Multiple entropy production for multitime quantum processes

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Entropy production and the detailed fluctuation theorem are of fundamental importance for thermodynamic processes. In this paper we study the multiple entropy production for multitime quantum processes in a unified framework. For closed quantum systems and Markovian open quantum systems, the given entropy productions all satisfy the detailed fluctuation relation. This also shows that the entropy production rate under these processes is non-negative. For non-Markovian open quantum systems, the memory effect can lead to a negative-entropy production rate. Thus, in general, the entropy production of the marginal distribution does not satisfy the detailed fluctuation theorem relation. Our framework can be applied to a wide range of physical systems and dynamics. It provides a systematic tool for studying entropy production and its rate under arbitrary quantum processes.

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I. INTRODUCTION

The fluctuation theorem can give a generalization of the second law of thermodynamics and imply the Green-Kubo relations. It applies to fluctuations far from equilibrium and is of fundamental importance to nonequilibrium statistical mechanics. Roughly speaking, fluctuation theorems (FTs) are closely related to time-reversal symmetry and the relations between the probabilities of forward and backward processes. For isolated quantum systems, the forward and backward processes can be described by unitary evolution. In addition, there is always a widely held detailed FT [1]. For open quantum systems, one can assume that the entire system-environment combination is a large closed system and make use of the detailed FT of the closed system. Within this framework, the state of the environment must be detectable. If this is not the case, the backward mapping cannot fully recover the system state due to the lack of environment information. One approach is to use the Petz recovery map as the backward process and establish the relation between the quantum channel and its Petz recovery map [2]. The FTs for closed systems and the FTs for open systems give the same entropy production when dealing with maps with global fixed points [3].

The FTs focus mainly on entropy production. If the entropy production satisfies the detailed FT, we call it a fluctuation quantity here. In the single-shot scenario, the entropy production depends on two-point measurements. For the multitime processes, the intermediate measurements may affect the system state and subsequent evolution, so the entropy production may depend on multipoint measurements. The process in the presence of feedback control is a typical multitime process. Further, the corresponding FTs need to be modified to take into account the information gained from the measurement [4–6]. With the multipoint measurements, it becomes more natural to study the entropy production rate. An important observation is that the non-negativity of the entropy production does not guarantee the non-negativity of the entropy production rate. The entropy production rate is determined by the entropy production relation between the $(k+1)$-step processes and the k-step processes. Since non-negative average entropy production is a natural consequence of the FT, studying the entropy production and the detailed FT for multitime processes can help in the understanding of the relationship between the sign of the entropy production rate and the occurrence of non-Markovian effects.

It should be noted that in the single-shot scenario, the entropy production rate can also be discussed by comparing the average entropy production for different evolutionary times. However, as we will explain later, evolution with different evolution times can be described by the same evolution process, but the measurement process is completely different. Therefore, entropy production at different evolutionary times does not correspond to the same overall process and cannot be described by the same joint distribution.

Usually the FT is directly related to the actual observation and the multipoint measurements can give a joint probability distribution that can reflect the multitime properties of the system. However, in the quantum regime, the measurements are generally invasive: The measurements are invasive not only to the system itself, but also to the subsequent dynamics of the open system. On the one hand, since the measurement is invasive to the state, the measurement contributes directly to the entropy production. The cost of quantum measurement in a thermodynamic process is addressed in [7,8]. Reference [7] tried to use a single measurement and obtained a Jarzynski-like equality. Since there is only a single-point measurement, the properties it gives must also depend only on a single point. There will be neither the concept of the backward processes nor a fluctuation-dissipation theorem related to the properties of two-point measurements. On the other hand, because the measurement is invasive to the subsequent dynamics of the open system, it must have other indirect effects on the entropy production of multitime processes. Therefore, in this paper we

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TABLE I. Average entropy production relation of unitary evolution requires the Kolmogorov consistency condition. For Markovian evolution, even if the measurement is noninvasive, due to the irreversibility of evolution, there will generally be $\langle R\rangle > \langle R_1\rangle + \langle R_2\rangle$. One can find an entropy production such that $R_1' + R_2 = R$, but the fluctuation relation of R_1' is established in another process (61).

Time evolution	Satisfy the FT relation	Average relation	Subsequent entropy production
unitary	$R, R_1, R_2, \dots$	$\langle R\rangle = \langle R_1\rangle + \langle R_2\rangle$	$\langle R\rangle \geqslant \langle R_{\rm sub}\rangle$
Markovian	$R, R_1, R_2, \dots$	$\langle R\rangle \leqslant \langle R_1\rangle + \langle R_2\rangle$	$\langle R\rangle \geqslant \langle R_{\rm sub}\rangle$
non-Markovian	R, others to be determined	depends on conditions	depends on conditions

consider the combined effects of evolution and measurement on the entropy production and the FT.

The quantum dynamical semigroups are standard Markovian quantum processes. Their entropy production and detailed FT have been studied in [9,10]. For non-Markovian quantum processes, the memory effect makes the evolution much more complex. The process tensors [11,12] are powerful operational tools for studying various temporally extended properties of general quantum processes. Using these tools, Ref. [13] set up a framework for quantum stochastic thermodynamics and discussed the entropy production of the Markovian processes. In our previous work [14] we used an equivalent form of process tensors to obtain the FTs for non-Markovian processes. In this work we continue to use this form and consider the marginal distribution of the multitime quantum processes. The detailed FT of the joint probability does not guarantee that the marginal distribution also satisfies these relations. If these relations are indeed satisfied, then there can be several compatible fluctuation quantities in the same processes. As we will show later, the existence of multiple compatible fluctuations is directly related to the issue of the entropy production rate.

Due to the invasiveness of the measurements, the marginal distributions do not correspond to derived processes, in which some measurements are not performed, since the memory effect can lead to a negative-entropy production rate. Thus, in general, the entropy production of the marginal distribution does not satisfy the detailed FT relation. Only if the Kolmogorov consistency condition is satisfied, then not performing a measurement is the same as averaging over its probabilities [15]. In addition, the entropy production of the corresponding marginal distributions will be the same as that of the derived processes. The entropy production of the derived processes should satisfy the detailed FT relation, so the entropy production of the marginal distributions should also do so. Another interesting relationship between the Kolmogorov condition and quantum thermodynamics is work extraction [16]. Work extraction itself is not a fluctuation quantity and the proof of [16] has nothing to do with backward processes. So the relation between the sum of each intermediate amount of entropy production and the total entropy production is still unclear, which we will discuss briefly in this paper.

This paper is organized as follows. In Sec. II we first briefly introduce the general framework of operator states and process states. Then, for closed quantum systems, Markovian open quantum systems, and non-Markovian open quantum systems, we try to derive the entropy production and the detailed FT of the joint probability and marginal distributions (see Table I). We show that the Kolmogorov condition will make the sum of each intermediate amount of entropy production equal to the total entropy production in the closed system but fails for other systems. We also show how multiple compatible fluctuations are related to the entropy production rate. In Sec. III we discuss the average entropy production of a simple Jaynes-Cummings model. Section IV summarizes.

II. ENTROPY PRODUCTION FOR MULTITIME QUANTUM PROCESSES

In the operator-state formalism [14,17], operators are treated as states. The inner product of these states is defined as

$$(O_1|O_2) = {\rm Tr}(O_1^\dagger O_2). \tag{1}$$

The operator vector space is orthonormalized as

$$(\Pi_{kl}|\Pi_{ij}) = \delta_{ik}\delta_{jl}, \tag{2}$$

where $\Pi_{ij} = |i\rangle\langle j|$. The completeness relation is

$$\hat{I} = \sum_{ij} |\Pi_{ij})(\Pi_{ij}|. \tag{3}$$

The evolution of the system can be generally described with the quantum channel $\mathcal{N}$, which is a superoperator that maps a density matrix $|\rho)$ to another density matrix $|\rho') = \mathcal{N}|\rho)$. The operator state $|\Phi^{AS}) = \sum_{ij} |\Pi_{ij}^A \otimes \Pi_{ij}^S)$ is often used to link the input and output of the state. It differs from the maximally entangled state by only a normalization factor N. Hence, the state $|\Psi_{\mathcal{N}}^{AS}) \equiv \mathcal{N}^S|\Phi^{AS})/N$ is nothing but the Choi state obtained from the Choi-Jamiłkowski isomorphism. We use the Choi-state form of the process tensors, which we call the process states. This form can help us separate the measurement from evolution, and it is more convenient to use the results for the Choi state of the quantum channel.

A. Closed quantum system

In general, the time evolution of closed quantum systems is described by unitary operators acting on the system. Under $(n-1)$-step evolution, the forward process state is

$$|\mathcal{S}_{n:1}) := \mathcal{U}_{n-1}^{S_n} \circ \dots \circ \mathcal{U}_1^{S_2} \left| \rho_0^{S_1} \otimes \left[\bigotimes_{j=2}^{n} \Phi^{A^{(j-1)}S_j} \right] \right). \tag{4}$$

In such processes, it is natural to measure the state and use the measurement results as input for the next step. Therefore, we define the n-point measurement operation as

$$(\mathcal{O}_{n:1}^{(x_n:x_1)}| := \left(I^{S_n} \otimes \left[\bigotimes_{i=1}^{n-1} \Phi^{S_i A^{(i)}} \right] \right| \mathcal{M}_n^{(x_n)} \otimes \dots \otimes \mathcal{M}_1^{(x_1)}, \tag{5}$$

where the operation $\mathcal{M}_i^{(x_i)} = |\Pi^{x_i})(\Pi^{x_i}|$ acts on S_i. With this notation in hand, the joint probability for n-point

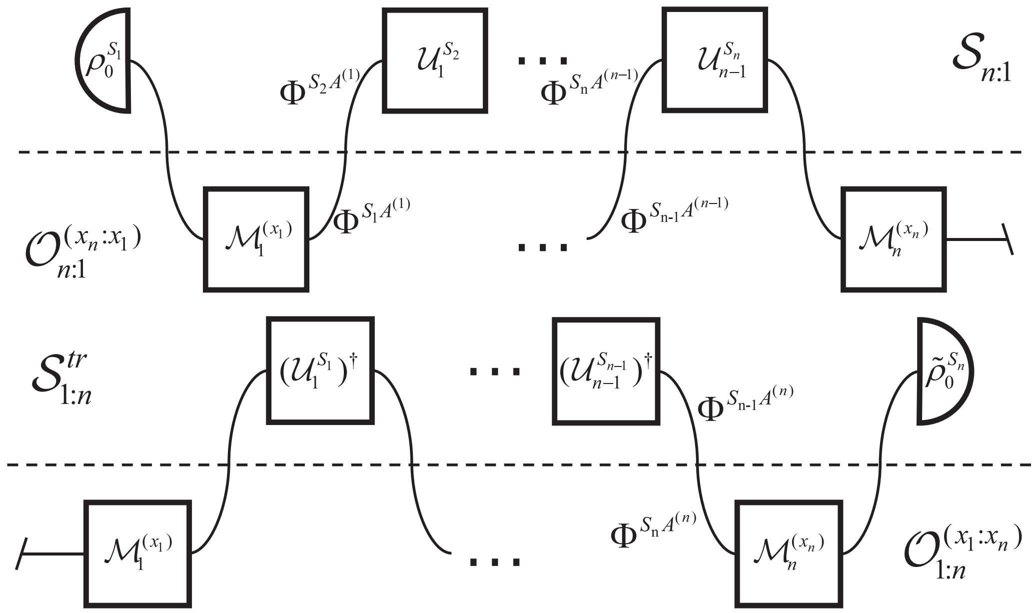

FIG. 1. Forward process state $\mathcal{S}_{n:1}$ and backward process state $\mathcal{S}^{tr}_{1:n}$ for $(n-1)$-step unitary evolution. The inner product of n-point measurement operation $\mathcal{O}$ and the process state gives a joint probability distribution. The dashed line represents the separation of measurements and evolution.

measurements can be expressed as (see Fig. 1)

$$\begin{aligned}\mathcal{P}_{n:1}(x_{n:1}|\mathcal{M}_{n:1}) &= \big(\mathcal{O}_{n:1}^{(x_n:x_1)}\big|\mathcal{S}_{n:1}\big)\\ &= (\Pi^{x_n}|\mathcal{U}_{n-1}|\Pi^{x_{n-1}})\\ &\quad\times\cdots(\Pi^{x_2}|\mathcal{U}_1|\Pi^{x_1})(\Pi^{x_1}|\rho_0).\end{aligned} \tag{6}$$

The unitary evolution is invertible with the time-reversed evolution. So the backward process state is

$$\big|\mathcal{S}^{tr}_{1:n}\big) := \big(\mathcal{U}_{n-1}^{S_{n-1}}\big)^\dagger\circ\cdots\circ\big(\mathcal{U}_1^{S_1}\big)^\dagger\left|\tilde{\rho}_0^{S_n}\otimes\left[\bigotimes_{j=1}^{n-1}\Phi^{A^{(j+1)}S_j}\right]\right),\tag{7}$$

where $\tilde{\rho}_0$ is the initial state of the backward process and $\mathcal{U}^\dagger$ is the adjoint map. For single-shot evolution, the final state of the forward process is usually chosen as the initial state of the backward process. For open quantum systems, there are also some other choices [3]. Here we choose

$$|\tilde{\rho}_0) = \mathcal{U}_{n-1}\circ\cdots\circ\mathcal{U}_1|\rho_0),\tag{8}$$

which is the final state of the forward process without any control operations. The backward n-point measurement operation can be defined as

$$\big(\mathcal{O}_{1:n}^{(x_1:x_n)}\big| := \left(I^{S_1}\otimes\left[\bigotimes_{i=1}^{n-1}\Phi^{S_{i+1}A^{(i+1)}}\right]\right|\mathcal{M}_n^{(x_n)}\otimes\cdots\otimes\mathcal{M}_1^{(x_1)}.\tag{9}$$

The joint probability for the backward process can be expressed as

$$\begin{aligned}\mathcal{P}^{tr}_{1:n}(x_{1:n}|\mathcal{M}_{1:n}) &= \big(\mathcal{O}_{1:n}^{(x_1:x_n)}\big|\mathcal{S}^{tr}_{1:n}\big)\\ &= \big(\Pi^{x_1}\big|\mathcal{U}_1^{-1}\big|\Pi^{x_2}\big)\\ &\quad\times\cdots\big(\Pi^{x_{n-1}}\big|\mathcal{U}_{n-1}^{-1}\big|\Pi^{x_n}\big)(\Pi^{x_n}|\tilde{\rho}_0).\end{aligned}\tag{10}$$

It is easy to see that

$$(\Pi^{x_j}|\mathcal{U}_{j-1}|\Pi^{x_{j-1}}) = (\Pi^{x_{j-1}}|\mathcal{U}_{j-1}^\dagger|\Pi^{x_j}).\tag{11}$$

The entropy production is defined as usual as the logarithm of the ratio of the forward and backward probabilities

$$R(x_{n:1}) := \ln\frac{\mathcal{P}_{n:1}(x_{n:1}|\mathcal{M}_{n:1})}{\mathcal{P}^{tr}_{1:n}(x_{1:n}|\mathcal{M}_{1:n})}.\tag{12}$$

It follows from Eqs. (6), (10), and (11) that

$$R(x_{n:1}) = R(x_n, x_1) = \ln\frac{(\Pi^{x_1}|\rho_0)}{(\Pi^{x_n}|\tilde{\rho}_0)},\tag{13}$$

which depends only on local measurements of the initial and final states. The distribution of entropy production is given by

$$p(R) = \sum_{x_{1:n}}\mathcal{P}_{n:1}(x_{n:1}|\mathcal{M}_{n:1})\delta(R-R(x_n,x_1)),\tag{14}$$

$$p^{tr}(R) = \sum_{x_{1:n}}\mathcal{P}^{tr}_{1:n}(x_{1:n}|\mathcal{M}_{1:n})\delta(R+R(x_n,x_1)).\tag{15}$$

It then follows that

$$p(R) = e^R p^{tr}(-R),\tag{16}$$

which gives the detailed FT.

Now consider the marginal distribution of the forward and backward probabilities. Without loss of generality, we divide the overall process into two parts: the first $n-2$ steps and the $(n-1)$th step. Summing over outcomes of the last measurement $\mathcal{M}_n$, we obtain a marginal distribution of the forward probability

$$\mathcal{P}_{n:1}(x_{n-1:1}|\mathcal{M}_{n:1}) := \sum_{x_n}\mathcal{P}_{n:1}(x_{n:1}|\mathcal{M}_{n:1}).\tag{17}$$

The backward one can be similarly defined. With these two probabilities, we can define another entropy production

$$R(x_{n-1:1}) := \ln \frac{\mathcal{P}_{n:1}(x_{n-1:1}|\mathcal{M}_{n:1})}{\mathcal{P}^{tr}_{1:n}(x_{1:n-1}|\mathcal{M}_{1:n})}. \tag{18}$$

Similar to Eq. (13), it is easy to show that

$$R(x_{n-1:1}) = R(x_{n-1}, x_1) = \ln \frac{(\Pi^{x_1}|\rho_0)}{(\Pi^{x_{n-1}}|\tilde{\rho}_1)}, \tag{19}$$

where $|\tilde{\rho}_1) = \mathcal{U}^{-1}_{n-1} \circ \mathcal{M}_n|\tilde{\rho}_0)$. In addition, $\mathcal{M}_k = \sum_{x_k} \mathcal{M}^{(x_k)}_k$ is a dephasing map. The entropy production $R(x_{n-1:1})$ depends only on local measurements of ρ_0 and $\tilde{\rho}_1$. The corresponding distribution of entropy production is given by

$$p(R_1) = \sum_{x_{1:n}} \mathcal{P}_{n:1}(x_{n:1}|\mathcal{M}_{n:1})\delta(R - R(x_{n-1}, x_1)), \tag{20}$$

$$p^{tr}(R_1) = \sum_{x_{1:n}} \mathcal{P}^{tr}_{1:n}(x_{1:n}|\mathcal{M}_{1:n})\delta(R + R(x_{n-1}, x_1)). \tag{21}$$

The entropy production R_1 also satisfies the detailed FT

$$p(R_1) = e^{R_1} p^{tr}(-R_1). \tag{22}$$

If summing over outcomes of measurements $\mathcal{M}_{n-2:1}$, we obtain another marginal distribution of the forward probability

$$\mathcal{P}_{n:1}(x_{n:n-1}|\mathcal{M}_{n:1}) := \sum_{x_{n-2:1}} \mathcal{P}_{n:1}(x_{n:1}|\mathcal{M}_{n:1}). \tag{23}$$

Similar to previous procedures, we can define

$$R(x_{n:n-1}) := \ln \frac{\mathcal{P}_{n:1}(x_{n:n-1}|\mathcal{M}_{n:1})}{\mathcal{P}^{tr}_{1:n}(x_{n-1:n}|\mathcal{M}_{1:n})} \tag{24}$$

and show that

$$R(x_{n:n-1}) = \ln \frac{(\Pi^{x_{n-1}}|\rho_{n-1})}{(\Pi^{x_n}|\tilde{\rho}_0)}, \tag{25}$$

where

$$|\rho_{n-1}) = \mathcal{U}_{n-2} \circ \mathcal{M}_{n-2} \circ \cdots \circ \mathcal{U}_1 \circ \mathcal{M}_1|\rho_0). \tag{26}$$

The corresponding distribution of entropy production is given by

$$p(R_2) = \sum_{x_{1:n}} \mathcal{P}_{n:1}(x_{n:1}|\mathcal{M}_{n:1})\delta(R - R(x_n, x_{n-1})), \tag{27}$$

$$p^{tr}(R_2) = \sum_{x_{1:n}} \mathcal{P}^{tr}_{1:n}(x_{1:n}|\mathcal{M}_{1:n})\delta(R + R(x_n, x_{n-1})). \tag{28}$$

The entropy production R_2 also satisfies the detailed FT. As stated above, the multitime quantum processes allow multiple entropy production terms. They all satisfy the detailed FT in a common multitime process. The previous procedures are actually applicable to all the marginal distributions. The corresponding entropy production also depends on local measurements.

If the joint probabilities $\mathcal{P}_{n:1}$ and $\mathcal{P}^{tr}_{1:n}$ satisfy the Kolmogorov consistency condition [15], then measuring but summing the measurements is equivalent to not measuring. In such cases, the probability distributions for all subsets of times can be obtained by marginalization. In addition, the detailed FTs mentioned above are all equivalent to two-point measurement FTs of some processes. In addition, when the Kolmogorov condition is met, there will be $|\rho_{n-1}) = \mathcal{U}_{n-2} \circ \cdots \circ \mathcal{U}_1|\rho_0)$ and $|\tilde{\rho}_1) = \mathcal{U}^{-1}_{n-1}|\tilde{\rho}_0)$. If choosing the initial state of the backward process as Eq. (8), then we have

$$|\tilde{\rho}_1) = \left(\mathcal{U}^{-1}_{n-1} \circ \mathcal{U}_{n-1}\right) \circ \cdots \circ \mathcal{U}_1|\rho_0) = |\rho_{n-1}) \tag{29}$$

and $(\Pi^{x_{n-1}}|\rho_{n-1}) = (\Pi^{x_{n-1}}|\tilde{\rho}_1)$. Under these circumstances, the previously mentioned entropy production terms have the relation

$$\begin{aligned} R(x_{n-1:1}) + R(x_{n:n-1}) &= \ln \frac{(\Pi^{x_{n-1}}|\rho_{n-1})(\Pi^{x_1}|\rho_0)}{(\Pi^{x_n}|\tilde{\rho}_0)(\Pi^{x_{n-1}}|\tilde{\rho}_1)} \\ &= \ln \frac{(\Pi^{x_1}|\rho_0)}{(\Pi^{x_n}|\tilde{\rho}_0)} = R(x_{n:1}). \end{aligned} \tag{30}$$

With this relation, one can easily see that $\langle R_1 \rangle + \langle R_2 \rangle = \langle R \rangle$, which is very similar to the relation of average work done shown in [16]. However, the work itself is not the entropy production. The relation of average work has nothing to do with the backward processes. Therefore, the two relations are very different.

Combining Eq. (30) and the detailed FT of $R(x_{n:n-1})$, we have

$$\langle e^{-[R(x_{n:1}) - R(x_{n-1:1})]} \rangle = 1. \tag{31}$$

The condition (30) is strong, but it is not a necessary condition for Eq. (31). In fact, if both R and R_1 satisfy the detailed FT, then we have

$$\begin{aligned} &\langle e^{-[R(x_{n:1}) - R(x_{n-1:1})]} \rangle \\ &= \sum_{x_{1:n}} \mathcal{P}_{n:1}(x_{n:1}|\mathcal{M}_{n:1}) e^{-[R(x_{n:1}) - R(x_{n-1:1})]} \\ &= \sum_{x_{1:n}} \mathcal{P}^{tr}_{1:n}(x_{1:n}|\mathcal{M}_{1:n}) e^{R(x_{n-1:1})} \\ &= \sum_{x_{1:n-1}} \mathcal{P}^{tr}_{1:n}(x_{1:n-1}|\mathcal{M}_{1:n}) e^{R(x_{n-1:1})} \\ &= \sum_{x_{1:n-1}} \mathcal{P}_{n:1}(x_{n-1:1}|\mathcal{M}_{n:1}) = 1. \end{aligned} \tag{32}$$

Using Jensen's inequality $\langle e^X \rangle \geqslant e^{\langle X \rangle}$, Eq. (31) implies

$$0 \leqslant \langle R(x_{n:1}) - R(x_{n-1:1}) \rangle = \langle R \rangle - \langle R_1 \rangle, \tag{33}$$

which means the total average entropy production is not less than an intermediate average entropy production. This also implies that the entropy production rate at step $n-1$ is nonnegative. The proof (32) applies to all cases where the joint probability is well defined.

Now let us calculate the average of the entropy production. The aforementioned Kolmogorov consistency condition and Eq. (8) are mainly used to give $(\Pi^{x_{n-1}}|\rho_{n-1}) = (\Pi^{x_{n-1}}|\tilde{\rho}_1)$. Without these conditions, using Eqs. (6), (13), and (26), the average entropy production R can be expressed generally as

$$\begin{aligned}\langle R\rangle &= \mathrm{Tr}[(\mathcal{M}_1\rho_0)\ln(\mathcal{M}_1\rho_0)] - \mathrm{Tr}[(\mathcal{M}_n\rho_n)\ln(\mathcal{M}_n\tilde{\rho}_0)]\\ &= S(\mathcal{M}_n\rho_n||\mathcal{M}_n\tilde{\rho}_0) + S(\mathcal{M}_n\rho_n) - S(\mathcal{M}_1\rho_0)\\ &= S(\mathcal{M}_n\rho_n||\mathcal{M}_n\tilde{\rho}_0) + \sum_{m=2}^{n}[S(\mathcal{M}_m\rho_m) - S(\rho_m)],\end{aligned} \tag{34}$$

where $S(\rho||\sigma) := \mathrm{Tr}[\rho(\ln\rho - \ln\sigma)]$ is the quantum relative entropy. In the third equality of Eq. (34) we exploit the fact that unitary transformations do not change entropy. For single-shot unitary evolution, if two-point measurements do not invade ρ_0, ρ_n, and $\tilde{\rho}_0$, then we can obtain the commonly used average entropy production [1]

$$\langle R\rangle = S(\rho(t))||\tilde{\rho}_0). \tag{35}$$

The average of the entropy productions R_1 and R_2 is

$$\begin{aligned}\langle R_1\rangle &= S(\mathcal{M}_{n-1}\rho_{n-1}||\mathcal{M}_{n-1}\tilde{\rho}_1)\\ &\quad + S(\mathcal{M}_{n-1}\rho_{n-1}) - S(\mathcal{M}_1\rho_0),\\ \langle R_2\rangle &= S(\mathcal{M}_n\rho_n||\mathcal{M}_n\tilde{\rho}_0)\\ &\quad + S(\mathcal{M}_n\rho_n) - S(\mathcal{M}_{n-1}\rho_{n-1}).\end{aligned} \tag{36}$$

Using the non-negativity of relative entropy and the data processing inequality, it is easy to find that these average entropy productions are non-negative. Combining Eqs. (34) and (36), it is easy to find that

$$\langle R_1\rangle + \langle R_2\rangle - \langle R\rangle = S(\mathcal{M}_{n-1}\rho_{n-1}||\mathcal{M}_{n-1}\tilde{\rho}_1) \geqslant 0. \tag{37}$$

If the Kolmogorov consistency condition is not satisfied, then the sum of the segmental entropy production is generally greater than the total average entropy production.

B. Markovian open quantum system

The time evolution of the Markovian open quantum system can be described by a sequence of independent completely positive and trace-preserving (CPTP) maps [18]. Under $(n-1)$-step evolution, the forward process state is

$$|\mathcal{S}_{n:1}) := \mathcal{N}_{n-1}^{S_n}\circ\cdots\circ\mathcal{N}_1^{S_2}\left|\rho_0^{S_1}\otimes\left[\bigotimes_{j=2}^{n}\Phi^{A^{(j-1)}S_j}\right]\right). \tag{38}$$

With the measurements (5), the joint probability for n-point measurements can still be expressed as

$$\mathcal{P}_{n:1}(x_{n:1}|\mathcal{M}_{n:1}) = \big(\mathcal{O}_{n:1}^{(x_n:x_1)}\big|\mathcal{S}_{n:1}\big). \tag{39}$$

For open quantum systems, the lack of information about the environment makes the evolution irreversible. It is common to use the Petz recovery map as the backward map [19]. For $(n-1)$-step evolution, the backward process state can be written as

$$\big|\mathcal{S}_{1:n}^{tr}\big) := \mathcal{R}_{n-1}^{S_{n-1}}\circ\cdots\circ\mathcal{R}_1^{S_1}\left|\tilde{\rho}_0^{S_n}\otimes\left[\bigotimes_{j=1}^{n-1}\Phi^{A^{(j+1)}S_j}\right]\right), \tag{40}$$

where $\mathcal{R}_m := \mathcal{J}_{\gamma_m}^{1/2}\circ\mathcal{N}_m^{\dagger}\circ\mathcal{J}_{\mathcal{N}_m(\gamma_m)}^{-1/2}$ is the Petz recovery map of $\mathcal{N}_m$, $\mathcal{J}_O^{\alpha}(\cdot) := O^{\alpha}(\cdot)O^{\alpha\dagger}$ is the rescaling map, and γ_m is the reference state that can be freely chosen. The Petz recovery map is a CPTP map and fully recovers the reference state. With the measurements (9), the joint probability for the backward process can be expressed as (see Fig. 2)

$$\mathcal{P}_{n:1}(x_{n:1}|\mathcal{M}_{n:1}) = \big(\mathcal{O}_{n:1}^{(x_n:x_1)}\big|\mathcal{S}_{n:1}\big). \tag{41}$$

Similar to Eq. (11), the adjoint map $\mathcal{N}^{\dagger}$ and $\mathcal{N}$ obey the following relation:

$$(\Pi_{k'l'}|\mathcal{N}|\Pi_{ij}) = (\Pi_{ij}|\mathcal{N}^{\dagger}|\Pi_{k'l'})^*. \tag{42}$$

The rescaling map in Petz recovery will make the joint probabilities (39) and (41) generally unable to establish a relation similar to Eq. (16). Only with the following operation can we obtain a detailed FT:

$$\begin{aligned}\big(\mathcal{O}_{n:1}^{(x_{n:1},i_{n-1:1},\ldots,l'_{n-1:1})}\big| &:= \left(I^{S_n}\otimes\left[\bigotimes_{m=1}^{n-1}\Pi_{i_mj_m}^{S_m}\Pi_{i_mj_m}^{A^{(m)}}\right]\right|\\ &\quad\times\mathcal{M}_n^{(x_n)}\mathcal{M}_n^{(k'_{n-1}l'_{n-1})}\otimes\\ &\quad\times\cdots\otimes\mathcal{M}_2^{(x_2)}\mathcal{M}_2^{(k'_1l'_1)}\otimes\mathcal{M}_1^{(x_1)}.\end{aligned} \tag{43}$$

Here $\mathcal{M}_{m+1}^{(k'_ml'_m)} := |\Pi_{k'_ml'_m}^{S_{m+1}})(\Pi_{k'_ml'_m}^{S_{m+1}}|$. The basis $|i_m\rangle$ is chosen such that it diagonalizes the reference state γ_m and $|k'_m\rangle$ is chosen as the eigenbasis of $\mathcal{N}_m(\gamma_m)$. On this basis, we have $\mathcal{J}_{\gamma_m}^{\alpha/2}|\Pi_{i_mj_m}) = Z_{ij}^{\gamma^{\alpha}}|\Pi_{i_mj_m})$, where

$$Z_{ij}^{\gamma^{\alpha}} := \big\|\mathcal{J}_{\gamma}^{\alpha/2}\Pi_{ij}\big\|_2 = \sqrt{(\Pi_i|\gamma^{\alpha})(\gamma^{\alpha}|\Pi_j)}. \tag{44}$$

With the operation (43), we obtain a quasiprobability distribution [2]

$$\begin{aligned}\mathcal{P}_{n:1}&\big(x_{n:1}, i_{n-1:1}, j_{n-1:1}, k'_{n-1:1}, l'_{n-1:1}|\mathcal{M}_{n:1}^{qs}\big)\\ &:= \big(\mathcal{O}_{n:1}^{(x_{n:1},\ldots,l'_{n-1:1})}\big|\mathcal{S}_{n:1}\big).\end{aligned} \tag{45}$$

This distribution is not positive, but it can be obtained from observable quantities [2]. The operation (43) can also be fully reconstructed as a linear combination of the n-point positive-operator-valued measurement operation. Moreover, the joint probability can be directly derived from the quasiprobability

$$\sum_{i_{n-1:1},\ldots}\mathcal{P}_{n:1}(x_{n:1},\ldots,l'_{n-1:1}|\mathcal{M}_{n:1}^{qs}) = \mathcal{P}_{n:1}(x_{n:1}|\mathcal{M}_{n:1}). \tag{46}$$

The backward quasimeasurement operation can be defined as

$$\begin{aligned}\big(\mathcal{O}_{1:n}^{(x_{1:n},\ldots,l'_{n-1:1})}\big| &:= \left(I^{S_1}\otimes\left[\bigotimes_{m=1}^{n-1}\Pi_{k'_ml'_m}^{S_{m+1}}\Pi_{k'_ml'_m}^{A^{(m+1)}}\right]\right|\\ &\quad\times\mathcal{M}_n^{(x_n)}\otimes\mathcal{M}_{n-1}^{(x_{n-1})}\mathcal{M}_{n-1}^{(i_{n-1}j_{n-1})}\otimes\\ &\quad\times\cdots\otimes\mathcal{M}_1^{(x_1)}\mathcal{M}_1^{(i_1j_1)},\end{aligned} \tag{47}$$

with which we obtain the quasiprobability distribution of backward processes

$$\mathcal{P}_{1:n}^{tr}\big(x_{1:n},\ldots,l'_{1:n-1}\big|\mathcal{M}_{1:n}^{qs}\big) := \big(\mathcal{O}_{1:n}^{(x_{n:1},\ldots,l'_{n-1:1})}\big|\mathcal{S}_{1:n}^{tr}\big)^*. \tag{48}$$

The joint probability of backward processes can also be directly derived from the quasiprobability of backward processes like Eq. (46). Similar to (12), the entropy production of which can be defined as

$$R(x_{n:1},\ldots,l'_{n-1:1}) := \ln\frac{\mathcal{P}_{n:1}\big(x_{n:1},\ldots,l'_{n-1:1}\big|\mathcal{M}_{n:1}^{qs}\big)}{\mathcal{P}_{1:n}^{tr}\big(x_{1:n},\ldots,l'_{1:n-1}\big|\mathcal{M}_{1:n}^{qs}\big)}, \tag{49}$$

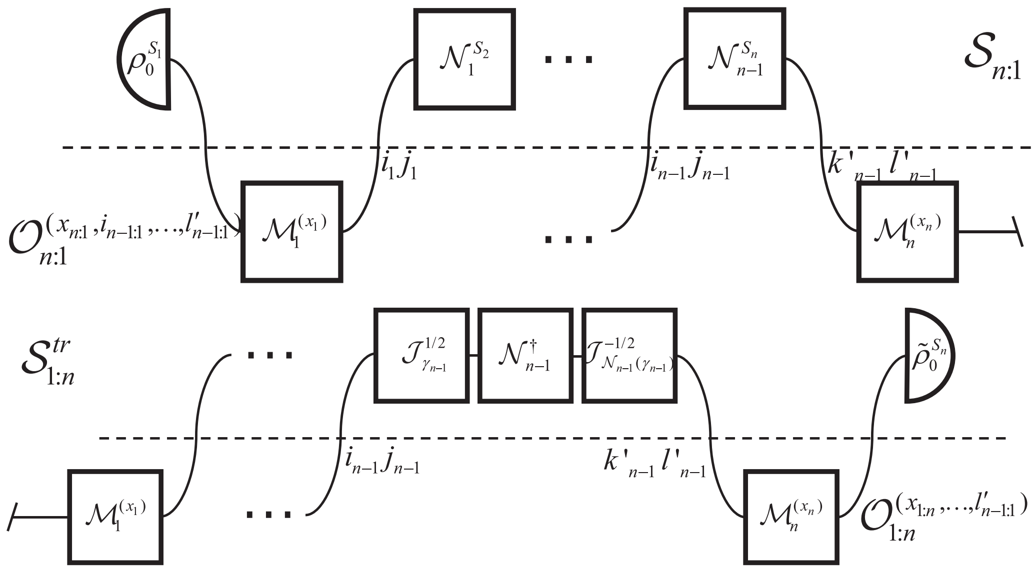

FIG. 2. Forward and backward processes of $(n-1)$-step Markovian quantum evolution. The quasimeasurements make the link Φ^{AS} become $\Pi_{ij}^{A} \otimes \Pi_{ij}^{S}$ and $\Pi_{k'l'}^{A} \otimes \Pi_{k'l'}^{S}$, which are abbreviated as ij and $k'l'$ in the figure.

with Eq. (42) it is easy to show that

$$\big(\Pi_{k'_m l'_m}\big|\mathcal{N}_m\big|\Pi_{i_m j_m}\big)^* = \big(\Pi_{i_m j_m}\big|\mathcal{R}_m\big|\Pi_{k'_m l'_m}\big) Z_{i_m j_m}^{\gamma_m^{-1}} Z_{k'_m l'_m}^{\mathcal{N}_m(\gamma_m)}. \quad (50)$$

Combining this with Eqs. (45) and (48), we find that

$$R(x_{n:1}, \ldots, l'_{n-1:1}) = R(x_n, x_1, \ldots, l'_{n-1:1}) = \ln \frac{(\Pi^{x_1}|\rho_0)}{(\Pi^{x_n}|\tilde{\rho}_0)} + \sum_{m=1}^{n-1} \ln \big(Z_{i_m j_m}^{\gamma_m^{-1}} Z_{k'_m l'_m}^{\mathcal{N}_m(\gamma_m)}\big) \quad (51)$$

is independent of intermediate measurements $\{x_{n-1:2}\}$. The distribution of entropy production for forward processes is given by

$$p(R) = \sum_{x_{n:1}, \ldots, l'_{n-1:1}} \mathcal{P}_{n:1}\big(x_{n:1}, \ldots, l'_{n-1:1}\big|\mathcal{M}_{n:1}^{qs}\big) \times \delta(R - R(x_n, x_1, \ldots, l'_{n-1:1})). \quad (52)$$

The backward one can be similarly defined. The detailed FT

$$p(R) = e^R p^{tr}(-R) \quad (53)$$

has been shown in Ref. [14].

Now following the same procedure used in Sec. II A, the marginal distribution can be defined as

$$\mathcal{P}_{n:1}\big(x_{n-1:1}, i_{n-2:1}, \ldots, l'_{n-2:1}\big|\mathcal{M}_{n:1}^{qs}\big) := \sum_{x_n, i_{n-1}, \ldots, l'_{n-1}} \mathcal{P}_{n:1}\big(x_{n:1}, \ldots, l'_{n-1:1}\big|\mathcal{M}_{n:1}^{qs}\big). \quad (54)$$

The corresponding entropy production is

$$R(x_{n-1:1}, \ldots, l'_{n-2:1}) := \ln \frac{\mathcal{P}_{n:1}\big(x_{n-1:1}, \ldots, l'_{n-2:1}\big|\mathcal{M}_{n:1}^{qs}\big)}{\mathcal{P}_{1:n}^{tr}\big(x_{1:n-1}, \ldots, l'_{1:n-2}\big|\mathcal{M}_{1:n}^{qs}\big)}, \quad (55)$$

which is independent of intermediate measurements $\{x_{n-2:2}\}$,

$$R(x_{n-1:1}, \ldots, l'_{n-2:1}) = R(x_{n-1}, x_1, \ldots, l'_{n-2:1}) = \ln \frac{(\Pi^{x_1}|\rho_0)}{(\Pi^{x_{n-1}}|\tilde{\rho}_1)} + \sum_{m=1}^{n-2} \ln \big(Z_{i_m j_m}^{\gamma_m^{-1}} Z_{k'_m l'_m}^{\mathcal{N}_m(\gamma_m)}\big), \quad (56)$$

where $|\tilde{\rho}_1) = \mathcal{R}_{n-1} \circ \mathcal{M}_n|\tilde{\rho}_0)$. The distributions of entropy production can be defined similarly to the previous one, and the detailed FT also holds (see Fig. 3).

After summing over outcomes of measurements $\mathcal{M}_{n-2:1}$, we obtain marginal distribution

$$\mathcal{P}_{n:1}\big(x_{n:n-1}, i_{n-1}, \ldots, l'_{n-1}\big|\mathcal{M}_{n:1}^{qs}\big) := \sum_{x_{n-2:1}, i_{n-2:1}, \ldots, l'_{n-2:1}} \mathcal{P}_{n:1}\big(x_{n:1}, \ldots, l'_{n-1:1}\big|\mathcal{M}_{n:1}^{qs}\big). \quad (57)$$

The corresponding entropy production is

$$R(x_{n:n-1}, \ldots, l'_{n-1}) := \ln \frac{\mathcal{P}_{n:1}\big(x_{n:n-1}, \ldots, l'_{n-1}\big|\mathcal{M}_{n:1}^{qs}\big)}{\mathcal{P}_{1:n}^{tr}\big(x_{n-1:n}, \ldots, l'_{n-1}\big|\mathcal{M}_{1:n}^{qs}\big)}. \quad (58)$$

It then follows that

$$R(x_{n:n-1}, \ldots, l'_{n-1}) = \ln \frac{(\Pi^{x_{n-1}}|\rho_{n-1})}{(\Pi^{x_n}|\tilde{\rho}_0)} + \ln \big(Z_{i_{n-1} j_{n-1}}^{\gamma_{n-1}^{-1}} Z_{k'_{n-1} l'_{n-1}}^{\mathcal{N}_{n-1}(\gamma_{n-1})}\big), \quad (59)$$

where

$$|\rho_{n-1}) = \mathcal{N}_{n-2} \circ \mathcal{M}_{n-2} \circ \cdots \circ \mathcal{N}_1 \circ \mathcal{M}_1|\rho_0). \quad (60)$$

The distributions of entropy production can be defined similarly to the previous one, and the detailed FT also holds.

Due to the irreversibility of open-system evolution, there is no relation like Eq. (29) for $\tilde{\rho}_1$ and ρ_{n-1}, even if the Kolmogorov condition is met. Further, Eq. (30) no longer holds for the $\{R, R_1, R_2\}$ defined here. The backward process of R_1 is

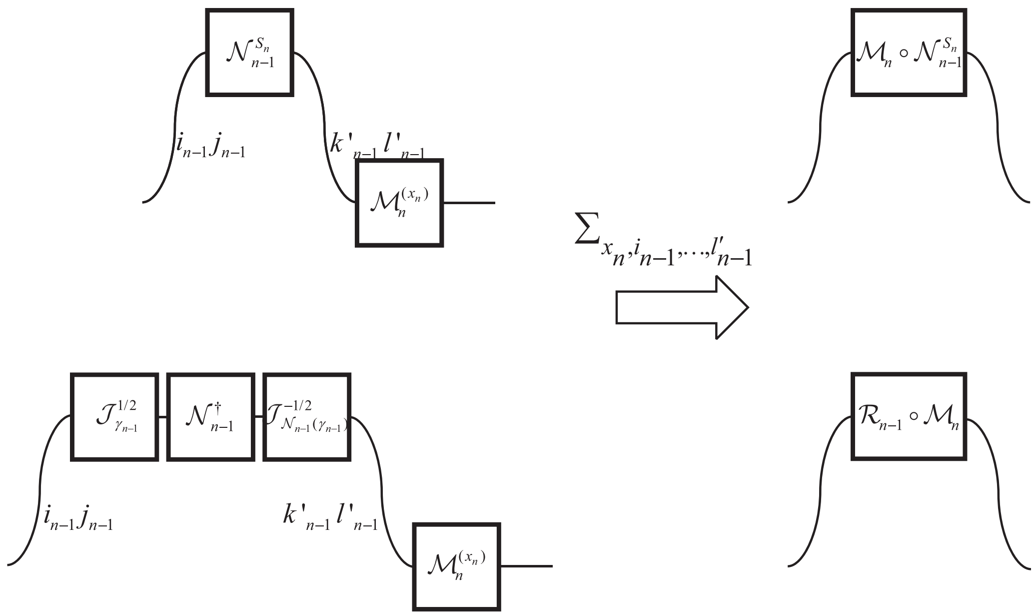

FIG. 3. For Markovian processes, summing over outcomes of (quasi)measurements gives a dephasing map.

derived from $\mathcal{S}^{tr}_{1:n}$. If we use instead the $(n-2)$-step backward processes

$$\left|\mathcal{S}^{tr}_{1:n-1}\right) := \mathcal{R}^{S_{n-2}}_{n-2}\circ\cdots\circ\mathcal{R}^{S_1}_1\left|\rho^{S_{n-1}}_{n-1}\otimes\left[\bigotimes_{j=1}^{n-2}\Phi^{A^{(j+1)}S_j}\right]\right), \tag{61}$$

then we can obtain the quasiprobability distribution

$$\begin{aligned}&\mathcal{P}^{tr}_{1:n-1}\left(x_{1:n-1},\ldots,l'_{1:n-2}\middle|\mathcal{M}^{qs}_{1:n-1}\right)\\&:=\left(\mathcal{O}^{(x_{n-1:1},\ldots,l'_{n-2:1})}_{1:n-1}\middle|\mathcal{S}^{tr}_{1:n-1}\right)^*\end{aligned} \tag{62}$$

and another entropy production

$$\begin{aligned}&R'(x_{n-1:1},\ldots,l'_{n-2:1})\\&:=\ln\frac{\mathcal{P}_{n:1}\left(x_{n-1:1},\ldots,l'_{n-2:1}\middle|\mathcal{M}^{qs}_{n:1}\right)}{\mathcal{P}^{tr}_{1:n-1}\left(x_{1:n-1},\ldots,l'_{1:n-2}\middle|\mathcal{M}^{qs}_{1:n-1}\right)},\end{aligned} \tag{63}$$

which satisfies

$$\begin{aligned}&R'(x_{n-1:1},\ldots,l'_{n-2:1})\\&=R'(x_{n-1},x_1,\ldots,l'_{n-2:1})\\&=\ln\frac{(\Pi^{x_1}|\rho_0)}{(\Pi^{x_{n-1}}|\rho_{n-1})}+\sum_{m=1}^{n-2}\ln\left(Z^{\gamma_m^{-1}}_{i_mj_m}Z^{\mathcal{N}_m(\gamma_m)}_{k'_ml'_m}\right).\end{aligned} \tag{64}$$

We can similarly define a distribution of entropy production $p(R'_1)$ and prove the detailed FT. However, the obtained fluctuation relation is related to the processes (61), not to the processes (40). The entropy productions R_1 and R'_1 share the same forward processes but use different initial states in the backward processes. Obviously,

$$\begin{aligned}&R'(x_{n-1:1},\ldots,l'_{n-2:1})+R(x_{n:n-1},\ldots,l'_{n-1})\\&=R(x_{n:1},\ldots,l'_{n-1:1}).\end{aligned} \tag{65}$$

Similarly, we can use different initial states in the forward processes

$$|\mathcal{S}_{n:n-1}) := \mathcal{N}^{S_n}_{n-1}\left|\tilde{\rho}^{S_{n-1}}_1\otimes\left[\Phi^{A^{(n-1)}S_n}\right]\right) \tag{66}$$

to obtain another entropy production R'_2, which satisfies

$$\begin{aligned}&R(x_{n-1:1},\ldots,l'_{n-2:1})+R'(x_{n:n-1},\ldots,l'_{n-1})\\&=R(x_{n:1},\ldots,l'_{n-1:1}).\end{aligned} \tag{67}$$

Similar to Eq. (32), if both R and R_1 satisfy the detailed FT, then

$$\langle e^{-[R(x_{n:1},\ldots,l'_{n-1:1})-R(x_{n-1:1},\ldots,l'_{n-2:1})]}\rangle = 1. \tag{68}$$

With Jensen's inequality, the conclusion that the total average entropy production is not less than the intermediate average entropy production still holds. Also, the entropy production rate is still non-negative. Note that the forward process of R'_2 is different from that of R and R_1, so we have in general

$$\begin{aligned}\langle R\rangle-\langle R_1\rangle&=\langle R(x_{n:1},\ldots,l'_{n-1:1})-R(x_{n-1:1},\ldots,l'_{n-2:1})\rangle\\&=\sum\mathcal{P}_{n:1}\left(x_{n:1},\ldots,l'_{n-1:1}\middle|\mathcal{M}^{qs}_{n:1}\right)\\&\quad\times R'(x_{n:n-1},\ldots,l'_{n-1})\\&\neq\sum\mathcal{P}_{n:n-1}\left(x_{n:n-1},\ldots,l'_{n-1}\middle|\mathcal{M}^{qs}_{n:n-1}\right)\\&\quad\times R'(x_{n:n-1},\ldots)\\&=:\langle R'_2\rangle',\end{aligned} \tag{69}$$

where $\mathcal{P}_{n:n-1}$ is the probability distribution from the two-point measurement of the processes (66) and $\langle\cdot\rangle'$ is the corresponding probability average.

Similar to Eq. (34), using Eqs. (45), (51), and (60), the average of entropy production R here can be generally expressed as

$$\begin{aligned}\langle R\rangle &= S(\mathcal{M}_n\rho_n||\mathcal{M}_n\tilde{\rho}_0) + S(\mathcal{M}_n\rho_n) - S(\mathcal{M}_1\rho_0)\\ &\quad + \sum_{m=1}^{n-1}[\mathrm{Tr}(\rho_{m+1}\ln\mathcal{N}_m\gamma_m) - \mathrm{Tr}(\mathcal{M}_m\rho_m\ln\gamma_m)]\\ &= S(\mathcal{M}_n\rho_n||\mathcal{M}_n\tilde{\rho}_0) + \sum_{m=2}^{n}[S(\mathcal{M}_m\rho_m) - S(\rho_m)]\\ &\quad + \sum_{m=1}^{n-1}[S(\mathcal{M}_m\rho_m||\gamma_m) - S(\rho_{m+1}||\mathcal{N}_m\gamma_m)]. \end{aligned} \tag{70}$$

Comparing Eqs. (34) and (70), it is easy to see that both of them contain the term $\Delta S_{\mathcal{M}} = S(\mathcal{M}\rho) - S(\rho)$, which is the direct contribution of the measurements to the entropy production [7]. If we choose $\rho_n = \tilde{\rho}_0$, set the reference states $\gamma_{m+1} = \mathcal{N}_m\gamma_m$, and assume that all measurements are noninvasive, then the average of the entropy production R can be simplified to

$$\langle R\rangle = S(\rho_0||\gamma_0) - S(\mathcal{N}_{n-1:1}\rho_0||\mathcal{N}_{n-1:1}\gamma_0), \tag{71}$$

where the evolution map

$$\mathcal{N}_{n-1:1} = \mathcal{N}_{n-1}\circ\cdots\circ\mathcal{N}_1. \tag{72}$$

For single-shot CPTP evolution, the average entropy production is [2]

$$\langle R\rangle = S(\rho_0||\gamma_0) - S(\mathcal{N}\rho_0||\mathcal{N}\gamma_0), \tag{73}$$

where the reference state γ_0 can be freely chosen according to the needs, and the Gibbs states are a common choice. Since the Gibbs states are fixed points of many processes, the following average entropy production formula is often used [20–22]:

$$\langle R\rangle = S(\rho_0||\rho^{(\beta)}) - S(\rho_\tau||\rho^{(\beta)}). \tag{74}$$

However, if Gibbs states are not fixed points, then Eq. (74) is inappropriate (see [23] for related discussion).

If we choose $\rho_n = \tilde{\rho}_0$ and assume that all measurements are noninvasive, then from Eq. (70) we can get

$$\langle R\rangle = \sum_{m=1}^{n-1}[S(\rho_m||\gamma_m) - S(\rho_{m+1}||\mathcal{N}_m\gamma_m)]. \tag{75}$$

When the evolution of the system state can be described through an exact time-convolutionless master equation

$$\rho_S(t) = \exp\left(\int_0^t \mathcal{L}(\tau)d\tau\right)\rho_S(0), \tag{76}$$

one can choose the instantaneous fixed point γ_t as the reference state. Here γ_t is a null eigenvector of the generator of the dynamics. For continuous evolution and measurement, the total average entropy production can be written as

$$\langle R\rangle = -\int_0^t d\tau \frac{d}{ds}\bigg|_{s=0} S(\rho_S(\tau+s)||\gamma_\tau). \tag{77}$$

It is just the entropy production used in Refs. [23,24].

The averages of the entropy productions R_1 and R_2 are

$$\begin{aligned}\langle R_1\rangle &= S(\mathcal{M}_{n-1}\rho_{n-1}||\mathcal{M}_{n-1}\tilde{\rho}_1)\\ &\quad + \sum_{m=2}^{n-1}[S(\mathcal{M}_m\rho_m) - S(\rho_m)]\\ &\quad + \sum_{m=1}^{n-2}[S(\mathcal{M}_m\rho_m||\gamma_m) - S(\rho_{m+1}||\mathcal{N}_m\gamma_m)],\\ \langle R_2\rangle &= S(\mathcal{M}_n\rho_n||\mathcal{M}_n\tilde{\rho}_0) + S(\mathcal{M}_n\rho_n) - S(\rho_n)\\ &\quad + S(\mathcal{M}_{n-1}\rho_{n-1}||\gamma_{n-1}) - S(\rho_n||\mathcal{N}_{n-1}\gamma_{n-1}). \end{aligned} \tag{78}$$

Combining Eqs. (70) and (78), we have

$$\langle R_1\rangle + \langle R_2\rangle - \langle R\rangle = S(\mathcal{M}_{n-1}\rho_{n-1}||\mathcal{M}_{n-1}\tilde{\rho}_1) \geqslant 0. \tag{79}$$

Different from the case of unitary evolution, even if the Kolmogorov consistency condition is satisfied, due to the irreversibility of the evolution $\mathcal{N}$, the sum of the segmental entropy production is generally greater than the total average entropy production.

C. Non-Markovian open quantum system

In general, the non-Markovian processes is indivisible. Under $(n-1)$-step evolution, the forward process state can be written as

$$|\mathcal{S}_{n:1}) := \mathcal{N}^{S_{n:2}}\left|\rho_0^{S_1}\otimes\left[\bigotimes_{j=2}^{n}\Phi^{A^{(j-1)}S_j}\right]\right), \tag{80}$$

where

$$\mathcal{N}^{S_{n:2}} = \left(I^E\middle|\mathcal{U}_{n-1}^{S_nE}\circ\cdots\circ\mathcal{U}_1^{S_2E}\middle|\rho_0^E\right). \tag{81}$$

With the measurements (5), the joint probability for n-point measurements can still be expressed as Eq. (39). If we use the Petz recovery map of $\mathcal{N}^{S_{n:2}}$ to give backward processes

$$\left|\mathcal{S}_{1:n}^{tr}\right) := \mathcal{R}_{\gamma_{2:n}}^{S_{n:2}}\left|\tilde{\rho}_0^{S_n}\otimes\left[\bigotimes_{j=1}^{n-1}\Phi^{A^{(j-1)}S_j}\right]\right), \tag{82}$$

then the backward process is not linkable [14]. For such processes, measuring with the measurement (9) will lead to an ill-defined probability distribution. One can freshly prepare the system state at each step and obtain the detailed FT for the many-body channels or insert a special operation and obtain the detailed FT for the derived channel [14]. The former has no connection with the joint probability distribution $(\mathcal{O}_{n:1}^{(x_n:x_1)}|\mathcal{S}_{n:1})$. The latter has $n-1$ input states, which are independent and there is no time-ordered relationship among them. Therefore, neither of them is suitable for studying multiple entropy production under the same processes.

For non-Markovian processes (see Fig. 4), the intermediate measurements can influence subsequent evolution of the system and cause entropy increases themselves. Hence, only by incorporating the intermediate measurements into the processes themselves can we guarantee that the entropy production is due to the processes rather than the measurements.

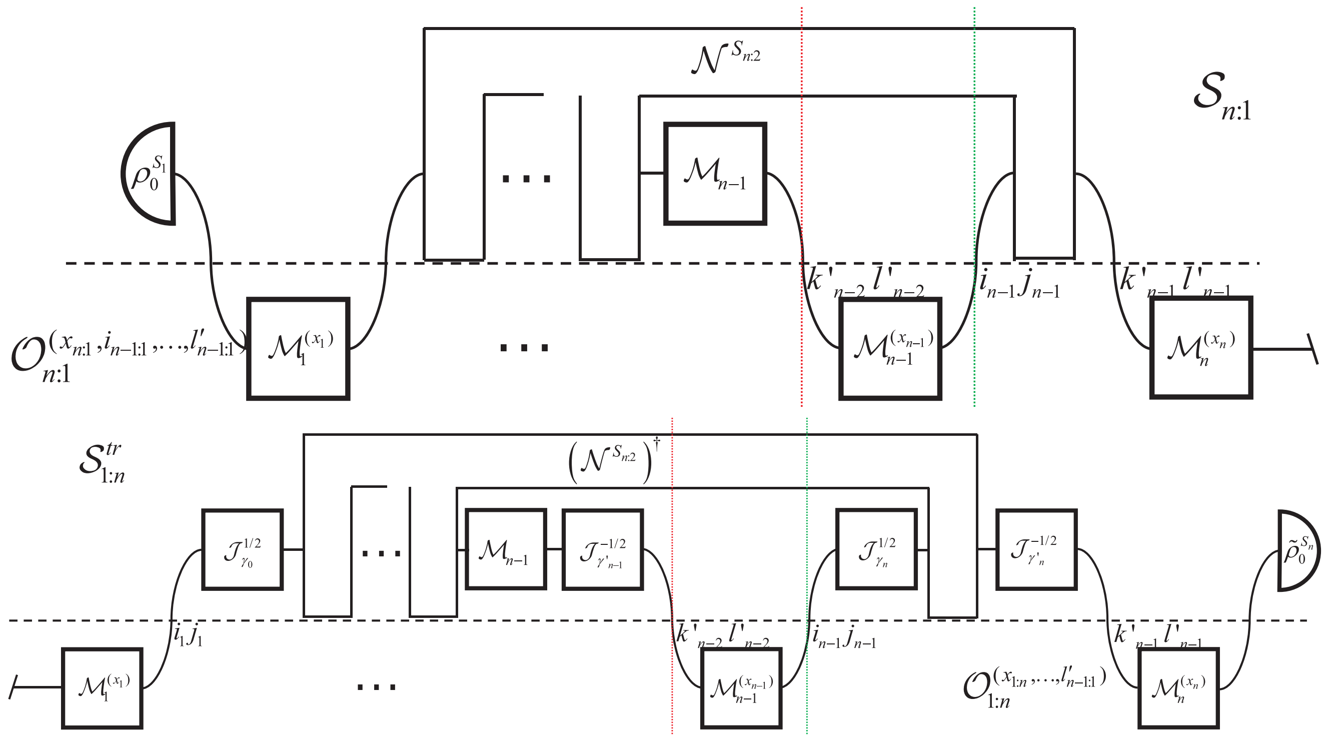

FIG. 4. Forward and backward processes of $(n-1)$-step non-Markovian quantum evolution. The dotted lines represent summing a fraction of the (quasi)measurements and then obtaining the marginal distributions. Unlike the Markovian cases, the subsequent evolution depends on the state of the environment, which in turn depends on the previous measurement results of the system. Therefore, we need $\mathcal{N}_{\mathcal{M}}^{S,E}$ to help us find the relation between marginal distributions. The green dotted line corresponds to the division used by $R(x_{n-1:1},\ldots,l'_{n-2:1})$. One needs to sum the measurements to the right of the green dotted line. The red dotted line corresponds to the division used by $R(x_{n:n-1},\ldots,l'_{n-1})$. One needs to sum the measurements to the left of the red dotted line.

In this manner, the forward process state is

$$\left|\mathcal{S}_{n:1}^{\mathcal{M}}\right) := \mathcal{N}_{\mathcal{M}}^{S_{n:2}}\left|\rho_0^{S_1}\otimes\left[\bigotimes_{j=2}^{n}\Phi^{A^{(j-1)}S_j}\right]\right), \tag{83}$$

where $\mathcal{N}_{\mathcal{M}}^{S_{n:2}} = (\bigotimes_{k=2}^{n-1}\mathcal{M}_k)\circ\mathcal{N}^{S_{n:2}}$. Note that the induction condition is naturally satisfied in the present framework; the present state and the measurement outcome of the present state cannot be affected by future measurements. Hence, we can obtain the first $(m-1)$-step evolution by ignoring the results of subsequent evolution $\mathcal{N}_{\mathcal{M}}^{S_{m:2}} = \mathrm{Tr}_{S_{n:m+1}}\mathcal{N}_{\mathcal{M}}^{S_{n:2}}$. Performing a measurement and discarding the outcomes will not affect the remeasurement of the intermediate state

$$\mathcal{M}_k^{(x_k)}\circ\mathcal{M}_k = \mathcal{M}_k^{(x_k)}. \tag{84}$$

Thus, with the measurements (5), the joint probability distribution given by $|\mathcal{S}_{n:1}^{\mathcal{M}})$ will be the same as that given by Eq. (80). Since the $|\mathcal{S}_{n:1}^{\mathcal{M}})$ has incorporated intermediate measurements into the processes itself, even without further measurements, the final state is also different from that given by Eq. (80),

$$|\rho_n) = \left(\left[\bigotimes_{i=1}^{n-1}\Phi^{S_iA^{(i)}}\right]\middle|\mathcal{S}_{n:1}^{\mathcal{M}}\right) \neq \left(\left[\bigotimes_{i=1}^{n-1}\Phi^{S_iA^{(i)}}\right]\middle|\mathcal{S}_{n:1}\right). \tag{85}$$

For the same reason, their reference final states are also different.

Here we propose the backward process state

$$\left|\mathcal{S}_{1:n}^{\mathcal{M},tr}\right) := \mathcal{R}_{\mathcal{M}}^{S_{n:2}}\left|\tilde{\rho}_0^{S_n}\otimes\left[\bigotimes_{j=1}^{n-1}\Phi^{A^{(j-1)}S_j}\right]\right), \tag{86}$$

where

$$\mathcal{R}_{\mathcal{M}}^{S_{n:2}} = \left(\bigotimes_{l=2}^{n}\mathcal{J}_{\gamma_l}^{1/2}\right)\circ(\mathcal{N}_{\mathcal{M}}^{S_{n:2}})^{\dagger}\circ\left(\bigotimes_{m=2}^{n}\mathcal{J}_{\gamma'_m}^{-1/2}\right) \tag{87}$$

and

$$\gamma'_m = \left(\left[\bigotimes_{i=1}^{m-1}\Phi^{S_iA^{(i)}}\right]\middle|\left(\bigotimes_{k=2}^{\min\{m,n-1\}}\mathcal{M}_k\right)\right.$$
$$\left.\times\circ\mathcal{N}^{S_{m:2}}\middle|\gamma_0^{S_1}\otimes\left[\bigotimes_{j=2}^{m}\Phi^{A^{(j-1)}S_j}\right]\right), \tag{88}$$

where γ'_m is the output of γ_0 after $m-1$ steps of evolution. According to this, for $m\leqslant n-1$, we know that γ'_m is the output state of the measurement $\mathcal{M}_m$, so they share the same basis. We set $\gamma_2=\gamma_0$ and $\gamma_l=\gamma'_{l-1}$ for the other l. We still use the quasimeasurements (43) for forward processes and Eq. (47) for backward processes. Now the basis $|i_m\rangle$ is chosen such that it diagonalizes γ_m and $|k'_m\rangle$ is chosen as the eigenbasis of γ'_m. The relation (46) still holds and so does the one of backward processes.

Before further discussion of the entropy production, we briefly analyze the probability distribution of backward processes. Since γ'_m and $\mathcal{M}_m$ share the same basis, we have

$$\mathcal{J}^{1/2}_{\gamma_{m+1}} \circ \mathcal{M}^{(x_m)}_m \circ \mathcal{J}^{-1/2}_{\gamma'_m} \circ \mathcal{M}_m = \mathcal{M}^{(x_m)}_m = \mathcal{M}^{(x_m)}_m \circ \mathcal{M}_m. \tag{89}$$

Combining this with the definition of $\mathcal{P}^{tr}_{1:n}(x_{1:n}|\mathcal{M}^{qs}_{1:n})$, we obtain

$$\sum_{x_1} \mathcal{P}^{tr}_{1:n}(x_{1:n}|\mathcal{M}^{qs}_{1:n}) = \sum_{x_1} \left(\mathcal{O}^{(x_{n:1})}_{1:n}\middle|\mathcal{S}^{\mathcal{M},tr}_{1:n}\right)^* = \left(\gamma'_{n,x_{2:n-1}}\middle|\mathcal{J}^{-1/2}_{\gamma'_n} \circ \mathcal{M}^{(x_n)}_n\middle|\tilde{\rho}_0\right)^*, \tag{90}$$

where

$$\gamma'_{n,x_{2:n-1}} = \left(\left[\bigotimes_{i=1}^{n-1} \Phi^{S_i A^{(i)}}\right]\middle|\left(\bigotimes_{m=2}^{m-1} \mathcal{M}^{(x_m)}_m\right) \circ \left(\bigotimes_{k=2}^{n-1} \mathcal{M}_k\right) \times \circ \mathcal{N}^{S_{n:2}}\middle|\gamma_0^{S_1} \otimes \left[\bigotimes_{j=2}^{n} \Phi^{A^{(j-1)}S_j}\right]\right) \tag{91}$$

is related to the intermediate measurement results. It is easy to show that $\sum_{x_{2:n-1}} \gamma'_{n,x_{2:n-1}} = \gamma'_n$, which leads to

$$\sum_{x_{1:n-1}} \mathcal{P}^{tr}_{1:n}(x_{1:n}|\mathcal{M}^{qs}_{1:n}) = \left(\gamma'_n\middle|\mathcal{J}^{-1/2}_{\gamma'_n} \circ \mathcal{M}^{(x_n)}_n|\tilde{\rho}_0\right)^* = (\Pi_{x_n}|\tilde{\rho}_0). \tag{92}$$

So the joint probability distribution $\mathcal{P}^{tr}_{1:n}(x_{1:n}|\mathcal{M}^{qs}_{1:n})$ is normalized.

Now we continue to discuss the FTs. The quasiprobability, entropy production, and probability distribution can be defined similarly to the definitions in Sec. II B. The entropy production reads

$$R(x_{n:1}, \ldots, l'_{n-1:1}) = \ln \frac{(\Pi^{x_1}|\rho_0)}{(\Pi^{x_n}|\tilde{\rho}_0)} + \sum_{m=1}^{n-1} \ln \left(Z^{\gamma^{-1}_{m+1}}_{i_m j_m} Z^{\gamma'_{m+1}}_{k'_m l'_m}\right). \tag{93}$$

If we choose $\tilde{\rho}_0 = \rho_n$, the average of the entropy production is

$$\langle R(x_{n:1}, \ldots, l'_{n-1:1})\rangle = S(\rho_0||\gamma_0) - S(\rho_n||\gamma'_n). \tag{94}$$

Since the intermediate measurements are already included in the processes $\mathcal{N}^{S_{n:2}}_{\mathcal{M}}$, the applied intermediate measurements are noninvasive. Furthermore, the intermediate reference states cannot be chosen arbitrarily. These make the entropy production (94) quite similar to the form (71). However, note that their evolutions are completely different. The corresponding evolution (85) in entropy production (94) contains measurements and cannot be divided due to memory effects, which are quite different from the evolution (72).

Before considering the marginal distribution $\mathcal{P}_{n:1}(x_{n-1:1}, i_{n-2:1}, \ldots, l'_{n-2:1}|\mathcal{M}^{qs}_{n:1})$, we need to clarify the relation between $\mathcal{N}^{S_{n-1:2}}_{\mathcal{M}}$ and $\mathcal{N}^{S_{n:2}}_{\mathcal{M}}$. Unlike the cases in Sec. II B, the memory effect will make the subsequent evolution depend on the previous measurement results, so the relation is more complex here. From the definition (81), we can utilize

$$\mathcal{N}^{S_{n-1:2},E}_{\mathcal{M}} = \left(\bigotimes_{k=2}^{n-1} \mathcal{M}_k\right) \circ \mathcal{U}^{S_{n-1}E}_{n-2} \circ \cdots \circ \mathcal{U}^{S_2E}_1 \tag{95}$$

to bridge $\mathcal{N}^{S_{n-1:2}}_{\mathcal{M}}$ and $\mathcal{N}^{S_{n:2}}_{\mathcal{M}}$. Since the measurements on the system will destroy quantum correlations between the system and environment [25], we have

$$\left(\bigotimes_{m=1}^{n-2} \Pi^{S_{m+1}}_{k'_m l'_m}\middle|\mathcal{N}^{S_{n-1:2},E}_{\mathcal{M}}\middle|\left[\bigotimes_{m=1}^{n-2} \Pi^{S_{m+1}}_{i_m j_m}\right] \otimes \rho_0^E\right) = \left(\bigotimes_{m=1}^{n-2} \delta_{k'_m l'_m} \Pi^{S_{m+1}}_{k'_m}\middle|\mathcal{N}^{S_{n-1:2}}_{\mathcal{M}}\middle|\bigotimes_{m=1}^{n-2} \Pi^{S_{m+1}}_{i_m j_m}\right) \times \left|\sigma^E_{i_{n-2:1}, j_{n-2:1}, k'_{n-2:1}}\right), \tag{96}$$

where σ^E represents the state of the environment. Since both $\mathcal{M}_k$ and $\mathcal{U}$ are CPTP maps, $\mathcal{N}^{S_{n-1:2},E}_{\mathcal{M}}$ is also a CPTP map and this requires

$$\left(I^E\middle|\sigma^E_{i_{n-2:1}, j_{n-2:1}, k'_{n-2:1}}\right) = \bigotimes_{m=1}^{n-2} \delta_{i_m j_m}. \tag{97}$$

Equation (96) can relate $\mathcal{R}^{S_{n-1:2}}_{\mathcal{M}}$ to $\mathcal{R}^{S_{n:2}}_{\mathcal{M}}$. According to the definition (87), the recovery map can be rewritten as

$$\mathcal{R}^{S_{n:2}}_{\mathcal{M}} = \left(\rho_0^E\middle|\left(\bigotimes_{l=2}^{n} \mathcal{J}^{1/2}_{\gamma_l}\right) \circ \left(\mathcal{N}^{S_{n-1:2,E}}_{\mathcal{M}}\right)^\dagger \times \circ \left(\mathcal{U}^{S_nE}_{n-1}\right)^\dagger \circ \left(\bigotimes_{m=2}^{n} \mathcal{J}^{-1/2}_{\gamma'_m}\right)\middle|I^E\right). \tag{98}$$

Combining this with Eq. (96), we obtain

$$\left(\bigotimes_{m=1}^{n-1} \Pi^{S_{m+1}}_{i_m j_m}\middle|\mathcal{R}^{S_{n:2}}_{\mathcal{M}}\middle|\bigotimes_{m=1}^{n-1} \Pi^{S_{m+1}}_{k'_m l'_m}\right) = \left(\Pi^{S_n}_{i_{n-1} j_{n-1}}\middle|\mathcal{J}^{1/2}_{\gamma_n} \circ \left(\sigma^E_{i_{n-2:1}, j_{n-2:1}, k'_{n-2:1}}\middle|\left(\mathcal{U}^{S_nE}_{n-1}\right)^\dagger\middle|I^E\right) \times \circ \mathcal{J}^{-1/2}_{\gamma'_n}|\Pi^{S_n}_{k'_{n-1} l'_{n-1}}\right) \times \left(\bigotimes_{m=1}^{n-2} \Pi^{S_{m+1}}_{i_m j_m}\middle|\mathcal{R}^{S_{n-1:2}}_{\mathcal{M}}\middle|\bigotimes_{m=1}^{n-2} \Pi^{S_{m+1}}_{k'_m l'_m}\right). \tag{99}$$

The quasiprobability, entropy production, and probability distribution can also be defined similarly to the definition in Sec. II B. However, unlike Eq. (56), according to Eq. (99), the entropy production becomes

$$R(x_{n-1:1}, \ldots, l'_{n-2:1}) = R(x_{n-1}, x_1, \ldots, l'_{n-2:1}) = \ln \frac{(\Pi^{x_1}|\rho_0)}{\left(\Pi^{x_{n-1}}\middle|\tilde{\rho}_1^{i_{n-2:1}, k'_{n-2:1}}\right)} + \sum_{m=1}^{n-2} \ln \left(Z^{\gamma^{-1}_{m+1}}_{i_m j_m} Z^{\gamma'_{m+1}}_{k'_m l'_m}\right). \tag{100}$$

The density matrix will depend on the historical measurements

$$\tilde{\rho}_1^{i_{n-2:1}, k'_{n-2:1}} = \mathcal{J}^{1/2}_{\gamma_n} \circ \mathcal{N}^\dagger_{i_{n-2:1}, k'_{n-2:1}} \circ \mathcal{J}^{-1/2}_{\gamma'_n} \circ \mathcal{M}_n(\tilde{\rho}_0),$$

where the evolution depends on the state of the environment

$$\mathcal{N}_{i_{n-2:1}, k'_{n-2:1}} = \left(I^E\middle|\mathcal{U}^{S_nE}_{n-1}\middle|\sigma^E_{i_{n-2:1}, i_{n-2:1}, k'_{n-2:1}}\right). \tag{101}$$

The detailed FT relation still holds, but the entropy production (100) is no longer a combination of time-localized measurements but contains temporal nonlocal measurements. Only when the evolution $\mathcal{N}_{i_{n-2:1},k'_{n-2:1}}$ does not change with historical measurements, such as the Markovian processes, will the entropy production be related to the two-point measurement.

Now let us discuss another marginal distribution. Summing over $\{x_{n-2:1}, i_{n-2:1}, \ldots, l'_{n-2:1}\}$ is related to the process

$$\mathcal{N}_{\mathcal{M}}^{S_{n-1},E} = \mathcal{M}_{n-1} \circ \mathcal{U}_{n-2}^{SE} \circ \cdots \circ \mathcal{M}_2 \circ \mathcal{U}_1^{SE}. \tag{102}$$

Similar to Eq. (96), the measurement makes

$$\begin{aligned}&\big(\Pi_{k'_{n-2}l'_{n-2}}\big|\mathcal{N}_{\mathcal{M}}^{S_{n-1},E}\big|\Pi_{i_1j_1}\otimes\rho_0^E\big)\\&\quad=\big|\sigma^E_{i_1,j_1,k'_{n-2}}\big)\delta_{k'_{n-2}l'_{n-2}}\big(\Pi_{k'_{n-2}}\big|\mathcal{N}_{\mathcal{M}}^{S_{n-1}}\big|\Pi_{i_1j_1}\big),\end{aligned} \tag{103}$$

where $\mathcal{N}_{\mathcal{M}}^{S_{n-1}} = (I^E|\mathcal{N}_{\mathcal{M}}^{S_{n-1},E}|\rho_0^E)$. For the same reason as in Eq. (97), it requires $(I^E|\sigma^E_{i_1,j_1,k'_{n-2}}) = \delta_{i_1j_1}$. With $\mathcal{N}_{\mathcal{M}}^{S_{n-1},E}$, the quasiprobability distribution of forward processes can be written as

$$\begin{aligned}\mathcal{P}_{n:1}\big(x_{n:n-1},\ldots,l'_{n-1}\big|\mathcal{M}^{qs}_{n:1}\big) &\propto \big(I^E\otimes\Pi_{k'_{n-1}l'_{n-1}}\big|\mathcal{U}^{SE}_{n-1}\big|\Pi_{i_{n-1}j_{n-1}}\big)\big(\Pi^{x_{n-1}}\big|\mathcal{N}_{\mathcal{M}}^{S_{n-1},E}\circ\mathcal{M}_1\big|\rho_0\otimes\rho_0^E\big)\\&=\sum_{i_1,j_1,k'_{n-2}}\big(\Pi_{k'_{n-1}l'_{n-1}}\big|\mathcal{N}^{S_n}_{i_1,k'_{n-2}}\big|\Pi_{i_{n-1}j_{n-1}}\big)\big(\Pi^{x_{n-1}}\big|\Pi_{k'_{n-2}}\big)\big(\Pi_{k'_{n-2}}\big|\mathcal{N}_{\mathcal{M}}^{S_{n-1}}\big|\Pi_{i_1j_1}\big)[\Pi_{i_1j_1}|\mathcal{M}_1(\rho_0)],\end{aligned} \tag{104}$$

where $\mathcal{N}^{S_n}_{i_1,k'_{n-2}} = (I^E|\mathcal{U}^{SE}_{n-1}|\sigma^E_{i_1,i_1,k'_{n-2}})$ and the proportional to symbol is used because we ignore some common terms like $(\Pi^x|\Pi_{ij})$, which also exist in the quasiprobability distribution of backward processes. The quasiprobability distribution of backward processes is

$$\begin{aligned}&\mathcal{P}^{tr}_{1:n}\big(x_{n-1:n},\ldots,l'_{n-1}\big|\mathcal{M}^{qs}_{1:n}\big)\\&\propto\Big[\big(I\otimes\rho_0^E\big|J^{1/2}_{\gamma_2}\circ\big(\mathcal{N}_{\mathcal{M}}^{S_{n-1},E}\big)^\dagger\circ J^{-1/2}_{\gamma'_{n-1}}\big|\Pi^{x_{n-1}}\big)\big(\Pi_{i_{n-1}j_{n-1}}\big|\mathcal{J}^{1/2}_{\gamma_n}\circ\big(\mathcal{U}^{SE}_{n-1}\big)^\dagger\circ\mathcal{J}^{-1/2}_{\gamma'_n}\big|\Pi_{k'_{n-1}l'_{n-1}}\big)\big(\Pi^{x_n}\big|\tilde\rho_0\otimes I^E\big)\Big]^*\\&=\sum_{i_1,j_1,k'_{n-2}}\Big[(\gamma_0|\Pi_{i_1j_1})\big(\Pi_{i_1j_1}\big|\big(\mathcal{N}_{\mathcal{M}}^{S_{n-1}}\big)^\dagger\circ J^{-1/2}_{\gamma'_{n-1}}\big|\Pi_{k'_{n-2}}\big)\big(\Pi_{k'_{n-2}}\big|\Pi^{x_{n-1}}\big)\big(\Pi_{i_{n-1}j_{n-1}}\big|\mathcal{J}^{1/2}_{\gamma_n}\circ\big(\mathcal{N}^{S_n}_{i_1,k'_{n-2}}\big)^\dagger\circ\mathcal{J}^{-1/2}_{\gamma'_n}\big|\Pi_{k'_{n-1}l'_{n-1}}\big)\big(\Pi^{x_n}|\tilde\rho_0\big)\Big]^*.\end{aligned} \tag{105}$$

From Eqs. (104) and (105) we find that the summing over $\{i_1, j_1\}$ cannot be eliminated or be attributed to a local measurement. So there is no detailed FT for $R(x_{n:n-1},\ldots,l'_{n-1})$ in general. Only when $\mathcal{N}^{S_n}_{i_1,k'_{n-2}}$ is independent of $\{i_1\}$, which is also satisfied in the Markovian processes, do we have

$$\begin{aligned}R(x_{n:n-1},\ldots,l'_{n-1}) = \ln\frac{(\Pi^{x_{n-1}}|\rho_{n-1})}{(\Pi^{x_n}|\tilde\rho_0)}\\+\ln\big(Z^{\gamma_n^{-1}}_{i_{n-1}j_{n-1}}Z^{\gamma'_n}_{k'_{n-1}l'_{n-1}}\big),\end{aligned} \tag{106}$$

where

$$|\rho_{n-1}) = \mathcal{N}_{\mathcal{M}}^{S_{n-1}}\circ\mathcal{M}_1(\rho_0). \tag{107}$$

The proof uses that γ'_m and $\mathcal{M}_m$ share the same basis. From the previous analysis we can see that if we want the entropy production of the marginal distribution to satisfy the fluctuation theorem with time-localized measurements, we must require that the evolution does not change with historical measurements, that is to say, the measurement is not invasive to evolution.

Let us briefly discuss why the marginal distribution in the non-Markovian processes does not necessarily give a detailed FT, since the proof (32) also applies here. If both R and $R_{\rm sub}$ satisfy the detailed FT, then we have $\langle R\rangle \geqslant \langle R_{\rm sub}\rangle$. For non-Markovian processes, the state of the system could be fully recovered. Then according to Eq. (94), there must be $\langle R\rangle = 0$. However, $\langle R_{\rm sub}\rangle > 0$ is very natural when there is a contraction in the state space of the system. So there is a conflict here, which makes the detailed FT for marginal distributions conditional.

Since the proof (32) applies to all cases here, it leads to the following conclusion: For the marginal distribution, if its corresponding entropy production satisfies the detailed fluctuation theorem, then its average gives a lower bound on the total average entropy production.

III. EXAMPLE

To illustrate our framework we briefly discuss the Jaynes-Cummings model. For simplicity, here we consider a two-state atom coupled to a single harmonic-oscillator mode. Suppose the atom is our system of interest. The non-Markovian dynamics and the memory effects of this model have been discussed in [26,27]. Reference [24] took it as an example to calculate the entropy production of a single-shot evolution under different evolution durations so as to obtain the entropy production rate. The total Hamiltonian is given by

$$H_{\rm JC} = \omega_a\frac{\sigma_z}{2} + \omega_c a^\dagger a + \frac{\Omega}{2}(a\sigma_+ + a^\dagger\sigma_-). \tag{108}$$

Its eigenstates are

$$\begin{aligned}|n,+\rangle &= \cos\left(\frac{\alpha_n}{2}\right)|n,1\rangle + \sin\left(\frac{\alpha_n}{2}\right)|n+1,0\rangle,\\|n,-\rangle &= \sin\left(\frac{\alpha_n}{2}\right)|n,1\rangle - \cos\left(\frac{\alpha_n}{2}\right)|n+1,0\rangle,\end{aligned} \tag{109}$$

where n denotes the number of radiation quanta in the mode, 1 and 0 denote the excited and ground states, respectively, $\alpha_n = \tan^{-1}(\Omega\sqrt{n+1}/\Delta)$, and $\Delta = \omega_a - \omega_c$. The energy eigenvalues associated with the eigenstates $|n,\pm\rangle$ are given by

$$E_{n,\pm} = \omega_c\left(n+\tfrac{1}{2}\right) \pm \tfrac{1}{2}\Omega_n, \tag{110}$$

where $\Omega_n = \sqrt{\Delta^2 + \Omega^2(n+1)}$. The unitary evolution operator $U_t = e^{-iH_{JC}t}$ in the basis $\{|0\rangle, |1\rangle\}$ is given by the matrix [26]

$$U_t = e^{-i\omega_c \hat{n} t}\begin{pmatrix} c_{\hat{n}}^\dagger(t) & -b^\dagger d_{\hat{n}+1}^\dagger(t) \\ d_{\hat{n}+1}(t) b & c_{\hat{n}+1}(t) \end{pmatrix}, \tag{111}$$

where the operators

$$c_{\hat{n}}(t) = e^{-i\omega_c t/2}\left[\cos\left(\sqrt{\hat{\varphi}}\frac{t}{2}\right) - i\Delta \sin\left(\sqrt{\hat{\varphi}}\frac{t}{2}\right)\Big/\sqrt{\hat{\varphi}}\right],$$

$$d_{\hat{n}}(t) = -ie^{-i\omega_c t/2} 2g \sin\left(\sqrt{\hat{\varphi}}\frac{t}{2}\right)\Big/\sqrt{\hat{\varphi}}, \tag{112}$$

with $\hat{\varphi} = \Delta^2 + 4g^2\hat{n}$. All functions related to the particle number operator satisfy the relations

$$b f_{\hat{n}}(t) = f_{\hat{n}+1}(t) b, \quad b^\dagger f_{\hat{n}+1}(t) = f_{\hat{n}}(t) b^\dagger. \tag{113}$$

Using these relations and

$$c_{\hat{n}}^\dagger(t) c_{\hat{n}}(t) + \hat{n} d_{\hat{n}}^\dagger(t) d_{\hat{n}}(t) = 1, \tag{114}$$

one can easily verify the unitarity of $U(t)$. In addition, using

$$c_{\hat{n}}^\dagger(\tau_2) c_{\hat{n}}^\dagger(\tau_1) - \hat{n} d_{\hat{n}}^\dagger(\tau_2) d_{\hat{n}}(\tau_1) e^{i\omega_c \tau_1} = c_{\hat{n}}^\dagger(\tau_1 + \tau_2),$$

$$c_{\hat{n}}(\tau_2) d_{\hat{n}}(\tau_1) + d_{\hat{n}}(\tau_2) c_{\hat{n}}^\dagger(\tau_1) e^{-i\omega_c \tau_1} = d_{\hat{n}}(\tau_1 + \tau_2), \tag{115}$$

it is easy to verify that $U(\tau_2)U(\tau_1) = U(\tau_1 + \tau_2)$.

Suppose the initial state of the system and the environment is factorized $\rho_0^{SE} = \rho_0^S \otimes \rho_0^E$ and the environment is initially in the thermal equilibrium (Gibbs) state $\rho_0^E = \exp(-\beta H_c)/Z_c$. Then the dynamical map of the system is

$$\mathcal{N}_{0\to t}(\rho_0^S) = \mathrm{Tr}_E\left(\mathcal{U}_t \rho_0^S \otimes \rho_0^E\right). \tag{116}$$

Since the unitary evolution obeys the energy conservation relation $[H_0, H_I] = 0$, the thermal equilibrium state of the system $\rho_S^{(\beta)} = \exp(-\beta H_a)/Z_a$ is the fixed point of the map $\mathcal{N}_{0\to t}$. For single-shot evolution, the corresponding forward process state is

$$|\mathcal{S}_{2:1}) = \mathcal{N}_{0\to t}^{S_2}\left|\rho_0^{S_1} \otimes \Phi^{A^{(1)}S_2}\right) = \sum_{ijk'l'} \Phi_{ij\to k'l'}(t)\left|\rho_0^{S_1} \otimes \Pi_{ij}^{A^{(1)}} \otimes \Pi_{k'l'}^{S_2}\right), \tag{117}$$

where $\Phi_{ij\to k'l'}(t) = \langle \Phi_0(t)\rangle_{ij\to k'l'}$. See Eq. (A3) for its specific matrix form.

If considering two-step evolution, the corresponding channel is

$$\mathcal{N}_{0\to t_1\to t_2}^{S_2S_3}\left(\rho_0^{S_2} \otimes \rho_{t_1}^{S_3}\right) = \mathrm{Tr}_E\left(\mathcal{U}_{t_2-t_1}^{S_2E}\mathcal{U}_{t_1}^{S_1E}\rho_0^{S_2} \otimes \rho_{t_1}^{S_3} \otimes \rho_0^E\right). \tag{118}$$

The corresponding forward process state is given by

$$|\mathcal{S}_{3:1}) = \mathcal{N}_{0\to t_1\to t_2}^{S_2S_3}\left|\rho_0^{S_1} \otimes \Phi^{A^{(1)}S_2} \otimes \Phi^{A^{(2)}S_3}\right) = \sum_{i_{2:1}j_{2:1}k'_{2:1}l'_{2:1}} \Phi_{i_{2:1}j_{2:1}\to k'_{2:1}l'_{2:1}}(t_1, t_2) \times \left|\rho_0^{S_1} \otimes \Pi_{i_1j_1}^{A^{(1)}} \otimes \Pi_{k_1l_1}^{S_2} \otimes \Pi_{i_2j_2}^{A^{(2)}} \otimes \Pi_{k_2l_2}^{S_3}\right). \tag{119}$$

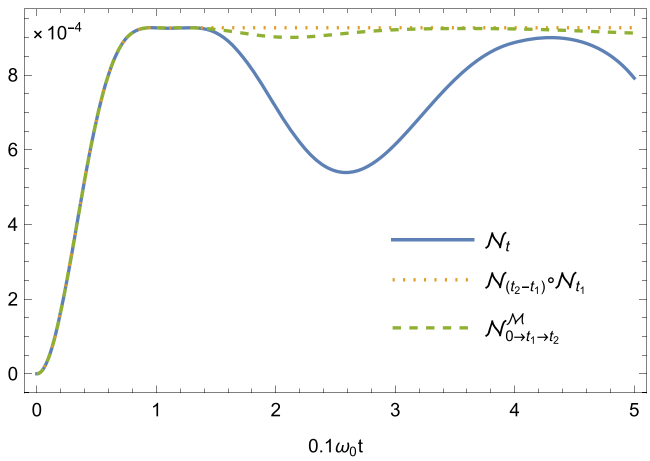

FIG. 5. Average entropy production for the Jaynes-Cummings model under different processes. Here $\mathcal{N}_t$ is single-shot evolution and $\mathcal{N}_{t_2-t_1}\circ\mathcal{N}_{t_1}$ is two-step Markovian evolution. We applied a measurement to the system at time $\omega_0 t_1 = 10$ while simultaneously refreshing the environment to the initial thermal state. Here $\mathcal{N}^{\mathcal{M}}_{0\to t_1\to t_2}$ is two-step non-Markovian evolution. We applied a measurement to the system at time $\omega_0 t_1 = 10$, but did no additional operations on the environment. The parameters are $\omega_a = \omega_0$, $g = 0.1\omega_0$, $\omega_c = 0.1\omega_0$, $k_B T = \omega_0$, $\rho_S^{11}(0) = 0.25$, and $\rho_S^{01}(0) = 0$. All the measurements are in the basis $\{|0\rangle, |1\rangle\}$.

The specific matrix form of $\Phi_{i_{2:1}j_{2:1}\to k'_{2:1}l'_{2:1}}(t)$ can be found in Eq. (A5).

Here we plot the comparison between the entropy productions (71), (73), and (94) (see Fig. 5). Comparing the entropy production of $\mathcal{N}_t$ with $\mathcal{N}_{t_2-t_1}\circ\mathcal{N}_{t_1}$, we can find that the memory effect reduces entropy production. When the memory is erased (by refreshing the environment state), the restoration of the state will be prevented. Comparing $\mathcal{N}_t$ with $\mathcal{N}^{\mathcal{M}}_{0\to t_1\to t_2}$, it can be found that the recovery of the state is weakened, which means that the measurement can destroy part of the memory. This is consistent with the fact that memory can be divided into quantum and classical parts. Comparing $\mathcal{N}_{t_2-t_1}\circ\mathcal{N}_{t_1}$ with $\mathcal{N}^{\mathcal{M}}_{0\to t_1\to t_2}$, it can be found that the negative-entropy production still exists, which means that the measurement will not destroy all the memories. In addition, for the process $\mathcal{N}^{\mathcal{M}}_{0\to t_1\to t_2}$, we can see that $\langle R_{\omega_0 t=10}\rangle > \langle R_{\omega_0 t=20}\rangle$. This makes it impossible for the corresponding marginal distribution to give a detailed FT.

IV. CONCLUSION

In this paper we have studied the entropy production and the detailed FT for multitime quantum processes in the framework of the operator-state formalism. For closed quantum systems and Markovian open quantum systems, the entropy production of the joint probability and marginal distributions all satisfy the detailed FT relation. This also leads to a nonnegative-entropy production rate. In addition, the total average entropy production is not less than the intermediate average entropy production. For closed quantum systems, we further showed that the sum of each intermediate amount of entropy production can be equal to the total entropy production if the Kolmogorov condition is satisfied. For non-Markovian open quantum systems, the memory effect can lead to a negative-entropy production rate. Therefore, the entropy production of the marginal distribution generally does not satisfy the detailed FT relation.

To illustrate the framework, we briefly calculated the total average entropy production of the three processes in the Jaynes-Cummings model. The results show that memory effects do reduce the average entropy production, while measurements destroy part of the memory. For the non-Markovian open quantum system, the intermediate measurements can destroy the system-environment quantum correlations and thus part of the memory. If these intermediate measurements are not performed, then the full memory effect can be preserved. If the environment state is refreshed at each step, as in [28], then the evolution is completely Markovian and the memory effect is completely destroyed. A further discussion of the effect of measurement on different memories and the effect of different memories on entropy production will help us fully understand the influence of memory effects on the fluctuation theorems.

This paper mainly focused on the fluctuation theorem of multiple entropy production under the same process and the same measurement. The multiple fluctuations here were obtained by calculating the marginal distribution of measurements at different times. Some previous studies such as [29] considered the fluctuation theorem under the single-shot process and the multibody measurement, where the multiple fluctuations were obtained by calculating the marginal distribution of different measurement objects (such as the system and reference). The conditional mutual information obtained by introducing the auxiliary system can be used to measure the memory effect. Therefore, studying the fluctuation theorem of many-body measurements under multitime evolution can give some interesting results.
In [30,31] the entropy production and the detailed FT relation of multiple channels were used to develop an arrow of time statistics associated with the measurement dynamics. Evolution with memory effects is clearly beyond these frameworks, so irreversibility may be violated. The framework of this paper allows us to discuss path probabilities of non-Markovian processes and thus may help us to deepen our understanding of the relation between the Poincaré recurrence theorem and the statistical arrow of time.

The FTs considered here are completely general but only useful when R can be expressed exclusively in terms of physical and measurable quantities. For closed quantum systems and Markovian open quantum systems, the entropy production defined here is consistent with previous work, so we do not go into detail here. The main difference is in the non-Markovian cases, where memory effects cause the entropy production to include temporally nonlocal measurements. This issue needs further investigation.

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APPENDIX: OVERALL PROCESS STATE AND OPEN-SYSTEM PROCESS STATE

Here we use the system-environment unitary evolution process state to obtain the open-system process state. The overall process state corresponding to a single-shot unitary evolution is

$$\begin{aligned}|\mathcal{S}_{2:1}^{U}) &:= \mathcal{U}_t^{S_2E}\big|\rho_0^{S_1}\otimes\Phi^{A^{(1)}S_2}\otimes\rho_0^{E}\big) \\ &= \sum_{ijk'l'}\big|\rho_0^{S_1}\otimes\Pi_{ij}^{A^{(1)}}\otimes\Pi_{k'l'}^{S_2}\otimes\mathcal{T}_t^{ij,k'l'}(\rho_0^E)\big),\end{aligned} \tag{A1}$$

where $\mathcal{T}_t^{ij,k'l'}(\rho)$ can be expressed in the matrix

$$\mathcal{T}_t(\rho)=e^{-i\omega_c\hat{n}t}\begin{pmatrix} c_{\hat{n}}^\dagger(t)\rho c_{\hat{n}}(t) & c_{\hat{n}}^\dagger(t)\rho b^\dagger d_{\hat{n}+1}^\dagger(t) & d_{\hat{n}+1}(t)b\rho c_{\hat{n}}(t) & d_{\hat{n}+1}(t)b\rho b^\dagger d_{\hat{n}+1}^\dagger(t)\\ -c_{\hat{n}}^\dagger(t)\rho d_{\hat{n}+1}(t)b & c_{\hat{n}}^\dagger(t)\rho c_{\hat{n}+1}^\dagger(t) & -d_{\hat{n}+1}(t)b\rho d_{\hat{n}+1}(t)b & d_{\hat{n}+1}(t)b\rho c_{\hat{n}+1}^\dagger(t)\\ -b^\dagger d_{\hat{n}+1}^\dagger(t)\rho c_{\hat{n}}(t) & -b^\dagger d_{\hat{n}+1}^\dagger(t)\rho b^\dagger d_{\hat{n}+1}^\dagger(t) & c_{\hat{n}+1}(t)\rho c_{\hat{n}}(t) & c_{\hat{n}+1}(t)\rho b^\dagger d_{\hat{n}+1}^\dagger(t)\\ b^\dagger d_{\hat{n}+1}^\dagger(t)\rho d_{\hat{n}+1}(t)b & -b^\dagger d_{\hat{n}+1}^\dagger(t)\rho c_{\hat{n}+1}^\dagger(t) & -c_{\hat{n}+1}(t)\rho d_{\hat{n}+1}(t)b & c_{\hat{n}+1}(t)\rho c_{\hat{n}+1}^\dagger(t)\end{pmatrix}e^{i\omega_c\hat{n}t}, \tag{A2}$$

where the rows are ij, the columns are $k'l'$, and the value is $\{00, 01, 10, 11\}$. According to Eq. (81), we can obtain the process state of the open system from the overall process state. When the environmental state commutes with the number operator, we have $(I^E|\mathcal{T}_t(\rho_0^E)) = \Phi_0(t)$, where

$$\Phi_m(t)=\begin{pmatrix}\alpha_m(t) & 0 & 0 & 1-\alpha_m(t)\\ 0 & \gamma_m^\dagger(t) & 0 & 0\\ 0 & 0 & \gamma_m(t) & 0\\ 1-\alpha_{m+1}(t) & 0 & 0 & \alpha_{m+1}(t)\end{pmatrix}, \tag{A3}$$

with $\alpha_i(t) = c_{\hat{n}+i}^\dagger(t)c_{\hat{n}+i}(t)$ and $\gamma_i(t) = c_{\hat{n}+i}(t)c_{\hat{n}+i+1}(t)$. For two-step unitary evolution, the following overall process state can be obtained:

$$|\mathcal{S}_{3:1}^{U}) = \sum_{i_{2:1}j_{2:1}k'_{2:1}l'_{2:1}}\big|\rho_0^{S_1}\otimes\Pi_{i_1j_1}^{A^{(1)}}\otimes\Pi_{k'_1l'_1}^{S_2}\otimes\Pi_{i_2j_2}^{A^{(2)}}\otimes\Pi_{k_2l_2}^{S_3}\otimes\mathcal{T}_{t_2-t_1}^{i_2j_2,k'_2l'_2}\big[\mathcal{T}_{t_1}^{i_1j_1,k'_1l'_1}(\rho_0^E)\big]\big). \tag{A4}$$

Again, we can use $(I^E|\mathcal{T}_{t_2-t_1}[\mathcal{T}_{t_1}(\rho_0^E)])$ to obtain the specific form of the map $\Phi_{i_{2:1}j_{2:1}\to k'_{2:1}l'_{2:1}}(t_1,t_2)$. Using Eqs. (113)–(115) repeatedly, one can get

$$
\begin{aligned}
\Phi_{i_1j_1\to k'_1l'_1,X_1} &= \langle \Phi_0(t_1)\Phi_{\Delta_1}(t_2-t_1)\rangle_{i_1j_1\to k'_1l'_1,X_1},\\
\Phi_{i_1j_1\to k'_1l'_1,X_2} &= \delta_{\Delta_2+i_1-j_1-k'_1+l'_1,0}\,\langle \Phi^{s_1}(t_1,t_2)\Phi^{s_2}_{\Delta_1+1/2}(t_2-t_1)\rangle_{i_1j_1\to k'_1l'_1,X_2},\\
\Phi_{i_1j_1\to k'_1l'_1,X_3} &= \delta_{i_1,i_2^\perp}\delta_{j_1,j_2^\perp}\delta_{k'_1,k_2'^\perp}\delta_{l'_1,l_2'^\perp}\,\langle \xi_0^\dagger(t_1,t_2)\xi_1(t_1,t_2)\rangle,
\end{aligned}
\tag{A5}
$$

where $X_1=\{i_2j_2\to i_2j_2, i_2i_2\to i_2^\perp i_2^\perp\}$, $X_2=\{i_2j_2\to i_2^\perp j_2, i_2j_2\to i_2j_2^\perp\}$, $X_3=\{i_2i_2^\perp\to i_2^\perp i_2\}$, $\Delta_1=(i_1+j_1-k'_1-l'_1)/2$, $\Delta_2=(i_2-j_2-k'_2+l'_2)$, and

$$
\Phi^{s_1}(t_1,t_2)=\begin{pmatrix}
0 & \xi_0(t_1,t_2)c^\dagger_{\hat n}(t_1) & \xi_0^\dagger(t_1,t_2)c_{\hat n}(t_1) & 0\\
-\xi_1^\dagger(t_1,t_2)c^\dagger_{\hat n}(t_1) & 0 & 0 & \xi_0^\dagger(t_1,t_2)c^\dagger_{\hat n+1}(t_1)\\
-\xi_1(t_1,t_2)c_{\hat n}(t_1) & 0 & 0 & \xi_0(t_1,t_2)c_{\hat n+1}(t_1)\\
0 & -\xi_1(t_1,t_2)c^\dagger_{\hat n+1}(t_1) & -\xi_1^\dagger(t_1,t_2)c_{\hat n+1}(t_1) & 0
\end{pmatrix},
\tag{A6}
$$

$$
\Phi^{s_2}_m(t)=\begin{pmatrix}
0 & c^\dagger_{\hat n+m-1}(t) & c_{\hat n+m-1}(t) & 0\\
-c^\dagger_{\hat n+m}(t) & 0 & 0 & c^\dagger_{\hat n+m}(t)\\
-c_{\hat n+m}(t) & 0 & 0 & c_{\hat n+m}(t)\\
0 & -c^\dagger_{\hat n+m+1}(t) & -c_{\hat n+m+1}(t) & 0
\end{pmatrix},
\tag{A7}
$$

where

$$
\xi_i(t_1,t_2)=c_{\hat n+i}(t_1)c_{\hat n+i}(t_2-t_1)-c_{\hat n+i}(t_2)=e^{-i\omega_c t_2}\xi_i^\dagger(t_1,t_2).
\tag{A8}
$$

When $t_2=t_1$, there is $\xi_i(t_1,t_2)=0$, and it is easy to verify that

$$
\Phi_{i_1j_1\to k'_1l'_1,i_2j_2\to k'_2l'_2}=\langle\Phi_0(t_1)\rangle_{i_1j_1\to k'_1l'_1}\,\delta_{i_2,k'_2}\delta_{j_2,l'_2}.
\tag{A9}
$$

This mapping corresponds to the channel $\mathrm{Tr}_E[\mathcal{I}^{S_2E}\circ\mathcal{U}^{S_1E}_{0\to t_1}(\cdot\otimes\rho_0^E)]$. In addition, one-step evolution can be derived from two-step evolution when intermediate states are directly connected. Accordingly, it can be verified that

$$
\sum_{k'_2,i_2,l'_2,j_2}\delta_{k'_2,i_2}\delta_{l'_2,j_2}\Phi_{i_{2:1}j_{2:1}\to k'_{2:1}l'_{2:1}}(t_1,t_2)=\Phi_{i_1j_1\to k'_2l'_2}(t_2).
\tag{A10}
$$

If the intermediate states are measured and we assume that the measurement is in the basis $\{|0\rangle,|1\rangle\}$, then from

$$
\sum_{k'_2,i_2,l'_2,j_2}\delta_{k'_2,i_2}\delta_{l'_2,j_2}\delta_{k'_2,l'_2}\Phi_{i_{2:1}j_{2:1}\to k'_{2:1}l'_{2:1}}(t_1,t_2)=\Phi^{\mathcal{M}}_{i_1j_1\to k'_2l'_2}(t_1,t_2)
\tag{A11}
$$

we can get the mapping corresponding to Eq. (85). Its specific matrix form is

$$
\Phi^{\mathcal{M}}(t_1,t_2)=\begin{pmatrix}
1-\chi_0(t_1,t_2) & 0 & 0 & \chi_0(t_1,t_2)\\
0 & \eta^\dagger(t_1,t_2) & 0 & 0\\
0 & 0 & \eta(t_1,t_2) & 0\\
\chi_1(t_1,t_2) & 0 & 0 & 1-\chi_1(t_1,t_2)
\end{pmatrix},
\tag{A12}
$$

where

$$
\begin{aligned}
\chi_i(t_1,t_2)&=\alpha_i(t_1)+\alpha_i(t_2-t_1)-2\alpha_i(t_1)\alpha_i(t_2-t_1),\\
\eta(t_1,t_2)&=-\xi_0(t_1,t_2)c_{\hat n+1}(t_1)c_{\hat n+1}(t_2-t_1)-\xi_1(t_1,t_2)c_{\hat n}(t_1)c_{\hat n}(t_2-t_1).
\end{aligned}
\tag{A13}
$$

The mapping (A12) can be used to calculate the entropy production (94).

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