Effect of quantum jumps on non-Hermitian systems

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One possible realization of non-Hermitian systems is based on open quantum systems by omitting quantum jumping terms in the master equation. This is a good approximation at short times where the effects of quantum jumps can be ignored. However, the jumps can affect the long-time dynamics of the system, motivating us to take the jumps into account in these studies. In this paper, by treating the quantum jumps as perturbations, we examine the effect of the quantum jumps on the non-Hermitian system. For this purpose, we first derive an effective Hamiltonian to describe the dynamics of the open quantum system based on the master equation, then expand the eigenstates and eigenenergies up to the first and second order in the quantum jumps. Finally, we apply our theory to a dissipative two-level system and dissipative fermionic superfluids. The effect of the quantum jump on the dynamics and the nonequilibrium phase transition is demonstrated and discussed.

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I. INTRODUCTION

In recent years, non-Hermitian (NH) systems [1] have attracted much attention [2] from both sides of theoretical and experimental studies. Without the restriction of Hermiticity, non-Hermitian Hamiltonians have been applied to reexamine well-known quantum systems ranging from single-particle to many-body systems [3–7]. Interesting features and novel observations are found, including phase transitions [8], exceptional points (EPs) [9–12], quantum skin effect [13–15], non-Bloch bulk-boundary correspondence [5,16], unidirectional zero reflection [17,18], and so on.

Non-Hermitian systems differ from their Hermitian counterparts in many aspects, such as the nonconservation of probability, complex-valued eigenenergies, and biorthonormal eigenstates [19]. In order to obtain an effective NH Hamiltonian, many works suggest using open quantum system [3,4,6,7,20,21] by neglecting quantum jumps, adding reciprocal terms into Hermitian systems [5,14,22], or using parametric-amplifier type interactions [23]. Among them, the most popular scheme is the open system approach, which neglects the quantum jumps in the Lindblad master equation and is valid at the short-time limit defined by the loss rate $1/\gamma$ [3,7,20,24]. It is worth addressing that the validity of ignoring the jumps also depends on initial states of the dynamics.

The quantum jumps are associated with the terms in the quantum master equation that act on both left and right sides of the density matrix. From the viewpoint of measurement, the environment can be treated as a device which continuously measures the system and the quantum jumps cause an abrupt change in the state of the system. According to the quantum trajectory theory, the quantum jumps are the terms responsible for the abrupt stochastic change of the wave function. The quantum jumps can also result in different properties of the EPs. In fact, for Lindbladians with and without quantum jumps [25,26], the EPs can be remarkably different. Connections between the two types of EPs are established by introducing a hybrid-Liouvillian superoperator, where "hybrid" denotes Liouvillians with different strengths of jumping terms, which are capable of describing the passage from a non-Hermitian Hamiltonian to a true Liouvillian including quantum jumps [27].

Generally speaking, an analytical solution to the master equation is difficult to obtain due to the huge size of the Hilbert space. Several stochastic approaches, for instance, Monte Carlo [28,29] and quantum trajectory [30], are put forward. These approaches apply randomness and statistical laws to simulate the occurrence of quantum jumps, which reduce the complexity from H_N^2 to H_N with H_N being the size of the Hilbert space of the Hamiltonian. However, their numerical simulations are time consuming and lack analytical results. To this extent, the non-Hermitian Hamiltonian is a convenient approach to describe open systems. However, dropping the quantum jump terms might lead to a wrong result. Therefore, the examination on the validity of neglecting the quantum jump terms is an urgent task.

In this paper, by using the effective Hamiltonian approach [31], we propose a method to approximately solve the master equation. The effective Hamiltonian approach can transform the Lindblad master equation into a Schrödinger-like equation with an effective Hamiltonian, which describes the dynamics of a composite system consisting of the system and an auxiliary system. Thus the dynamics governed by the master equation is transformed into an evolution of a pure state governed by the effective Hamiltonian, and the pure state can be mapped back to the density matrix of the system. Here we develop the mapping rule with a biorthonormal basis. By combining the effective Hamiltonian approach with NH-perturbation theory [32,33], we formally derive a higher-order

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approximate solution to the master equation, and illustrate our theory with examples.

This paper is organized as follows. In Sec. II, we introduce the effective NH Hamiltonian approach and combine it with the perturbation theory to derive a solution for the density operator. We first assume that the quantum jump terms in the master equation are negligible, then treat these terms as perturbations. In Secs. III and IV we apply our method to a two-level system with decoherence and a dissipative Bardeen-Cooper-Schrieffer (BCS) system [34]. We calculate the approximate density operator, energy, and the fidelity of the initial state by the present theory. The results are discussed and the effect of quantum jumps on the BCS state is analyzed. Finally, we conclude in Sec. V.

II. FORMALISM

The Markov master equation (ME) is one of the most fundamental descriptions for open systems in quantum theory [35], which was derived with the weak-coupling assumption and the Markov approximation. The master equation is valid in many circumstances, and the solution to the equation obeys the basic rules of quantum mechanics such as trace preserving, complete positivity, and Hermiticity. This is the reason why a wide range of applications has been developed in various fields including quantum state preparation [36], excitation transfer in light-harvesting systems, quantum measurement [37], and quantum computation [38]. Due to the complexity of the master equation, many methods have been introduced to approximately solve the equation. For example, in Ref. [39] the authors developed a short-time perturbative expansion method, and Ref. [40] presented a perturbation theory by treating the small loss as the perturbation. Reference [41] decomposes the Liouvillian superoperator into two parts, treating part of the dissipators as the dominant contribution to the system, while the other parts of the dissipators were treated as perturbations. Based on this, Ref. [42] considered a practical model of damping Jaynes-Cumming lattices, in which the interaction between the resonator mode and the qubit was viewed as a perturbation. Reference [43] introduced a perturbation theory for a time-dependent Lindbladian master equation with the help of Dyson expansions and linear response theory.

In the following, we will develop an approach to solve the master equation based on the perturbation theory for non-Hermitian systems; the difference is that the jumping terms in the master equation are treated as the perturbation terms. Let us start with the master equation in the Lindblad form [35,44]

$$\dot{\rho} = -i[H_0, \rho] + \sum_m \frac{\kappa_m}{2}(2F_m\rho F_m^\dagger - F_m^\dagger F_m\rho - \rho F_m^\dagger F_m), \quad (1)$$

where H_0 is the free Hamiltonian of the system, κ_m is the decay rate for the mth decay channel, and F_m stands for the eigenoperator of the system, usually named as Lindblad operators. The reduced density matrix ρ remains completely positive and trace preserving [45]. However, these break down when the jumping terms are neglected and an effective non-Hermitian Hamiltonian $H = H_0 - i/2\sum_m \kappa_m F_m^\dagger F_m$ is obtained to describe the system. Suppose H is diagonalizable and the eigenvectors satisfy

$$H|r_n\rangle = E_n|r_n\rangle, H^\dagger|l_n\rangle = E_n^*|l_n\rangle, \quad (2)$$

where $|r_n\rangle$ and $|l_n\rangle$ are the right and left eigenvectors of H. As the system is diagonalizable or nondefective, we can follow the biorthonormal relation $\langle l_m|r_n\rangle = \delta_{mn}$, as well as the completeness relation [19]

$$\sum_n |r_n\rangle\langle l_n| = \sum_n |l_n\rangle\langle r_n| = I. \quad (3)$$

Since the biorthonormal eigenvectors are also complete [19], we can use them to expand the density matrix. Following Ref. [31], we can obtain an effective Hamiltonian as long as the mapping between the composite system and the density operator is specified. Here we generalize this theory taking a different set of eigenstates as the basis. The details of the generalization can be found in Appendix A. We should note that once the relation is established the effective Hamiltonian is unique:

$$\widetilde{H} = H - H^{A*} + i\sum_m \kappa_m F_m F_m^{A*}, \quad (4)$$

where the superscript A denotes the auxiliary system, whose matrix representation satisfies

$$\langle r_m|O^\dagger|l_n\rangle = ({}^A\langle L_n|O^A|R_m\rangle^A)^*. \quad (5)$$

Under the mapping rules, the density operator matrix element is now defined as $\rho_{mn} = \langle l_m|\rho|l_n\rangle$ and the Schrödinger-like state $|\psi_\rho\rangle$ reads

$$|\psi_\rho\rangle = \sum_{mn}^N \rho_{mn}|r_m\rangle|R_n\rangle^{A*} \to \rho = \sum_{mn} \rho_{mn}|r_m\rangle\langle r_n|, \quad (6)$$

where the element of ρ is defined in basis $\{|r_n\rangle\}$ as $\rho_{mn} = \langle l_m|\rho|l_n\rangle$, which is slightly different from the earlier definition $\rho_{mn} = \langle l_m|\rho|r_n\rangle$ [19]. The trace of the density matrix shall be taken as $\mathrm{Tr}(\rho) = \sum_n \langle l_n|\rho|r_n\rangle$, and the average values of a physical observable O could thus be calculated as $\langle O\rangle = \mathrm{Tr}(\rho O) = \sum_n \langle l_n|\rho O|r_n\rangle$. Both expansions are feasible, but the matrix representation is slightly different. Actually, the right and left eigenvectors can be connected via an invertible matrix A, i.e., [46]

$$|r_m\rangle = A|m\rangle, |l_m\rangle = (A^{-1})^\dagger|m\rangle, \quad (7)$$

where $\{|m\rangle\}$ is a set of complete orthonormal bases (see Appendix B for more details).

The effective Hamiltonian $\widetilde{H}$ in Eq. (4) can be regarded as a composite system, whose Hilbert space is hence enlarged from N to N^2, and the jumping terms in the master equation now describe the coupling between the system and the ancilla (see Appendix A). When κ_m is small, the interaction term $\widetilde{V} = i\sum_m \kappa_m F_m F_m^A$ can be treated as a perturbation. Following the perturbation theory [33] for non-Hermitian systems [47–49], we find the first-order correction to the nth energy and first-order correction to the nth eigenvector:

$$e_n^{(1)} = \langle\widetilde{\psi}_n^{(0)}|\widetilde{V}|\psi_n^{(0)}\rangle,$$
$$|\psi_n^{(1)}\rangle = \sum_{k\neq n} \frac{\langle\widetilde{\psi}_k^{(0)}|\widetilde{V}|\psi_n^{(0)}\rangle}{e_n^{(0)} - e_k^{(0)}}|\psi_k^{(0)}\rangle, \quad (8)$$

where $\{|\psi_n^{(0)}\rangle\}$ and $\{|\widetilde{\psi}_n^{(0)}\rangle\}$ are the right and left eigenvectors of the effective Hamiltonian without interaction terms, corresponding to eigenenergy $e_n^{(0)}$. Apparently the eigenvectors are actually a direct product of the basis of the two systems. The corresponding $e_n^{(0)}$ is also easy to calculate, because the two subsystems are independent of each other.

Now we are in a position to discuss the dynamics of the open quantum system. By using Eq. (A4) we can obtain the state $|\psi_\rho(t)\rangle$ at time t with an initial state $|\psi_\rho(0)\rangle$ (given by the corresponding initial density matrix). Straightforward calculations show that

$$|\psi_\rho(t)\rangle = e^{-i\widetilde{H}t}|\psi_\rho(0)\rangle = e^{-i\widetilde{H}t}\sum_n |\psi_n\rangle\langle\widetilde{\psi}_n|\psi_\rho(0)\rangle$$
$$= \sum_n e^{-ie_nt}|\psi_n\rangle\langle\widetilde{\psi}_n|\psi_\rho(0)\rangle, \tag{9}$$

leading to the state of the system at time t:

$$\rho(t) = \sum_n e^{-ie_nt}\langle\widetilde{\psi}_n|\psi_\rho(0)\rangle\rho_n, \tag{10}$$

where $\{|\psi_n\rangle\}$ and $\{|\widetilde{\psi}_n\rangle\}$ are the exact right and left eigenvectors of $\widetilde{H}$ with corresponding eigenvalue e_n. $|\psi_n\rangle$ can be expanded by the complete basis vector $\{|r_i\rangle|R_j\rangle^{A*}\}$ and ρ_n can be obtained by the mapping role in Eq. (6). Namely, $|\psi_n\rangle = \sum_{ij} d_{ij}|r_i\rangle|R_j\rangle^{A*} \to \rho_n = \sum_{ij} d_{ij}|r_i\rangle\langle r_j|$.

From this decomposition, we can find the decay feature of the system [25], since it relates closely to the eigenvalues of the effective Hamiltonian $\widetilde{H}$. In other words, the eigenenergies characterize the decay rates of different eigenstates. $\widetilde{H}$ has one zero eigenenergy in general, which corresponds to the steady state of the master equation. As time evolves, the coefficient e^{-ie_nt} is vanishing for $e_n \neq 0$, and the system reaches its steady state in the long-time limit. Also, when $e_n \neq 0$, we must have $\mathrm{Tr}(\rho_n) = 0$, whereas $\mathrm{Tr}(\rho_n) = 1$ when $e_n = 0$. This property protects the density operator to preserve its trace.

As aforementioned, most of the earlier studies focus on the differences between the spectra of the Liouvillian with and without quantum jumps. Here we shall emphasize that both the eigenenergies and eigenvectors are important for the dynamics. Take the model in Ref. [22] as an example, where the authors proposed an implementation scheme in optical lattices for the asymmetric hopping Hatano-Helson model. The free Hamiltonian can be written as $H_0 = -J\sum_j(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1})$, where J is the hopping strength of the lattice and c_j stands for the fermion annihilation operator at site j. When the lattice suffers from the collective one-body loss, the dynamics is described by a master equation with a Lindblad operator $F = c_j - ic_{j+1}$ with loss rate κ. Postselection is used to guarantee that there are no quantum jumps at any time and the system conserves particle numbers. After neglecting the overall loss, we obtain an effective Hamiltonian $H = \sum_j(J_R c_{j+1}^\dagger c_j + J_L c_j^\dagger c_{j+1})$ with the asymmetric hopping strengths $J_R = -J + \kappa/2, J_L = -J - \kappa/2$. By exact diagonalization [50], we numerically solve the Liouvillian spectrum of the system in both cases with and without quantum jumps which are illustrated respectively. Here, both the open boundary condition [Fig. 1(a)] and period boundary condition [Fig. 1(b)] are considered. From the figures, we find that the quantum jumps have no effect on the spectrum of the Liouvillians. In other words, the Liouvillians with and without quantum jumps have the same spectrum. Mathematically, this can be understood as that the quantum jumps contribute only to the block-upper-triangular elements, while the Liouvillian without quantum jumps is of block-diagonal form [51–53].

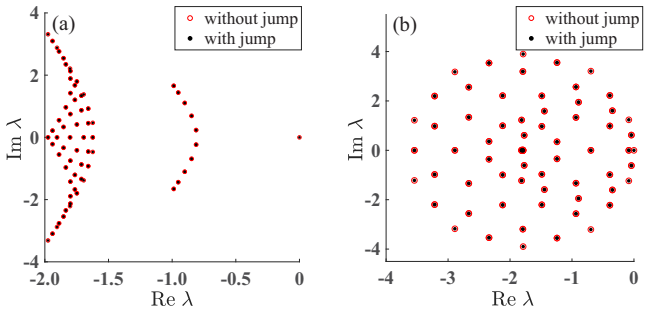

FIG. 1. The Liouvillian spectrum of the Hatano-Nelson model for (a) the open boundary condition and (b) the periodic boundary condition. The red circles represent the Liouvillian without jumps, while the black dots are for the full Liouvillian. We find that they are perfectly overlapping. Although our numerical calculation is restricted to having two particles at most, the observation holds for more particles. In both figures, the parameters are chosen as $J/\kappa = 1$, and the number of lattice sites is $n = 10$.

Despite the two Liouvillians holding the same spectrum, the dynamics governed by them are totally different. To be specific, in Fig. 2 we show the average particle number N as a function of time t with initial state $c_1^\dagger c_2^\dagger|0\rangle$, where $|0\rangle$ is the vacuum state of the fermion. The blue dash-dotted line and red line in the figure are plotted for the system governed by Liouvillians with and without quantum jumps, respectively. It is obvious that the NH Hamiltonian commutes with the particle number $[H, N] = 0$, so that the particle number is conserved. On the other side, the particle number decreases with time due to the quantum jumps. From these observations, we find that the same Liouvillian spectrum might lead to different dynamics because the eigenstates of the Liouvillians are different. In the other words, start from an initial state $\rho(0)$ and evolve under a non-Hermitian Hamiltonian H_{eff}, and after a tiny time interval δt the density matrix $\rho(\delta t)$ becomes

$$\rho(\delta t) = \frac{e^{-iH_{\mathrm{eff}}\delta t}\rho(0)e^{iH_{\mathrm{eff}}^\dagger\delta t}}{\mathrm{Tr}[e^{-iH_{\mathrm{eff}}\delta t}\rho(0)e^{iH_{\mathrm{eff}}^\dagger\delta t}]}. \tag{11}$$

Clearly, the eigenenergies and eigenvectors together determine the evolution of the system.

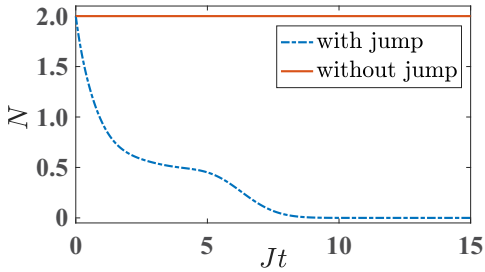

FIG. 2. The average particle number N as a function of time. The blue dash-dotted line and red solid line stand for the Liouvillians with and without quantum jumps.

Note that we can also employ the perturbation expansion Eq. (10) to calculate $\rho(\delta t)$ and a normalization is necessary because ρ_n in Eq. (10) is not traceless in the present perturbation theory.

Before closing this section, we would like to point out that our scheme is different from approximations in the literature on the following points. The first point is that we only treat the jumping terms as perturbations—this means that without perturbations the system is governed by a non-Hermitian Hamiltonian, and we focus on whether some of the results and phenomena in the various existing references are sustainable over time. And the second is that the definition of the zeroth-order steady state may not be so intuitive, because the zeroth-order equation in general does not satisfy $-i(H\rho_0 - \rho_0 H^\dagger) = 0$, where $\rho_0 = |r\rangle\langle l|$ and $\{|r\rangle\}, \{|l\rangle\}$ is a set of right and left eigenvectors of the effective non-Hermitian Hamiltonian H. In our scheme the perturbation of the energy and eigenvectors of the Hamiltonian in Eq. (4) is actually used for the building of an evolution equation, as shown in Eq. (10).

III. APPLICATION 1: TWO-LEVEL SYSTEM

In this section, we illustrate our theory with a dissipative two-level atom. We consider three decoherence channels, including the bit-flip and phase-flip channels, and their decoherence rates are γ_p, γ_x, and γ_z, respectively. The dynamics of the system can be described by the following master equation:

$$\dot{\rho} = -i(H\rho - \rho H^\dagger) + \gamma_p \sigma_+ \rho \sigma_- + \gamma_x \sigma_x \rho \sigma_x + \gamma_z \sigma_z \rho \sigma_z, \tag{12}$$

with $H = \frac{\omega}{2}\sigma_z - i\frac{\gamma_p}{2}\sigma_+\sigma_- - i\frac{\gamma_x}{2}\sigma_x^2 - i\frac{\gamma_z}{2}\sigma_z^2$, where σ_x, σ_y, and σ_z are Pauli matrices, and $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ are the rising and lowering operators.

Based on the effective Hamiltonian approach, we introduce an ancillary two-level system with σ^A denoting its Pauli matrix, with the basis spanned by the eigenvectors of σ_z and σ_z^A, with spin-up state $|0\rangle$ for the system and $|0\rangle^A$ for the ancilla, while the spin-down states are $|1\rangle$ and $|1\rangle^A$. We can first write out the matrix representation of H:

$$H = \begin{bmatrix} \frac{\omega}{2} - i\frac{\gamma_p}{2} - i\frac{\gamma_x}{2} - i\frac{\gamma_z}{2} & 0 \\ 0 & -\frac{\omega}{2} - i\frac{\gamma_x}{2} - i\frac{\gamma_z}{2} \end{bmatrix}, \tag{13}$$

where the order of the basis is $\{|0\rangle, |1\rangle\}$ and they diagonalize the Hamiltonian H. Apparently, ${}^A\langle 0|H^A|0\rangle^A = (\langle 0|H^\dagger|0\rangle)^* = \frac{\omega}{2} - i\frac{\gamma_p}{2} - i\frac{\gamma_x}{2} - i\frac{\gamma_z}{2}$, ${}^A\langle 0|H^A|1\rangle^A = (\langle 1|H^\dagger|0\rangle)^* = {}^A\langle 1|H^A|0\rangle^A = (\langle 0|H^\dagger|1\rangle)^* = 0$, and ${}^A\langle 1|H^A|1\rangle^A = (\langle 1|H^\dagger|1\rangle)^* = -\frac{\omega}{2} - i\frac{\gamma_x}{2} - i\frac{\gamma_z}{2}$. Thus H^{A*} takes

$$H^{A*} = \begin{bmatrix} \frac{\omega}{2} + i\frac{\gamma_p}{2} + i\frac{\gamma_x}{2} + i\frac{\gamma_z}{2} & 0 \\ 0 & -\frac{\omega}{2} + i\frac{\gamma_x}{2} + i\frac{\gamma_z}{2} \end{bmatrix}, \tag{14}$$

under its basis $\{|0\rangle^A, |1\rangle^A\}$.

With the above consideration, we can obtain the matrix representation of the free Hamiltonian of the composite system $H - H^{A*}$, which will be treated as the zeroth-order Hamiltonian:

$$H - H^{A*} = \begin{bmatrix} -i(\gamma_p + \gamma_x + \gamma_z) & 0 & 0 & 0 \\ 0 & \omega - i\left(\frac{\gamma_p}{2} + \gamma_x + \gamma_z\right) & 0 & 0 \\ 0 & 0 & -\omega - i\left(\frac{\gamma_p}{2} + \gamma_x + \gamma_z\right) & 0 \\ 0 & 0 & 0 & -i(\gamma_x + \gamma_z) \end{bmatrix},$$

and the order of the basis is $\{|0\rangle|0\rangle^A, |0\rangle|1\rangle^A, |1\rangle|0\rangle^A, |1\rangle|1\rangle^A\}$. Similarly, the quantum jumps, i.e., the third terms in Eq. (4), which describe the coupling between the two systems, read

$$\begin{bmatrix} i\gamma_z & 0 & 0 & i\gamma_x \\ 0 & -i\gamma_z & i\gamma_x & 0 \\ 0 & i\gamma_x & -i\gamma_z & 0 \\ i\gamma_p + i\gamma_x & 0 & 0 & i\gamma_z \end{bmatrix},$$

and we will treat this coupling as a perturbation. By the perturbation theory given in Eq. (8), the first- and second-order corrections to the eigenenergies and the right eigenvectors can be given by

$$E_1^{(0)} = -i(\gamma_p + \gamma_x + \gamma_z), \quad E_2^{(0)} = \omega - i\left(\frac{\gamma_p}{2} + \gamma_x + \gamma_z\right),$$
$$E_3^{(0)} = -\omega - i\left(\frac{\gamma_p}{2} + \gamma_x + \gamma_z\right), \quad E_4^{(0)} = -i(\gamma_x + \gamma_z),$$
$$E_1^{(1)} = i\gamma_z, \quad E_2^{(1)} = -i\gamma_z, \quad E_3^{(1)} = -i\gamma_z, \quad E_4^{(1)} = i\gamma_z,$$
$$E_1^{(2)} = \frac{-i\gamma_x(\gamma_p + \gamma_x)}{\gamma_p}, \quad E_2^{(2)} = -\frac{\gamma_x^2}{2\omega},$$
$$E_3^{(2)} = \frac{\gamma_x^2}{2\omega}, \quad E_4^{(2)} = \frac{i\gamma_x(\gamma_p + \gamma_x)}{\gamma_p},$$
$$|\psi_1^{(1)}\rangle = -\frac{(\gamma_p + \gamma_x)}{\gamma_p}|\psi_4^{(0)}\rangle, \quad |\psi_2^{(1)}\rangle = \frac{i\gamma_x}{2\omega}|\psi_3^{(0)}\rangle,$$
$$|\psi_3^{(1)}\rangle = -\frac{i\gamma_x}{2\omega}|\psi_2^{(0)}\rangle, \quad |\psi_4^{(1)}\rangle = \frac{\gamma_x}{\gamma_p}|\psi_1^{(0)}\rangle,$$
$$|\psi_1^{(2)}\rangle = |\psi_2^{(2)}\rangle = |\psi_3^{(2)}\rangle = |\psi_4^{(2)}\rangle = 0. \tag{15}$$

To show the validity of the perturbation theory, we present in Fig. 3 the comparison between the numerical results given by ME, NH, and the perturbation theory. We find that the NH approximation is close to the numerical result in the short-time limit, but it gradually deviates and finally reaches its steady state, which is totally different from the steady state of the master equation. The results given by the perturbation theory are in good agreement with those given by the master

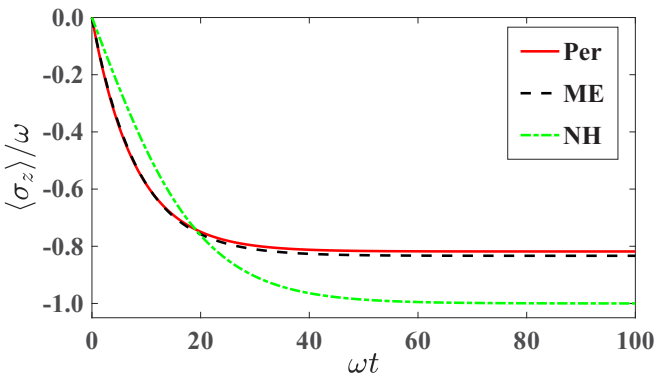

FIG. 3. The average of σ_z at different times t. The results are given by the master equation without any approximation (Per, black dashed line), by the non-Hermitian Hamiltonian (NH, green dash-dotted line), and by the perturbation theory (ME, red solid line). The parameters chosen are $\gamma_p = 0.1\omega$, $\gamma_x = 0.01\omega$, $\gamma_z = 0.5\omega$.

equation (or the Liouvillian). Thus we can claim that the perturbation theory based on the non-Hermitian Hamiltonian might be a good method to deal with non-Hermitian systems. Of course this example is easy to solve exactly as the Hilbert space is small. In the next section, we will present a many-body system to exemplify the perturbation theory.

IV. APPLICATION 2: EFFECT OF QUANTUM JUMPS ON THE NON-HERMITIAN BCS STATES

In recent years, open many-body systems have become active in various fields from quantum optics to condensed matter. However the master equation of many-body open systems is difficult to solve [5,54]. Here we apply our perturbation theory to such systems, taking the NH BCS model as an example [20].

Due to inelastic collisions, the atoms in the BCS system suffer from two-body loss and atoms will leave the system with time; such a system can be described by an effective Markovian master equation [24]. In the case that the quantum jumps could be neglected, the earlier study [20] showed that when the interaction strength is not so strong, the superfluid suffers a breakdown and restoration transition occurs as the dissipation increases, whereas in the strong-dissipation limit the superfluid phase would never be broken. This gives rise to a question: what happens if the quantum jumps cannot be neglected?

To answer this question concretely, we consider the one-dimensional model in Ref. [20]. Here $H_S = \sum_{k\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma}$ describes the free Hamiltonian of the lattice, where $\xi_k = \epsilon_k - \mu$, ϵ_k stands for the energy dispersion and μ is the chemical potential. The interaction Hamiltonian $H_I = -U_0 \sum_i c^\dagger_{i\uparrow} c^\dagger_{i\downarrow} c_{i\downarrow} c_{i\uparrow}$ (take $\hbar = 1$). $c_{i\sigma}$ ($c_{k\sigma}$) denote the annihilation operators of a spin-$\sigma \in \{\uparrow, \downarrow\}$ fermion at site i (with momentum $\boldsymbol{k}$). Consider that the system undergoes inelastic collisions; the dynamics of the system is governed by

$$\dot{\rho} = -i(H_{\text{eff}}\rho - \rho H^\dagger_{\text{eff}}) + \kappa \sum_i L_i \rho L^\dagger_i, \tag{16}$$

where κ is the loss rate and $L_i = c_{i\downarrow} c_{i\uparrow}$, and $H_{\text{eff}} = \sum_{k\sigma} \xi_k c^\dagger_{k\sigma} c_{k\sigma} - \sum_{kk'} U_1/N c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$. Here N is the number of lattice sites and $U_1 = U_0 + i\kappa/2$ is the complex interaction strength.

In order to diagonalize the Hamiltonian, the mean-field (MF) approximation is applied [20], where the quasiparticles obey neither Fermi nor Bose statistics, since a NH Hamiltonian cannot be diagonalized by unitary transformations. Under the MF approximation, Hamiltonian H_{eff} reduces to

$$H_{\text{MF}} = \sum_k E_k(\bar{\gamma}_{k\uparrow}\gamma_{k\uparrow} + \bar{\gamma}_{-k\downarrow}\gamma_{-k\downarrow}) - \sum_k E_k, \tag{17}$$

with $E_k = \sqrt{\xi_k^2 + \Delta_0^2}$, where $\Delta_0 = -U_1/N \sum_k \langle c_{-k\downarrow} c_{k\uparrow}\rangle$ is the order parameter (gap function) of the superfluid. In the $\beta \to 0$ limit, the order parameter can be established with the NH path-integral approach [20] or self-consistency method [55], which is given by the NH gap equation

$$\frac{N}{U_1} = \sum_k \frac{1}{2\sqrt{\xi_k^2 + \Delta_0^2}}, \tag{18}$$

when Δ_0 takes zero; such phase is denoted as a "normal state," where the gap equation has only a trivial solution. For most cases Δ_0 is a complex number.

The quasiparticle operators in Eq. (17) can be written as

$$\bar{\gamma}_{k\uparrow} = u_k c^\dagger_{k\uparrow} - v_k c_{-k\downarrow},\ \bar{\gamma}_{-k\downarrow} = v_k c_{k\uparrow} + u_k c^\dagger_{-k\downarrow}, \tag{19}$$

$$\gamma_{k\uparrow} = u_k c_{k\uparrow} - v_k c^\dagger_{-k\downarrow},\ \gamma_{-k\downarrow} = v_k c^\dagger_{k\uparrow} + u_k c_{-k\downarrow}, \tag{20}$$

with $u_k = \sqrt{\frac{E_k+\xi_k}{2E_k}}$, $v_k = \sqrt{\frac{E_k-\xi_k}{2E_k}}$. In the above derivation, the symmetry $H^*_{\text{MF}} = H^\dagger_{\text{MF}}$ has been used, which could be found from the matrix representation in terms of Fock states. In fact, under such representation, the non-Hermiticity of the system is attributed to the complex diagonal elements, which implies that the left eigenvector $|L_n\rangle$ of the Hamiltonian is exactly the complex conjugation of the right one $|R_n\rangle$ [20].

In the following discussion, we will focus on the ground state of the system. To clarify the discussion, we write down the right and left ground state:

$$\begin{aligned} |r_0\rangle &= |\text{BCS}\rangle_R = \prod_k (u_k + v_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})|0\rangle, \\ |l_0\rangle &= |\text{BCS}\rangle_L = \prod_k (u^*_k + v^*_k c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow})|0\rangle, \end{aligned} \tag{21}$$

where $|0\rangle$ is the vacuum state of the fermions. It is easy to find that $H\bar{\gamma}_{k\sigma}|\text{BCS}\rangle_R = E_k\bar{\gamma}_{k\sigma}|\text{BCS}\rangle_R$, $H^\dagger\gamma^\dagger_{k\sigma}|\text{BCS}\rangle_L = E^*_k\gamma^\dagger_{k\sigma}|\text{BCS}\rangle_L$. Here H is Hamiltonian H_{MF} except the constant $-\sum_k E_k$ is neglected. In this way, all left and right eigenvectors $\{|l_n\rangle\}$, $\{|r_n\rangle\}$ of the effective Hamiltonian H can be constructed.

In the Hermitian case, the superfluidity of the system arises from the nonzero gap function, since the energy spectrum will always have a gap in order to excite quasiparticles, even if ξ_k takes zero [55]. However, for NH systems, such defined superfluid may be metastable, distinguished by the sign of the real part of the condensation energy E_c:

$$E_c = \frac{N}{U_1}\Delta_0^2 - \sum_k \left(\sqrt{\xi_k^2 + \Delta_0^2} - |\xi_k|\right), \tag{22}$$

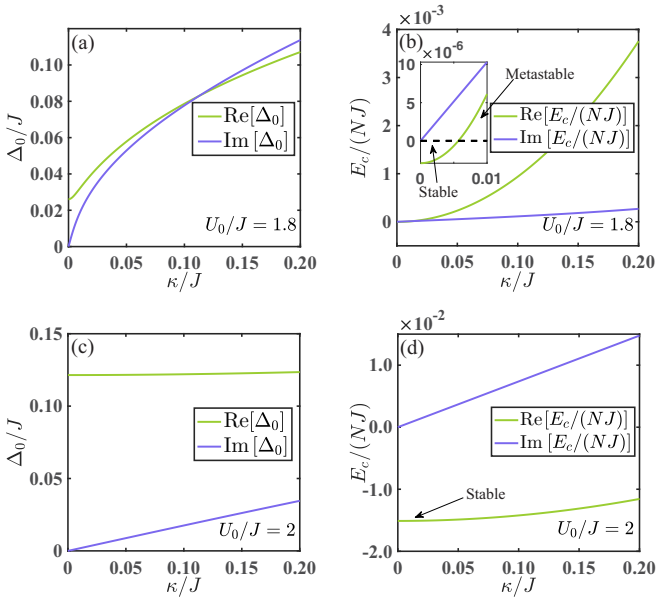

FIG. 4. Numerical solution of (a, c) nonzero superfluid gap Δ_0 and (b, d) condensation energy E_c, calculated by Eqs. (18) and (22), respectively, as a function that changes with loss rate κ. Parameters are chosen as (a, b) $N = 10, \kappa = 0.1J, U_0/J = 1.8$ and (c, d) $U_0/J = 2$. The former system is metastable at most parameters, with stable superfluidity only in the weak dissipation limit. The latter one is always stable for strong attractive interaction U_0.

where in fact E_c represents the difference in the ground-state energy between the superfluid and normal states. For positive $\mathrm{Re}(E_c)$, the system is metastable, while a negative $\mathrm{Re}(E_c)$ leads to a stable superfluid solution [20]. Figures 4(a)–4(d) show real and imaginary parts of the gap function Δ_0 and the condensation energy E_c when $U_1 = 1.8J$ and $2J$, respectively. Apparently, the NH gap equations have nontrivial solutions; for small U_0, the system is stable only at the small dissipation limit. As attractive interaction strength U_0 gets stronger, the system remains stable.

In order to analyze the jump involved circumstance, suppose the system is initially prepared in a NH steady state ρ_0, satisfying $\dot{\rho}_0 = -i(H_{\mathrm{eff}}\rho_0 - \rho_0 H_{\mathrm{eff}}^\dagger) = 0$ and $\mathrm{Tr}(\rho_0) = 1$. Clearly we can choose $\rho_0 = \mathcal{N}|r_0\rangle\langle r_0|$ as an initial state that meets the requirement [56]. Here $\mathcal{N}$ is the normalization coefficient that satisfies $\mathrm{Tr}(\rho_0) = \sum_n \langle l_n|\rho_0|r_n\rangle = 1$. Now we apply the perturbation theory into the NH BCS system. Consider that our system has only two-body loss; we then restrict the Hilbert space enclosing at most two quasiparticles. The corresponding eigenenergies are [the constant in Eq. (17) is omitted] $\{0, 0 - 2E^*_{\boldsymbol{k}_1}, 0 - 2E^*_{\boldsymbol{k}_2} \ldots, 2E_{\boldsymbol{k}_1} - 0, 2E_{\boldsymbol{k}_1} - 2E^*_{\boldsymbol{k}_1} \ldots, \ldots, 2E_{\boldsymbol{k}_n} - 0, 2E_{\boldsymbol{k}_n} - 2E^*_{\boldsymbol{k}_1} \ldots, 2E_{\boldsymbol{k}_n} - 2E^*_{\boldsymbol{k}_1}\}$. In this Hilbert space, the matrix representation of the quantum jumps $L_{\boldsymbol{k}_0}$ takes

$$\begin{bmatrix} u_{\boldsymbol{k}_0}v_{\boldsymbol{k}_0} & 0 & \cdots & u^2_{\boldsymbol{k}_0} & \cdots & 0 \\ 0 & u_{\boldsymbol{k}_0}v_{\boldsymbol{k}_0} & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & 0 \\ -v^2_{\boldsymbol{k}_0} & 0 & \cdots & -u_{\boldsymbol{k}_0}v_{\boldsymbol{k}_0} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \ldots & u_{\boldsymbol{k}_0}v_{\boldsymbol{k}_0} \end{bmatrix}, \tag{23}$$

where the order of the basis is arranged as $\{|r_0\rangle, \bar{\gamma}_{\boldsymbol{k}_1\uparrow}\bar{\gamma}_{-\boldsymbol{k}_1\downarrow}|r_0\rangle, \ldots, \bar{\gamma}_{\boldsymbol{k}_n\uparrow}\bar{\gamma}_{-\boldsymbol{k}_n\downarrow}|r_0\rangle\}\{\langle l_0|, \langle l_0|\gamma^\dagger_{-\boldsymbol{k}_1\downarrow}\gamma^\dagger_{\boldsymbol{k}_1\uparrow}, \ldots, \langle l_0|\gamma^\dagger_{-\boldsymbol{k}_n\downarrow}\gamma^\dagger_{\boldsymbol{k}_n\uparrow}\}$. Notice that the diagonal elements with a minus sign result from the contribution of $\bar{\gamma}_{\boldsymbol{k}_0\uparrow}\bar{\gamma}_{-\boldsymbol{k}_0\downarrow}|r_0\rangle, \langle l_0|\gamma^\dagger_{-\boldsymbol{k}_0\downarrow}\gamma^\dagger_{\boldsymbol{k}_0\uparrow}$.

By the definition given in Eq. (4), the first two terms $H - H^{A*}$ can be taken as the zeroth-order Hamiltonian and the third term $i\sum_m \kappa_m F_m F_m^{A*}$ is the perturbation. Collecting the results in Ref. [20], we find that the zeroth energy of the ground state is

$$E_0^{(0)} = -2\sum_{\boldsymbol{k}} \mathrm{Im}(E_{\boldsymbol{k}}), \tag{24}$$

and by the non-Hermitian perturbation theory the first-order correction to the energy of the ground state is given by

$$E_0^{(1)} = -\kappa \sum_{\boldsymbol{k}} |u_{\boldsymbol{k}}|^2|v_{\boldsymbol{k}}|^2. \tag{25}$$

Here we want to emphasize that the complex constant eigenenergy of the system can be safely ignored, similar to Hermitian systems in which constants cannot affect their dynamical features. This can be interpreted as a gauge shift $H \to H + cI$, where I is the identity operator and c is a complex c number. In other words, the dynamics under H and $H + cI$ is the same [57]. By our theory, the first-order corrections to the left and right ground states are (unnormalized)

$$\begin{aligned} |r_0^{(1)}\rangle &= \sum_{\boldsymbol{k}} \frac{-i\kappa u_{\boldsymbol{k}}v_{\boldsymbol{k}}v^{*2}_{\boldsymbol{k}}}{2E^*_{\boldsymbol{k}}}|r_0\rangle\gamma^\dagger_{\boldsymbol{k}\uparrow}\gamma^\dagger_{-\boldsymbol{k}\downarrow}|l_0\rangle + \sum_{\boldsymbol{k}} \frac{i\kappa u^*_{\boldsymbol{k}}v^{*}_{\boldsymbol{k}}v^2_{\boldsymbol{k}}}{2E_{\boldsymbol{k}}}\bar{\gamma}_{\boldsymbol{k}\uparrow}\bar{\gamma}_{-\boldsymbol{k}\downarrow}|r_0\rangle|l_0\rangle - \sum_{\boldsymbol{k}} \frac{\kappa|v_{\boldsymbol{k}}|^4}{4\mathrm{Im}(E_{\boldsymbol{k}})}\bar{\gamma}_{\boldsymbol{k}\uparrow}\bar{\gamma}_{-\boldsymbol{k}\downarrow}|r_0\rangle\gamma^\dagger_{\boldsymbol{k}\uparrow}\gamma^\dagger_{-\boldsymbol{k}\downarrow}|l_0\rangle, \\ |l_0^{(1)}\rangle &= \sum_{\boldsymbol{k}} \frac{-i\kappa u^*_{\boldsymbol{k}}v^{*}_{\boldsymbol{k}}v^2_{\boldsymbol{k}}}{2E_{\boldsymbol{k}}}|l_0\rangle\bar{\gamma}_{\boldsymbol{k}\uparrow}\bar{\gamma}_{-\boldsymbol{k}\downarrow}|r_0\rangle + \sum_{\boldsymbol{k}} \frac{i\kappa u^{*2}_{\boldsymbol{k}}u_{\boldsymbol{k}}v_{\boldsymbol{k}}}{2E^*_{\boldsymbol{k}}}\gamma^\dagger_{\boldsymbol{k}\uparrow}\gamma^\dagger_{-\boldsymbol{k}\downarrow}|l_0\rangle|r_0\rangle - \sum_{\boldsymbol{k}} \frac{\kappa|v_{\boldsymbol{k}}|^4}{4\mathrm{Im}(E_{\boldsymbol{k}})}\gamma^\dagger_{\boldsymbol{k}\uparrow}\gamma^\dagger_{-\boldsymbol{k}\downarrow}|l_0\rangle\bar{\gamma}_{\boldsymbol{k}\uparrow}\bar{\gamma}_{-\boldsymbol{k}\downarrow}|r_0\rangle. \end{aligned} \tag{26}$$

The normalization condition is $|\mathcal{N}_0|^2(\langle l_0^{(0)}| + \langle l_0^{(1)}|)(|r_0^{(0)}\rangle + |r_0^{(1)}\rangle) = 1$, where $\mathcal{N}_0$ is the normalization constant of the right ground state. Simple algebra yields $|\mathcal{N}_0|^2 = 1/(\sum_{\boldsymbol{k}} \frac{\kappa^2|u_{\boldsymbol{k}}|^4|v_{\boldsymbol{k}}|^4}{4E^{*2}_{\boldsymbol{k}}} + \sum_{\boldsymbol{k}} \frac{\kappa^2|u_{\boldsymbol{k}}|^4|v_{\boldsymbol{k}}|^4}{4E^2_{\boldsymbol{k}}} + \sum_{\boldsymbol{k}} \frac{\kappa^2|u_{\boldsymbol{k}}|^4|v_{\boldsymbol{k}}|^4}{16\mathrm{Im}(E_{\boldsymbol{k}})^2} + 1)$. The first-order corrections to the other eigenvectors can be computed in the same way. These eigenvectors form a new biorthonormal and complete basis, and the whole dynamic can be predicted by Eq. (10). The calculation is tedious

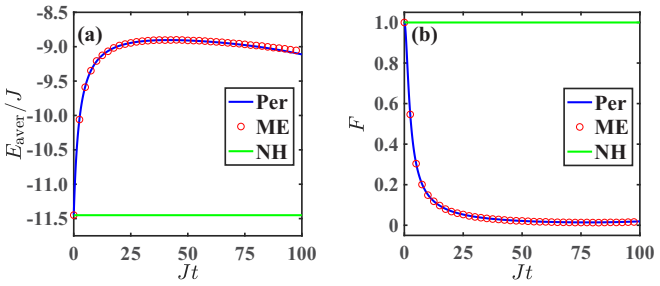

FIG. 5. (a) The average energy $E_{\rm aver} = \langle H_S \rangle$ and (b) the fidelity of the initial state governed by the perturbation method (blue solid line), master equation (red circles), and non-Hermitian Hamiltonian (green solid line), respectively. Here we take $\epsilon_{\boldsymbol{k}} = -2J\cos\boldsymbol{k}$, where J is the hopping amplitude and the loss rate κ is set to be $0.1J$ and $\mu = 0$. The interaction strength is set to be $U_0 = 1.8J$, and the number of lattice sites $N = 10$. The order parameter $\Delta_0/J = 0.0786 + 0.0777i$ [obtained by solving Eq. (18)], and the system is assumed to stay in the superfluid state (metastable).

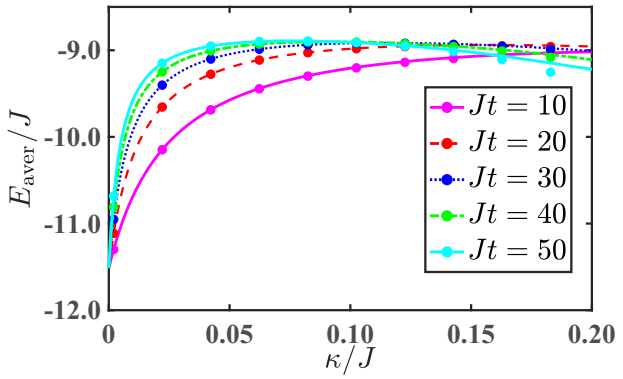

FIG. 6. The average energy $E_{\rm aver}$ as a function of loss rate κ at different times t. We calculate this energy by both the master equation (solid lines) and the perturbation theory (dotted lines). The purple solid line, red dashed line, blue dotted line, green dash-dotted line, and cyan solid line are for different times $Jt = 10, 20, 30, 40, 50$, respectively. Parameter $U_0 = 1.8J$ and the number of lattice sites is 10.

and expression is involved, so we do not present them here.

In Fig. 5(a), we plot the average energy of the system at time t. The results are given by first-order perturbation (blue solid line) and by numerical simulations with the master equation (red circles). For the purpose of comparison, the results given by the non-Hermitian Hamiltonian (green solid line) are also shown. We can find that the results given by perturbation theory match well with that by the master equation even for long-time evolution. The other interesting observation is that the energy has a large degree of deviation from the prediction based on the non-Hermitian Hamiltonian. Recall that energy by non-Hermitian evolution is always unchanged because the initial state (i.e., the ground state) is a steady state of the system. This feature can be understood by examining the fidelity of the ground state $F = |\mathrm{Tr}(\rho_0\rho)|/\sqrt{\mathrm{Tr}(\rho_0^2)\mathrm{Tr}(\rho^2)}$ [58]. As shown in Fig. 5(b), at $Jt = 100$ the system is almost all excited to the excited states, resulting in a low fidelity between the initial states. This result suggests that the quantum jumps play an important role in this system, and the state $|r_0\rangle$ is unstable under the effect of quantum jumps.

From the other point of view, the non-Hermitian BCS model does not conserve the number of fermions and thus the Liouvillians without and with quantum jumps cannot be written into a block-diagonal and block-upper-triangular form, respectively. This leads to different spectra for the Liouvillians with and without quantum jumps [51,52]. This observation is quite different from that of the non-Hermitian Hatano-Nelson model. From the viewpoint of the quasiparticle, the quasiparticle number is conserved because $[H_{\rm MF}, \sum_{k\sigma} \bar{\gamma}_{k\sigma}\gamma_{k\sigma}] = 0$. However, in the basis spanned by the quasiparticles, the jumping terms $\sum_{\boldsymbol{k}} \kappa c_{-k\downarrow} c_{k\uparrow}$ still cannot be written as a block-upper-triangular form, which is in agreement with the aforementioned analysis.

In Fig. 6, we perform a comparison between the results of the master equation and by the first-order perturbation with different loss rate κ. We observe that the bigger the loss rate is, the more intensity the excitation will be at the beginning. Moreover, even when κ is small, after a long time, the average energy of the system suffers an abrupt change. As κ increase, the whole dynamic is totally different from NH ones [the same property as the green solid line in Fig. 5(a)].

Figures 7(a) and 7(b) show the time evolution of the average energy with different system sizes. The results are calculated with both the master equation and the perturbation theory. The fact that the two results (one from the master equation and another from the perturbation theory) match well even for a system of large size suggests the validity of our theory, which is also essential in many-body systems.

Finally, we consider the condensation energy E_c of the effective model that changes with the interaction strength U_0, with a fixed loss rate $\kappa = 0.1J$, as shown in Fig. 8. As U_0 increases from $1.8J$ [as in Fig. 4(b)] to $2J$ [as in Fig. 4(d)], the real part of E_c crosses the zero point, which leads to the transition of the superfluidity from the metastable (red area) to the stable region (green area). A similar result is also addressed in Ref. [20]. However, at $Jt = 50$, when quantum jumps are introduced, there is no significant variation in the

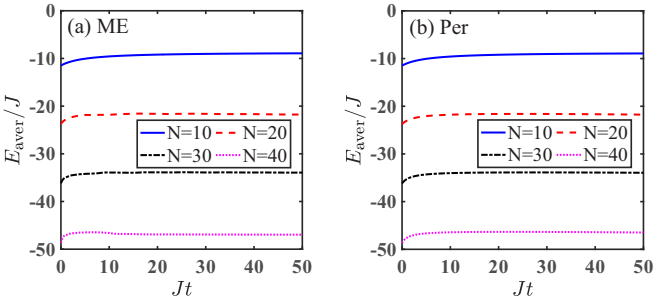

FIG. 7. The time evolution of average energy $E_{\rm aver}$ for different N. $U_0 = 1.8J$ and the loss rate $\kappa = 0.05J$ were set for these plots. (a) Results from the master equation. (b) Corresponding results from the perturbation theory. The blue solid line, red dashed line, black dash-dotted line, and cyan dotted line represent $N = 10, 20, 30, 40$, respectively.

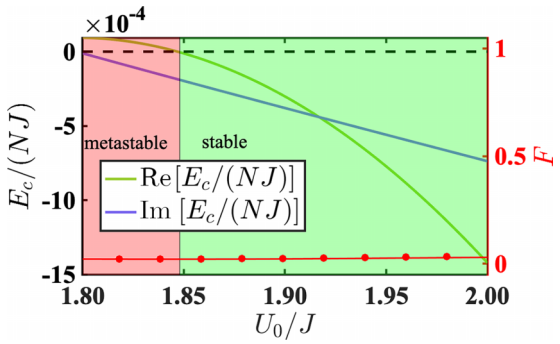

FIG. 8. Left axis: Condensation energy E_c as a function of U_0/J, $\kappa/J = 0.1$; as U_0/J increases, the superfluid undergoes a phase transition, from metastable area (red) to stable area (green). Right axis: Fidelity of the ground state at time $Jt = 50$; the red solid line represents ME simulation, while the red dots stand for the perturbation; there is no significant difference in system dynamics.

fidelity F from the ground state. It illustrates that in the study of NH systems, the presence of quantum jump events may challenge certain properties of NH systems, which warrants further investigation. Our proposed method can serve as a solid foundation for such research and verification.

V. CONCLUSION

Based on the effective Hamiltonian approach, we developed a perturbation theory to study open quantum systems governed by the Lindblad master equation. Treating the quantum jumps as perturbations, we derived a set of corrections up to the first and second order in the jumps to the eigenenergies and corresponding eigenfunctions. This development is not trivial since our perturbation theory is based on the non-Hermitian Hamiltonian and then the basis of the Hilbert space behaves differently from its Hermitian counterpart. We applied our theory to two examples, a decoherence two-level system and the non-Hermitian BCS model. The results show that the present theory is in good agreement with the results obtained by solving the master equation. Also, the present theory saves computing time and in most cases analytical expressions can be found. We believe that the present theory opens a door to study the non-Hermitian physics and paves a way to check the validity of the description of open systems by non-Hermitian Hamiltonians.

ACKNOWLEDGMENTS

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APPENDIX A: DERIVATION OF THE EFFECTIVE HAMILTONIAN APPROACH

In this Appendix, we will introduce the effective Hamiltonian approach, which was proposed to exactly solve the master equation of open quantum systems.

According to the master equation Eq. (1), the differential equation of matrix element $\rho_{mn} = \langle l_m|\rho|l_n\rangle$ can be written as

$$i\dot{\rho}_{mn} = \langle l_m|(H\rho - \rho H^\dagger + i\kappa F\rho F^\dagger)|l_n\rangle, \tag{A1}$$

where $H = H_0 - i\kappa/2F^\dagger F$ is the effective non-Hermitian Hamiltonian. Here for simplicity we assume that there is only one Lindblad operator in the master equation Eq. (1).

We introduce an auxiliary system A (the ancilla, which is the same as the system), and then extend the N-dimensional Hilbert space to an N^2-dimensional Hilbert space. The Hilbert space of the composite system (the open quantum system plus the ancilla) can be expanded by a set of biorthonormal bases $\{|l_m\rangle \otimes |L_n\rangle^{A*}, |r_m\rangle \otimes |R_n\rangle^{A*}\}$, where $|R_n\rangle^A$ and $|L_n\rangle^A$ are the right and left basis for the ancilla Hilbert space. At this time, a density matrix ρ of the open quantum system can be mapped into a pure bipartite state in N^2-dimensional Hilbert space, i.e.,

$$|\psi_\rho(t)\rangle = \sum_{mn}^{N} \rho_{mn}(t)|r_m\rangle|R_n\rangle^{A*}. \tag{A2}$$

Taking the time derivative on the pure state $|\psi_\rho(t)\rangle$ yields

$$i\partial_t|\psi_\rho(t)\rangle = \sum_{mn}^{N}(\langle l_m|(H\rho - \rho H^\dagger + i\kappa F\rho F^\dagger)|l_n\rangle)|r_m\rangle|R_n\rangle^{A*}. \tag{A3}$$

For an arbitrary system operator O, we can define a corresponding operator O^A of the ancilla, which satisfies $\langle r_m|O^\dagger|l_n\rangle = ({}^A\langle L_n|O^A|R_m\rangle^A)^*$. Inserting the complete relation Eq. (3) into Eq. (A3), it is not difficult to obtain

$$\sum_{mn}\langle l_m|H\rho|l_n\rangle|r_m\rangle|R_n\rangle^{A*} = H\otimes I^A|\psi_\rho(t)\rangle,$$
$$\sum_{mn}\langle l_m|\rho H^\dagger|l_n\rangle|r_m\rangle|R_n\rangle^{A*} = I\otimes (H^A)^*|\psi_\rho(t)\rangle,$$
$$\sum_{mn}\langle l_m|F\rho F^\dagger|l_n\rangle|r_m\rangle|R_n\rangle^{A*} = F\otimes (F^A)^*|\psi_\rho(t)\rangle.$$

Thus, the dynamical equation for the composite system can be rewritten as

$$i\partial_t|\psi_\rho(t)\rangle = \widetilde{H}|\psi_\rho(t)\rangle, \tag{A4}$$

with the effective Hamiltonian $\widetilde{H} = H\otimes I - I\otimes (H^A)^* + i\kappa F\otimes (F^A)^*$. Therefore, the master equation is equivalent to the evolution of a pure state of the composite system. The jumping terms in the master equation describe the interactions between the open quantum system and the ancilla. Further, if we choose the basis of the ancilla with the same representation as the system, the effective Hamiltonian returns to the Liouvillian superoperator (a $-i$ factor is neglected) [25]:

$$\widetilde{H} = \left(H_0 - \frac{i\kappa F^\dagger F}{2}\right)\otimes I - I\otimes\left(H_0^{\mathrm{TR}} + \frac{i\kappa F^{\mathrm{TR}}F^*}{2}\right) + i\kappa F\otimes F^*, \tag{A5}$$

where TR denotes transpose operation.

APPENDIX B: SHORT DERIVATION OF THE EQUIVALENCE OF BIORTHOGONAL BASES

Consider an arbitrary non-Hermitian system with a diagonalizable and nondegenerate Hamiltonian $H_{\rm NH}$, whose right and left biorthonormal bases are $\{|r_n\rangle\}, \{|l_n\rangle\}$ and satisfy biorthonormal relation $\langle l_n|r_m\rangle = \delta_{mn}$. In the following, we divide our discussion into two cases, i.e., the real spectrum and the complex spectrum [59].

Now we first prove that $H_{\rm NH}$ with a real spectrum $\{E_n\}$ can be written as $H_{\rm NH} = A^{-1}H_{\rm H}A$, where $H_{\rm H}$ is a Hermitian Hamiltonian and A is a nonunitary operator. According to the completeness relation $\sum_n |r_n\rangle\langle l_n| = \sum_n |l_n\rangle\langle r_n| = I$, the left eigenvector $|l_n\rangle$ can be expressed as an expansion of the right eigenvectors, i.e.,

$$|l_n\rangle = \sum_{n'} M_{nn'}|r_{n'}\rangle, \tag{B1}$$

taking a Hermitian conjugate operation on the above equation and acting $|r_{n'}\rangle$ on the result from the right-hand side; employing biorthonormal relation $\langle l_n|r_m\rangle = \delta_{mn}$ yields

$$\langle l_n|r_{n'}\rangle = \sum_{n''} M^*_{nn''}\langle r_{n''}|r_{n'}\rangle = \delta_{nn'}. \tag{B2}$$

From Eq. (B2), it is easy to obtain $M^*X = I$, where $X_{mn} = \langle r_m|r_n\rangle$ and I is the identity matrix. Thus

$$(M^{-1})^*_{mn} = \langle r_m|r_n\rangle, \tag{B3}$$

$$(M^{-1})_{mn} = \langle r_n|r_m\rangle. \tag{B4}$$

Combing Eqs. (B3) and (B4), it is straightforward to find that $(M^{-1})^*_{nm} = (M^{-1})_{mn}$ always holds for arbitrary m, n. Recall the definition of the Hermitian matrix; M^{-1} is a Hermitian matrix, which implies that M is Hermitian. In other words, the left eigenvectors $\{|l_n\rangle\}$ can be obtained from the right eigenvectors $\{|r_n\rangle\}$ via a Hermitian transformation. Considering an arbitrary set of complete orthonormal bases $\{|n\rangle\}$, we have $(M^{-1})_{mn} = \langle r_n|r_m\rangle = \sum_{n'}\langle r_n|n'\rangle\langle n'|r_m\rangle$. By defining a matrix A, whose matrix elements satisfy $A_{mn} = \langle n|r_m\rangle$, or $\langle r_m|n\rangle = (A)^*_{mn} = (A^\dagger)_{nm}$, it is easy to prove that $M^{-1} = (A^*A^T)^T$ and thus

$$M = A^{\dagger-1}A^{-1}, \tag{B5}$$

and by using this definition we can determine the relationship between the basis vectors, because

$$|m\rangle = \sum_n |l_n\rangle\langle r_n|m\rangle = \sum_n (A^\dagger)_{mn}|l_n\rangle, \tag{B6}$$

whose matrix form is written

$$|n\rangle = A^\dagger|l_n\rangle. \tag{B7}$$

Apparently we will have $|l_n\rangle = A^{\dagger-1}|n\rangle$ and $|r_n\rangle = A|n\rangle$. As a result, we obtain Eq. (7) from Eqs. (B5) and (B1). The proof presented above is applicable to the complex spectrum $H_{\rm NH}$ as well, as it does not rely on the eigenvalues. Based on the characteristic equations $H_{\rm NH}|r_n\rangle = E_n|r_n\rangle$ and $H_{\rm NH}|l_n\rangle = E_n|l_n\rangle$, we can derive the following equations: $A^{-1}H_{\rm NH}A|n\rangle = E_n|n\rangle$ and $A^\dagger H^\dagger_{\rm NH}A^{\dagger-1}|n\rangle = E_n|n\rangle$. Take $H_H = A^{-1}H_{\rm NH}A = H^\dagger_H$; it can be proven that $H_{\rm NH}$ becomes a Hermitian operator H_H under a nonunitary transformation, and vice versa.

Additionally, for a non-Hermitian $H_{\rm NH}$ with a complex energy spectrum $\{E_n\}$, we can always find two Hermitian Hamiltonians $H_1 = H_1^\dagger$ and $H_2 = H_2^\dagger$ which satisfy $H_{\rm NH} = H_1 + iH_2$. We still have two cases such as $H_{\rm NH1} : [H_1, H_2]=0$ and $H_{\rm NH2} : [H_1, H_2]\neq 0$. For the former situation, despite the fact that the $H_{\rm NH}$ is non-Hermitian, H_1 and H_2 share common orthonormal basis $\{|n\rangle\}$. For the latter case, it can be proved that $H_{\rm NH2}$ is of an equivalence relation with $H_{\rm NH1}$. According to Eq. (B5), we have $|l_n\rangle = M|r_n\rangle$ and $A^\dagger|l_n\rangle = A^{-1}|r_n\rangle = |n\rangle$. $\{|n\rangle\}$ is a set of orthonormal bases which relate to a non-Hermitian Hamiltonian $H_{\rm NH1}$. Therefore, it yields $A^{-1}H_{\rm NH2}A|n\rangle = E_n|n\rangle$ and $A^\dagger H^\dagger_{\rm NH2}A^{\dagger-1}|n\rangle = E^*_n|n\rangle$, which indicates that $H_{\rm NH1} = A^{-1}H_{\rm NH2}A$ and $H_{\rm NH2}$ are linked through a nonunitary transformation.

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