Switchable superradiant phase transition with Kerr magnons

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The superradiant phase transition (SPT) has been widely studied in cavity quantum electrodynamics (CQED). However, this SPT is still subject to ongoing debates due to the no-go theorem induced by the so-called $\mathbf{A}^2$ term (AT). We propose a hybrid quantum system, consisting of a single-mode cavity simultaneously coupled to both a two-level system and yttrium-iron-garnet sphere supporting magnons with Kerr nonlinearity, to restore the SPT against the AT. The Kerr magnons here can effectively introduce an additional AT tunable and strong enough to counteract the intrinsic AT, via adiabatically eliminating the degrees of freedom of the magnons. We show that the Kerr-magnon-induced SPT can exist in both cases of ignoring and including the intrinsic AT. Without the intrinsic AT, the critical coupling strength can be dramatically reduced by introducing the Kerr magnons, which greatly relaxes the experimental conditions for observing the SPT. With the intrinsic AT, the forbidden SPT can be recovered with the Kerr magnons in a reversed way. Our work paves a potential way to manipulate the SPT against the AT in hybrid systems combining CQED and nonlinear magnonics.

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I. INTRODUCTION

With the experimental advances in the era of ultrastrong coupling in light-matter interactions [1,2] and the theoretical efforts studying the fundamental quantum Rabi model (QRM) [3–44], finite-component quantum phase transitions (QPTs) have recently received increasing attention [5–21]. Practical applications of the finite-component QPTs have been exploited in critical quantum metrology and quantum information science [42–48].

A QPT occurs in the ground state in the variation of some nonthermal parameter and is regarded as being driven by quantum fluctuations [21,49]. Besides topological transitions under symmetry protection [12–15] and various transitions in symmetry breaking [11,16], the superradiant phase transition (SPT) is a typical QPT in light-matter interactions. The SPT was first predicted in the Dicke model [50,51] consisting of an ensemble of N two-level systems (TLSs) coupled to photons in a cavity [52]. By varying the coupling strength, the ground state of the system changes abruptly from the normal phase (NP) to the superradiant phase (SP) with a boost of photon number. Due to this exotic behavior, the SPT has been widely studied [5–12,15–21,53–59]. Besides the Dicke model, the QRM [33–35,60,61], describing the interaction between a single TLS (with level splitting Ω) and a single-mode field (with frequency ω), can also predict the SPT in the low-frequency limit (i.e., $\omega/\Omega \to 0$) as a replacement of thermodynamic limit [5–12,15–21,62]. Both the Dicke model and the QRM can be experimentally realized in cavity quantum electrodynamics (CQED) or circuit systems [63–67]. However, the no-go theorem induced by the so-called $\mathbf{A}^2$ term (AT) (i.e., the squared electromagnetic vector potential) in realistic CQED forbids the SPT [68]. Actually, whether or not the SPT is prohibited by the AT has been a subject of much debate for several decades [57,68–88]. Indeed, both no-go theorems [68,70,83,86,87] and counter no-go theorems [72,75,84,85,88] have been raised, essentially depending on different situations such as adopting the conventional two-level approximation (qubit) or nontruncated Hilbert space for the matter part [68,70,82–84,86,87], applying the minimal replacement rule of the gauge to kinetic terms and also to nonlocal potentials [82,84,85,88], considering a spatially uniform cavity field [68,70] or a spatially varying electromagnetic field [81,86,87]. An arbitrary-gauge approach suggests that the conflicting no-go and counter no-go theorems may be reconciled in many-dipole cavity QED systems as views of different quantum subsystems [85,88]. Besides natural atomic systems, the debate has been extended to artificial atomic systems as well [57,72–74,79,88], as in circuit systems the existence of an equivalent AT is also controversial and depends on specific circuit designs [57,79,80].

Another way to circumvent the difficulty of reaching a consensus on the existence of the AT is to find a possibility to cancel the AT [89–91]. It has been suggested that the disappearing SPT of the Rabi (Dicke) model in the presence of the AT can be regained by combining the optomechanics and CQED [90,91], where the customarily used Coulomb gauge is chosen under a two-level approximation. The optomechanics there actually provides an auxiliary AT at the single-photon level to compensate for the intrinsic AT. However, the required strong quadratic optomechanical coupling at the single-photon level is still a challenge within current

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fabrication techniques [92]. In such a situation, an alternative quantum system capable of introducing a strong and tunable auxiliary AT is highly desirable, although the arbitrary-gauge approach [84,85,88] has been proposed to recover the SPT in CQED systems with the AT.

In this regard, a feasible way may lie in nonlinear cavity magnonics. With the advancement of quantum materials, magnons (i.e., quanta of spin wave) in a yttrium iron garnet (YIG) sphere with flexible controllability and high spin density have received much attention theoretically and experimentally [93–96], especially in magnon dark modes [97], spin currents [98,99], entanglement [100–104], nonreciprocity [105], and microwave-to-optical transduction [106,107]. Moreover, the magnons in the YIG sphere can have tunable Kerr nonlinearity via controlling the external magnetic field, originating from the magnetocrystalline anisotropy [108,109]. This nonlinearity has been demonstrated in experiment [110] and used to study the bi- and multistabilities [111–113], quantum entanglement [114], quantum phase transition [115], long-range spin-spin coupling [116,117], and nonreciprocal entanglement in cavity-magnon optomechanics [118].

In the present work we open a possible avenue to gain a strong and tunable auxiliary AT which is capable of switching on or manipulating the SPT, with the AT whether present or not as in the aforementioned debate on the no-go theorem. Based on the advances in the study of magnons, we propose a hybrid system consisting of a microwave cavity simultaneously coupled to both a TLS and Kerr magnons (i.e., magnons with Kerr nonlinearity) in a YIG sphere to restore the SPT. Here the coupled-TLS-cavity subsystem and the coupled-magnon-cavity subsystem form the Rabi model and cavity magnonics, respectively. By adiabatically eliminating the degrees of freedom of the magnons, we demonstrate that the Kerr magnons finally play a role to effectively introduce an additional AT which can counteract the original one. Owing to the experimental controllability and the nonlinearity-enhanced effect, the additional AT can be tunable and strong enough to switch on and manipulate the SPT. In the absence of the AT, the critical coupling of the SPT can be significantly reduced when the Kerr magnons are included, while in the presence of the AT the vanishing SPT can be restored by the Kerr magnons. Our proposal shows that the combination of CQED and nonlinear cavity magnonics can provide a potential platform to study quantum critical physics.

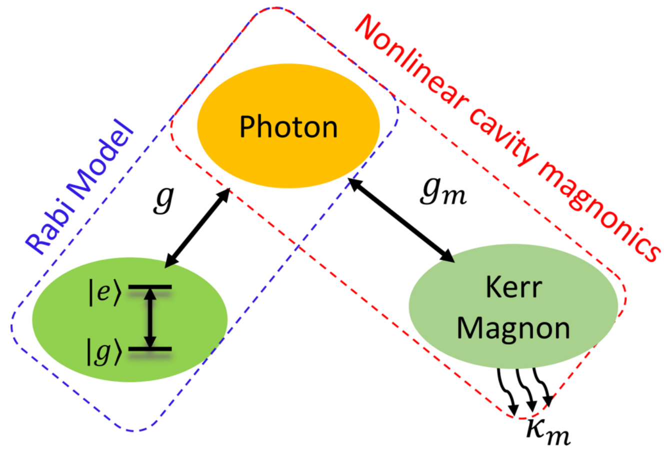

FIG. 1. Sketch of a hybrid quantum model including a Rabi model coupled to a Kerr magnon mode. Here g (g_m) is the Rabi (magnon) coupling strength of the two atomic levels (the magnon mode), respectively. Here the cavity mode is represented by a photon and κ_m is the magnon decay rate.

II. MODEL OF THE HYBRID SYSTEM

We consider a hybrid quantum system consisting of a microwave cavity [110] or superconducting resonator [119]) simultaneously coupled to a TLS, such as superconducting qubits [120,121], and Kerr magnons in a YIG sphere (see Fig. 1), where the Kerr nonlinearity of the magnons stems from the magnetocrystalline anisotropy. The Hamiltonian of the proposed hybrid system can be written as (setting $\hbar = 1$)

$$H = H_{\rm Rabi} + H_{A^2} + H_K + H_I, \tag{1}$$

where $H_{\rm Rabi} = \omega a^\dagger a + \frac{1}{2}\Omega\hat{\sigma}_z + g\sigma_x(a + a^\dagger)$ is the QRM, describing the interaction between the TLS and the cavity. The parameter ω (Ω) is the frequency of the cavity (TLS) and g is the coupling strength. The operators $\sigma_x = |e\rangle\langle g| + |g\rangle\langle e|$ and $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$ represent the Pauli matrices of the TLS, while a and $a^\dagger$ denote the annihilation and creation operators of the cavity, respectively. The second term $H_{A^2} = (\alpha g^2/\Omega)(a + a^\dagger)^2$ is the AT, where $\alpha \geqslant 1$ for the achieved Rabi model (Dicke model) in cavity quantum electrodynamics, which is dependent on the Thomas-Reiche-Kuhn sum rule [72]. The Kerr Hamiltonian $H_K = \omega_m m^\dagger m + K m^\dagger m m^\dagger m$, with the frequency $\omega_m = \gamma B_0 + 2\mu_0 K_{\rm an}\gamma^2/M^2V_m^2 - 2\mu_0\rho_s s K_{\rm an}\gamma^2/M^2$ and the Kerr coefficient $K/\hbar = 2\mu_0 K_{\rm an}\gamma^2/(M^2V_m^2)$, represents the interaction among magnons and provides the anharmonicity of the magnons. Here $\gamma/2\pi = g_e\mu_B/\hbar$ is the gyromagnetic ration with the g factor g_e and the Bohr magneton μ_B, $s = \hbar/2$ is the spin quantum number, $\rho_s = 2.1 \times 10^{-22}$ cm^{-3} is the spin density of the YIG sphere, μ_0 is the vacuum permeability, $K_{\rm an}$ is the first-order anisotropy constant of the YIG sphere, B_0 is the amplitude of a bias magnetic field in the z direction, M is the saturation magnetization, and V_m is the volume of the YIG sphere. Note that the Kerr coefficient K can be either positive or negative by tuning the angle between the crystallographic axis [100] or [110] of the YIG sphere and the bias magnetic field [109–111]. The last term $H_I = g_m(am^\dagger + a^\dagger m)$ characterizes the interaction between the photons in the cavity and the magnons in the YIG sphere with coupling strength g_m.

III. EFFECTIVE HAMILTONIAN OF THE HYBRID SYSTEM

Taking into account the dissipations of the system considered in Eq. (1), its dynamics is governed by the Heisenberg-Langevin equation [122], i.e.,

$$\frac{d\mathcal{O}}{dt} = -i[\mathcal{O}, H] + \mathcal{L}^\dagger\mathcal{O}\mathcal{L} - \frac{1}{2}(\mathcal{O}\mathcal{L}^\dagger\mathcal{L} + \mathcal{L}^\dagger\mathcal{L}\mathcal{O}), \tag{2}$$

where $\mathcal{O}$ represents the system operator and $\mathcal{L}$ is the Lindblad operator. For the magnon-cavity subsystem of interest, the relaxation operators can be defined as $\mathcal{L}_a = \sqrt{\kappa}a$ and $\mathcal{L}_m = \sqrt{\kappa_m}m$, where κ (κ_m) is the decay rate of the cavity (magnon).

Specifically, the dynamics of the magnon-cavity subsystem can be written as

$$\begin{aligned}\dot{a}(t) =& -\left(\frac{\kappa}{2}+i\omega\right)a - ig\sigma_x - ig_m m\\&-2i\left(\frac{\alpha g^2}{\Omega}\right)(a+a^\dagger)+\sqrt{\kappa}a_{\rm in},\\ \dot{m}(t) =& -\left(\frac{\kappa_m}{2}+i\omega_m\right)m - 2iKm^\dagger mm\\&-ig_m a+\sqrt{\kappa_m}m_{\rm in},\end{aligned} \quad (3)$$

where $a_{\rm in}$ and $m_{\rm in}$ are the vacuum input noise operators of the cavity and magnon, respectively. The corresponding mean values are zero, i.e., $\langle a_{\rm in}\rangle = \langle m_{\rm in}\rangle = 0$. By rewriting each operator of the magnon-cavity subsystem as its expectation value plus the corresponding fluctuation, i.e., $a \to \langle a\rangle + a$ and $m \to \langle m\rangle + m$, the nonlinear term, i.e., $m^\dagger mm$, in Eq. (3) can be reexpressed as

$$\begin{aligned}m^\dagger mm \to& |\langle m\rangle|^2\langle m\rangle + 2|\langle m\rangle|^2 m + \langle m^\dagger\rangle m^2\\&+\langle m\rangle^2 m^\dagger + 2\langle m\rangle m^\dagger m + m^\dagger m^2.\end{aligned} \quad (4)$$

By neglecting the high-order fluctuation terms, we can linearize the dynamics in Eq. (3) as

$$\begin{aligned}\dot{a}(t) =& -\left(\frac{\kappa}{2}+i\omega\right)a - ig\sigma_x - ig_m m\\&-2i\left(\frac{\alpha g^2}{\Omega}\right)(a+a^\dagger)+\sqrt{\kappa}a_{\rm in},\end{aligned} \quad (5)$$

$$\dot{m}(t) = -\left(\frac{\kappa_m}{2}+i\Delta_m\right)m - 2iK\langle m\rangle^2 m^\dagger - ig_m a + \sqrt{\kappa_m}m_{\rm in},$$

where $\Delta_m = \omega_m + 4KN_m$, with the mean magnon number $N_m = |\langle m\rangle|^2$, is the effective frequency of the magnon induced by its Kerr nonlinearity.

Noting that the magnon-cavity coupling g_m can be tuned by the displacement of the YIG sphere [123], we can assume a weak coupling $\kappa_m \gg g_m$ so that the decoherence time of magnons is much shorter than that of photons in the cavity. In such a situation, the degrees of freedom of the magnon can be adiabatically eliminated by setting $\dot{m}(t) = 0$, which directly gives rise to

$$m = \frac{2K\langle m\rangle^2 g_m}{W}a^\dagger - \frac{g_m\left(\Delta_m + i\frac{\kappa_m}{2}\right)}{W}a, \quad (6)$$

where $W = \Delta_m^2 + \frac{\kappa_m^2}{4} - 4K^2N_m^2$. Substitution of Eq. (6) back into Eq. (5) leads us to

$$\begin{aligned}\dot{a} =& -\left(\frac{\kappa_{\rm eff}}{2}+i(\omega-\eta\Delta_m)\right)a - ig\sigma_x - 2i\left(\frac{\alpha g^2}{\Omega}\right)(a+a^\dagger)\\&-2iK\eta\langle m\rangle^2 a^\dagger + \sqrt{\kappa_{\rm eff}}a_{\rm in},\end{aligned} \quad (7)$$

where $\eta = g_m^2/W$ is a dimensionless parameter related to the cavity frequency shift $\eta\Delta_m$ and $\kappa_{\rm eff} = \kappa + \kappa_m\eta$ is the effective decay rate of the cavity. By rewriting the equation of motion in Eq. (7) as $\dot{a} = -i[a, H_{\rm eff}] - (\kappa_{\rm eff}/2)a + \sqrt{\kappa_{\rm eff}}a_{\rm in}$, we obtain the effective Hamiltonian after eliminating the degrees of freedom of the Kerr magnons

$$H_{\rm eff} = H_{\rm Rabi} + H_{A^2} + K\eta\langle m\rangle^2\left(a^\dagger - \frac{\Delta_m}{2K\langle m\rangle^2}a\right)^2. \quad (8)$$

As Δ_m, K, and $\langle m\rangle$ can be tuned by the amplitude and the direction of the bias field, the YIG sphere size, and the drive power, with the sign of K also being adjustable [109–111], we can assume $K < 0$ and set $\Delta_m = -2K\langle m\rangle^2$. Thus, the Hamiltonian (8) reduces to

$$H_{\rm eff} = H_{\rm Rabi} + H_{A^2} + H'_{A^2}, \quad (9)$$

where $H'_{A^2} = -\chi(a+a^\dagger)^2$, with $\chi \equiv -K\eta\langle m\rangle^2$, and a reduced $\eta = 4g_m^2/\kappa_m^2$ is an additional AT induced by the Kerr magnons. One sees that the strength of the additional AT χ is highly enhanced by the nonlinear Kerr effect in proportion to $\langle m\rangle^2$, thus being competent to counteract the effect of the original AT H_{A^2} and restore the SPT of the QRM in the presence of the AT.

IV. SWITCHABLE SPT BY KERR MAGNONIC COUPLING

To study the ground-state properties of the Hamiltonian in Eq. (9), a squeezing transformation, i.e., $S(r) = \exp[r(a^{\dagger 2} - a^2)/2]$, is imposed, leading to $a \to a\cosh(r) + a^\dagger\sinh(r)$. By choosing the squeezing parameter

$$r = -\frac{1}{4}\ln\left(1 + \alpha\bar{g}^2 - 4\frac{\chi}{\omega}\right), \quad (10)$$

with rescaled coupling $\bar{g} = g/g_c$, where $g_c = \sqrt{\omega\Omega}/2$ is the critical coupling of the QRM without the AT [6,7,11], Hamiltonian (9) is transformed to an effective QRM

$$H_S = U^\dagger H_{\rm eff}U = \tilde{\omega}a^\dagger a + \frac{\Omega}{2}\sigma_z + \tilde{g}(a^\dagger + a)\sigma_x, \quad (11)$$

where $\tilde{\omega} = \omega e^{-2r}$ is the renormalized photon frequency in the cavity and $\tilde{g} = ge^r$ is the effective coupling strength. Compared to the original QRM, the parameters $\tilde{\omega}$ and $\tilde{g}$ in Eq. (11) are tunable via χ. Correspondingly, with renormalized critical coupling $\tilde{g}_c = \sqrt{\tilde{\omega}\Omega}/2$, the rescaled coupling strength in Eq. (11) becomes

$$\tilde{\bar{g}} \equiv \tilde{g}/\tilde{g}_c = \bar{g}/\sqrt{1 + \alpha\bar{g}^2 - 4\chi/\omega}. \quad (12)$$

The critical point is decided by $\tilde{\bar{g}}_c = 1$, beyond which the ground state transits from the NP to the SP.

Obviously, when Kerr magnons are not included (i.e., $K = 0$ or equivalently $\chi = 0$), $\tilde{\bar{g}}$ reduces to $\tilde{\bar{g}} \to \bar{g}/\sqrt{1 + \alpha\bar{g}^2}$. For the realistic cavity QED, $\alpha \geqslant 1$ leads to $\tilde{\bar{g}} < 1$, indicating that the Rabi model with the AT is always in the normal phase, unable to reach the SPT. This is just the aforementioned no-go theorem [72].

When we have Kerr magnons, the critical coupling is explicitly determined by $\bar{g}_c = \sqrt{(1 - 4\chi/\omega)/(1 - \alpha)}$ in terms of the original system parameters. This indicates that the SPT is switchable via the Kerr magnons as long as $\chi/\omega > \frac{1}{4}$ for $\alpha > 1$, which is equivalent to $g_m/\kappa_m > \sqrt{3\omega/8\omega_m}$ by the constraint $\Delta_m = -2K\langle m\rangle^2$. Figure 2 illustrates the SPT-restored regime (red, marked by $4\chi/\omega > 1$). These conditions can be readily fulfilled experimentally [110–112,123,124] as $|K| = 0$–10^3 nHz and $\langle m\rangle = 0$–10^{17}, with a typical order

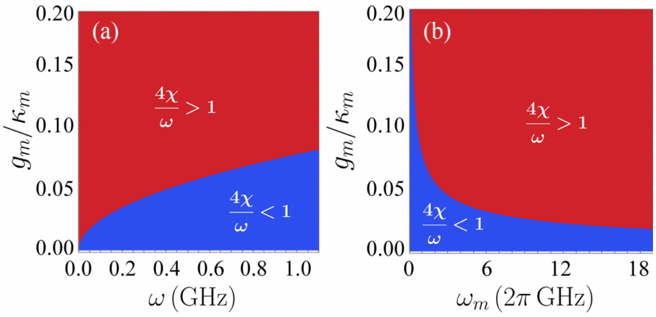

FIG. 2. SPT-restored regime (red, marked by $4\chi/\omega > 1$) in (a) the ω-g_m plane at $\omega_m/2\pi = 10$ GHz and (b) the ω_m-g_m plane at $\omega = 0.1$ GHz.

$2\pi \times 10$ GHz for ω_m and the low-frequency requirement of ω for SPT, while g_m can be tuned from weak couplings ($g_m/\kappa_m < 1$) to strong couplings ($g_m/\kappa_m > 1$) by the YIG sphere displacement [123].

V. REVERSED SPT AND PHASE DIAGRAM

In the absence of the AT, the SPT occurs in increasing the coupling strength as the small-g regime is the NP. However, in the presence of the AT, the SPT restored by Kerr magnons is reversed. In fact, the ground state is in the SP for the regime $\tilde{\bar{g}} > 1$, which gives rise to $\bar{g} < \bar{g}_c$. This means the SP lies below $\bar{g}_c$ instead of above $\bar{g}_c$. Conversely, the regime $\bar{g} > \bar{g}_c$ is in the NP, as derived from $\tilde{\bar{g}} < 1$. The reversed SPT would facilitate experimental study of the SP without the need to go beyond critical Rabi coupling.

In order to characterize the behavior of the SPT, we define the ground-state photon number as the order parameter, i.e., $\xi = A\langle a^\dagger a\rangle_g$, with the scaling factor $A = e^{-2r}\omega/\Omega$ to unify different value cases of ω and χ. In the limit of $\omega/\Omega \to 0$, Hamiltonian (11) can be readily diagonalized by expansion and unitary transformations [10–12,90,91]. Explicitly, $\xi = 0$ for $\bar{g} < e^{-2r}$ and $\xi = (\tilde{\bar{g}}^2 - \tilde{\bar{g}}^{-2})/4$ for $\bar{g} > e^{-2r}$. To show this clearly, we plot ξ as a function of $\bar{g}$ in Figs. 3(a) and 3(c).

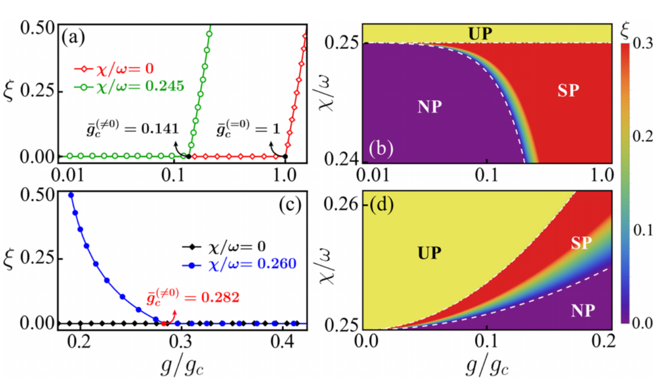

FIG. 3. SPT at (a) and (b) $\alpha = 0$ and (c) and (d) $\alpha = 1.5$ for (a) and (c) the order parameter ξ as a function of the coupling strength g and (b) and (d) the phase diagram of ξ in the g-χ plane. The dashed (dot-dashed) line denotes the analytical boundary between SP and NP (UP).

From Fig. 3(a) where the AT is not included, the SPT can be induced regardless of whether the magnon Kerr effect is taken into account or not. Without the magnon Kerr effect ($\chi = 0$), the SPT occurs at $\bar{g}_c^{(=0)} = 1$ (see the curve marked by diamonds). However, when the magnon Kerr effect is considered ($\chi = 0.245$), the SPT point is shifted to $\bar{g}_c^{(\neq 0)} = 0.141$ (curve marked by circles), which is much smaller than $\bar{g}_c^{(=0)}$, i.e., $\bar{g}_c^{(\neq 0)} = 0.141\bar{g}_c^{(=0)}$. This indicates that the introduced magnon Kerr effect can be utilized to dramatically reduce the critical coupling strength of the SPT, which greatly relaxes the parameter requirement in experiments. The AT is considered ($\alpha = 1.5$) in Fig. 3(c), where one sees that the SPT disappears in the absence of the magnon Kerr effect (black curve with closed diamonds). However, when the magnon Kerr effect is introduced, the SPT is restored at $\bar{g}_c^{(\neq 0)} = 0.282$ (blue curve with closed circles). Note here, as previously discussed, that the SPT is reversed with the transition direction from the SP to the NP with increasing coupling, which is opposite to the case in Fig. 3(a).

Figures 3(b) and 3(d) further show the ground-state phase diagram of ξ in the g-χ plane. In the absence of the AT but including the magnon Kerr effect [Fig. 3(c) with $\alpha = 0$], we can see that the transition from the NP ($\xi = 0$) to the SP ($\xi > 0$) can be observed with the increase of the coupling strength g in $\chi/\omega < \frac{1}{4}$ regime. The critical boundary is governed by $\chi/\omega = (1 - \bar{g}^2)/4$ (dashed line). In the $\chi/\omega > \frac{1}{4}$ regime, one has $\tilde{\bar{g}}^2 = 1 - 4\chi/\omega < 0$ spuriously; the system enters an unstable phase (UP) (yellow area). When both the AT and the magnon Kerr effect are included [Fig. 3(d) with $\alpha = 1.5$], we find that both the SP and the NP can recover in the previously unstable regime of $\chi/\omega > \frac{1}{4}$, now with $1 - 4\chi/\omega$ and $1 - \alpha$ both negative to fulfill $\tilde{\bar{g}}^2 > 0$. The critical NP-SP boundary is described by $\bar{g} = \sqrt{(1 - 4\chi/\omega)/(1 - \alpha)}$ [dashed line in Fig. 3(d)], while the SP-UP boundary is shifted from $\chi/\omega = \frac{1}{4}$ to $\chi/\omega = (1 + \alpha\bar{g}^2)/4$ [dot-dashed line in Fig. 3(d)].

The results in Fig. 3 are obtained from the effective Hamiltonian in Eq. (9). Although the Hamiltonian (9) is analytically derived from the original Hamiltonian (1) by adiabatically eliminating the magnon mode, one may wonder about a numerical cross-check. To confirm the validity of our results, we further compare the order parameter ξ with the result of the original Hamiltonian (1), as simulated by including the decay rates of the cavity and the magnons via $\mathcal{H}_{\rm eff} = H - i\frac{\kappa_m}{2}m^\dagger m - i\frac{\kappa_a}{2}a^\dagger a$ according to the quantum Langevin equation. An example is illustrated in Fig. 4 in the absence of the AT [$\alpha = 0$ in Fig. 4(a)] and in the presence of the AT [$\alpha = 1.5$ in Fig. 4(b)]. The parameters here meet the condition $\kappa_m \gg g_m$ as previously applied in adiabatically eliminating the magnon mode. Here, in both Figs. 4(a) and 4(b), the results of the effective Hamiltonian (9) and original Hamiltonian (1) are represented by the solid lines and dashed lines, respectively. We see in both cases, without the AT and with the AT, that the results from the effective Hamiltonian and original Hamiltonian are in good agreement, which shows that the mapping from the Hamiltonian (1) to the Hamiltonian (9) is reliable and the SPT can indeed be restored and reversed by the magnon Kerr effect, with the critical coupling strength significantly reduced.

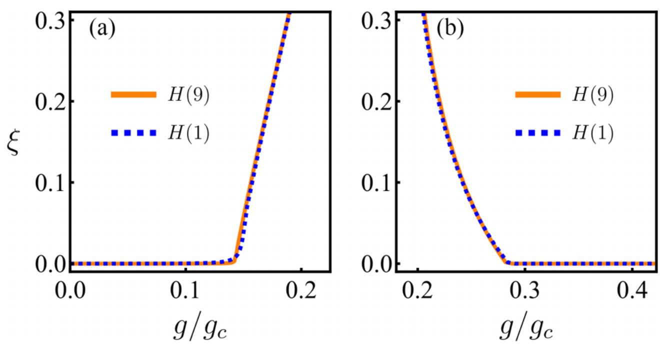

FIG. 4. Comparison of the order parameter ξ as a result of the effective Hamiltonian (9) (solid lines) and the original Hamiltonian (1) including the decay rates (dashed lines) for (a) $\alpha = 0$, with $K/\Omega = -6.125 \times 10^{-13}$ and $\omega_m/\Omega = 367.5$, and (b) $\alpha = 1.5$, with $K/\Omega = -6.5 \times 10^{-13}$ and $\omega_m/\Omega = 390$. The other parameters are $\omega/\Omega = 0.01$, $\Omega = 1$ MHz, $g_m/\Omega = 2\pi \times 0.01$, $\kappa_m/\Omega = 2\pi \times 100$, and $\kappa_a/\Omega = 2\pi \times 0.0001$ in both panels.

VI. CONCLUSION

In summary, we have proposed a hybrid quantum system combining nonlinear cavity magnonics and CQED to restore the SPT of the QRM, which has been thought to disappear in the presence of the AT due to the constraint of the no-go theorem or to dramatically reduce the critical coupling strength if the SPT is not prohibited as in the counter no-go theorem in the debate. Indeed, by adiabatically eliminating the degrees of freedom of the magnons, we have demonstrated that the Kerr magnons in a YIG sphere coupling with the Rabi cavity system effectively introduce an additional AT which can counteract the intrinsic AT. The additional AT is not only tunable via the Kerr magnon effect but also can be strong as in the nonlinear dependence of the magnon number, which can be very large, thus being capable of making the SPT switchable. We have analytically extracted the critical coupling generally in the presence of both the AT and the Kerr effect. The recovered SPT is illustrated by the transition in the photon number and shown in an overview by the determined phase diagram. We see that, when the AT is absent, our hybrid system can reduce the critical Rabi coupling and thus greatly relaxes the experimental conditions for observing the SPT. When the intrinsic AT is included, the unreachable SPT without Kerr magnons can be gained by turning on the Kerr magnon-photon coupling, while the superradiant phase is available in small Rabi couplings due to the revered transition direction. We have also shown the magnonic parameter regimes for restoring the SPT which are experimentally tunable and accessible. Note that an adjustable critical point is more favorable and provides more flexibility for applications such as in critical quantum metrology [42–48], where a wide range of critical couplings would greatly enlarge the critical sensing regime [44]. In such a trend, our proposal provides a promising path to manipulate the quantum phase transition with a hybrid system, combining the advantages of nonlinear cavity magnonics and CQED.

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[1] P. Forn-Díaz, L. Lamata, E. Rico, J. Kono, and E. Solano, Rev. Mod. Phys. **91**, 025005 (2019).
[2] A. F. Kockum, A. Miranowicz, S. De Liberato, S. Savasta, and F. Nori, Nat. Rev. Phys. **1**, 19 (2019).
[3] D. Braak, Phys. Rev. Lett. **107**, 100401 (2011).
[4] See a review of theoretical methods for light-matter interactions in A. Le Boité, Adv. Quantum Technol. **3**, 1900140 (2020).
[5] See a review of quantum phase transitions in light-matter interactions, e.g., in J. Liu, M. Liu, Z.-J. Ying, and H.-G. Luo, Adv. Quantum Technol. **4**, 2000139 (2021).
[6] S. Ashhab, Phys. Rev. A **87**, 013826 (2013).
[7] Z.-J. Ying, M. Liu, H.-G. Luo, H.-Q. Lin, and J. Q. You, Phys. Rev. A **92**, 053823 (2015).
[8] M.-J. Hwang, R. Puebla, and M. B. Plenio, Phys. Rev. Lett. **115**, 180404 (2015).
[9] M.-J. Hwang and M. B. Plenio, Phys. Rev. Lett. **117**, 123602 (2016).
[10] M. Liu, S. Chesi, Z.-J. Ying, X. Chen, H.-G. Luo, and H.-Q. Lin, Phys. Rev. Lett. **119**, 220601 (2017).
[11] Z.-J. Ying, Phys. Rev. A **103**, 063701 (2021).
[12] Z.-J. Ying, Adv. Quantum Technol. **5**, 2100088 (2022); **5**, 2270013 (2022).
[13] Z.-J. Ying, Adv. Quantum Technol. **5**, 2100165 (2022).
[14] Z.-J. Ying, Adv. Quantum Technol. **6**, 2200068 (2023); **6**, 2370011 (2023).
[15] Z.-J. Ying, Adv. Quantum Technol. **6**, 2200177 (2023); **6**, 2370071 (2023).
[16] Z.-J. Ying, L. Cong, and X.-M. Sun, J. Phys. A: Math. Theor. **53**, 345301 (2020).
[17] L. Cong, X.-M. Sun, M. Liu, Z.-J. Ying, and H.-G. Luo, Phys. Rev. A **95**, 063803 (2017); **99**, 013815 (2019).
[18] R. Grimaudo, A. S. M. de Castro, A. Messina, E. Solano, and D. Valenti, Phys. Rev. Lett. **130**, 043602 (2023).
[19] R. Grimaudo, D. Valenti, A. Sergi, and A. Messina, Entropy **25**, 187 (2023).
[20] F. Minganti, L. Garbe, A. Le Boité, and S. Felicetti, Phys. Rev. A **107**, 013715 (2023).
[21] J. Larson and E. K. Irish, J. Phys. A: Math. Theor. **50**, 174002 (2017).
[22] E. K. Irish and A. D. Armour, Phys. Rev. Lett. **129**, 183603 (2022).
[23] J. Casanova, R. Puebla, H. Moya-Cessa, and M. B. Plenio, npj Quantum Inf. **4**, 47 (2018).
[24] D. Braak, Symmetry **11**, 1259 (2019).
[25] Q.-H. Chen, C. Wang, S. He, T. Liu, and K.-L. Wang, Phys. Rev. A **86**, 023822 (2012).

[26] Q.-T. Xie, S. Cui, J.-P. Cao, L. Amico, and H. Fan, Phys. Rev. X **4**, 021046 (2014).
[27] Y. Yan, Z. Lü, L. Chen, and H. Zheng, Adv. Quantum Technol. **6**, 2200191 (2023).
[28] D. Fallas Padilla, H. Pu, G.-J. Cheng, and Y.-Y. Zhang, Phys. Rev. Lett. **129**, 183602 (2022).
[29] J. Peng, E. Rico, J. Zhong, E. Solano, and I. L. Egusquiza, Phys. Rev. A **100**, 063820 (2019).
[30] M. T. Batchelor and H.-Q. Zhou, Phys. Rev. A **91**, 053808 (2015).
[31] Z.-M. Li, D. Ferri, D. Tilbrook, and M. T. Batchelor, J. Phys. A: Math. Theor. **54**, 405201 (2021).
[32] L. Yu, S. Zhu, Q. Liang, G. Chen, and S. Jia, Phys. Rev. A **86**, 015803 (2012).
[33] D. Braak, Q. H. Chen, M. T. Batchelor, and E. Solano, J. Phys. A: Math. Theor. **49**, 300301 (2016).
[34] H.-P. Eckle, *Models of Quantum Matter* (Oxford University Press, Oxford, 2019).
[35] J. Larson and T. Mavrogordatos, *The Jaynes-Cummings Model and Its Descendants* (IOP, London, 2021).
[36] H. P. Eckle and H. Johannesson, J. Phys. A: Math. Theor. **50**, 294004 (2017).
[37] Y.-Q. Shi, L. Cong, and H.-P. Eckle, Phys. Rev. A **105**, 062450 (2022).
[38] B.-L. You, X.-Y. Liu, S.-J. Cheng, C. Wang, and X.-L. Gao, Acta Phys. Sin. **70**, 100201 (2021).
[39] L. Cong, S. Felicetti, J. Casanova, L. Lamata, E. Solano, and I. Arrazola, Phys. Rev. A **101**, 032350 (2020).
[40] K. K. W. Ma, Phys. Rev. A **102**, 053709 (2020).
[41] L.-T. Shen, Z.-B. Yang, H.-Z. Wu, and S.-B. Zheng, Phys. Rev. A **95**, 013819 (2017).
[42] L. Garbe, M. Bina, A. Keller, M. G. A. Paris, and S. Felicetti, Phys. Rev. Lett. **124**, 120504 (2020).
[43] L. Garbe, O. Abah, S. Felicetti, and R. Puebla, Phys. Rev. Res. **4**, 043061 (2022).
[44] Z.-J. Ying, S. Felicetti, G. Liu, and D. Braak, Entropy **24**, 1015 (2022).
[45] T. Ilias, D. Yang, S. F. Huelga, and M. B. Plenio, PRX Quantum **3**, 010354 (2022).
[46] Y. Chu, S. Zhang, B. Yu, and J. Cai, Phys. Rev. Lett. **126**, 010502 (2021).
[47] N. Wang, G.-Q. Liu, W.-H. Leong, H. Zeng, X. Feng, S.-H. Li, F. Dolde, H. Fedder, J. Wrachtrup, X.-D. Cui, S. Yang, Q. Li, and R.-B. Liu, Phys. Rev. X **8**, 011042 (2018).
[48] G. L. Zhu, X. Y. Lü, S. W. Bin, C. You, and Y. Wu, Front. Phys. **14**, 52602 (2019).
[49] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, Cambridge, 2011).
[50] K. Hepp and E. H. Lieb, Ann. Phys. (NY) **76**, 360 (1973).
[51] K. Hepp and E. H. Lieb, Phys. Rev. A **8**, 2517 (1973).
[52] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
[53] Y. Li, Z. D. Wang, and C. P. Sun, Phys. Rev. A **74**, 023815 (2006).
[54] K. Baumann, R. Mottl, F. Brennecke, and T. Esslinger, Phys. Rev. Lett. **107**, 140402 (2011).
[55] A. Baksic and C. Ciuti, Phys. Rev. Lett. **112**, 173601 (2014).
[56] L. J. Zou, D. Marcos, S. Diehl, S. Putz, J. Schmiedmayer, J. Majer, and P. Rabl, Phys. Rev. Lett. **113**, 023603 (2014).
[57] M. Bamba, K. Inomata, and Y. Nakamura, Phys. Rev. Lett. **117**, 173601 (2016).
[58] P. Kirton and J. Keeling, Phys. Rev. Lett. **118**, 123602 (2017).
[59] Y. Wang, M. Liu, W. L. You, S. Chesi, H. G. Luo, and H. Q. Lin, Phys. Rev. A **101**, 063843 (2020).
[60] I. I. Rabi, Phys. Rev. **49**, 324 (1936).
[61] I. I. Rabi, Phys. Rev. **51**, 652 (1937).
[62] L. Bakemeier, A. Alvermann, and H. Fehske, Phys. Rev. A **85**, 043821 (2012).
[63] T. Niemczyk, F. Deppe, H. Huebl, E. P. Menzel, F. Hocke, M. J. Schwarz, J. J. Garcia-Ripoll, D. Zueco, T. Hummer, E. Solano, A. Marx, and R. Gross, Nat. Phys. **6**, 772 (2010).
[64] A. A. Anappara, S. De Liberato, A. Tredicucci, C. Ciuti, G. Biasiol, L. Sorba, and F. Beltram, Phys. Rev. B **79**, 201303(R) (2009).
[65] J. A. Mlynek, A. A. Abdumalikov, C. Eichler, and A. Wallraff, Nat. Commun. **5**, 5186 (2014).
[66] J. D. Breeze, E. Salvadori, J. Sathian, N. McN. Alford, and C. W. M. Kay, npj Quantum Inf. **3**, 40 (2017).
[67] J. Braumöller, M. Marthaler, A. Schneider, A. Stehli, H. Rotzinger, M. Weides, and A. V. Ustinov, Nat. Commun. **8**, 779 (2017).
[68] K. Rzażewski, K. Wódkiewicz, and W. Zakowicz, Phys. Rev. Lett. **35**, 432 (1975).
[69] J. M. Knight, Y. Aharonov, and G. T. C. Hsieh, Phys. Rev. A **17**, 1454 (1978).
[70] I. Bialynicki-Birula and K. Rzążewski, Phys. Rev. A **19**, 301 (1979).
[71] J. Keeling, J. Phys.: Condens. Matter **19**, 295213 (2007).
[72] P. Nataf and C. Ciuti, Nat. Commun. **1**, 72 (2010).
[73] O. Viehmann, J. von Delft, and F. Marquardt, Phys. Rev. Lett. **107**, 113602 (2011).
[74] C. Ciuti and P. Nataf, Phys. Rev. Lett. **109**, 179301 (2012).
[75] A. Vukics and P. Domokos, Phys. Rev. A **86**, 053807 (2012).
[76] S. De Liberato, Phys. Rev. Lett. **112**, 016401 (2014).
[77] A. Vukics, T. Grießer, and P. Domokos, Phys. Rev. Lett. **112**, 073601 (2014).
[78] M. Leib and M. J. Hartmann, Phys. Rev. Lett. **112**, 223603 (2014).
[79] T. Jaako, Z. L. Xiang, J. J. Garcia-Ripoll, and P. Rabl, Phys. Rev. A **94**, 033850 (2016).
[80] M. Bamba and N. Imoto, Phys. Rev. A **96**, 053857 (2017).
[81] P. Nataf, T. Champel, G. Blatter, and D. M. Basko, Phys. Rev. Lett. **123**, 207402 (2019).
[82] O. Di Stefano, A. Settineri, V. Macr, L. Garziano, R. Stassi, S. Savasta, and F. Nori, Nat. Phys. **15**, 803 (2019).
[83] G. M. Andolina, F. M. D. Pellegrino, V. Giovannetti, A. H. MacDonald, and M. Polini, Phys. Rev. B **100**, 121109(R) (2019).
[84] A. Stokes and A. Nazir, Nat. Commun. **10**, 499 (2019).
[85] A. Stokes and A. Nazir, Phys. Rev. Lett. **125**, 143603 (2020).
[86] G. M. Andolina, F. M. D. Pellegrino, V. Giovannetti, A. H. MacDonald, and M. Polini, Phys. Rev. B **102**, 125137 (2020).
[87] G. M. Andolina, F. M. D. Pellegrino, A. Mercurio, O. Di Stefano, M. Polini, and S. Savasta, Eur. Phys. J. Plus **137**, 1348 (2022).
[88] A. Stokes and A. Nazir, Rev. Mod. Phys. **94**, 045003 (2022).
[89] X. Chen, Z. Wu, M. Jiang, X.-Y. Lü, X. Peng, and J. Du, Nat. Commun. **12**, 6281 (2021).

[90] X. Y. Lü, L. L. Zheng, G. L. Zhu, and Y. Wu, Phys. Rev. Appl. **9**, 064006 (2018).
[91] X. Y. Lü, G. L. Zhu, L. L. Zheng, and Y. Wu, Phys. Rev. A **97**, 033807 (2018).
[92] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, Rev. Mod. Phys. **86**, 1391 (2014).
[93] B. Z. Rameshti, S. V. Kusminskiy, J. A. Haigh, K. Usami, D. Lachance-Quirion, Y. Nakamura, C. M. Hu, H. X. Tang, G. E. W. Bauer, and Y. M. Blanter, Phys. Rep. **979**, 1 (2022).
[94] Y. P. Wang and C. M. Hu, J. Appl. Phys. **127**, 130901 (2020).
[95] H. Y. Yuan, Y. Cao, A. Kamra, R. A. Duine, and P. Yan, Phys. Rep. **965**, 1 (2022).
[96] D. Lachance-Quirion, Y. Tabuchi, A. Gloppe, K. Usami, and Y. Nakamura, Appl. Phys. Express **12**, 070101 (2019).
[97] X. Zhang, C. L. Zou, N. Zhu, F. Marquardt, L. Jiang, and H. X. Tang, Nat. Commun. **6**, 8914 (2015).
[98] L. Bai, M. Harder, Y. P. Chen, X. Fan, J. Q. Xiao, and C. M. Hu, Phys. Rev. Lett. **114**, 227201 (2015).
[99] L. Bai, M. Harder, P. Hyde, Z. Zhang, C. M. Hu, Y. P. Chen, and J. Q. Xiao, Phys. Rev. Lett. **118**, 217201 (2017).
[100] J. Li, S. Y. Zhu, and G. S. Agarwal, Phys. Rev. Lett. **121**, 203601 (2018).
[101] H. Y. Yuan, P. Yan, S. Zheng, Q. Y. He, K. Xia, and M. H. Yung, Phys. Rev. Lett. **124**, 053602 (2020).
[102] F. X. Sun, S. S. Zheng, Y. Xiao, Q. Gong, Q. He, and K. Xia, Phys. Rev. Lett. **127**, 087203 (2021).
[103] G. Q. Zhang, W. Feng, W. Xiong, Q. P. Su, and C. P. Yang, Phys. Rev. A **107**, 012410 (2023).
[104] S. F. Qi and J. Jing, Phys. Rev. A **105**, 022624 (2022).
[105] Y. Wang, W. Xiong, Z. Xu, G. Q. Zhang, and J. Q. You, Sci. China Phys. Mech. Astron. **65**, 260314 (2022).
[106] R. Hisatomi, A. Osada, Y. Tabuchi, T. Ishikawa, A. Noguchi, R. Yamazaki, K. Usami, and Y. Nakamura, Phys. Rev. B **93**, 174427 (2016).
[107] N. Zhu, X. Zhang, X. Han, C. L. Zou, C. Zhong, C. H. Wang, L. Jiang, and H. X. Tang, Optica **7**, 1291 (2020).
[108] D. D. Stancil and A. Prabhakar, *Spin Waves* (Springer, Berlin, 2009).
[109] G. Q. Zhang, Y. P. Wang, and J. Q. You, Sci. China Phys. Mech. Astron. **62**, 987511 (2019).
[110] Y. P. Wang, G. Q. Zhang, D. Zhang, X. Q. Luo, W. Xiong, S. P. Wang, T. F. Li, C.-M. Hu, and J. Q. You, Phys. Rev. B **94**, 224410 (2016).
[111] R. C. Shen, Y. P. Wang, J. Li, S. Y. Zhu, G. S. Agarwal, and J. Q. You, Phys. Rev. Lett. **127**, 183202 (2021).
[112] Y. P. Wang, G. Q. Zhang, D. Zhang, T. F. Li, C. M. Hu, and J. Q. You, Phys. Rev. Lett. **120**, 057202 (2018).
[113] J. M. P. Nair, Z. Zhang, M. O. Scully, and G. S. Agarwal, Phys. Rev. B **102**, 104415 (2020).
[114] Z. Zhang, M. O. Scully, and G. S. Agarwal, Phys. Rev. Res. **1**, 023021 (2019).
[115] G.-Q. Zhang, Z. Chen, W. Xiong, C.-H. Lam, and J. Q. You, Phys. Rev. B **104**, 064423 (2021).
[116] W. Xiong, M. Tian, G.-Q. Zhang, and J. Q. You, Phys. Rev. B **105**, 245310 (2022).
[117] M. Tian, M. Wang, G.-Q. Zhang, H.-C. Li, and W. Xiong, arXiv:2304.13553.
[118] J. Chen, X.-G. Fan, W. Xiong, D. Wang, and L. Ye, Phys. Rev. B **108**, 024105 (2023).
[119] Y.-P. Wang, J. W. Rao, Y. Yang, P.-C. Xu, Y. S. Gui, B. M. Yao, J. Q. You, and C.-M. Hu, Phys. Rev. Lett. **123**, 127202 (2019).
[120] J. Q. You and F. Nori, Nature (London) **474**, 589 (2011).
[121] Z. L. Xiang, S. Ashhab, J. Q. You, and F. Nori, Rev. Mod. Phys. **85**, 623 (2013).
[122] R. Benguria and M. Kac, Phys. Rev. Lett. **46**, 1 (1981).
[123] D. Zhang, X.-Q. Luo, Y.-P. Wang, T.-F. Li, and J. Q. You, Nat. Commun. **8**, 1368 (2017).
[124] A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, Phys. Rev. A **69**, 062320 (2004).