Comment on "Subluminality of relativistic quantum tunneling"

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The central claim of Gavassino and Disconzi [Gavassino and Disconzi, Phys. Rev. A **107**, 032209 (2023)] that relativistic quantum tunneling is an entirely subluminal process is shown to be incorrect. The Hartman effect can lead to superluminal tunneling, but not to superluminal signaling.

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The central thrust of the paper by Gavassino and Disconzi (GD) [1] is to prove the claim, as stated in their abstract, that "the relativistic quantum tunneling (if modeled using the Dirac equation) is an entirely subluminal process, and it is not instantaneous." The first part of this claim is not in agreement with numerical observations [2] and the second part is not new: We have shown in two recent papers [2,3] that the tunneling time, when appropriately defined, is just the well-known Wigner phase time [4].

In the caption of their Fig. 3, the authors claim that the "only way for a tunneled wave packet to exit at a time $t < L$ is that $\Psi \neq 0$ on the right of Q already at $t = 0$." They further clarify this comment in their text, stating "we *must* assume that the incoming wave packet had a long tail, which extended largely inside the barrier, and that the tunneled wave packet is just the (subluminal) evolution of such long tail." The numerical evidence presented in our paper [2] negates this claim of the authors. We stated that the initial density of the wave packets we used had a density at the left edge of the barrier which is 10^{-27} less than the peak value of the incident wave packet. The transmitted superluminal density as shown in Fig. 1 of our paper is greater than or equal to 2×10^{-12}. This does not negate the arguments and estimates of Appendix B of GD. Indeed, the initial density within the light cone is much larger than the transmitted density.

In our numerical experiments [2] we found a superluminal peak in the scattering of a tunneling particle through a thick barrier, attributed to the Hartman effect. Gavassino and Disconzi claim that the superluminal peak is not superluminal but is related purely to that part of the incident wave function which was within the light cone and that therefore any particle arriving early is not superluminal but comes from the front of the wave packet which as stated is within the light cone. In Fig. 1 we plot the time distribution of transmitted particles for three different widths of the incident wave packet, $15ƛ$, $10ƛ$, and $7ƛ$, with the other parameters the same as in Fig. 1c of Ref. [2]. One notes that the superluminal peak persists and changes very little, even though the width of the incident wave packet has changed significantly. The integrated density within the light cone is in all three cases greater than or equal to 10^{-2}, much larger than the transmission probability. However, whereas decreasing the wave-packet width decreases the density within the light cone, the integrated flux associated with the superluminal peak actually increases (note the increasing superluminal peak flux in Fig. 1). This is in accord with the MacColl-Hartman effect, which we believe causes the superluminality and not the existence of an initial tail under the barrier, as claimed by GD.

The authors claim on p. 4 of their paper that "we would like to point out that, when we say that the tunneled wave packet originates from the right tail, we are just making two rigorous mathematical statements. First, that if you change your initial data by removing the tail, i.e., by replacing $\Psi(t = 0)$ by $\Psi(t = 0)\Theta(Q - z)$, where Θ is the Heaviside step [function], the tunneled wave packet disappears." It is not clear what the authors mean by the "tunneled wave packet." Gavassino and Disconzi have not demonstrated that the superluminality observed in our numerical experiments will disappear.

Gavassino and Disconzi then continue to state the following: "Second, if you instead replace $\Psi(t = 0)$ with $\Psi(t = 0)\Theta(z - Q)$, leaving only the tail and cutting all the rest, the tunneled wave packet still remains, *and it is completely unaffected*" (emphasis ours). Quantum mechanics cannot predict the trajectory of a single particle. In the tunneling experiment we considered, one will observe particles that are transmitted at times that are earlier than those of transmitted free particles. Gavassino and Disconzi relate these particles to the front of the incident wave packet. However, this is not possible because the operator which projects out the component of a wave packet that transmits at a particular time does not commute with the position operator. One cannot relate the transmitted wave function to any particular part of the incident wave. Furthermore, multiplying any wave function with a step function fundamentally alters the properties of the wave function and therefore it is not at all clear that "the tunneled wave packet still remains" whatever that means. This is well known, but to prevent any possible misunderstanding we demonstrate this as follows.

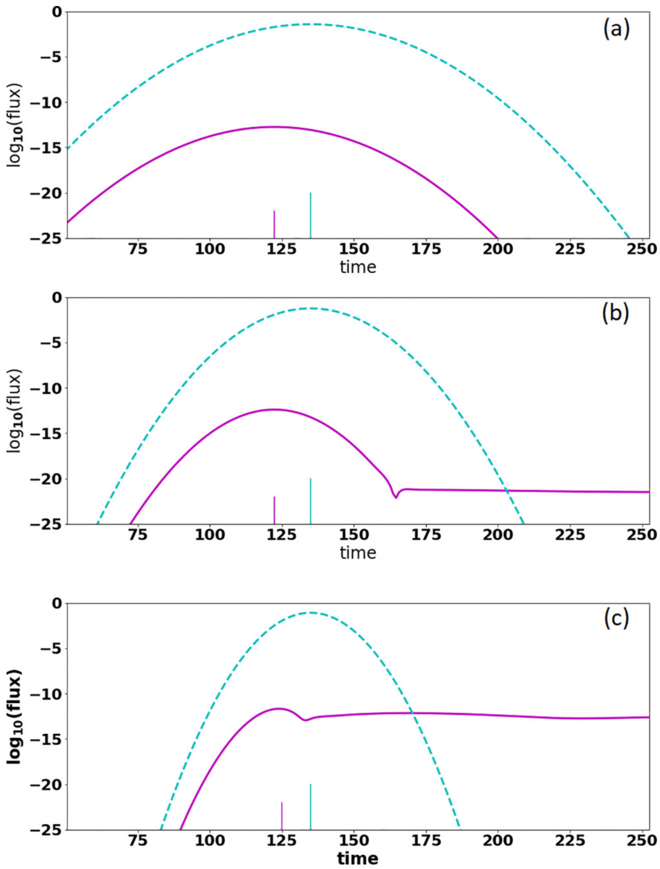

FIG. 1. Tunneled flux versus time for an electron tunneling through the barrier of Fig. 1c of Ref. [2]. The barrier height and width are $6.7mc^2$ and $15\lambda\!\!\!^{-}$. The particle wave-packet widths are (a) $15\lambda\!\!\!^{-}$, (b) $10\lambda\!\!\!^{-}$, and (c) $7\lambda\!\!\!^{-}$ ($1\,\lambda\!\!\!^{-} = 386$ fm, the reduced Compton wavelength of an electron). Also shown (dashed lines) are the corresponding distributions for photons in a vacuum. The times of maximum flux are indicated by the tick marks on the time axis. The unit of time is $\lambda\!\!\!^{-}/c = 1.29\times10^{-21}$ s.

The wave function we choose initially is a coherent state whose form is

$$\langle x|\Psi\rangle = \left(\frac{\Gamma}{\pi}\right)^{1/4} \exp\left(-\frac{\Gamma}{2}(x-x_0)^2 + \frac{i}{\hbar}p_0(x-x_0)\right), \quad (1)$$

where x_0 and p_0 are the initial means of the position and momentum of the normalized wave packet. Furthermore, this is a minimum-uncertainty wave packet, that is,

$$\langle(\hat{x}-x_0\hat{I})^2\rangle\langle(\hat{p}-p_0\hat{I})^2\rangle = \frac{\hbar^2}{4}, \quad (2)$$

where circumflexes denote operators and angular brackets mean, i.e., $\langle\hat{x}\rangle = \int_{-\infty}^{\infty} dx\, x|\langle x|\Psi\rangle|^2$. Then GD make claims on the properties of "cutoff wave packets," i.e.,

$$\langle x|\Psi_c\rangle = \sqrt{N_Q}\left(\frac{\Gamma}{\pi}\right)^{1/4} \times \exp\left(-\frac{\Gamma}{2}(x-x_0)^2 + \frac{i}{\hbar}p_0(x-x_0)\right)\Theta(x-Q), \quad (3)$$

where N_Q is a normalization constant and one readily sees that $N_Q = 2/\mathrm{erfc}(\sqrt{\Gamma}Q)$, where $\mathrm{erfc}(x)$ is the complementary error function. It is then a matter of straightforward algebra to show that the mean position of this wave packet is

$$\langle\Psi_c|\hat{x}|\Psi_c\rangle = \frac{1}{\sqrt{\pi\Gamma}\exp[\Gamma(Q-x_0)^2]\mathrm{erfc}[\sqrt{\Gamma}(\mathrm{Q}-\mathrm{x}_0)]} \quad (4)$$

and that its second spatial moment is

$$\langle\Psi_c|\hat{x}^2|\Psi_c\rangle = \frac{1}{2\Gamma} + (Q-x_0)\langle\Psi_c|\hat{x}|\Psi_c\rangle. \quad (5)$$

The resulting variance in the position is thus

$$\Delta x^2 = \frac{1}{2\Gamma} + \langle\Psi_c|\hat{x}|\Psi_c\rangle(Q - x_0 - \langle\Psi_c|\hat{x}|\Psi_c\rangle). \quad (6)$$

The same is now repeated for the momentum operator. Again, straightforward algebra shows that

$$\langle\Psi_c|\hat{p}|\Psi_c\rangle = p_0, \quad (7)$$

so cutting off the wave function in the configuration space does not change its mean momentum; however, it very much changes the second moment

$$\langle\Psi_c|\hat{p}^2|\Psi_c\rangle = \hbar^2|\langle Q|\Psi_c\rangle|^2 \lim_{x\to Q-x_0}\delta(Q-x_0-x) - \frac{\hbar^2\Gamma}{2}(Q-x_0)|\langle Q-x_0|\Psi_c\rangle|^2 + p_0^2 + \frac{\hbar^2\Gamma}{2}. \quad (8)$$

We see here that for any finite value of Q, the second moment of the momentum diverges. The divergence is of course independent of the normalization. Even if GD argue that we should cut off the front and look at it separately without normalization, we would still get the divergence. This then means that by cutting off the wave function one observes properties that are very different from those of the original wave function. The statement that the resulting tunneled wave packet associated with this incident cutoff wave packet is completely unaffected is therefore incorrect. The cutoff wave packet has initial momenta which are much higher than the barrier, thus necessarily changing the scattered wave packet and its properties.

To gain further insight into the properties of the cutoff wave packet, it is illuminating to consider its Wigner phase-space representation. To simplify, we set $q_0 = 0$. First, we write down the phase-space representation of the coherent state without a cutoff

$$\Psi_W(p,q;Q=-\infty) = \frac{1}{\pi\hbar}\exp\left(-\Gamma q^2 - \frac{(p-p_0)^2}{\Gamma\hbar^2}\right). \quad (9)$$

Then we work out the representation for the wave function with the cutoff

$$\Psi_{c,W}(p,q;Q) \quad (10)$$

$$= \frac{N_Q}{\pi\hbar}\exp\left(-\Gamma q^2 - \frac{(p-p_0)^2}{\Gamma\hbar^2}\right) \times\mathrm{Re}\left[\mathrm{erf}\left(\sqrt{\Gamma}(q-Q) + \frac{i(p-p_0)}{\hbar\sqrt{\Gamma}}\right)\right]\theta(q-Q). \quad (11)$$

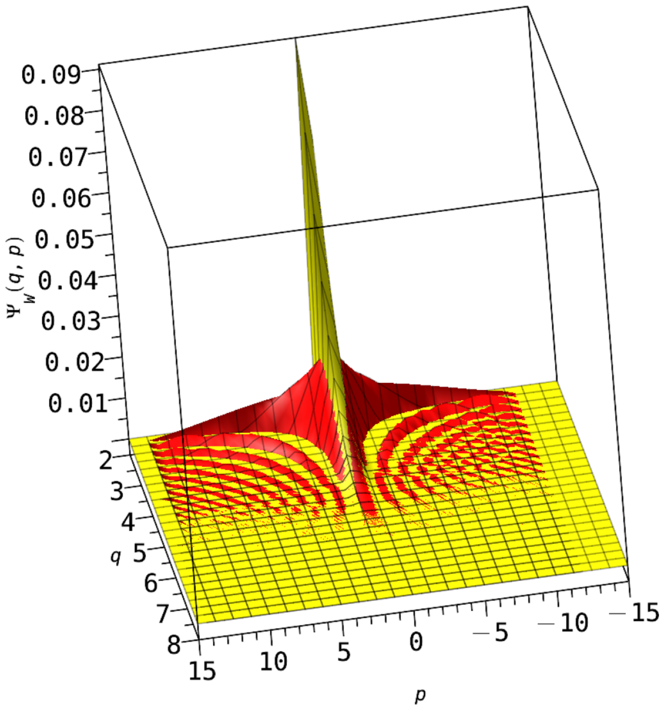

FIG. 2. Comparison of the phase-space distribution of the fully coherent state (yellow) with the cutoff coherent state without its normalization (red). Note how the cutoff induces a much broader distribution in the momentum space, introducing momenta that are not there in the full wave function.

This is an even function of $(p - p_0)$, so it is clear that the mean momentum is just p_0. However, the density itself is very different from that of the density of $\Psi_W(p, q; Q = -\infty)\Theta(q - Q)$, as shown in Fig. 2. Here we set the normalization constant $N_Q = 1$ to facilitate the comparison. The figure is plotted using the parameters (in atomic units) $Q = 2$, $\Gamma = 0.25$, and $p_0 = 1$. One notes how the cutoff induces oscillations in the momentum direction. It is true that when one compares the density in configuration space of the cutoff function with the density of the full wave function without a cutoff, leaving out the normalization N, the two densities are identical for any $q > Q$. However, the same is not true when considering the phase-space density. The two are simply not the same as implied by the claims of Gavassino and Disconzi.

Moreover, this figure points out an additional difficulty with the claims of Gavassino and Disconzi. The fact that such a wave packet would have a significant above-barrier component in energy makes it difficult to distinguish the effects of "tunneling" from the effects of free propagation slowed by a potential step. At this point, one may no longer speak about a tunneled wave packet. Since the cutoff wave packet in phase space has components with energy above the barrier energy, at least some part of the associated transmitted wave packet is not tunneling at all. This also complicates the comparison to a light-speed, free-traveling wave packet. If GD are claiming that there exists a "tunneling wave packet" which presumably is a part of the transmitted wave packet which remains the same, they must prove this.

Our computations show that particles may be transmitted superluminally. Particles exiting the barrier at times that precede those of photons cannot be associated only with the "front of the wave packet." Their origin may come from anywhere in the incident wave packet, whether within the light cone or outside it. However, how can one reconcile this with the theorem stated by GD? We claim that it is not relevant to our numerical experiment. Indeed, there are circumstances in which the propagation is limited by the light cone. As shown by Thaller [5], this result requires the cancellation of beyond-the-light-cone terms in the kernel components associated with the positive- and negative-energy subspaces. The two kernel components exhibit exponentially decaying tails beyond the light cone (note the modified Bessel function of the second kind in Eq. 1.100 on p. 17 in [5]). Thus, if [as done in our paper; see Eq. (2.7) in [2]] one restricts oneself to the positive-energy component in the incident wave packet, there is no *strict* cutoff beyond the light cone. There is instead an exponential decay, just like the wave-packet penetration into any other classically forbidden region. Such a restriction is readily accessible experimentally, by scattering only electrons, and therefore the superluminality would exist. Yet, as claimed qualitatively in our paper and proved elsewhere [6], the fact that there are superluminal particles is insufficient to transfer information faster than light.

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[1] L. Gavassino and M. M. Disconzi, Phys. Rev. A **107**, 032209 (2023).
[2] R. S. Dumont, T. Rivlin, and E. Pollak, New J. Phys. **22**, 093060 (2020).
[3] T. Rivlin, E. Pollak, and R. S. Dumont, Phys. Rev. A **103**, 012225 (2021).
[4] E. Wigner, Phys. Rev. **98**, 145 (1955).
[5] B. Thaller, *The Dirac Equation* (Springer, Berlin, 1992).
[6] R. S. Dumont and T. Rivlin, Phys. Rev. A **107**, 052212 (2023).