources. Another interesting question is to understand how our approach connects to the work of Ref. [15], where min-entropy sources are considered.

V. DISCUSSION

We explored the question of observing quantum nonlocality considering local models with relaxed measurement independence. Using the approach developed by Pütz *et al.* [19], we showed that the requirements in terms of measurement independence can still be considerably relaxed and presented an example of quantum correlations featuring nonlocality in the presence of strong measurement dependence. This suggests that nonlocality could be demonstrated with strings of measurement settings having only a very small amount of global min-entropy.

The main open question is arguably whether our work has now reached the absolute minimal requirements in terms of measurement independence allowing for quantum nonlocality. Of course, this question is only meaningful within a certain framework. For the approach developed by Pütz *et al.* [19], we believe that our work has identified the minimal requirements. An interesting question is now to see how this result may impact applications in randomness amplification.

Finally, our work has interesting consequences also from the point of view of network nonlocality. Following the construction of Ref. [23], one can embed our example of strong measurement-dependent quantum nonlocality in a triangle network without inputs. In this way, one obtains a quantum distribution that cannot be reproduced classically even when the three sources of the triangle network are perfectly correlated in almost all rounds of the Bell test.

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APPENDIX A: PROOF OF EQ. (9)

In this Appendix we prove the bound given in Eq. (9). The form of the functional $\tilde{\mathcal{I}}_{\xi,\eta}$ is given in Eq. (14). Taking into account Eq. (2), we get

$$\begin{aligned} \tilde{\mathcal{I}}_{\xi,\eta} = & \sum_{a,b,x,y} \left(\xi \omega_{a,b,x,y}^+ \int_{\lambda} d\lambda p(\lambda) p(x, y|\lambda) p(a, b|x, y, \lambda) \right. \\ & \left. - \eta \omega_{a,b,x,y}^- \int_{\lambda} d\lambda p(\lambda) p(x, y|\lambda) p(a, b|x, y, \lambda) \right). \end{aligned} \quad (\text{A1})$$

Given that the local hidden variable satisfies Eqs. (4) and (8), we obtain

$$\begin{aligned} \tilde{\mathcal{I}}_{\xi,\eta} & \leq \sum_{a,b,x,y} \left(\xi \omega_{a,b,x,y}^+ \int_{\lambda} d\lambda p(\lambda) \eta p(a, b|x, y, \lambda) \right. \\ & \quad \left. - \eta \omega_{a,b,x,y}^- \int_{\lambda} d\lambda p(\lambda) \xi p(a, b|x, y, \lambda) \right) \\ & = \xi \eta \sum_{a,b,x,y} \left(\omega_{a,b,x,y}^+ \int_{\lambda} d\lambda p(\lambda) p(a|x, \lambda) p(b|y, \lambda) \right. \\ & \quad \left. - \omega_{a,b,x,y}^- \int_{\lambda} d\lambda p(\lambda) p(a|x, \lambda) p(b|y, \lambda) \right) \\ & = \xi \eta \sum_{a,b,x,y} \omega_{a,b,x,y} p_{\text{loc}}(a, b|x, y) \\ & \leq \xi \eta \beta_{\text{loc}}. \end{aligned} \quad (\text{A2})$$

In the third and fourth lines we used conditions (8) and (4). In the fifth line we noticed that both integrals are just expressions for the local behavior given in (5). Finally, in the last line we used the local bound for the original Bell inequality, which we used to build the functional (14).

APPENDIX B: QUANTUM NONLOCALITY IN THE PRESENCE OF A STRONG MEASUREMENT DEPENDENCE ACHIEVED BY THE GREENBERGER-HORNE-ZEILINGER STATE

Here we show another example of quantum correlations featuring strong MDL for any parameter values $q > 0$ and $\xi > 0$. This is based on a tripartite Bell test, involving a third party, Charlie, whose input is denoted by z and output by c . The main idea is to start from the Mermin Bell inequality [26], which also has the property that the quantum bound is equal to the algebraic one. The optimal quantum strategy is that of the famous Greenberger-Horne-Zeilinger (GHZ) paradox.

Like our construction in the main text, we must however first modify the Mermin inequality in a form that is suitable for our purpose. This implies using suitable definitions for correlators. For a given set of inputs x, y, z we define the weighted correlators

$$\hat{C}_{x,y,z} = \sum_{a,b,c} 1^{a \oplus b \oplus c} (-3)^{a \oplus b \oplus c \oplus 1} p(a, b, c|x, y, z), \quad (\text{B1})$$

$$\bar{C}_{x,y,z} = \sum_{a,b,c} 3^{a \oplus b \oplus c} (-1)^{a \oplus b \oplus c \oplus 1} p(a, b, c|x, y, z), \quad (\text{B2})$$

with $a, b, c \in \{0, 1\}$. We observe that $-3 \leq \hat{C}_{x,y,z} \leq 1$ and $-1 \leq \bar{C}_{x,y,z} \leq 3$. From this we now construct an alternative version of the Mermin Bell inequality, namely,

$$\mathcal{I} = \hat{C}_{0,0,0} - \bar{C}_{1,1,0} - \bar{C}_{1,0,1} - \bar{C}_{0,1,1}. \quad (\text{B3})$$

The quantum bound for this inequality is $\mathcal{I} = 4$, which can be attained by considering a three-qubit GHZ state and each

party performing local measurements in the Pauli X and Y basis (similarly as in the well-known GHZ paradox, giving maximal violation of the Mermin Bell inequality). It turns out that this quantum bound coincides with the algebraic bound of the inequality, i.e., $\beta_q = \beta_{\text{alg}}$, similarly to the standard Mermin inequality. The notable difference, which is convenient for our purpose, is that the local bound is now $\beta_{\text{loc}} = 0$.

We can now apply the method outlined above to adapt this inequality to the measurement-dependent scenario, namely,

$$\tilde{\mathcal{I}} = \hat{C}_{0,0,0}^{\eta,\xi} - \bar{C}_{1,1,0}^{\eta,\xi} - \bar{C}_{1,0,1}^{\eta,\xi} - \bar{C}_{0,1,1}^{\eta,\xi}, \quad (\text{B4})$$

where $\hat{C}_{x,y,z}^{\eta,\xi}$ and $\bar{C}_{x,y,z}^{\eta,\xi}$ are now weighted correlators of joint distributions, defined as

$$\hat{C}_{x,y,z}^{\eta,\xi} = \sum_{a,b,c} (\xi)^{a \oplus b \oplus c} (-3\eta)^{a \oplus b \oplus c \oplus 1} p(a, b, c, x, y, z), \quad (\text{B5})$$

$$\bar{C}_{x,y,z}^{\eta,\xi} = \sum_{a,b,c} (3\eta)^{a \oplus b \oplus c} (-\xi)^{a \oplus b \oplus c \oplus 1} p(a, b, c, x, y, z). \quad (\text{B6})$$

Finally, we find the local and quantum bounds for this new inequality: $\beta_{\text{loc}}^{\text{SMD}} = (1-q)\xi$ and $\beta_q^{\text{SMD}} = \xi$. Hence we obtain a quantum violation for any parameter values $q > 0$ and $\xi > 0$.

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