Universal tetramer limit cycle at the unitarity limit

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We investigate the appearance of a four-boson limit cycle in Hamiltonian systems at the unitarity limit. The model interaction incorporates two-, three-, and four-body short-range potentials, which allow us to disentangle the interwoven dynamics of three- and four-boson energy levels. Through numerical evidence, we observe a correlation between the energies of two successive universal tetramer levels for a fixed weakly bound trimer, which is found to be largely model independent. Interestingly, this correlation cycle is consistent with the findings of Hadizadeh *et al.* [Phys. Rev. Lett. **107**, 135304 (2011)] using a zero-range model.

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I. INTRODUCTION

It is fascinating how bosonic quantum systems behave in the limit of zero-range forces, also known as the scaling limit. Thomas in 1935 [1] gave the first hint of its nontrivial properties showing the collapse of the three-boson system, meaning the Hamiltonian spectrum is not bounded from below. Later, Skorniakov and Ter-Matirosyan (STM) in 1956 [2] formulated the three-body integral equations with the zero-range interaction to be solved, which inspired Faddeev to formulate his famous coupled set of three-body integral equations [3] that overcame the nonuniqueness problem of the Lippman-Schwinger equations.

However, the solution of the STM equations was plagued by the Thomas collapse, which was tamed by Danilov in 1961 [4], who found the log-periodic solutions in the ultraviolet (UV) limit of those equations, as a consequence of its continuous scale invariance. He recognized the need to introduce a boundary condition to have a unique solution of the scattering by giving the three-body binding energy as input. Soon after that, Minlos and Faddeev [5] showed how the route to the Thomas collapse of the bound state proceeds to the "fall to center" [6] by solving the homogeneous form of the STM equation for three bosons in the UV limit, finding an infinite discrete spectrum of the equation that extends to $-\infty$, with levels geometrically separated by the factor $\exp(2\pi/s_0) \approx 515$. This was the first clear observation of the continuous scale symmetry breaking to a discrete one.

Efimov in 1970 [7,8] discovered the presence of those infinite number of geometrically spaced levels when a three-boson system interacts resonantly with any short-range potential, which has an infinite scattering length. Nowadays this is called the unitarity limit in the context of effective field theory (EFT) [9] (see also Ref. [10]). Furthermore, the existence of this infinite number of states can be linked to the presence of a renormalization-group limit cycle within the space of coupling constants, particularly in the space of three-body couplings, as a function of a momentum cutoff. For a comprehensive discussion, refer to Ref. [9] and the references therein.

The first experimental evidence of Efimov states came from the Innsbruck experiment, which used an ultracold gas of cesium atoms [11] near a Feshbach resonance. This discovery made Efimov states a reality, and since then, many other experiments have observed their presence. Efimov states have also been observed in mass-imbalanced atomic systems (see, e.g., Ref. [12]).

The Efimov cycle manifests, in practice, through correlations between observables [13]. It is associated with the three-boson limit cycle found in the context of EFT [14], as well as in other works [15]. Moreover, the existence of such correlations is not restricted to zero-range interactions, but persists in finite-range systems too [16,17].

However, a question arises: How do new cycles in addition to the Efimov one manifest themselves for a larger number of bosons in the unitarity limit, or in other words, for s-wave interactions in the zero-range limit?

The first nontrivial step is the four-boson system. It was proposed in Ref. [18], that a new limit cycle beyond the Efimov one can appear in the four-boson system at the unitarity limit. Such a result, expressed by a correlation between the energies of consecutive tetramers for a fixed trimer energy, was obtained by solving a regularized set of Faddeev-Yakubovsky (FY) equations in the limit of a zero-range interaction. The new cycles were revealed when a four-boson scale was forced to move independently of the three-body one.

In Ref. [19] it was shown, by an approximate analytic solution of the FY equations, how the continuous scale symmetry is broken to a discrete one, which is associated with a log periodicity different from the Efimov one. Such a qualitative view was confirmed within a Born-Oppenheimer approximation of the heavy-heavy-light-light system, where it was shown explicitly that the Efimov periodicity of the heavy-heavy-light system is distinct from the four-body case [20], indicating that new limit cycles for more than three particles are different from the Efimov cycle at the unitarity limit.

While the correlation between consecutive tetramers was supported by the just mentioned calculations, it remains an open question whether or not a four-boson Hamiltonian system, with short-range forces at the unitarity limit, could exhibit a typical four-boson correlation cycle independent of the three-boson one. The four-boson cycle was not seen before in Hamiltonian systems when using two- and three-body forces, which exhibited for each Efimov state two tetramers [21–28]. However, an indication that this independent correlation cycle could exist in Hamiltonian systems comes from the accurate calculations provided in Ref. [29], which also suggests that the property of the Efimov cycle being interwoven with the four-boson one is verified as pointed out in Ref. [18] and confirmed in Ref. [20] for heavy-heavy-light-light bosonic systems and beyond, within the Born-Oppenheimer treatment.

A crucial property associated with the independent three- and four-boson cycles is the necessity of the introduction of a four-boson scale unrelated to the three-boson one. Such a possibility introduced in Ref. [30] confronted previous findings within EFT [21] and further explored in Ref. [22], which by now was understood that at next-to-leading order a four-boson scale has to be introduced in the EFT approach of the universal system [31]. The appearance of a four-body scale was also confirmed in calculations up to five bosons with van der Waals interactions [32]. Therefore, it is appealing to study the eigenvalues of a four-boson Hamiltonian for short-range interactions at the unitarity limit, with two-, three-, and four-body potentials to disentangle the three- and four-boson cycles [33] by manipulating the three- and four-boson short-range scales in an independent way, to follow the path of the recognized two universal tetramer levels attached to an Efimov state found in Hamiltonian models [21,22,24,25].

Evidence of induced multiboson interactions [30,34] may be already found in cold atomic gases, where the positions of three-atom resonances for narrow Feshbach resonances [35] and also for intermediate ones [36,37] deviate significantly from the predictions based on the van der Waals universality (see, e.g., Refs. [38,39]).

In this paper, we demonstrate correlations between successive energies of universal tetramers, which are calculated within a Hamiltonian system while keeping the trimer energy fixed at unitarity. These correlations, when accounting for range corrections, align with the predictions presented in Ref. [18]. The aforementioned reference relies on the solution of the regularized four-boson Faddeev-Yakubovsky equations to identify a four-body limit cycle that is independent of the three-body limit cycle underlying the Efimov effect.

The Hamiltonian employed in our investigation incorporates two-, three-, and four-body short-range potentials. Specifically, the two-body potential is utilized to adjust the energy of two bosons at the unitarity point. On the other hand, the three- and four-body potentials are employed to disentangle the intertwined three-boson cycle and the elusive four-boson cycle, the existence of which has been a topic of debate within the field [18,21,22,30,31].

II. THE HAMILTONIAN MODEL

We solve variationally the Schrödinger equation for a system of three and four bosons finding the ground state and several excited states. We develop the states using a set of correlated Gaussian functions, which were previously optimized using the stochastic variational method (SVM) [40,41]. SVM is essential in adapting the basis functions to different scales, which is a crucial factor in the calculation of multiple excited states.

As we want to study the behavior of systems interacting via a short-range interaction at the unitarty limit, we choose the Gaussian potential as a representative of such interactions. It has been extensively shown that close to the unitarity limit the Gaussian potential gives a universal representation of the class of short-range potentials [42,43]. The potential we use has two-, three-, and four-particle Gaussian terms. The two-body term reads

$$V_{2b}(r) = V_2\, e^{-(r/r_2)^2}, \tag{1}$$

and depends on the relative distance r between two particles. The range of the force is fixed to $r_2 = 1$, which is used as the unit of length; in this way the energy unit is $\hbar^2/mr_2^2$. The two-body strength is set to $V_2 = -2.684\,005\hbar^2/mr_2^2$ to tune the two-body system as close to the unitarity limit as possible.

The three-body term is

$$V_{3b}(r) = V_3\, e^{-(\rho_3/r_3)^2}, \tag{2}$$

where $\rho_3^2 = r_{12}^2 + r_{13}^2 + r_{23}^2$ is proportional to the three-body hyper-radius, and r_3 is the potential range in units of r_2. In the unitarity limit, as mentioned in the Introduction, a fascinating phenomenon known as the Efimov effect arises, resulting in an infinite tower of three-body states [7,8]. For our purposes, we focus solely on the ground-state trimer, as our objective is to compute bound four-body states rather than resonances. The three-body force Eq. (2) is employed to modify the value of the ground-state energy for the three-body system.

To accomplish our goal, we introduce a four-body potential

$$V_{4b}(r) = V_4\, e^{-(\rho_4/r_4)^2}, \tag{3}$$

where $\rho_4^2 = \sum_{i<j=1}^{4} r_{ij}^2$ is proportional to the four-body hyper-radius, and r_4 is the potential range in units of r_2. We use this four-body force to change the energy of the four-body states below the three-body threshold. Moreover, it is also used to change the number of four-body states below the threshold. With the ensemble of these forces, we can address individuality at the two-, three-, and four-body energy levels.

III. RESULTS

Our Hamiltonian model with Gaussian potentials provides compelling evidence of correlations between the energies of consecutive tetramers. These finite-range correlations are consistent with the ones obtained in Ref. [18] using a zero-range model. We place emphasis on the significance of recognizing the universal tetramers, which are constructed through a complex avoided-crossing energy-level structure. This structure arises from the interplay between the local Gaussian interaction utilized to reveal the underlying dynamics and a long-range effective potential that gives rise to the universal

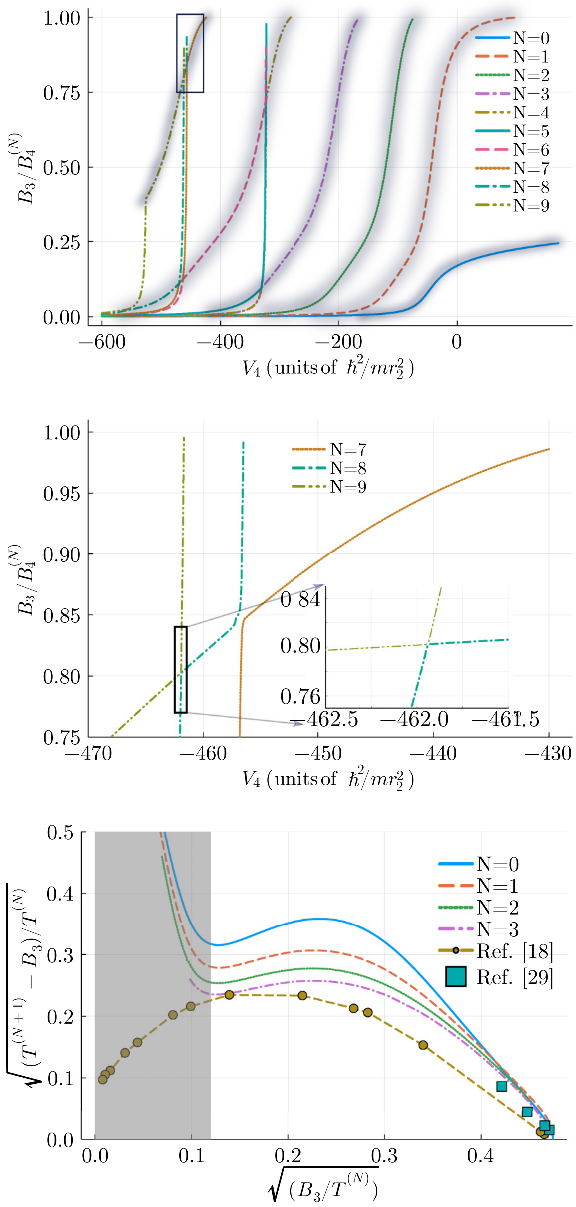

FIG. 1. Top panel: Evolution of the first ten four-boson energy levels as a function of the strength of the four-body force V_4. The three-boson bound-state energy is $B_3 = 0.238\,451\hbar^2/mr_2^2$, while the two-body system is at the unitarity limit. The four-body potential range is $r_4 = r_2$. Middle panel: Zoomed-in view of the region inside the rectangle shown in the top panel. Bottom panel: The correlation cycles for the four-boson successive universal levels. We have four cycles that have been obtained by following the universal states through the avoided crossing. The solid points correspond to the universal prediction of Ref. [18], and the solid squares to the separable-potential calculations of Ref. [29].

correlations, similar to the dynamics observed in the three-body sector. In the subsequent sections and figures, we present our findings.

The first set of calculations are shown in Fig. 1, where we set the two-body strength to $V_2 = -2.684\,005\hbar^2/mr_2^2$ to ensure $B_2 = 0$, and set the three-body force to zero ($V_3 = 0$). Using these parameter values, the binding energy of the three-boson bound state is $B_3 = 0.238\,451\hbar^2/mr_2^2$.

After fixing the two- and three-body sectors, we utilize the four-body force to manipulate the four-body spectrum, aiming to identify the universal levels and examine correlations among these states. Specifically, we vary the parameter V_4 while keeping the range of the four-body force fixed at $r_4 = r_2$. We calculate the four-body bound states with zero total angular momentum and observe the resulting spectrum. In the top panel of Fig. 1, we plot the ratio $B_3/B_4^{(N)}$ against V_4.

As we increase the strength of V_4, more states gradually appear in the spectrum, emerging from the three-body threshold. Interestingly, we observe two types of states: The first- ($N = 1$) and second-excited states ($N = 2$) smoothly move to deeper values after emerging from the threshold. However, there are states such as the third-excited state ($N = 3$) that also emerge smoothly from the threshold, but at a certain point, such as at $V_4 \approx -323\hbar^2/mr_2^2$ in this case, they exhibit a strong avoided crossing with other states, for instance, with $N = 4$, and $N = 5$. The state that emerges from the threshold as $N = 5$ evolves in a much narrower range of the four-body interaction, first interacting with the $N = 4$ and $N = 6$ states and then with $N = 3$ and $N = 4$. During this interaction, there is a role exchange, and we can still trace the smooth trajectory of the $N = 3$ state, but now as $N = 5$. This is just one example of the avoided-crossing structure of the spectrum, which becomes more evident as we increase the strength of the four-body force.

The avoided-crossing structure arises due to the interplay between the long-range effective (hyperradial) potential, similar to the three-body system [7], and the short-range nature of the four-body force used to reveal the universal states and their underlying correlations. This competition between the short- and long-range interactions results in the avoided-crossing structure of the spectrum.

The phenomenon of this nature was first observed in a two-body system by Zel'dovich in 1960. In that study, a variable local potential was added to fixed long-range potential. As the strength of the local interaction was varied, the spectrum underwent a significant rearrangement, resembling that of the long-range potential. This phenomenon is commonly referred to as the Zel'dovich phenomenon and has been extensively reviewed and expanded upon in Ref. [44] in the case of exotic atomes.

In the middle panel of Fig. 1, we present a zoomed-in view of the region inside the rectangle shown in the top panel. This zoomed-in view allows the reader to better discern the avoided-crossing structure of the energy levels, which might be less evident in the overall top panel.

In terms of studying correlations, we only consider the energy levels resulting from the long-range interaction and therefore having universal scaling properties. We use their energy, denoted as $T^{(N)}$, to construct the correlation, as shown in the bottom panel of Fig. 1. To clarify, in the case of Fig. 1, we have $T^{(0)} = B_4^{(0)}$, $T^{(1)} = B_4^{(1)}$, and $T^{(2)} = B_4^{(2)}$. However, for $T^{(3)}$, it is equal to $B_4^{(3)}$ only up to $V_4 \approx -323\hbar^2/mr_2^2$, after which it becomes equal to $B_4^{(5)}$. In the top panel of Fig. 1, these states have been highlighted.

The correlation function between the energies of two consecutive universal tetramers is plotted in the bottom panel of Fig. 1 using the $T^{(N)}$ levels. This function is constructed as suggested in Ref. [18] by plotting $\sqrt{(T^{(N+1)} - B_3)/T^{(N)}}$ as a function of $\sqrt{B_3/T^{(N)}}$. The resulting plot demonstrates the correlation cycle consistently. We display four such cycles, and we observe that they exhibit a convergence pattern as a function of N, even if it is not possible to extrapolate the limit. While the difference between the cycles is more pronounced for $\sqrt{B_3/T^{(N)}} \approx 0.25$, they collapse to the same curve for $\sqrt{B_3/T^{(N)}} \approx 0.45$, which corresponds to the point where a new tetramer emerges from the three-body threshold, and the state with energy $T^{(N)}$ is sufficiently shallow.

For the sake of comparison, in the same plot, we report the zero-range calculation of Hadizadeh *et al.* [18], and also the results from Deltuva [29] obtained with separable potentials for tetramers associated with different trimers at the unitarity limit. The general trend of the limit cycle obtained with the Gaussian potentials reproduces both the zero-range model and the separable-potential calculations, which are placed close to the point where one of the tetramers hits the trimer threshold. We believe that the difference between the zero-range cycle and the present results is possibly due to range corrections, even in the region where $\sqrt{B_3/T^{(N)}} \approx 0.45$ because the Nth tetramer is still compact enough to be influenced by the potential.

Moreover, we observe another noticeable discrepancy between the zero-range and finite-range calculations when examining values lower than $\sqrt{B_3/T^{(N)}} \approx 0.12$. Instead of continuing to decrease, the correlation cycles begin to grow in this region, which is covered by the shadow box. As a result, the $T^{(N)}$ state transitions from the universal window into a very tight four-body state. The same gray region is depicted in the top panel of Fig. 1. In Fig. 2, we investigate the influence of the trimer energy on the correlation cycle, which is one of the components contributing to finite-range effects. To manipulate the trimer energy, we introduce a nonzero three-body force with a strength of $V_3 = 30\hbar^2/mr_2^2$ and a range of $r_3 = r_2$. This modification shifts the trimer to a weakly bound state with an energy of $B_3 = 0.040\,01\hbar^2/mr_2^2$. The four-body range is now $r_4 = 2r_2$.

In the top panel of Fig. 2, we display the energy levels, which exhibit the same avoided-crossing pattern observed in the previous case. Additionally, we identify the universal tetramers $T^{(N)}$, highlighted within the same panel, which display avoided crossings starting from the third-excited level $N = 3$.

On the bottom panel of Fig. 2, we illustrate the correlation cycles, alongside the results from previous studies, specifically Refs. [18,29]. Although the correlations are similar to the case with the tighter trimer, there are differences, such as a lower maximum value, which can be attributed to the finite-range nature of the interaction. However, the trend of the correlation cycle obtained with the Gaussian potentials for the universal tetramer levels is closer to both the zero-range model and the separable-potential calculations, which are located near the point where one of the tetramers hits the trimer threshold. The better agreement with the zero-range results is evident in the interval where $\sqrt{(B_3/T^{(N)})} \gtrsim 0.3$, although range corrections are still present. The gray region in the top and bottom panels of Fig. 2 represents the strongly bound tetramers, which are outside the "window" where the universal states are.

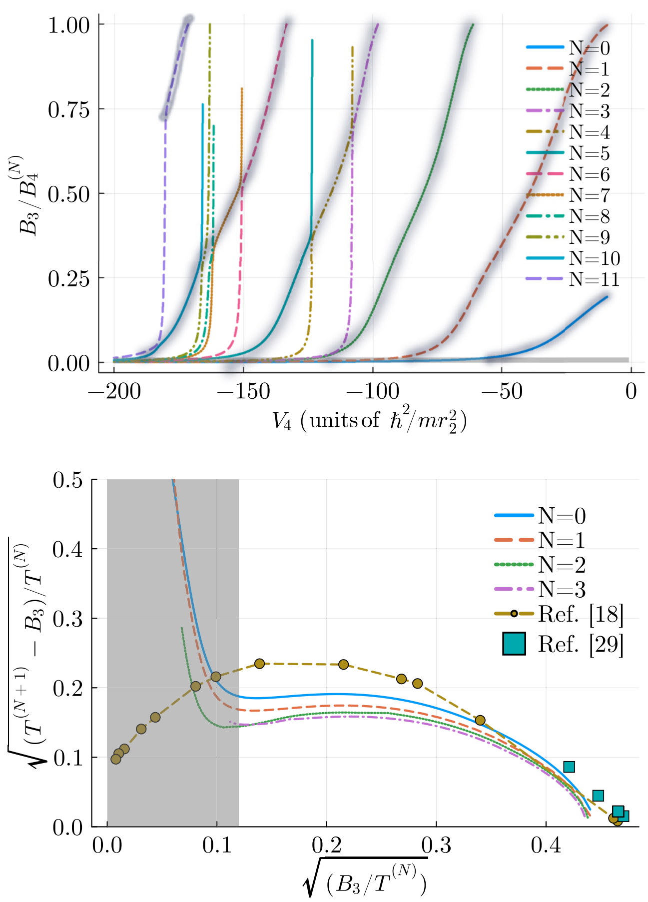

FIG. 2. Top panel: Evolution of the first 12 energy levels as a function of the strength V_4, obtained fixing the two-body system at the unitarity point, $B_2 = 0$. In this case, a three-body force has been introduced so that the three-body bound-state energy is $B_3 = 0.040\,01\hbar^2/mr_2^2$ and the four-body potential range is $r_4 = 2r_2$. We observe that starting from the third-excited four-boson state, $N = 3$, there is the appearance of the avoided crossing, and the evolution of the universal levels is evidenced by the shadow line. Bottom panel: The correlation cycles for the four-boson successive universal levels. We have four cycles that have been obtained by following the universal states through the avoided crossing. The solid points correspond to the universal prediction of Ref. [18], and the solid squares to the separable-potential calculations of Ref. [29].

In Fig. 3, we investigate the impact of changing the range of the four-body force r_4 on the finite-range effects. To maintain a consistent trimer energy, we employ the same three-body force as in the case of Fig. 2, ensuring that the trimer energy remains fixed at $B_3 = 0.040\,01\hbar^2/mr_2^2$.

In the top panel of Fig. 3, we observe that the details of the spectrum change, particularly the structure of the avoided-crossing levels. However, in the bottom panel, we can see that the first correlation cycle remains consistent with the cycles observed in Fig. 2. This suggests that the role of the four-body range is not significantly strong, as it does not significantly alter the observed correlations.

In the middle panel of Fig. 3, we present the variation of the mean radius $\langle r\rangle$ of the four-body states as a function of V_4. We specifically aim to highlight the different behavior of states that undergo significant changes compared to those that

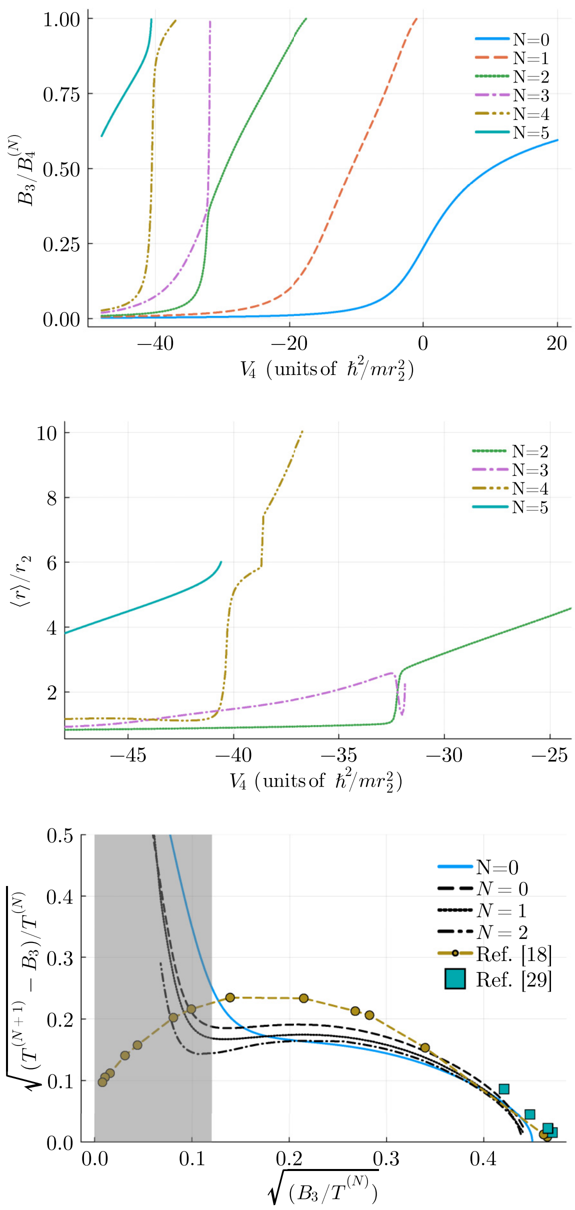

FIG. 3. Top panel: Evolution of the first eight energy levels as a function of the strength V_4, obtained by fixing the two-body system at the unitarity point, $B_2 = 0$. In this case, a three-body force has been introduced so that the three-body bound-state energy is $B_3 = 0.040\,01\hbar^2/mr_2^2$ and the four-body potential range is $r_4 = 3r_2$. Middle panel: The average radius $\langle r\rangle$ for specific states within a region of V_4 where avoided crossings occur. Notably, we observe that the states exhibiting significant variations in response to the strength V_4 emerge from the continuum with relatively small radii. Bottom panel: The first correlation cycle (solid line) compared with the cycles of Fig. 2 for the case $r_4 = 2r_2$ (dashed lines). The solid points correspond to the universal prediction of Ref. [18], and the solid squares to the separable-potential calculations of Ref. [29].

exhibit more gradual changes in response to variations in the four-body strength.

States that undergo rapid changes in the middle panel are characterized by a local nature, as evidenced by their smaller radii when emerging from the threshold. This observation suggests that the interplay between the local Gaussian potential and the emerging long-range hyperradial potential generates small wells from which these local states emerge. The presence of these small wells is likely responsible for the sensitivity of these states to variations in the four-body strength.

Despite these finite-range details, we have successfully proven the existence of a universal tetramer correlation cycle independent of the trimer, which can be related to the universal four-body limit cycle predicted in Ref. [18]. We leave for future research more refined Hamiltonian calculations that can control finite-range effects.

IV. SUMMARY

We have revealed the unexpected presence of a universal correlation cycle in Hamiltonian systems, which is compatible with the one predicted in Ref. [18] within a zero-range model. The observed cycle is independent of the one appearing in the three-boson system and related to Efimov physics. Our Hamiltonian system consists of two-, three-, and four-body short-range interactions, tuned to the unitarity limit. We have identified a series of universal tetramer states that maintain their model-independent properties, including the correlation between the energies of successive levels that converge toward the cycle. Our results suggest that the use of Gaussian interactions does not restrict the exploration of these cycles, as different parametrizations demonstrate the persistence of the universal levels and the associated energy correlation.

Our work opens the surprising perspective to search for interwoven correlation cycles in the $N \geqslant 5$ boson systems, by exploring the rich pattern arising when moving the universal levels through the control of many-body potentials. How this can be done in practice, for instance in cold-atom experiments, is an open question. Recently, the possibility of many-body forces manifesting themselves near relatively narrow Feshbach resonances has been suggested, which is attributed to the coupling between the open and closed channels when the atom-atom interaction is magnetically tuned in cold traps. However, further research is needed to explore this possibility, as highlighted in recent studies such as Refs. [34,45].

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