Reply to "Comment on 'Subluminality of relativistic quantum tunneling' "

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We show that the criticism raised by Dumont and Pollak [Dumont and Pollak, preceding Comment, Phys. Rev. A **108**, 036201 (2023)] is based on a flawed understanding of our analysis.

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Let us recall the main result of our paper (see [1], Sec. IV).

Proposition 1. According to the Dirac equation, the probability of detecting a particle in a region $\mathcal{R}$ at a time t cannot exceed the probability of detecting it inside the past lightcone of $\mathcal{R}$ at any earlier time $t' < t$ (see Fig. 1).

Every other statement of the article follows directly or indirectly from this simple property of the Dirac equation. It is in this sense that we say that quantum tunneling is subluminal: Probabilities cannot exit the lightcone. Let us remark that this mathematical result is rigorously proven in the paper and it has not been disproved (or even questioned) by the authors of the Comment [3]. Indeed, Fig. 1 of the Comment [3] fully corroborates Proposition 1. To check this, one only needs to make a quick exercise: Pick any time T in Fig. 1 of the Comment (e.g., $T = 225$) and integrate the flux for $t \leqslant T$ (note the logarithmic scale). The integral of the solid line provides the probability of finding the electron on the right of the barrier at time T, while the integral of the dashed line provides the probability inside the past lightcone at $t = 0$. Clearly, the former never exceeds the latter, in agreement with Proposition 1. Changing the width of the wave packet cannot lead to a violation of Proposition 1 because, as we said before, it is a rigorously proven mathematical result. If Dumont and Pollak claim otherwise, the burden of proof is on them.

The first sentence of our article that Dumont and Pollak criticize, namely "we *must* assume that the incoming wave packet had a long tail, which extended largely inside the barrier, and that the tunneled wave packet is just the (subluminal) evolution of such long tail," is an implication of Proposition 1[1] and it cannot be understood out of context. In fact, it refers to a specific Minkowski diagram (bottom panel of Fig. 3 in [1]) where we applied Proposition 1 with a precise choice of earlier time t' in mind, which does not coincide with the initial time step of the numerical experiment of Dumont *et al.* [4]. To understand this point, consider our Fig. 2 herein. The left panel shows the choice of time t' we were referring to in the quoted text: The main body of the wave packet is about to enter the barrier and the point Q (which marks the boundary of the lightcone) falls inside the barrier. The right panel takes t' to coincide with the initial time step of the numerical experiment of [4], where the wave packet is far from the barrier and Q falls on the far left of the barrier. Proposition 1 applies to both cases, but the quoted sentence above was referring only to the first choice of t' (left panel).

We would like to remark that this point is stated explicitly also in our article [1], at the end of Sec. IV A, as follows: "There is a subtlety that we need to mention. In Fig. 3 (lower panel), the point Q falls inside the barrier (i.e. $Q > 0$). But this is true only if we set our clocks in such a way that at $t = 0$ the incoming wave packet is about to enter the barrier. In numerical experiments like the one performed by Dumont *et al.* [4] [*sic*], the wave packet is on the far left of the barrier at $t = 0$. In this case, Eq. (10) still holds, but point Q will also be on the far left of the barrier ($Q \ll 0$). In Appendix B, we calculate the position of Q for the numerical experiment of Dumont *et al.* [4] [*sic*], and we verify explicitly that their numerical analysis corroborates Eq. (10)."

Let us now discuss the second part of the Comment.

The following is the portion of text of [1] that Dumont and Pollak have criticized: "[We] would like to point out that,

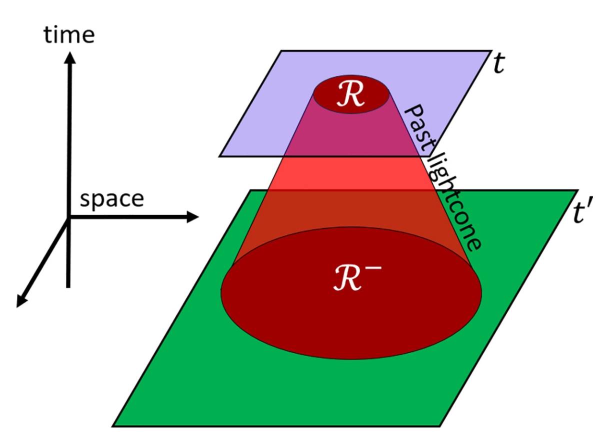

FIG. 1. Space-time geometry of Proposition 1. Consider a region of space $\mathcal{R}$ at time t. Color in red all the timelike and lightlike world lines that originate in $\mathcal{R}$ and travel to the past. This defines the past lightcone (or causal past [2]) of $\mathcal{R}$. Call $\mathcal{R}^-$ its section at a time $t' < t$. According to Proposition 1, the probability of detecting the particle in $\mathcal{R}$ at time t cannot exceed the probability of detecting it in $\mathcal{R}^-$ at time t'.

[1]Strictly speaking, in [1] the quoted text appear before Proposition 1 is introduced. In fact, it is an application of Theorem 1 of [1]. However, if one combines Proposition 1 with the fact that the Dirac equation is linear, Theorem 1 of [1] follows automatically.

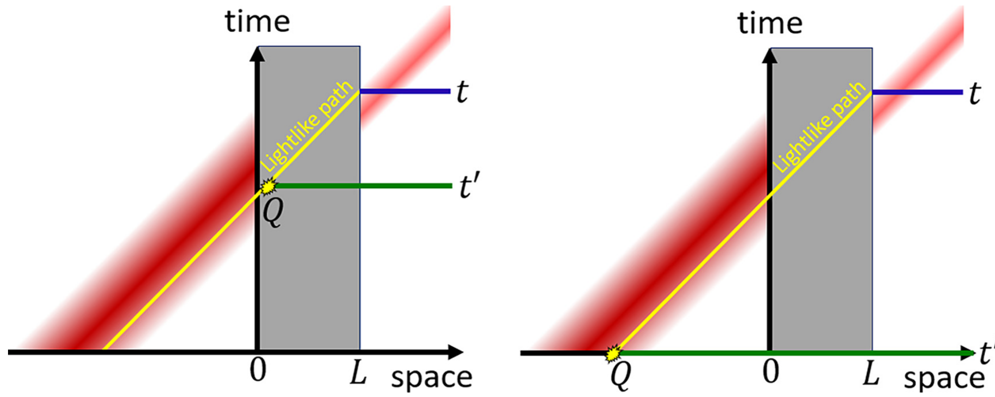

FIG. 2. Application of Proposition 1 to a schematic tunneling experiment (gray denotes the barrier and red the wave packet) for two different choices of $t' < t$. In both cases, the probability stored in the blue half-line cannot exceed the probability stored in the green one. In the left panel t' is the time when the incoming wave packet is about to enter the barrier (we do not show the reflected part for clarity) and it reproduces the situation described in the bottom panel of Fig. 3 of [1]. In the right panel t' coincides with the initial time step of the numerical experiment of Dumont *et al.* [4], when the incoming wave packet is far from the barrier.

when we say that the tunneled wave packet originates from the right tail, we are just making two rigorous mathematical statements. First, that if you change your initial data by removing the tail, i.e., by replacing $\Psi(t=0)$ with $\Psi(t=0)\Theta(Q-z)$, where Θ is the Heaviside step [function], the tunneled wave packet disappears. Second, if you instead replace $\Psi(t=0)$ with $\Psi(t=0)\Theta(z-Q)$, leaving only the tail and cutting all the rest, the tunneled wave packet still remains, and it is completely unaffected." Again, this statement is hard to understand without a Minkowski diagram. Let us consider, for clarity, the right panel of Fig. 2 above herein. Theorem 1 of [1] establishes a causal relationship between the green half-line and the blue half-line. In particular, the value of Ψ on the blue half-line is uniquely determined by the restriction of Ψ to the green half-line. In other words, if we change the initial data for Ψ on the left of Q, the value of Ψ on the blue half-line is completely unaffected. This is just the standard formulation of the principle of causality that we meet in general relativity textbooks (see, e.g., [5], Theorem 10.1.3) and it is a direct implication of the Holmgren uniqueness theorem [6].

The two cases in the quoted text above are two extreme examples. If we set to zero the part of the wave function on the right of Q, then the wave function vanishes on the green half-line. Consequently, it must also vanish on the blue half-line. If instead we set to zero the part of the wave function on the left of Q (leaving the right part unchanged), then the initial profile on the green half-line is the same. Hence, the value of Ψ on the blue half-line is also unaffected.

We are now ready to address the comments of Dumont and Pollak one by one.

(i) Theorem 1 does not say that if we detect the electron on the blue half-line, then the electron "comes from" the green half-line. Before the measurement, the electron did not have a well-defined position and we are not claiming otherwise. Theorem 1 only establishes a causal dependence between regions of space-time. In particular, it tells us that (according to the Dirac equation) the portion of wave function inside the blue half-line *depends* only on the portion of initial wave function inside the green half-line.

(ii) Unitary evolution allows one to establish causal relationships between parts of wave functions. If $|\Psi\rangle = |L\rangle + |R\rangle$, then $e^{-iHt}|\Psi\rangle = e^{-iHt}|L\rangle + e^{-iHt}|R\rangle$ and we can say that the part $e^{-iHt}|R\rangle$ depends on $|R\rangle$.

(iii) The fact that truncated wave functions like $\Psi(t=0)\Theta(z-Q)$ have some high-momentum part that is above the barrier energy is completely irrelevant for our purposes. Again, we are only establishing causal connections.

(iv) Dumont and Pollak have devoted much of their Comment to discussing the phase-space distribution of "cutoff wave packets" like $\Psi(t=0)\Theta(z-Q)$. However, this is due to a misunderstanding of the quoted sentence. When we say that "the tunneled wave packet is completely unaffected" we just mean that the part of wave function inside the blue half-line is unchanged by the cutoff, i.e., the cutoff information travels subluminally. Of course, nonlocal observables like the linear momentum depend also on the part of wave function outside the blue segment. Hence, it is obvious that the phase-space distribution is affected by the cutoff.

In conclusion, our analysis does not contradict the numerical study of Dumont *et al.* [4]. Rather, they are complementary. While in [4] superluminality is defined as a mere convention (through the Wigner phase time), in [1] we focused on three more pragmatic definitions of superluminality (which are consistent with the textbook definition [2,5,7]) and we rigorously proved that none of them occurs. The authors of the Comment are free to disagree with us about the proper definition of superluminality, in which case they are more than welcome to discuss the matter in a later work. However, we urge any reader of our work to focus on the content of our theorems and their implications for physics and to place our statements within the context of the complete discussion.

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