# Quantifying environment nonclassicality in dissipative open quantum dynamics

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Open quantum systems are inherently coupled to their environments, which in turn also obey quantum dynamical rules. Restricting ourselves to dissipative dynamics, here we propose a measure that quantifies how far the action of an environment on a system departs from the influence of classical noise fluctuations. It is based on the lack of commutativity between the initial reservoir state and the total system-environment Hamiltonian. Independently of the nature of the dissipative system evolution, Markovian or non-Markovian, the measure can be written in terms of the dual propagator that defines the evolution of system operators. The physical meaning and properties of the proposed definition are discussed in detail and also characterized through different paradigmatic dissipative Markovian and non-Markovian open quantum dynamics.

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## I. INTRODUCTION

Open quantum systems are inherently coupled to their supporting environments [1,2]. This interaction induces time-irreversible behaviors such as dissipation and decoherence. These phenomena have been studied in a wide range of systems such as, for example, in quantum optics [3], magnetic resonance [4,5], solid state devices [6], and quantum sensing [7].

In a full microscopic description not only the system but also the environment obeys quantum dynamical rules. Nevertheless, depending on system and environment properties, as well as on the studied regimes, the system fluctuations induced by the environment influence can be well approximated by the action of classical stochastic fields. In fact, open quantum systems driven by classical noises is a well-established physical modeling [4–7] that has been characterized from different perspectives. Many specific studies rely on assuming, for example, Gaussian [8–12] or telegraphic noises [13–15].

In the context of open quantum system theory it is of interest to establish the conditions under which the environment action can be approximated by classical noises. For example, the possibility of representing the system evolution in terms of a statistical superposition of unitary dynamics has been recently explored [16–19]. When the open system dynamics leads only to dephasing [20–26], the possibility of detecting quantum entanglement between the system and the environment [22–25] provides a strong criterion for determining when a classical noise representation is appropriate or not. In addition, many related contributions have focused on this problem, providing general or particular conditions under which a classical representation is a valid approximation [27–41].

The main goal of this work is to introduce a measure that quantifies how much the environment action departs from the influence of classical noise fluctuations. This result provides an interesting insight and contribution in the described research line. Unlike previous analysis (see Refs. [16–25]), here we are mainly interested in dissipative open quantum dynamics; that is, the environment not only induces decoherence but is also able to induce (energy) transitions between the system states.

The proposed measure has a clear physical motivation related to the quantumness of the microscopic dynamics. In fact, the main ingredient is the lack of commutativity between the initial reservoir state and the total *system-environment* Hamiltonian. In contrast, there are alternative proposals where the quantumness of a qubit channel is established by the noncommutativity of two *system* output states [42]. Due to its alternative microscopic definition, the present approach leads to predictions consistent with former results [5,43] in which the influence of a quantum reservoir, in a high-temperature limit, is approximated by noisy (commuting) scalar functions.

The formalism applies independently of the system dynamical regime, that is, Markovian or non-Markovian. In fact, regardless of which approach is used to define memory effects, operational [44,45] or nonoperational [46,47], the proposed measure can be written in terms of the dual evolution associated with system operators. Thus, it can be consistently defined for Markovian Lindblad equations [48] but also, in the same way, outside this regime. The proposal is characterized in detail through its general properties and also through specific dissipative Markovian and non-Markovian dynamics.

The paper is outlined as follows. In Sec. II we motivate and formulate the environment nonclassicality measure. In Sec. III the measure is characterized for some general classes of open quantum dynamics. In Sec. IV we study its behavior for specific Markovian and non-Markovian dissipative open system dynamics. In Sec. V we provide the conclusions.

## II. MEASURE OF ENVIRONMENT NONCLASSICALITY

Here we introduce the environment nonclassicality measure, providing some of its general properties.

 

### A. Physical motivation

We consider a system ($s$) that interacts with its environment ($e$). Their quantum dynamics is set by a total Hamiltonian $H = H_s + H_e + H_I$, where $H_s$ and $H_e$ are the system and environment Hamiltonians, respectively, while $H_I$ defines their mutual interaction. The system density matrix $\rho_t$ can be written as

$$\rho_t = \mathbb{G}_{t,0}[\rho_0] \equiv \mathrm{Tr}_e[e^{-iHt}(\rho_0 \otimes \sigma_0)e^{+iHt}]. \quad (1)$$

Here, $\mathrm{Tr}[\cdots]$ is the trace operation. In addition, $\rho_0$ and $\sigma_0$ are the initial system and environment states, respectively. To avoid introducing intrinsic quantum features that cannot be recovered in a classical noise approximation we only consider uncorrelated $s$-$e$ initial conditions.

The quantum nature of the system and its environment can be read straightforwardly from Eq. (1). In fact, all objects appearing in this expression can be written as matrices that, in general, do not commute. Focusing on the environment, we argue that the nature of its influence over the system is inherently quantum because, in general, its initial state does not commute with the total Hamiltonian, that is, $[H, \sigma_0] \neq 0$. Supporting this argument, when the initial environment state approaches the identity matrix $\sigma_0 \simeq \mathrm{I}_e$, which implies $[H, \sigma_0] \approx 0$, its action over the system can be represented by classical noises. This result, which is rederived in Appendix A, is well known in the context of magnetic resonance [5] and also has been characterized when expressing the system evolution in terms of stochastic wave vectors [43]. For systems coupled to thermal environments, the property $[H, \sigma_0] \approx 0$ becomes valid in a high-temperature limit.

Under the motivation of the above perspective, we rewrite the system state [Eq. (1)] as

$$\rho_t = \mathrm{Tr}_e[(e^{-iHt}\rho_0 e^{+iHt})\sigma_0] + \mathrm{Tr}_e[(e^{-iHt}\rho_0 e^{+iHt})\Delta\sigma_t], \quad (2)$$

where $\Delta\sigma_t \equiv e^{-iHt}\sigma_0 e^{+iHt} - \sigma_0$. We identify the first term in the right-hand side of Eq. (2) with the "classical" contribution of the environment influence. In fact, when $[H, \sigma_0] \approx 0$, the first contribution does not vanish while the second one fades out (vanishes) correspondingly. Interestingly, when describing the system evolution in a weak interaction and Markovian limits, the previous splitting recovers the structure of quantum master equations [5,49,50] proposed for dealing with a high-temperature approximation [5].

Even when the property $\mathrm{Tr}_s[\rho_t] = 1$ is satisfied, each contribution in Eq. (2) does not preserve trace of the system by itself. Thus, for measuring the incompatibility of the environment action with respect to the action of classical noises we arrive at the (dimensionless) time-dependent "noncommutativity measure"

$$Q_t \equiv \mathrm{Tr}_{se}[(e^{-iHt}\rho_0 e^{+iHt})\sigma_0]. \quad (3)$$

It corresponds to the trace over the system degrees of freedom of the first term in the right-hand side of Eq. (2).

### B. Degree of environment quantumness

The noncommutativity measure $Q_t$ has some desirable properties. For example, when $[H, \sigma_0] = 0$ it follows $Q_t = 1$. Therefore, this value is associated with classicality. On the other hand, it is straightforward to obtain

$$\frac{dQ_t}{dt} = +(i)\mathrm{Tr}_{se}[(e^{-iHt}\rho_0 e^{+iHt})[H, \sigma_0]], \quad (4)$$

while its $n$ time derivative reads

$$\frac{d^n Q_t}{dt^n} = +(i)^n \mathrm{Tr}_{se}[(e^{-iHt}\rho_0 e^{+iHt})[H^{(n)}, \sigma_0]], \quad (5)$$

where $[H^{(1)}, \sigma_0] = [H, \sigma_0]$, $[H^{(2)}, \sigma_0] = [H, [H, \sigma_0]]$, and, in general, $[H^{(n)}, \sigma_0] = [H, [H^{(n-1)}, \sigma_0]]$. From these expressions we conclude that the time derivatives of $Q_t$ are proportional to the lack of commutativity of $\sigma_0$ with higher nested commutators of the total Hamiltonian $H$.

In spite of the previous properties, the definition of $Q_t$ [Eq. (3)] is symmetrical in the initial system and environment states. In particular, when $\rho_0 = \mathrm{I}_s/\dim(\mathcal{H}_s)$, where $\mathrm{I}_s$ is the identity matrix and $\dim(\mathcal{H}_s)$ is the dimension of the system Hilbert space, it follows that $Q_t = 1$.

The difference between the roles played by the system and the environment is introduced as follows. As usual, we assume that the system state [obtained from the unitary microscopic dynamics, Eq. (1)] always achieves a stationary regime. This property, in general, demands a reservoir with an infinite number of degrees of freedom such that its eigenfrequencies form a continuous spectrum [1]. Thus, we define a "degree of environment quantumness," denoted as $D_Q$, which reads

$$D_Q \equiv \max_{[\rho_0]} \left| \lim_{t\to\infty} \int_0^t dt' \frac{dQ(t')}{dt'} \right| = \max_{[\rho_0]} |\lim_{t\to\infty} Q_t - 1|. \quad (6)$$

The stationary limit of $Q_t$ is granted by the above assumptions and it has been used that $Q_0 = 1$. The maximization is over the initial system state $\rho_0$. With this definition at hand, environment classicality is associated with $D_Q = 0$. Moreover, this parameter allows us to study the (time-dependent) noncommutativity measure $Q_t$ [Eq. (3)] by choosing system initial conditions that maximize $D_Q$.

### C. Definition in terms of the operator dual evolution

Given a system operator $A$, by definition its expectation value reads $\langle A\rangle_t \equiv \mathrm{Tr}_s[\rho_t A]$. It can alternatively be written as $\langle A\rangle_t = \mathrm{Tr}_s[\rho_0 A_t] = \mathrm{Tr}_s[\rho_0 \mathbb{G}^{\star}_{t,0}[A_0]]$, where the dual propagator $\mathbb{G}^{\star}_{t,0}$ for the operator evolution, from Eq. (1), is given by

$$A_t = \mathbb{G}^{\star}_{t,0}[A_0] \equiv \mathrm{Tr}_e[e^{+iHt}A_0 e^{-iHt}\sigma_0]. \quad (7)$$

Therefore, from this expression and Eq. (3) it follows that the noncommutativity measure $Q_t$ can be written as

$$Q_t = \mathrm{Tr}_s[\mathbb{G}^{\star}_{-t,0}[\rho_0]], \quad (8)$$

which only depends on the operator dual evolution and the initial system state. From this expression it follows that $Q_t/\dim(\mathcal{H}_s)$ can be read as the expectation value of the "operator" $\rho_0$ at time $t$ given that the "initial system state" is $\mathrm{I}_s/\dim(\mathcal{H}_s)$ [see Eqs. (7) and (8)]. This result also ensures that $\lim_{t\to\infty} Q_t$ exists [Eq. (6)] whenever $\lim_{t\to\infty} \rho_t$ exists [Eq. (1)].

Taking into account that $\rho_0$ is a positive definite operator, the previous equivalent interpretation of $Q_t$ allow us to obtain

$$0 \leqslant Q_t \leqslant \dim(\mathcal{H}_s). \quad (9)$$

This inequality is valid independently of the system and environment initial conditions and also of the particular microscopic model. In addition, Eq. (9) implies

$$0 \leqslant D_Q \leqslant \dim(\mathcal{H}_s) - 1, \tag{10}$$

where classicality corresponds to $D_Q = 0$. Both constraints [Eqs. (9) and (10)] imply that the quantumness of the environment influence is bounded by the dimension of the system Hilbert space.

### D. Optimal initial states

By analyzing the stationary regime of Eq. (3), or alternatively Eq. (8), it follows that

$$\lim_{t\to\infty} Q_t = \dim(\mathcal{H}_s)\text{Tr}_s[\tilde{\rho}_\infty \rho_0], \tag{11}$$

where the stationary system state is $\tilde{\rho}_\infty \equiv \lim_{t\to\infty} \tilde{\rho}_t$. The upper tilde symbol represents a time-reversal operation, $t \leftrightarrow -t$. Equations (6) and (11) imply that

$$\frac{D_Q}{\dim(\mathcal{H}_s)} = \max_{[\rho_0]} \left| \text{Tr}_s[\tilde{\rho}_\infty \rho_0] - \frac{1}{\dim(\mathcal{H}_s)} \right|. \tag{12}$$

Thus, $D_Q$ can be seen as a functional of $\rho_0$ that is parametrized by the stationary state $\rho_\infty$. The optimal state $\rho_0$ that maximizes $D_Q$ is obtained below.

The expression (12) provides a clear geometric interpretation of $D_Q$. Introducing a basis of vectors $\{|i\rangle\}$ where the stationary system state is a diagonal matrix, $\tilde{\rho}_\infty = \sum_i \lambda_i |i\rangle\langle i|$, with $i = 1, \ldots, \dim(\mathcal{H}_s)$, it follows that $D_Q = \max_{\{p_i\}} |\dim(\mathcal{H}_s) \sum_i \lambda_i p_i - 1|$, where $p_i \equiv \langle i|\rho_0|i\rangle$. Therefore, $D_Q$ is the maximal (absolute) value assumed by the hyperplane defined by the variables $\{p_i\}$ when restricted to the domain $\sum_i p_i = 1$. It is straightforward to bound the main contribution to $D_Q$ as $\sum_i \lambda_i p_i \leqslant (\sum_i p_i) \max(\{\lambda_i\}) = \max(\{\lambda_i\})$. This boundary is always achieved by choosing $\rho_0$ as the eigenprojector of $\tilde{\rho}_\infty$ with the maximal eigenvalue. Thus, we conclude that

$$D_Q = \dim(\mathcal{H}_s) \max(\{\lambda_i\}) - 1, \quad \rho_0 = |i_{\max}\rangle\langle i_{\max}|, \tag{13}$$

where $\max(\{\lambda_i\})$ is the largest eigenvalue of the stationary state $\tilde{\rho}_\infty \equiv \lim_{t\to\infty} \tilde{\rho}_t$, while $|i_{\max}\rangle$ is the corresponding eigenstate $\tilde{\rho}_\infty |i_{\max}\rangle = \max(\{\lambda_i\})|i_{\max}\rangle$. This expression for $D_Q$ is valid when the stationary state does not depend on the initial condition. Moreover, we note that, in general, more than one initial state, $\rho_0 \neq |i_{\max}\rangle\langle i_{\max}|$, can lead to this extreme value (see Sec. IV). On the other hand, it is simple to realize that when the time-reversal operation is equivalent to conjugation Eq. (13) is valid with $\tilde{\rho}_\infty \to \rho_\infty$.

## III. CLASSICALITY FOR DIFFERENT CLASSES OF OPEN SYSTEM DYNAMICS

In this section we characterize the previous proposal for different classes of open quantum system dynamics where the noncommutativity measure indicates classicality $Q_t = 1\ \forall t$, which, in turn, implies $D_Q = 0$.

### A. Hamiltonian ensembles

In Refs. [16–19] the classicality of the environment action was related to the possibility of representing the open system dynamics in terms of Hamiltonian ensembles, that is, a statistical superposition of different system unitary dynamics. This kind of dynamics is recovered in the present approach after assuming that $[H, \sigma_0] = 0$. In fact, introducing a complete basis of environment states $\{|e\rangle\}$ where the initial state is diagonal, $\sigma_0 = \sum_e p_e |e\rangle\langle e|$, with $p_e = \langle e|\sigma_0|e\rangle$, the system density matrix [Eq. (1)] can be written as

$$\rho_t = \sum_e p_e e^{-itH_s^{(e)}} \rho_0 e^{+itH_s^{(e)}}, \quad \Rightarrow \quad Q_t = 1, \tag{14}$$

where the system Hamiltonians are $H_s^{(e)} \equiv H_s + \langle e|(H_e + H_I)|e\rangle$. The equality $Q_t = 1$ follows from Eq. (3) and is valid independently of the system initial condition. The degree of quantumness [Eq. (6)] also indicates the presence of a classical environment influence, $D_Q = 0$.

### B. Stochastic Hamiltonians

The coupling of a quantum system to classical noises is usually modeled by (system) stochastic Hamiltonians $H_{\text{st}}(t)$. Their time dependence takes into account the action of classical fluctuating external fields. The noises can have arbitrary statistical properties (see, for example, Refs. [8–15]). Even more, their correlation can also be arbitrary, that is, $\delta$ correlated (white noises) or color ones (finite correlation times).

For each realization of the noises, we introduce the stochastic propagator $\mathcal{T}_{\text{st}}(t) = \lceil \exp -i \int_0^t dt' H_{\text{st}}(t') \rceil$, where $\lceil \cdots \rceil$ means a time-ordering operation. The system density matrix can then be written as

$$\rho_t = \overline{\mathcal{T}_{\text{st}}(t)\rho_0 \mathcal{T}_{\text{st}}^\dagger(t)}, \quad \Rightarrow \quad Q_t = 1, \tag{15}$$

where the overline denotes an average over noise realizations. The equality $Q_t = 1$ is valid for arbitrary system initial conditions. It can be derived by using the alternative definition in terms of the dual evolution [Eq. (8)]. Consistently, in this case $D_Q = 0$.

### C. Collisional dynamics

Open quantum systems dynamics, in Markovian and non-Markovian regimes, can also be modeled through collisional models [51,52]. The underlying stochastic dynamics consists of free propagation with the system Hamiltonian added to the action of instantaneous transformations occurring at successive random times. The (stochastic) state of the system conditioned on the occurrence of $n$ collisional events can be written as

$$\rho_t^{(n)} = \mathcal{G}_{t-t_n} \mathcal{E} \mathcal{G}_{t_n - t_{n-1}} \cdots \mathcal{E} \mathcal{G}_{t_2 - t_1} \mathcal{E} \mathcal{G}_{t_1}[\rho_0]. \tag{16}$$

Here, $\mathcal{G}_t$ is the propagator of the free evolution, $\mathcal{G}_t[\bullet] \equiv \exp[-itH_s] \bullet \exp[+itH_s]$, while $\mathcal{E}$ is an arbitrary completely positive trace-preserving superoperator. The times $\{t_i\}_{i=1}^n$ are random variables in the interval $(0, t)$. The state of the system follows as

$$\rho_t = \sum_{n=0}^{\infty} \overline{\rho_t^{(n)}}, \tag{17}$$

where the overline means an average over the random collisional times. The dual operator dynamics can be written in a similar way. In Appendix B we develop a formal derivation.

Using the definition (8), it is possible to conclude (see Appendix B) that

$$\mathrm{Tr}_s[\mathcal{E}^{\star}[A]] = \mathrm{Tr}_s[A], \quad \Rightarrow \quad Q_t = 1, \tag{18}$$

where the dual superoperator is defined from the relation $\mathrm{Tr}_s[A\mathcal{E}[\rho]] = \mathrm{Tr}_s[\rho\mathcal{E}^{\star}[A]]$, with $A$ being an arbitrary system operator. Thus, when the dual superoperator $\mathcal{E}^{\star}$ preserves trace the environment influence is classical, $D_Q = 0$. The condition (18) is fulfilled when $\mathcal{E}$ corresponds to a unitary transformation and, in general, is satisfied by unital maps (see below).

### D. Lindblad equations

When the system-environment coupling is weak and the time correlations of environment operators define the minor timescale of the problem, a Born-Markov approximation applies. Discarding nonsecular terms, the system evolution can be written as a Lindblad equation [1,48],

$$\frac{d\rho_t}{dt} = -i[\bar{H}, \rho_t] + \sum_{\mu\nu} a_{\mu\nu}\left(V_\mu \rho_t V_\nu^\dagger - \frac{1}{2}\{V_\nu^\dagger V_\mu, \rho_t\}_+\right). \tag{19}$$

Here, $\bar{H}$ is an effective system Hamiltonian that may include contributions induced by the interaction with the environment. $\{V_\mu\}$ are system operators, while the matrix of rate coefficients $\{a_{\mu\nu}\}$ defines a semipositive definite matrix. The anticommutator operation is defined as $\{a, c\}_+ \equiv (ac + ca)$.

The noncommutativity measure $Q_t$ can be calculated for the previous quantum master equation by using its definition in terms of the dual evolution [Eq. (8)]. For an arbitrary operator $A_t$ it reads [48]

$$\frac{dA_t}{dt} = +i[\bar{H}, A_t] + \sum_{\mu\nu} a_{\mu\nu}\left(V_\nu^\dagger A_t V_\mu - \frac{1}{2}\{V_\nu^\dagger V_\mu, A_t\}_+\right). \tag{20}$$

Consistently, notice that this evolution does not preserve trace. By solving Eq. (20) with the initial condition $A_t|_{t=0} = \rho_0$, the noncommutativity measure can be written as $Q_t = \mathrm{Tr}_s[\tilde{A}_t]$, where the tilde symbol takes into account the time-reversal operation $t \leftrightarrow -t$. In this way, the present approach can be applied in the Markovian regime where a Lindblad equation approximates the open system dynamics.

Interestingly, Lindblad equations that are compatible with the influence of classical noises have been characterized from a rigorous mathematical point of view [21,53]. The proposed structures were derived as commutative dilations of dynamical semigroups. Specifically, they correspond to Eq. (19) written in a diagonal base of operators ($a_{\mu\nu} = \delta_{\mu\nu}a_\mu$) with the constraints of Hermitian operators, $V_\mu = V_\mu^\dagger$, or, alternatively, unitary ones, $V_\mu^\dagger V_\mu = \mathrm{I}_s$. These solutions can be put in one-to-one correspondence with models based respectively on stochastic Hamiltonians [Eq. (15)] with white-noise fluctuations and collisional models [Eqs. (17) and (18)] with Poisson statistics between collisional events.

### E. Unital open system dynamics

A completely positive open system dynamics can always be written in a Kraus representation as [1]

$$\rho_t = \sum_\alpha T_\alpha \rho_0 T_\alpha^\dagger, \quad \sum_\alpha T_\alpha^\dagger T_\alpha = \mathrm{I}_s, \tag{21}$$

where the system operators $\{T_\alpha\}$ are time dependent, $T_\alpha = T_\alpha(t)$. The dynamics is defined as unital when, in addition, it is fulfilled that

$$\sum_\alpha T_\alpha T_\alpha^\dagger = \mathrm{I}_s, \quad \Rightarrow \quad Q_t = 1. \tag{22}$$

We notice that the result $Q_t = 1$, valid for arbitrary system initial conditions, follows from Eq. (8) and after noting that the dual operator evolution can be written as $A_t = \sum_\alpha T_\alpha^\dagger A_0 T_\alpha$, which implies that $\mathrm{Tr}_s[A_t] = \sum_\alpha \mathrm{Tr}_s[T_\alpha T_\alpha^\dagger A_0] = \mathrm{Tr}_s[A_0]$.

In general, it is possible to argue that any open quantum dynamics induced by coupling the system with stochastic classical degrees of freedom is always unital, which consistently implies $Q_t = 1$. In fact, the dynamics defined by Eqs. (15) and (17) can be read as "non-Markovian" extensions of the commutative dilations of dynamical semigroups obtained in Ref. [53]. With non-Markovian, here we mean considering non-white noises or non-Poisson statistics, respectively.

On the other hand, the inverse implication is not valid in general; that is, there exist unital dynamics that cannot be obtained by considering the action of classical stochastic fields. This property emerges, for example, in dephasing dynamics with $\dim(\mathcal{H}_s) \geqslant 3$ [21]. In addition, this feature has been related to the break of a time-reversal symmetry [26]. While these cases imply a limitation on the applicability of the indicator $Q_t$ and the related measure $D_Q$, the corresponding class of dynamics is well characterized. On the other hand, the examples studied in the next section explicitly demonstrate the consistence of the proposed approach.

## IV. EXAMPLES

Here we characterize the noncommutativity measure $Q_t$ and the degree of environment quantumness $D_Q$ for some specific dissipative Markovian and non-Markovian open quantum dynamics.

### A. Two-level system in contact with a thermal environment

We consider a two-level system interacting with a bosonic bath at temperature $T$. Its density matrix $\rho_t$ evolves as [1]

$$\frac{d\rho_t}{dt} = \frac{-i\omega_0}{2}[\sigma_z, \rho_t] + \kappa\left(\sigma\rho_t\sigma^\dagger - \frac{1}{2}\{\sigma^\dagger\sigma, \rho_t\}_+\right) + \zeta\left(\sigma^\dagger\rho_t\sigma - \frac{1}{2}\{\sigma\sigma^\dagger, \rho_t\}_+\right). \tag{23}$$

With $\sigma_z$ we denote the $z$ Pauli matrix. $\sigma$ and $\sigma^\dagger$ are the standard lowering and raising operators with respect to the eigenvectors of $\sigma_z$. Furthermore, $\kappa = \gamma(n_{\mathrm{th}} + 1)$ and $\zeta = \gamma n_{\mathrm{th}}$, where $\gamma$ is the natural decay rate and $n_{\mathrm{th}} = \exp(-\beta\hbar\omega_0)/[1 - \exp(-\beta\hbar\omega_0)]$ is the average number of thermal boson excitations at the natural frequency of the system, with $\beta = 1/kT$.

Using the alternative definition (8), jointly with the dual evolution (20), for arbitrary system initial conditions it is possible to obtain

$$Q_t = 1 + \langle\sigma_z\rangle_\infty \langle\sigma_z\rangle_0 [1 - e^{-t(\kappa+\zeta)}], \tag{24}$$

where the operator mean values are $\langle\sigma_z\rangle_0 = \mathrm{Tr}_s[\sigma_z\rho_0]$ and $\langle\sigma_z\rangle_\infty = \lim_{t\to\infty}\mathrm{Tr}_s[\sigma_z\rho_t] = (\zeta-\kappa)/(\zeta+\kappa) \leqslant 0$. In general, depending on the initial condition, as a function of time, $Q_t$ decays or grows in a monotonic way. In any of these cases, consistently with Eq. (9), it is fulfilled that $0 \leqslant Q_t \leqslant 2$.

From Eq. (24) it follows that $\lim_{t\to\infty} Q_t = 1 + \langle\sigma_z\rangle_\infty\langle\sigma_z\rangle_0$. This stationary value has maximal departure from the unit value when $\langle\sigma_z\rangle_0 = \pm 1$. Thus, the initial conditions that maximize the definition of $D_Q$ [Eq. (6)] are pure states, which in turn are eigenvectors of $\sigma_z$. This result is consistent with Eq. (13). The degree of environment quantumness finally reads

$$D_Q = |\langle\sigma_z\rangle_\infty| = \left|\frac{\zeta-\kappa}{\zeta+\kappa}\right| = \tanh\left(\beta\frac{\hbar\omega_0}{2}\right). \tag{25}$$

In the last equality we have used the dependence on temperature of the characteristic rates.

Equation (25) defines the degree of environment quantumness corresponding to the evolution (23). As a function of the inverse temperature $\beta$, it has the expected behaviors. In fact, in the limit of high temperatures it follows $\lim_{\beta\to 0} D_Q = 0$, which correctly means that the environment influence can be represented through classical noises [5,43] (see also Appendix A). In the limit of vanishing temperatures, $D_Q$ assumes its maximal value [Eq. (10)], $\lim_{\beta\to\infty} D_Q = 1$.

### B. Non-Markovian decay at zero temperature

Unlike the previous case, here we consider a dynamics where the Born-Markov approximation does not apply in general. The microscopic dynamics is defined by the Hamiltonians $H_s = (\omega_0/2)\sigma_z$ and $H_e = \sum_j \omega_k a_k^\dagger a_k$, while the interaction is set by $H_I = \sum_k (g_k\sigma^\dagger a_k + g_k^* \sigma a_k^\dagger)$. With $a_k$ and $a_k^\dagger$ we denote the annihilation and creation operators associated with each mode of the bosonic environment. Memory effects for this open dynamics has been studied from both nonoperational [54] and operational [55] approaches to quantum non-Markovianity.

The (two-level) system dynamics can be solved in an exact way by assuming that all modes of the environment start in their ground states, which is equivalent to a vanishing temperature assumption. The system density matrix reads [1]

$$\rho_t = \begin{pmatrix} \rho_0^{++}|c_t|^2 & \rho_0^{+-}\,c_t \\ \rho_0^{-+}\,c_t^* & \rho_0^{--} + \rho_0^{++}(1-|c_t|^2) \end{pmatrix}. \tag{26}$$

Here, $\rho_0^{ss'} \equiv \langle s|\rho_0|s'\rangle$, where $\{|s\rangle\} = |\pm\rangle$ are the eigenvectors of $\sigma_z$. The function $c_t$ is defined by $(d/dt)c(t) = -\int_0^t f(t-t')c(t')dt'$, where the memory kernel corresponds to the bath correlation function $f(t) \equiv \sum_k |g_k|^2 \exp[+i(\omega_0-\omega_k)t]$.

Using that $\langle A\rangle_t = \mathrm{Tr}_s[\rho_t A] = \mathrm{Tr}_s[\rho_0 A_t]$, from Eq. (26) it is possible to obtain the operator dual dynamics, which explicitly reads

$$A_t = \begin{pmatrix} A_0^{++}|c_t|^2 + A_0^{--}(1-|c_t|^2) & A_0^{+-}\,c_t^* \\ A_0^{-+}\,c_t & A_0^{--} \end{pmatrix}, \tag{27}$$

where $A_0^{ss'} \equiv \langle s|A_0|s'\rangle$. The noncommutativity measure $Q_t$ can be obtained from the relation (8), which here delivers

$$Q_t = 1 - \langle\sigma_z\rangle_0[1-|c_t|^2], \tag{28}$$

where $\langle\sigma_z\rangle_0 = \mathrm{Tr}_s[\sigma_z\rho_0]$.

From Eq. (28) it follows that $\lim_{t\to\infty} Q_t = 1 - \langle\sigma_z\rangle_0$. This limit assumes extreme values when $\langle\sigma_z\rangle_0 = \pm 1$. Therefore, the degree of environment quantumness is maximal [Eq. (10)],

$$D_Q = 1. \tag{29}$$

Consistently, this value also emerges from the Lindblad modeling Eq. (23) when the environment temperature vanishes [see Eq. (25)]. In addition, the expressions for $Q_t$, Eq. (24) with $\langle\sigma_z\rangle_\infty = -1$ and Eq. (28), assume the same structure. Non-Markovian effects appear through the temporal behavior of the decay function $|c_t|^2$, which, unlike the Markovian case, may develop oscillatory behaviors [1]. For example, assuming a Lorentzian spectral density, which implies the exponential correlation $f(t) = (\gamma/2\tau_c)\exp[-|t|/\tau_c]$, it follows that $c_t = e^{-t/2\tau_c}[\cosh(t\chi/2\tau_c) + \chi^{-1}\sinh(t\chi/2\tau_c)]$, where $\chi \equiv \sqrt{1-2\gamma\tau_c}$. In a weak-coupling limit $\gamma \ll 1/\tau_c$, where the correlation time $\tau_c$ of the bath is the minor timescale of the problem, a monotonic exponential decay is recovered, $c_t \simeq \exp(-\gamma t/2)$ [Eq. (24)].

### C. Resonance fluorescence

An optical two-level transition submitted to the action of a resonant external laser field can be well approximated by the evolution [3]

$$\frac{d\rho_t}{dt} = -i\frac{\Omega}{2}[\sigma_x, \rho_t] + \gamma\left(\sigma\rho_t\sigma^\dagger - \frac{1}{2}\{\sigma^\dagger\sigma, \rho_t\}_+\right). \tag{30}$$

Here, $\gamma$ is the natural decay rate while the frequency $\Omega$ is proportional to the intensity of the external excitation. With $\sigma_j (j = x, y, z)$ we denote the $j$ Pauli matrix. As before, $\sigma$ and $\sigma^\dagger$ are the standard lowering and raising operators. We observe that the effective environment action corresponds to a thermal bath at zero temperature. Next we study how the previous result $D_Q = 1$ [Eqs. (25)] is affected by the presence of the external excitation.

From Eqs. (8) and (20), in a Laplace domain $[f(u) = \int_0^\infty dt e^{-ut} f(t)]$, $Q_t$ is defined by the exact expression

$$Q_u = \frac{1}{u} - \langle\sigma_z\rangle_0 \frac{\gamma(2u+\gamma)}{u[(u+\gamma)(2u+\gamma)+2\Omega^2]} + \langle\sigma_y\rangle_0 \frac{2\gamma\Omega}{u[(u+\gamma)(2u+\gamma)+2\Omega^2]}, \tag{31}$$

where $\langle\sigma_j\rangle_0 = \mathrm{Tr}_s[\sigma_j\rho_0]$. Explicitly,

$$\langle\sigma_z\rangle_0 = (\rho_0^{++} - \rho_0^{--}), \quad \langle\sigma_y\rangle_0 = i(\rho_0^{+-} - \rho_0^{-+}). \tag{32}$$

In the time domain, from Eq. (31) it is possible to write

$$Q_t = 1 + \int_0^t \gamma dt' e^{-\frac{3}{4}\gamma t'}[\langle\sigma_z\rangle_0\, z(t') + \langle\sigma_y\rangle_0\, y(t')], \tag{33}$$

where the auxiliary functions are

$$y(t) \equiv 4\frac{\Omega}{\Gamma}\sinh\left(\frac{t\Gamma}{4}\right), \tag{34a}$$

$$z(t) \equiv \cosh\left(\frac{t\Gamma}{4}\right) - \frac{\gamma}{\Gamma}\sinh\left(\frac{t\Gamma}{4}\right), \tag{34b}$$

with $\Gamma \equiv \sqrt{\gamma^2 - (4\Delta)^2}$.

The stationary value of $Q_t$ can be obtained straightforwardly from Eq. (31) as $Q_\infty = \lim_{t\to\infty} Q_t = \lim_{u\to 0} uQ_u$, which leads to

$$Q_\infty = 1 + \langle\sigma_z\rangle_\infty\langle\sigma_z\rangle_0 + \langle\sigma_y\rangle_\infty\langle\sigma_y\rangle_0, \tag{35}$$

where the stationary mean values are

$$\langle\sigma_z\rangle_\infty = -\frac{\gamma^2}{\gamma^2 + 2\Omega^2}, \quad \langle\sigma_y\rangle_\infty = \frac{2\gamma\Omega}{\gamma^2 + 2\Omega^2}. \tag{36}$$

Consistently, these expressions also follow as $\langle\sigma_j\rangle_\infty = \lim_{t\to\infty} \mathrm{Tr}_s[\sigma_j\rho_t]$.

The system initial conditions that lead to extreme values of $(Q_\infty - 1)$ can be determined from Eq. (35). It is found that the initial state must be pure, $\rho_0 = |\psi_0\rangle\langle\psi_0|$, where the state $|\psi_0\rangle$ is parametrized in terms of angles $(\theta_0, \phi_0)$ on the Bloch sphere, $|\psi_0\rangle = |\psi(\pm, \theta_0, \phi_0)\rangle$ [56]. Maximization implies that

$$\tan(\theta_0) = -\frac{2\Omega}{\gamma} = \frac{\langle\sigma_y\rangle_\infty}{\langle\sigma_z\rangle_\infty}, \quad \phi_0 = \frac{\pi}{2}. \tag{37a}$$

We remark that there are two different orthogonal states $\{|\psi(\pm, \theta_0, \phi_0)\rangle\}$ associated with the direction defined by these angles. They correspond to the basis where the stationary state, $\rho_\infty = \lim_{t\to\infty}\rho_t$, is a diagonal matrix. In addition, it is found that maximization is achieved with

$$\tan(\tilde{\theta}_0) = \frac{2\Omega}{\gamma} = -\frac{\langle\sigma_y\rangle_\infty}{\langle\sigma_z\rangle_\infty}, \quad \tilde{\phi}_0 = \frac{3\pi}{2}. \tag{37b}$$

These angles define the basis where the (time-reversed) stationary system density matrix is a diagonal operator, $\tilde{\rho}_\infty = \lim_{t\to\infty}\rho_t^*$, where conjugation is taken in the basis defined by the eigenvectors of $\sigma_z$. These last solutions are consistent with Eq. (13).

With the previous choice of initial conditions, from the definition (6) and Eq. (35), the degree of environment quantumness associated with the dynamics (30) can be written as

$$D_Q = \frac{\gamma\sqrt{\gamma^2 + 4\Omega^2}}{\gamma^2 + 2\Omega^2}. \tag{38}$$

In Fig. 1(a) we plot the time dependence of $Q_t$ [Eq. (33)] assuming system initial conditions that maximize the degree of environment quantumness [Eq. (37)]. When $\Omega/\gamma = 0$, that is, in the absence of the external excitation, a monotonic behavior is obtained, $Q_t = 1 \mp [1 - e^{-t\gamma}]$. Consistently, this case can be recovered from Eq. (24) after taking a vanishing environment temperature, $\langle\sigma_z\rangle_\infty = -1$, and $\langle\sigma_z\rangle_0 = \pm 1$. On the other hand, as $\Omega/\gamma$ increases, oscillations arise in the behavior of $Q_t$. Furthermore, the asymptotic values $\lim_{t\to\infty} Q_t$ begin to approach the unitary value. Even more, when $\Omega/\gamma \gg 1$ it follows that $\lim_{t\to\infty} Q_t \approx 1$.

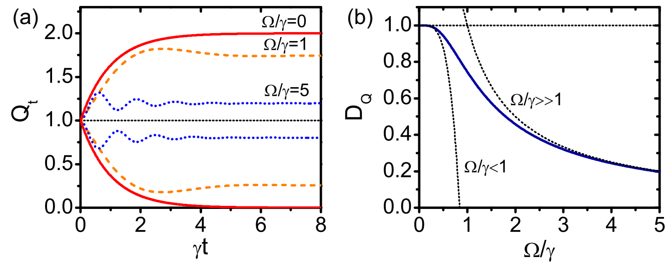


FIG. 1. (a) $Q_t$ [Eq. (33)] as a function of time for different values of $\Omega/\gamma$. The system initial conditions fulfill Eq. (37). The curves above and below $Q_t = 1$ correspond to the lower and upper initial states, respectively. (b) Degree of quantumness $D_Q$ [Eq. (38)] (solid line) jointly with the weak- and strong-intensity approximations [Eqs. (39) and (40)] (dotted lines). By definition both $Q_t$ and $D_Q$ are dimensionless quantities.

Since for each value of $\Omega/\gamma$ the initial states of the system meet the condition (37), the asymptotic values of $Q_t$ shown in Fig. 1(a) are related to the degree of environment quantumness [Eq. (6)] as $D_Q = |\lim_{t\to\infty} Q_t - 1|$. In Fig. 1(b) we plot $D_Q$ as a function of the amplitude of the external field $\Omega/\gamma$. When the external excitation is weak, Eq. (38) can be well approximated by

$$D_Q \simeq 1 - 2(\Omega/\gamma)^4, \quad (\Omega/\gamma) < 1. \tag{39}$$

Thus, in this regime the quantumness of the environment influence is maximal. In fact, the deviation from $D_Q = 1$ depends on the fourth power of $\Omega/\gamma$. On the other hand, when the external excitation is strong enough, it follows that

$$D_Q \simeq 1/(\Omega/\gamma) \to 0, \quad (\Omega/\gamma) \gg 1. \tag{40}$$

This result means that, in this extreme regime, the environment influence can be well approximated by classical noises. This is a nonintuitive result. In fact, some quantum features of the dynamics (30) emerge by increasing the external coherent field [3]. This apparent contradiction is raised up when realizing that the proposed measure quantifies how much the environment action departs from the influence of classical noise fluctuations by considering the full open quantum dynamics, that is, reservoir and external fields.

By finding the explicit solutions of the matrix elements of $\rho_t$, when $\Omega/\gamma \gg 1$ it is possible to approximate the Lindblad evolution (30) by

$$\frac{d\rho_t}{dt} \approx -i\frac{\Omega}{2}[\sigma_x, \rho_t] + \frac{3}{4}\gamma(\sigma_z\rho_t\sigma_z - \rho_t), \quad (\Omega/\gamma) \gg 1. \tag{41}$$

Thus, the combined action of the reservoir (whose effective temperature is zero) and the external excitation can be represented by a *dephasing* mechanism, that is, $(3\gamma/4)(\sigma_z\rho_t\sigma_z - \rho_t)$. This contribution can always be obtained by coupling the system to external white noises such as, for example, Gaussian [8] or Poisson noises [21]. Consequently, the classicality indicated by the result (40) is completely consistent, which in turn also shows the physical meaning of the developed approach. On the other hand, when considering measures based on the commutativity of two output states, due to a different underlying physical motivation, the quantumness of *dephasing*

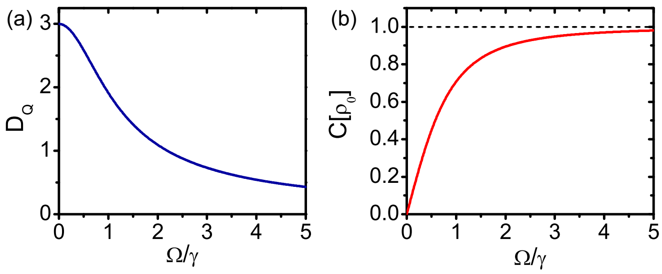


FIG. 2. Degree of environment quantumness $D_Q$ [Eq. (44)] and concurrence of the initial optimal state $C[\rho_0] = C[|i_{\max}\rangle\langle i_{\max}|]$ [Eq. (45)], corresponding to the bipartite evolution (42). The plotted objects are dimensionless.

*(qubits) maps* becomes proportional to the system coherence decay [42]. Thus, classicality is only achieved in a stationary regime (instead, here $Q_t = 1\ \forall t$ and $D_Q = 0$).

### D. Optimal states for two interacting qubits

We consider two qubits ($a$ and $b$) whose bipartite density matrix $\rho_t^{ab}$ evolves as

$$\frac{d\rho_t^{ab}}{dt} = -i\frac{\Omega}{2}\left[\sigma_x \otimes \sigma_x, \rho_t^{ab}\right] + \gamma\mathcal{L}_a\left[\rho_t^{ab}\right] + \gamma\mathcal{L}_b\left[\rho_t^{ab}\right]. \quad (42)$$

The frequency $\Omega$ scales the Hamiltonian interaction between both systems. In addition, $\mathcal{L}_a$ and $\mathcal{L}_b$ define the dissipative dynamics of each subsystem. They are defined by the dissipative contribution in Eq. (30), here written in each Hilbert space. We study the relationship between the proposed noncommutativity measure $Q_t$ and the optimal initial conditions $\rho_0^{ab}$ that lead to its maximal value in the stationary regime [Eq. (6)].

The stationary state $[\rho_\infty^{ab} = \lim_{t\to\infty} \rho_t^{ab}]$ of the dynamics (42) can be obtained in an exact analytical way. Introducing the standard base of states $\{|+ +\rangle, |+ -\rangle, |- +\rangle, |- -\rangle\}$, it follows that

$$\rho_\infty^{ab} = \frac{1}{4\Gamma^2}\begin{pmatrix} \Omega^2 & 0 & 0 & -i2\gamma\Omega \\ 0 & \Omega^2 & 0 & 0 \\ 0 & 0 & \Omega^2 & 0 \\ i2\gamma\Omega & 0 & 0 & 4\gamma^2+\Omega^2 \end{pmatrix}, \quad (43)$$

where $\Gamma \equiv \sqrt{\gamma^2+\Omega^2}$. By using Eq. (13), the degree of environment quantumness can be written in terms of the largest eigenvalue of $\tilde{\rho}_\infty^{ab} = \rho_\infty^{*ab}$. We get

$$D_Q = \frac{\gamma(\gamma + 2\sqrt{\gamma^2+\Omega^2})}{\gamma^2+\Omega^2}. \quad (44)$$

In Fig. 2(a) we plot $D_Q$ as a function of $\Omega/\gamma$. We notice that by increasing the influence of the Hamiltonian contribution classicality is achieved, $\lim_{\Omega/\gamma\to\infty} D_Q = 0$. Similarly to the previous case [Eqs. (38) and (41)], in this limit the combined action of the environment and the subsystems interaction Hamiltonian can be written in terms of dephasing (unital) mechanisms.

Equation (13) also characterizes the initial condition $\rho_0^{ab} = |i_{\max}\rangle\langle i_{\max}|$ that leads to maximal stationary values of the noncommutativity measure $Q_t$. $|i_{\max}\rangle$ is the eigenstate of the stationary state $\tilde{\rho}_\infty^{ab}$ with the largest eigenvalue. It reads

$$|i_{\max}\rangle = \frac{1}{\sqrt{2\Gamma(\Gamma-\gamma)}}[i(\Gamma-\gamma)|+ +\rangle + \Omega|- -\rangle], \quad (45)$$

where, as before, $\Gamma = \sqrt{\gamma^2+\Omega^2}$, and $\langle i_{\max}|i_{\max}\rangle = 1$.

In general $|i_{\max}\rangle$ is an entangled state. This feature can be quantified through its concurrence [57] $C[\rho_0] = C[|i_{\max}\rangle\langle i_{\max}|]$. In Fig. 2(b) we plot its dependence on $\Omega/\gamma$. In the limit of a vanishing unitary coupling, from Eq. (45) it follows that

$$\lim_{\Omega/\gamma\to 0} |i_{\max}\rangle = |- -\rangle. \quad (46)$$

This is an unentangled state, implying $C[\rho_0] = 0$. In contrast, in the limit of strong coupling we get

$$\lim_{\Omega/\gamma\to\infty} |i_{\max}\rangle = \frac{1}{\sqrt{2}}(i|+ +\rangle + |- -\rangle), \quad (47)$$

which is a maximal entangled state, $C[\rho_0] = 1$.

The previous behaviors have an interesting physical implication. In the weak-coupling limit $[\Omega/\gamma \approx 0]$, a (nearly) unentangled initial state leads to the maximal departure from classicality of the environment action (quantified by $D_Q$). By increasing the unitary coupling $[\Omega/\gamma > 0]$, an increasing initial entanglement between both subsystems is necessary to obtain the maximal deviation from classicality.

For this model the measure $Q_t$ assumes a simple form [Eq. (8)]. When maximizing its stationary value with respect to the initial conditions, it follows that

$$Q_t = 1 + \frac{\gamma^2(1+e^{-2\gamma t})}{\Gamma^2} + 2\frac{\gamma}{\Gamma}[1 - \lambda e^{-2\gamma t}\cos(\Omega t)], \quad (48)$$

where $\lambda \equiv 1 + (\gamma/\Gamma)$. Consistently, the initial state that leads to this expression is $\rho_0^{ab} = |i_{\max}\rangle\langle i_{\max}|$ [Eq. (45)].

The previous results rely on taking both subsystems as the system of interest. One can also deal with the partial dynamics $\rho_t^a = \mathrm{Tr}_b[\rho_t^{ab}]$ or, alternatively, $\rho_t^b = \mathrm{Tr}_a[\rho_t^{ab}]$. The corresponding stationary states read

$$\rho_\infty^s = \frac{1}{2\Gamma^2}\begin{pmatrix} \Omega^2 & 0 \\ 0 & 2\gamma^2+\Omega^2 \end{pmatrix}, \quad s = a, b. \quad (49)$$

Performing similar calculations, the degree of environment quantumness and the optimal state are

$$D_Q = \frac{\gamma^2}{\gamma^2+\Omega^2}, \quad |i_{\max}\rangle = |-\rangle. \quad (50)$$

Given the symmetry of Eq. (42), this results applies to both subsystems. Furthermore, assuming $\rho_0^{ab} = |i_{\max}\rangle\langle i_{\max}| \otimes \rho_0^b$, where $\rho_0^b$ is an arbitrary state, it follows that

$$Q_t = 1 + \frac{\gamma^2}{\Gamma^2} + \frac{\gamma e^{-\gamma t}}{\Gamma^2}[\Omega\sin(\Omega t) - \gamma\cos(\Omega t)]. \quad (51)$$

The same expression follows from $\rho_0^{ab} = \rho_0^a \otimes |i_{\max}\rangle\langle i_{\max}|$. These results differ from those obtained starting from a bipartite representation [Eqs. (44), (45), and (48)]. This feature shows that the environment's influence over a system cannot, in general, be related in a simple way with the action over the constitutive subsystems.

### E. Quantum harmonic oscillator coupled to a thermal environment

The developed approach applies consistently to systems with a Hilbert space of finite dimension [see Eqs. (9) and (10)]. Complementarily, here we study the case of a quantum harmonic oscillator coupled to a thermal environment at a finite temperature.

The density matrix evolution can be written as in Eq. (23) under the replacements $\sigma^\dagger \to a^\dagger$ and $\sigma \to a$, where $a^\dagger$ and $a$ are the creation and annihilation bosonic operators of the system, respectively [1]. Alternatively, the evolution can be written using a Wigner function. It is defined as the Fourier transform $W(\alpha,\alpha^*,t) \equiv (1/\pi^2)\int d^2z\chi(z,z^*)e^{-iz^*\alpha^*}e^{-iz\alpha}$, where the characteristic function is $\chi(z,z^*) \equiv \mathrm{Tr}_s[\rho_t \exp(iz^*a^\dagger + iza)]$. Denoting $W_t = W(\alpha,\alpha^*,t)$, its time evolution reads [3]

$$\frac{\partial W_t}{\partial t} = \left\{\varphi\frac{\partial}{\partial\alpha}\alpha + \varphi^*\frac{\partial}{\partial\alpha^*}\alpha^* + \left(\frac{\kappa+\zeta}{2}\right)\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right\}W_t, \quad (52)$$

where $\varphi \equiv i\omega_0 + (\kappa-\zeta)/2$. Here, $\omega_0$ is the natural frequency of the system. Note that dissipative contributions (first-order derivatives) are present whenever the underlying rates are different, $\kappa \neq \zeta$. On the other hand, diffusion (second-order derivatives) always develops, being scaled by $(\kappa+\zeta)/2$.

The operator evolution can be obtained in a similar way from the dual dynamics associated with the system density matrix. Alternatively, it can be deduced using that operator expectation values can be written as $\langle A\rangle_t = \int d\alpha d\alpha^* W_t A_0(\alpha,\alpha^*) = \int d\alpha d\alpha^* W_0 A(\alpha,\alpha^*,t)$, where $A_0(\alpha,\alpha^*)$ is the "scalar representation" of the system operator $A$. From Eq. (52) we get $[A_t = A(\alpha,\alpha^*,t)]$

$$\frac{\partial A_t}{\partial t} = -\left\{\varphi\alpha\frac{\partial}{\partial\alpha} + (\varphi\alpha)^*\frac{\partial}{\partial\alpha^*} - \left(\frac{\kappa+\zeta}{2}\right)\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right\}A_t. \quad (53)$$

Using this representation, from Eq. (8) we get $Q_t = \int d\alpha d\alpha^* A_t$, where $A_t$ is the solution of the dual evolution with the initial condition $A_0 = W_0$. By integration by parts of Eq. (53), it is simple to arrive at $(d/dt)Q_t = (\kappa-\zeta)Q_t$, which leads to

$$Q_t = \exp[(\kappa-\zeta)t]. \quad (54)$$

This result is valid regardless of the initial system density matrix. Therefore, no maximization procedure is available. The same expression for $Q_t$ follows by using a Glauber-Sudarshan $P$ representation or $Q$ representation [3], or even from a (diagonal) Fock-number characteristic function approach [58].

Since $\kappa \geqslant \zeta$ [$\kappa = \gamma(n_{\mathrm{th}}+1)$ and $\zeta = \gamma n_{\mathrm{th}}$], the indicator $Q_t$ develops an exponential divergence in time for any finite bath temperature. Only when the reservoir temperature is infinite ($\kappa = \zeta$) does a classical noise representation apply, $Q_t = 1$ and $D_Q = 0$. This behavior has a clear interpretation. In fact, for any finite reservoir temperature, the Wigner function involves dissipative contributions [see Eq. (52)]. These (trace-preserving) effects develop in the system Hilbert space and cannot be reproduced by any classical external influence. Dissipative contributions only disappear when $\kappa = \zeta$, consistently supporting the (discontinuous) temperature dependence of the degree of quantumness in this case.

Although the previous result is consistent, it differs strongly from the case of the two-level system [see Eq. (25)], where $D_Q$ has a continuous dependence on the reservoir temperature. Interestingly, for infinite-dimensional Hilbert spaces, the expression (12) allows us to define a *renormalized degree of environment quantumness* as $D_{Q_R} \equiv \max_{[\rho_0]} \mathrm{Tr}_s[\tilde{\rho}_\infty\rho_0]$. In terms of the Wigner function, it is

$$D_{Q_R} = \max_{[W_0]} \int d\alpha d\alpha^* \tilde{W}_\infty W_0. \quad (55)$$

Here, maximization must be performed over all possible (normalized) initial conditions $W_0$. Furthermore, Eq. (52) implies that $\tilde{W}_\infty = W_\infty = \lim_{t\to\infty} W_t = (1/\pi\sigma_\infty)\exp(-|\alpha|^2/\sigma_\infty)$, where $\sigma_\infty = (1/2)(\kappa+\zeta)/(\kappa-\zeta) = n_{\mathrm{th}} + (1/2) = (1/2)(\tanh[\beta\hbar\omega_0/2])^{-1}$.

Since $D_{Q_R}$ is a linear functional of $W_0$, the maximization problem cannot be solved by using standard functional derivative techniques. As an ansatz we assume that $W_0$ is also a Gaussian function. In such a case, it follows that $W_0$ must have the smallest possible width. Therefore, it must be the Wigner function of the ground state of the system, which in turn from Eq. (55) delivers

$$D_{Q_R} = \frac{1}{\sigma_\infty + (1/2)} = 1 - \exp(-\beta\hbar\omega_0). \quad (56)$$

The same result is obtained by performing a similar ansatz in the energy eigenbasis representation. $D_{Q_R}$ has the expected dependence on the environment temperature. In particular, classicality [$D_{Q_R} = 0$] is approached in a high-temperature limit. In contrast to the two-level system case [Eq. (25)], here the renormalized degree of quantumness cannot be associated with the time-dependent noncommutativity measure $Q_t$ [Eq. (54)].

## V. SUMMARY AND CONCLUSIONS

We have developed a consistent proposal that allows us to quantify to what extent the influence of a given environment on an open quantum system departs from the action of classical stochastic fields. Its physical ground is based on associating the quantumness of the environment influence with the lack of commutativity between the initial state of the reservoir and the system-environment total Hamiltonian. On this basis we introduced a (time-dependent) noncommutativity measure [Eq. (3)]. Its stationary value (long time-regime) when maximized over all possible initial conditions of the system defines a degree of environment quantumness [Eq. (6)]. For dissipative dynamics it can be determined from the largest eigenvalue of the (time-reversal) stationary system density matrix [Eq. (13)]. Independently of the system dynamical regime, the noncommutativity measure can be written in terms of the dual evolution of operators [Eq. (8)]. This alternative definition provides a powerful tool to characterize the environment quantumness in both Markovian and non-Markovian regimes.

Consistently the nonclassicality measure vanishes identically for a broad class of quantum dynamics, which include Hamiltonian ensembles [Eq. (14)], stochastic Hamiltonians [Eq. (15)], and a class of collisional dynamics [Eq. (18)]. All these dynamics can be obtained by considering the action of underlying classical stochastic processes. Despite the consistency of this result, the quantumness indicator also vanishes when the open system dynamics is defined by a unital map [Eq. (22)]. Hence, the proposed indicator can also be read as a measure of departure from this dynamical property.

The consistency of the developed approach was supported by studying different dissipative open system dynamics. For two-level systems coupled to a thermal bath, the degree of environment quantumness decreases monotonically with the reservoir temperature [Eq. (25)]. For an optical transition (resonant fluorescence) the amplitude of the external coherent excitation monotonically drives the environment influence toward classicality [Eq. (38)]. The consistency of this result derives from the possibility of describing the high-intensity regime in terms of a dephasing quantum master equation that can be represented by the action of classical noises. On the other hand, by analyzing two interacting qubits it was found that quantum entanglement may become a necessary resource for detecting the quantumness of the environment influence when approaching a regime in which a classical noise representation becomes a valid approximation. Application to systems endowed with a Hilbert space of infinite dimension was also established.

The present formalism leaves open some interesting questions. For example, it is unknown what dynamical features determine the presence or absence of revivals in the temporal behavior of the noncommutativity measure. On the other hand, an operational definition and experimental measurability are also interesting issues that can be addressed from the proposed approach.

## ACKNOWLEDGMENTS

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## APPENDIX A: CLASSICAL NOISE APPROXIMATION

The microscopic derivation of a classical noise representation of the environment influence is usually performed in a weak-coupling limit [5] or, more generally, using projectors techniques [43]. Here we sketch the derivation. Given the total Hamiltonian $H = H_s + H_e + H_I$, in an interaction representation with respect to $H_s + H_e$, the system state ($\rho_t \to \rho_t^I$) can be written as

$$\rho_t^I = \mathrm{Tr}_e\Big(\Big\lceil e^{-i\int_0^t dt' H_I(t')}\Big\rceil \rho_0 \otimes \sigma_0 \Big\lfloor e^{+i\int_0^t dt' H_I(t')}\Big\rfloor\Big), \tag{A1}$$

where $H_I(t) = \exp[+it(H_s + H_e)]H_I \exp[-it(H_s + H_e)]$. In addition, $\lceil \cdots \rceil$ and $\lfloor \cdots \rfloor$ denote a time ordering operation and an antichronological ordering operation, respectively. Without loss of generality, the interaction is taken as

$$H_I(t) = \lambda V_t \otimes B_t, \tag{A2}$$

where $V_t$ and $B_t$ are respectively system and reservoir operators written in the interaction representation, $V_t = \exp[+itH_s]V \exp[-itH_s]$ and $B_t = \exp[+itH_e]B \exp[-itH_e]$. The dimensionless parameter $\lambda$ measures the interaction strength.

The *environment action over the system is classical* when the (interaction) environment operator can be replaced by a scalar noisy function,

$$B_t \to \xi_t. \tag{A3}$$

Notice that this replacement implies that the environment operator $B_t$ can be treated as a communting scalar function. Correspondingly, the system state (A1), under the change $\mathrm{Tr}_e(\cdots) \to \overline{(\cdots)}$, can be approximated as

$$\rho_t^I \approx \overline{\Big\lceil e^{-i\lambda\int_0^t dt' V_{t'}\xi_{t'}}\Big\rceil \rho_0 \Big\lfloor e^{+i\lambda\int_0^t dt' V_{t'}\xi_{t'}}\Big\rfloor}, \tag{A4}$$

where the overline denotes an average over realizations of the classical noise $\xi_t$. In general, Eqs. (A1) and (A4) are *incompatible* or inconsistent. The conditions under which this mapping is valid are obtained by performing an expansion on the interaction parameter $\lambda$ [5,43]. Next, we derive the same conditions in an alternative simplified way.

In terms of operator commutation properties, the replacement (A3) implies that $[H_I(t), \sigma_0] \approx 0$. This relation, in turn, implies that there exists an environment basis $\{|\omega\rangle\}$ where both objects, $H_I(t)$ and $\sigma_0$, are diagonal. Thus, Eq. (A1) can be approximated as

$$\rho_t^I \approx \sum_\omega \sigma_\omega \Big\lceil e^{-i\lambda\int_0^t dt' V_{t'}\langle\omega|B_{t'}|\omega\rangle}\Big\rceil \rho_0 \Big\lfloor e^{+i\lambda\int_0^t dt' V_{t'}\langle\omega|B_{t'}|\omega\rangle}\Big\rfloor, \tag{A5}$$

where $\sigma_\omega \equiv \langle\omega|\sigma_0|\omega\rangle$. In this expression the influence of the reservoir has been reduced to a *classical* one. In fact, each contribution $V_{t'}\langle\omega|B_{t'}|\omega\rangle$ can be read as a random (system) Hamiltonian that participates with weight $\sigma_\omega$. Given this feature, the (unknown) statistical properties of $\xi_t$ (all time-correlations) can be obtained by developing Eqs. (A4) and (A5) as series in the interaction strength.

To first order in $\lambda$, it follows that the condition

$$\overline{\xi_t} = \sum_\omega \sigma_\omega \langle\omega|B_t|\omega\rangle \approx \mathrm{Tr}_e[\sigma_0 B_t]. \tag{A6}$$

This relation can always be satisfied consistently. On the other hand, to second order in $\lambda$, we get

$$\overline{\xi_t\xi_{t'}} = \sum_\omega \sigma_\omega \langle\omega|B_t|\omega\rangle\langle\omega|B_{t'}|\omega\rangle \approx \mathrm{Tr}_e[\sigma_0 B_t B_{t'}], \tag{A7}$$

where $t \neq t'$. While this correlation mapping seems consistent, it implies that $\mathrm{Tr}_e[\sigma_0 B_t B_{t'}] \approx \mathrm{Tr}_e[\sigma_0 B_{t'} B_t]$. Due to the intrinsic quantum nature of the environment, this last property is not valid in general. The correlation inequality $\mathrm{Tr}_e[\sigma_0 B_t B_{t'}] \neq \mathrm{Tr}_e[\sigma_0 B_{t'} B_t]$ is avoided when $\sigma_0 \simeq \mathrm{I}_e$, which in turn for thermal environments is fulfilled in a high-temperature limit [5,43]. On the other hand, contributions proportional to higher orders in $\lambda$ define higher noise correlations. For Gaussian fluctuations [43], all of them can be reduced to Eq. (A7).

## APPENDIX B: RENEWAL COLLISIONAL MODELS

In these models the statistics of the collisional times are defined by a "waiting time distribution" $w(t)$. It gives the probability density for the time interval between consecutive collisional events. The corresponding survival probability is defined as $P_0(t) = 1 - \int_0^t dt' w(t')$. Poisson statistics corresponds to $w(t) = \gamma \exp(-\gamma t)$ and $P_0(t) = \exp(-\gamma t)$, an assumption that leads to Markovian Lindblad equations for the system dynamics.

In correspondence with Eq. (17), the system density matrix can be written in general as [51]

$$\rho_t = \sum_{n=0}^{\infty} \int_0^t dt' \mathcal{P}_0(t - t') \mathcal{W}^{(n)}(t') \rho_0. \tag{B1}$$

The involved superoperators are written in the Laplace domain $[f(u) = \int_0^\infty dt e^{-ut} f(t)]$ as

$$\mathcal{P}_0(u) \equiv P_0(u - \mathcal{L}_s), \quad \mathcal{W}^{(n)}(u) \equiv [\mathcal{E} w(u - \mathcal{L}_s)]^n. \tag{B2}$$

Here, $P_0(u) = [1 - w(u)]/u$. Furthermore, the free propagator between events was written as $\mathcal{G}_t = \exp(t\mathcal{L}_s)$. Notice that in a time domain $\mathcal{W}^{(n)}(t)$ consists in the convolution of free propagation and $n$ collisional events. Consistently, the function $P_0(u) w^n(u)$ gives the probability of occurring $n$ events up to time $t$.

Using that $\langle A \rangle_t = \mathrm{Tr}_s[A_0 \rho_t] = \mathrm{Tr}_s[\rho_0 A_t]$, from Eq. (B1) the operator dual evolution reads

$$A_t = \sum_{n=0}^{\infty} \int_0^t dt' \mathcal{W}^{\star(n)}(t') \mathcal{P}_0^{\star}(t - t') A_0. \tag{B3}$$

Here, the involved superoperators are defined as $\mathcal{P}_0^{\star}(z) = P_0(z - \mathcal{L}_s^{\star})$ and $\mathcal{W}^{\star(n)}(z) = [w(z - \mathcal{L}_s^{\star}) \mathcal{E}^{\star}]^n$. These expressions rely on the definitions $\mathrm{Tr}_s[A\mathcal{E}[\rho]] = \mathrm{Tr}_s[\rho \mathcal{E}^{\star}[A]]$ and $\mathrm{Tr}_s[A \exp(t\mathcal{L}_s)[\rho]] = \mathrm{Tr}_s[\rho \exp(t\mathcal{L}_s^{\star})[A]]$.

The time-dependent noncommutativity measure $Q_t$ can be calculated as the trace of dual dynamics [Eq. (8)]. The property $\mathrm{Tr}_s[A_t] = \mathrm{Tr}_s[A_0]$ leads to the condition $\mathrm{Tr}_s[\mathcal{E}^{\star}[A]] = \mathrm{Tr}_s[A]$, which recovers Eq. (18).


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