General method for solving electromagnetic radiation problems in an arbitrary linear medium

Farhang Loran [1,*] and Ali Mostafazadeh [2,3,†]

[1] *Department of Physics, Isfahan University of Technology, Isfahan 84156-83111, Iran*
[2] *Department of Mathematics and Department of Physics, Koç University, 34450 Sarıyer, Istanbul, Türkiye*
[3] *TÜBİTAK Research Institute for Fundamental Sciences, 41470 Gebze, Kocaeli, Türkiye*

(Received 25 May 2023; accepted 11 August 2023; published 15 September 2023)

Numerical transfer matrices have been widely used in the study of wave propagation and scattering. These may be viewed as discretizations of a recently introduced fundamental notion of transfer matrix which admits a representation in terms of the evolution operator for an effective nonunitary quantum system. We use the fundamental transfer matrix to develop a general method for the solution of the problem of radiation of an oscillating source in an arbitrary, possibly nonhomogeneous, anisotropic, and active or lossy linear medium. This allows us to obtain an analytic solution to this problem for an oscillating source located in the vicinity of a planar collection of possibly anisotropic and active or lossy point scatterers such as those modeling a two-dimensional photonic crystal.

DOI: 10.1103/PhysRevA.108.032211

I. INTRODUCTION

Electromagnetic radiation of an oscillating source is a physical phenomenon of great importance. By definition, a system of charges and currents radiates if it generates waves reaching spatial infinities, i.e., if they are detectable by detectors located far away from the source [1]. This is in contrast to the basic setup for a scattering problem, where not only the detectors but also the source of the wave reside at spatial infinities [2]. A more realistic situation is when the waves generated by a source interact with nearby scatterers before reaching the detectors. The purpose of this article is to develop a general method of dealing with this problem which is particularly effective for the description of the effects of the point scatterers on the emitted radiation.

The term "point scatterer" refers to an interaction with a negligibly small (zero) range [3]. The best-known examples are the interactions modeled by δ-function potentials. These have been extensively studied since the 1930s [4–12]. In one dimension, they provide useful exactly solvable toy models with interesting physical applications [13,14]. In two and higher dimensions, their standard treatment leads to divergent terms whose removal requires a coupling-constant renormalization [15–22]. The same problem arises in the study of the scattering of electromagnetic waves by δ-function permittivity profiles and leads to more serious complications even when they are isotropic [23,24].

Recently, we have developed an alternative approach to the scattering of scalar and electromagnetic waves which avoids the singularities of the standard treatment of point scatterers provided that they lie along a line in two dimensions and on a plane in three dimensions [25–27]. This approach is based on a fundamental notion of transfer matrix which, unlike the transfer matrices employed in earlier publications [28–30], allows for performing analytic calculations. This has so far led to the discovery of exact broadband unidirectional invisibility in two dimensions [31] and the construction of potentials for which the first Born approximation is exact [32]. See also [33]. These developments together with the remarkable effectiveness of the fundamental transfer matrix in dealing with point scatterers provide the basic motivation for exploring its utility in dealing with radiation problems.

The outline of this article is as follows. In Sec. II, we give the definition of the fundamental transfer matrix for a general (possibly nonhomogeneous, anisotropic, active, or lossy) stationary linear medium that contains an oscillating localized distribution of charges and currents. In Sec. III, we discuss the application of the fundamental transfer matrix to address the radiation problem for this setup. In Sec. IV, we address the problem of radiation of an oscillating source in the presence of a finite planar array of nonmagnetic point interactions. In Sec. V we confine our attention to the case where the source is a perfect dipole. Here we also explore in some detail the special case where the radiation of the dipole is affected by the presence of a single point scatterer. In Sec. VI we present our concluding remarks.

II. FUNDAMENTAL TRANSFER MATRIX FOR ELECTROMAGNETIC WAVES

Consider a stationary linear medium that contains an oscillating localized distribution of charges and currents (the source). Let $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ denote the permittivity and permeability tensors of the medium, ε_0 and μ_0 be the permittivity and permeability of vacuum, and ω be the angular frequency of the source. Then $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ are 3×3 matrix-valued functions of space, and we can respectively express the free charge and

*loran@iut.ac.ir
†amostafazadeh@ku.edu.tr

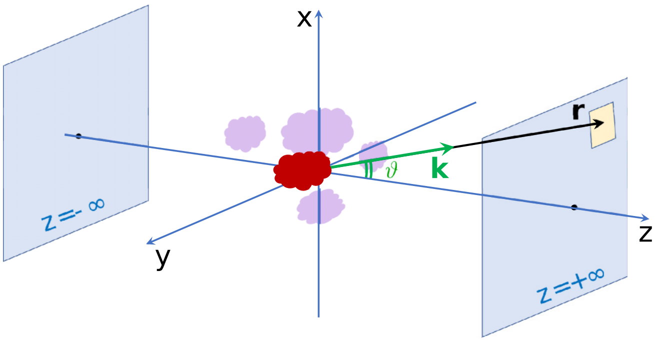

FIG. 1. Schematic view of the setup for the radiation of an oscillating source (in red) in a linear medium containing scatterers (nonhomogeneous, anisotropic, active or lossy regions marked in purple). The detectors are placed on the planes $z = \pm\infty$. $\mathbf{r}$ is the position of a detector's screen (in yellow) that lies on the plane $z = +\infty$. $\mathbf{k}$ is the wave vector for the detected wave. ϑ is the polar angle of the spherical coordinates.

current densities of the source and the electric and magnetic fields of the generated wave as

$$\rho(\mathbf{r},t) = \sqrt{\varepsilon_0}\, e^{-i\omega t}\varrho(\mathbf{r}), \quad \mathbf{J}(\mathbf{r},t) = \frac{e^{-i\omega t}\boldsymbol{\mathcal{J}}(\mathbf{r})}{\sqrt{\mu_0}}, \tag{1}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{e^{-i\omega t}\boldsymbol{\mathcal{E}}(\mathbf{r})}{\sqrt{\varepsilon_0}}, \quad \mathbf{H}(\mathbf{r},t) = \frac{e^{-i\omega t}\boldsymbol{\mathcal{H}}(\mathbf{r})}{\sqrt{\mu_0}}, \tag{2}$$

where $\mathbf{r}$ stands for the position vector, ϱ is a scalar function, and $\boldsymbol{\mathcal{J}}$, $\boldsymbol{\mathcal{E}}$, and $\boldsymbol{\mathcal{H}}$ are vector-valued functions. In terms of these, Maxwell's equations [1] take the form

$$\nabla\cdot(\hat{\boldsymbol{\varepsilon}}\,\boldsymbol{\mathcal{E}}) = \varrho, \quad \nabla\cdot(\hat{\boldsymbol{\mu}}\,\boldsymbol{\mathcal{H}}) = 0, \tag{3}$$

$$\nabla\times\boldsymbol{\mathcal{E}} = ik\hat{\boldsymbol{\mu}}\,\boldsymbol{\mathcal{H}}, \quad \nabla\times\boldsymbol{\mathcal{H}} = -ik\hat{\boldsymbol{\varepsilon}}\,\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{J}}, \tag{4}$$

where $\hat{\boldsymbol{\varepsilon}}(\mathbf{r}) := \varepsilon_0^{-1}\boldsymbol{\varepsilon}(\mathbf{r})$ and $\hat{\boldsymbol{\mu}}(\mathbf{r}) := \mu_0^{-1}\boldsymbol{\mu}(\mathbf{r})$ are, respectively, the relative permittivity and permeability tensors, $k := \omega/c$ is the wave number, and $c := (\varepsilon_0\mu_0)^{-1/2}$ is the speed of light in vacuum.[1]

Reference [27] defines the fundamental transfer matrix for a general linear medium in the absence of free charges and currents. We wish to extend this definition to linear media containing an oscillating source. To do this, we choose our coordinate system in such a way that the detectors measuring the radiation lie on the planes defined by $z = \pm\infty$, as depicted in Fig. 1. We also suppose that the last diagonal entries of $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\mu}}$ do not vanish; $\hat{\varepsilon}_{33} \neq 0 \neq \hat{\mu}_{33}$. This is a technical condition which we can satisfy with a proper choice of our coordinate system for nonexotic media.

In the following, we denote the zero and identity matrices of all sizes by $\mathbf{0}$ and $\mathbf{I}$, respectively, and use $\mathcal{E}_u$, $\mathcal{H}_u$, and $\mathcal{J}_u$, with $u \in \{x,y,z\}$, to denote the components of $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{H}}$, and $\boldsymbol{\mathcal{J}}$, i.e.,

$$\boldsymbol{\mathcal{E}} = \mathcal{E}_x\mathbf{e}_x + \mathcal{E}_y\mathbf{e}_y + \mathcal{E}_z\mathbf{e}_z, \quad \boldsymbol{\mathcal{H}} = \mathcal{H}_x\mathbf{e}_x + \mathcal{H}_y\mathbf{e}_y + \mathcal{H}_z\mathbf{e}_z,$$
$$\boldsymbol{\mathcal{J}} = \mathcal{J}_x\mathbf{e}_x + \mathcal{J}_y\mathbf{e}_y + \mathcal{J}_z\mathbf{e}_z,$$

where $\mathbf{e}_u$ is the unit vector pointing along the u axis. We also introduce the following quantities:[2]

$$\vec{\mathcal{E}} := \begin{bmatrix}\mathcal{E}_x\\ \mathcal{E}_y\end{bmatrix}, \quad \vec{\mathcal{H}} := \begin{bmatrix}\mathcal{H}_x\\ \mathcal{H}_y\end{bmatrix}, \quad \vec{\mathcal{J}} := \begin{bmatrix}\mathcal{J}_x\\ \mathcal{J}_y\end{bmatrix}, \quad \vec{\partial} := \begin{bmatrix}\partial_x\\ \partial_y\end{bmatrix}, \tag{5}$$

$$\vec{K}_{\mathcal{E}} := \begin{bmatrix}-\hat{\varepsilon}_{23}\\ \hat{\varepsilon}_{13}\end{bmatrix}, \quad \mathbf{K}_{\mathcal{E}} := \begin{bmatrix}-\hat{\varepsilon}_{21} & -\hat{\varepsilon}_{22}\\ \hat{\varepsilon}_{11} & \hat{\varepsilon}_{12}\end{bmatrix} = \begin{bmatrix}-\vec{\varepsilon}_2^{\,T}\\ \vec{\varepsilon}_1^{\,T}\end{bmatrix}, \tag{6}$$

$$\vec{K}_{\mathcal{H}} := \begin{bmatrix}-\hat{\mu}_{23}\\ \hat{\mu}_{13}\end{bmatrix}, \quad \mathbf{K}_{\mathcal{H}} := \begin{bmatrix}-\hat{\mu}_{21} & -\hat{\mu}_{22}\\ \hat{\mu}_{11} & \hat{\mu}_{12}\end{bmatrix} = \begin{bmatrix}-\vec{\mu}_2^{\,T}\\ \vec{\mu}_1^{\,T}\end{bmatrix}, \tag{7}$$

where the superscript T stands for the transpose of the corresponding matrix and

$$\vec{\varepsilon}_\ell := \begin{bmatrix}\hat{\varepsilon}_{\ell 1}\\ \hat{\varepsilon}_{\ell 2}\end{bmatrix}, \quad \vec{\mu}_\ell := \begin{bmatrix}\hat{\mu}_{\ell 1}\\ \hat{\mu}_{\ell 2}\end{bmatrix}, \quad \ell \in \{1,2,3\}.$$

We begin our analysis by using (4) to express $\mathcal{E}_z$ and $\mathcal{H}_z$ in the form

$$\begin{aligned}\mathcal{E}_z &= \hat{\varepsilon}_{33}^{-1}[-\hat{\varepsilon}_{31}\mathcal{E}_x - \hat{\varepsilon}_{32}\mathcal{E}_y + ik^{-1}(\partial_x\mathcal{H}_y - \partial_y\mathcal{H}_x - \mathcal{J}_z)]\\ &= -\hat{\varepsilon}_{33}^{-1}\big(\vec{\varepsilon}_3^{\,T}\vec{\mathcal{E}} + k^{-1}\vec{\partial}^{\,T}\boldsymbol{\sigma}_2\vec{\mathcal{H}} + ik^{-1}\mathcal{J}_z\big),\end{aligned} \tag{8}$$

$$\begin{aligned}\mathcal{H}_z &= \hat{\mu}_{33}^{-1}[-ik^{-1}(\partial_x\mathcal{E}_y - \partial_y\mathcal{E}_x) - \hat{\mu}_{31}\mathcal{H}_x - \hat{\mu}_{32}\mathcal{H}_y]\\ &= \hat{\mu}_{33}^{-1}\big(k^{-1}\vec{\partial}^{\,T}\boldsymbol{\sigma}_2\vec{\mathcal{E}} - \vec{\mu}_3^{\,T}\vec{\mathcal{H}}\big).\end{aligned} \tag{9}$$

With the help of (8) and (9), we can reduce (4) to a system of first-order differential equations for $\mathcal{E}_x, \mathcal{E}_y, \mathcal{H}_x$, and $\mathcal{H}_y$. This is equivalent to the nonhomogeneous time-dependent Schrödinger equation,

$$i\partial_z\boldsymbol{\Phi} = \widehat{\mathbf{H}}\,\boldsymbol{\Phi} + \mathbf{J}, \tag{10}$$

for the four-component field [27,34],

$$\boldsymbol{\Phi} := \begin{bmatrix}\mathcal{E}_x\\ \mathcal{E}_y\\ \mathcal{H}_x\\ \mathcal{H}_y\end{bmatrix} = \begin{bmatrix}\vec{\mathcal{E}}\\ \vec{\mathcal{H}}\end{bmatrix}, \tag{11}$$

where z plays the role of "time,"

$$\widehat{\mathbf{H}} := \begin{bmatrix}\widehat{\mathbf{H}}_{11} & \widehat{\mathbf{H}}_{12}\\ \widehat{\mathbf{H}}_{21} & \widehat{\mathbf{H}}_{22}\end{bmatrix}, \quad \mathbf{J} := \begin{bmatrix}k^{-1}\vec{\boldsymbol{\partial}}(\hat{\varepsilon}_{33}^{-1}\mathcal{J}_z)\\ i\hat{\varepsilon}_{33}^{-1}\mathcal{J}_z\vec{K}_{\mathcal{E}} - \boldsymbol{\sigma}_2\vec{\mathcal{J}}\end{bmatrix}, \tag{12}$$

$$\begin{aligned}\widehat{\mathbf{H}}_{11} &:= -i\vec{\partial}\,\frac{\vec{\varepsilon}_3^{\,T}}{\hat{\varepsilon}_{33}} + \frac{1}{\hat{\mu}_{33}}\vec{K}_{\mathcal{H}}\vec{\partial}^{\,T}\boldsymbol{\sigma}_2,\\ \widehat{\mathbf{H}}_{12} &:= -\frac{i}{k}\vec{\partial}\,\frac{1}{\hat{\varepsilon}_{33}}\vec{\partial}^{\,T}\boldsymbol{\sigma}_2 + k(\mathbf{K}_{\mathcal{H}} - \check{\mathbf{K}}_{\mathcal{H}}),\end{aligned} \tag{13}$$

---

[1] According to the continuity equation for the local charge conservation, which follows from the first equation in (3) and the second equation in (4), we have $\varrho = -ik^{-1}\nabla\cdot\boldsymbol{\mathcal{J}}$. Therefore, $\boldsymbol{\mathcal{J}}$ characterizes the source.

[2] In Ref. [27] we used $\vec{J}_{\mathcal{E}}, \mathbf{J}_{\mathcal{E}}, \vec{J}_{\mathcal{H}}$, and $\mathbf{J}_{\mathcal{H}}$ for what we call $\vec{K}_{\mathcal{E}}, \mathbf{K}_{\mathcal{E}}, \vec{K}_{\mathcal{H}}$, and $\mathbf{K}_{\mathcal{H}}$. This change in notation has been made to avoid giving the impression that these quantities are related to the current density $\mathbf{J}$.

$$\widehat{\mathbf{H}}_{21} := \frac{i}{k}\vec{\partial}\,\frac{1}{\hat{\mu}_{33}}\vec{\partial}^{\,T}\boldsymbol{\sigma}_2 + k(\check{\mathbf{K}}_{\mathcal{E}} - \mathbf{K}_{\mathcal{E}}),$$
$$\widehat{\mathbf{H}}_{22} := -i\vec{\partial}\frac{\vec{\mu}_3^{\,T}}{\hat{\mu}_{33}} + \frac{1}{\hat{\varepsilon}_{33}}\vec{K}_{\mathcal{E}}\vec{\partial}^{\,T}\boldsymbol{\sigma}_2, \qquad (14)$$

$$\check{\mathbf{K}}_{\mathcal{E}} := \frac{1}{\hat{\varepsilon}_{33}}\vec{K}_{\mathcal{E}}\vec{\varepsilon}_3^{\,T}, \quad \check{\mathbf{K}}_{\mathcal{H}} := \frac{1}{\hat{\mu}_{33}}\vec{K}_{\mathcal{H}}\vec{\mu}_3^{\,T}, \qquad (15)$$

and $\vec{\partial}$ and $\vec{\partial}^T$ act on all the terms appearing to their right.[3] According to (12)–(15), $\widehat{\mathbf{H}}$ is a "time-dependent" 4×4 matrix Hamiltonian with operator entries which represents the interaction of the electromagnetic waves with the medium, $\widehat{\mathbf{H}}_{ij}$ are the 2×2 blocks of $\widehat{\mathbf{H}}$, and $\mathbf{J}$ is a four-component function that contains the information about the source.

Because z plays a different role than x and y, we denote the projection of $\mathbf{r}$ onto the x-y plane by $\vec{r}$, i.e., set $\vec{r} := (x, y)$, and write $\boldsymbol{\Phi}(\mathbf{r})$ as $\boldsymbol{\Phi}(\vec{r}, z)$. This allows us to view $\boldsymbol{\Phi}(\cdot, z)$ as a function that maps $\mathbb{R}^2$ to $\mathbb{C}^{4\times 1}$, where $\mathbb{C}^{m\times n}$ denotes the set of $m \times n$ complex matrices. The Hamiltonian operator $\widehat{\mathbf{H}}$ is actually a z-dependent linear operator acting in the space of such functions. Equation (10) determines the dynamics in this space.

The four-component function $\boldsymbol{\Phi}(\cdot, z)$ plays the role of a position-space wave function in quantum mechanics. In the following, we employ the corresponding momentum-space wave function $\tilde{\boldsymbol{\Phi}}(\cdot, z)$, which is related to $\boldsymbol{\Phi}(\cdot, z)$ by the two-dimensional Fourier transformation $\mathcal{F}$:

$$\boldsymbol{\Phi}(\cdot, z) \xrightarrow{\mathcal{F}} \mathcal{F}\boldsymbol{\Phi}(\cdot, z) := \tilde{\boldsymbol{\Phi}}(\cdot, z),$$

where for all $\vec{p} := (p_x, p_y) \in \mathbb{R}^2$,

$$\tilde{\boldsymbol{\Phi}}(\vec{p}, z) := \int_{\mathbb{R}^2} d^2r\, e^{-i\vec{p}\cdot\vec{r}}\boldsymbol{\Phi}(\vec{r}, z)$$
$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-i(xp_x+yp_y)}\boldsymbol{\Phi}(x, y, z), \qquad (16)$$

and a dot stands for the dot product, i.e., $\vec{p}\cdot\vec{r} := xp_x + yp_y$. Applying $\mathcal{F}$ to both sides of (10) and evaluating the resulting equation at $(\vec{p}, z)$, we find

$$i\partial_z\tilde{\boldsymbol{\Phi}}(\vec{p}, z) = \widehat{\tilde{\mathbf{H}}}\,\tilde{\boldsymbol{\Phi}}(\vec{p}, z) + \tilde{\mathbf{J}}(\vec{p}, z), \qquad (17)$$

where $\widehat{\tilde{\mathbf{H}}} := \mathcal{F}\,\widehat{\mathbf{H}}\,\mathcal{F}^{-1}$, $\mathcal{F}^{-1}$ stands for the inverse Fourier transformation in two dimensions,[4] and $\tilde{\mathbf{J}}(\cdot, z) := \mathcal{F}\mathbf{J}(\cdot, z)$.[5] If we denote the space of d-component complex-valued functions of $\vec{p}$ by $\mathscr{F}^d$, so that $\tilde{\boldsymbol{\Phi}}(\cdot, z) \in \mathscr{F}^4$, we can view (17) as a dynamical equation in $\mathscr{F}^4$.

Consider an electromagnetic wave propagating in vacuum, so that $\hat{\boldsymbol{\varepsilon}} = \hat{\boldsymbol{\mu}} = \mathbf{I}$ and $\mathbf{J} = \mathbf{0}$, and let $\tilde{\boldsymbol{\Phi}}_0$ denote the corresponding momentum-space four-component wave function. Then (17) becomes

$$i\partial_z\tilde{\boldsymbol{\Phi}}_0(\vec{p}, z) = \tilde{\mathbf{H}}_0(\vec{p})\tilde{\boldsymbol{\Phi}}_0(\vec{p}, z), \qquad (18)$$

where

$$\tilde{\mathbf{H}}_0(\vec{p}) := \begin{bmatrix} \mathbf{0} & \tilde{\mathbf{L}}_0(\vec{p}) \\ -\tilde{\mathbf{L}}_0(\vec{p}) & \mathbf{0} \end{bmatrix},$$
$$\tilde{\mathbf{L}}_0(\vec{p}) := \frac{1}{k}\begin{bmatrix} -p_xp_y & p_x^2 - k^2 \\ -p_y^2 + k^2 & p_xp_y \end{bmatrix}. \qquad (19)$$

Because $\widehat{\tilde{\mathbf{H}}}_0(\vec{p})$ does not depend on z, we can write the general solution of (18) in the form

$$\tilde{\boldsymbol{\Phi}}_0(\vec{p}, z) = e^{-iz\tilde{\mathbf{H}}_0(\vec{p})}\boldsymbol{\mathcal{C}}(\vec{p}), \qquad (20)$$

where $\boldsymbol{\mathcal{C}} \in \mathscr{F}^4$ is arbitrary. To obtain a more explicit expression for the right-hand side of (20), we first note that

$$\tilde{\mathbf{L}}_0(\vec{p})^2 = -(k^2 - \vec{p}^{\,2})\mathbf{I}, \qquad (21)$$

which in turn implies

$$\tilde{\mathbf{H}}_0(\vec{p})^2 = (k^2 - \vec{p}^{\,2})\mathbf{I}. \qquad (22)$$

For $|\vec{p}| = k$, this becomes $\tilde{\mathbf{H}}_0(\vec{p})^2 = \mathbf{0}$, and we have

$$e^{-iz\tilde{\mathbf{H}}_0(\vec{p})} = \mathbf{I} - iz\tilde{\mathbf{H}}_0(\vec{p}). \qquad (23)$$

For $|\vec{p}| \neq k$, we can expand $e^{-iz\tilde{\mathbf{H}}_0(\vec{p})}$ in powers of $z\tilde{\mathbf{H}}_0(\vec{p})$ and use (22) to show that

$$e^{-iz\tilde{\mathbf{H}}_0(\vec{p})} = \cos[z\varpi(\vec{p})]\,\mathbf{I} - \frac{i\sin[z\varpi(\vec{p})]}{\varpi(\vec{p})}\tilde{\mathbf{H}}_0(\vec{p}), \qquad (24)$$

where $\varpi : \mathbb{R}^2 \to \mathbb{C}$ is the function defined by

$$\varpi(\vec{p}) := \begin{cases} \sqrt{k^2 - \vec{p}^{\,2}} & \text{for} \quad |\vec{p}| < k, \\ i\sqrt{\vec{p}^{\,2} - k^2} & \text{for} \quad |\vec{p}| \geqslant k. \end{cases} \qquad (25)$$

In the limit $|\vec{p}| \to k$, $\varpi(\vec{p})$ tends to zero, and (24) reproduces (23). Therefore, we can determine the value of $\tilde{\boldsymbol{\Phi}}_0(\vec{p}, z)$ for $|\vec{p}| = k$ by evaluating its $|\vec{p}| \to k$ limit. This observation allows us to confine our attention to $|\vec{p}| \neq k$, where $\tilde{\mathbf{H}}_0(\vec{p})$ is a diagonalizable matrix. In other words, without loss of generality we can restrict the domain of the definition of $\tilde{\boldsymbol{\Phi}}_0(\cdot, z)$ to $\mathbb{R}^2 \setminus S_k^1$, where $S_k^1 := \{\vec{p} \in \mathbb{R}^2 \,|\, |\vec{p}| = k\}$. Denoting the set of d-component functions defined on $\mathbb{R}^2 \setminus S_k^1$ by $\mathring{\mathscr{F}}_k^d$, we identify $\tilde{\boldsymbol{\Phi}}_0(\cdot, z)$ with an element of $\mathring{\mathscr{F}}_k^4$ and write (20) in the form

$$\tilde{\boldsymbol{\Phi}}_0(\cdot, z) = e^{-iz\widehat{\tilde{\mathbf{H}}}_0}\boldsymbol{\mathcal{C}}, \qquad (26)$$

where $\widehat{\tilde{\mathbf{H}}}_0 : \mathring{\mathscr{F}}_k^4 \to \mathring{\mathscr{F}}_k^4$ is the linear operator defined by $\widehat{\tilde{\mathbf{H}}}_0\mathbf{F}(\vec{p}) := \tilde{\mathbf{H}}_0(\vec{p})\mathbf{F}(\vec{p})$ and $\boldsymbol{\mathcal{C}} \in \mathring{\mathscr{F}}_k^4$ is arbitrary.

As we mentioned above, for $|\vec{p}| \neq k$, $\tilde{\mathbf{H}}_0(\vec{p})$ is a diagonalizable matrix. Equations (22) and (25) suggest that its spectrum consists of $\pm\varpi(\vec{p})$. We can construct the following projection matrices on its eigenspaces [27]:

$$\boldsymbol{\Pi}_j(\vec{p}) := \frac{1}{2}\left[\mathbf{I} + \frac{(-1)^j}{\varpi(\vec{p})}\tilde{\mathbf{H}}_0(\vec{p})\right]$$
$$= \frac{1}{2}\begin{bmatrix} \mathbf{I} & \frac{(-1)^j\tilde{\mathbf{L}}_0(\vec{p})}{\varpi(\vec{p})} \\ \frac{(-1)^{j+1}\tilde{\mathbf{L}}_0(\vec{p})}{\varpi(\vec{p})} & \mathbf{I} \end{bmatrix}, \qquad (27)$$

[3]For example, for every test function f, $\vec{\partial}\,\frac{\vec{\varepsilon}_3^{\,T}}{\hat{\varepsilon}_{33}}f$ stands for $\vec{\partial}(\frac{\vec{\varepsilon}_3^{\,T}}{\hat{\varepsilon}_{33}}f)$.

[4]Given a test function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{C}^{m\times n}$, $(\mathcal{F}^{-1}\mathbf{F})(\vec{r}) := \frac{1}{4\pi^2}\int_{\mathbb{R}^2} d^2p\, e^{i\vec{p}\cdot\vec{r}}\mathbf{F}(\vec{p})$.

[5]We can express $\widehat{\tilde{\mathbf{H}}}\tilde{\boldsymbol{\Phi}}$ in (17) as $\int_{\mathbb{R}^2} d^2q\, \boldsymbol{\mathcal{K}}(\vec{p}, \vec{q})\tilde{\boldsymbol{\Phi}}(\vec{q}, z)$, where $\boldsymbol{\mathcal{K}}(\vec{p}, \vec{q}) := \frac{1}{4\pi^2}\int_{\mathbb{R}^2} d^2r\, e^{-i\vec{p}\cdot\vec{r}}\widehat{\mathbf{H}}\, e^{i\vec{q}\cdot\vec{r}}$. Because $\widehat{\mathbf{H}}$ is a differential operator whose coefficients are functions of both $\vec{r}$ and z, the integral kernel $\boldsymbol{\mathcal{K}}(\cdot, \cdot)$ depends on z.

where $j \in \{1,2\}$. It is easy to check that

$$\tilde{\mathbf{H}}_0\boldsymbol{\Pi}_j(\vec{p}) = (-1)^j\varpi(\vec{p})\boldsymbol{\Pi}_j(\vec{p}), \tag{28}$$

$$\boldsymbol{\Pi}_i(\vec{p})\boldsymbol{\Pi}_j(\vec{p}) = \delta_{ij}\boldsymbol{\Pi}_j(\vec{p}), \tag{29}$$

$$\boldsymbol{\Pi}_1(\vec{p}) + \boldsymbol{\Pi}_2(\vec{p}) = \mathbf{I}, \tag{30}$$

where δ_{ij} stands for the Kronecker delta symbol. These in turn imply

$$e^{-iz\tilde{\mathbf{H}}_0(\vec{p})} = e^{iz\varpi(\vec{p})}\boldsymbol{\Pi}_1(\vec{p}) + e^{-iz\varpi(\vec{p})}\boldsymbol{\Pi}_2(\vec{p}). \tag{31}$$

Next, we let $\widehat{\varpi}, \widehat{\boldsymbol{\Pi}}_j, \widehat{\pi}_k : \mathring{\mathscr{F}}^4 \to \mathring{\mathscr{F}}^4$ be the operators defined by

$$(\widehat{\varpi}\,\mathbf{F})(\vec{p}) := \varpi(\vec{p})\mathbf{F}(\vec{p}), \tag{32}$$

$$(\widehat{\boldsymbol{\Pi}}_j\,\mathbf{F})(\vec{p}) := \boldsymbol{\Pi}_j(\vec{p})\mathbf{F}(\vec{p}), \tag{33}$$

$$(\widehat{\pi}_k\mathbf{F})(\vec{p}) := \begin{cases}\mathbf{F}(\vec{p}) & \text{for} \quad |\vec{p}| < k,\\ \mathbf{0} & \text{for} \quad |\vec{p}| > k,\end{cases} \tag{34}$$

where $\vec{p} \in \mathbb{R}^2 \setminus S_k^1$, $j \in \{1,2\}$ and $\mathbf{F} \in \mathring{\mathscr{F}}^4$. We can use (32)–(34) to establish the following identities:

$$[\widehat{\varpi}, \widehat{\boldsymbol{\Pi}}_j] = [\widehat{\varpi}, \widehat{\pi}_k] = [\widehat{\boldsymbol{\Pi}}_j, \widehat{\pi}_k] = \widehat{\mathbf{0}}. \tag{35}$$

Suppose that there are $a_\pm \in \mathbb{R}$, with $a_- < a_+$, such that the space outside the region bounded by the planes $z = a_\pm$ is empty. Then, $\hat{\boldsymbol{\varepsilon}}(\vec{r}, z) - \mathbf{I} = \hat{\boldsymbol{\mu}}(\vec{r}, z) - \mathbf{I} = \mathbf{J}(\vec{r}, z) = \mathbf{0}$ for $z \notin (a_-, a_+)$, and for every solution of (17) there are $\boldsymbol{\mathcal{C}}_\pm \in \mathring{\mathscr{F}}^4$ such that

$$\tilde{\boldsymbol{\Phi}}(\vec{p}, z) = \begin{cases} e^{iz\varpi(\vec{p})}\boldsymbol{\mathcal{A}}_-(\vec{p}) + e^{-iz\varpi(\vec{p})}\boldsymbol{\mathcal{B}}_-(\vec{p}) & \text{for} \quad z < a_-,\\ e^{iz\varpi(\vec{p})}\boldsymbol{\mathcal{A}}_+(\vec{p}) + e^{-iz\varpi(\vec{p})}\boldsymbol{\mathcal{B}}_+(\vec{p}) & \text{for} \quad z > a_+,\end{cases} \tag{36}$$

where

$$\boldsymbol{\mathcal{A}}_\pm := \widehat{\boldsymbol{\Pi}}_1\boldsymbol{\mathcal{C}}_\pm, \quad \boldsymbol{\mathcal{B}}_\pm := \widehat{\boldsymbol{\Pi}}_2\boldsymbol{\mathcal{C}}_\pm, \tag{37}$$

and for $\vec{p} = \vec{p}_0 \in S_k^1$, the right-hand side of (36) should be replaced by its $\vec{p} \to \vec{p}_0$ limit.[6]

If $\boldsymbol{\Phi}(\vec{r}, z)$ is a bounded function of z, the same applies to $\tilde{\boldsymbol{\Phi}}(\vec{p}, z)$. This condition together with Eqs. (25) and (36) and the facts that $\tilde{\boldsymbol{\Phi}}(\vec{p}, z)$ is a uniformly continuous function of $\vec{p}$ and $e^{\mp iz\varpi(\vec{p})}$ blows up for $|\vec{p}| > k$ as $z \to \pm\infty$ imply that

$$\boldsymbol{\mathcal{A}}_-(\vec{p}) = \boldsymbol{\mathcal{B}}_+(\vec{p}) = 0 \quad \text{for} \quad |\vec{p}| > k. \tag{38}$$

Let us introduce

$$\mathbf{C}_\pm := \widehat{\pi}_k\boldsymbol{\mathcal{C}}_\pm, \quad \mathbf{A}_\pm := \widehat{\pi}_k\boldsymbol{\mathcal{A}}_\pm = \widehat{\boldsymbol{\Pi}}_1\mathbf{C}_\pm,$$
$$\mathbf{B}_\pm := \widehat{\pi}_k\boldsymbol{\mathcal{B}}_\pm = \widehat{\boldsymbol{\Pi}}_2\mathbf{C}_\pm. \tag{39}$$

Then, by virtue of (29), (33), (34), and (39),

$$\widehat{\boldsymbol{\Pi}}_1\mathbf{A}_\pm = \mathbf{A}_\pm, \quad \widehat{\boldsymbol{\Pi}}_2\mathbf{A}_\pm = \widehat{\boldsymbol{\Pi}}_1\mathbf{B}_\pm = \mathbf{0}, \quad \widehat{\boldsymbol{\Pi}}_2\mathbf{B}_\pm = \mathbf{B}_\pm, \tag{40}$$

[6]In view of (28) and (37), $\boldsymbol{\mathcal{A}}_\pm(\vec{p})$ and $\boldsymbol{\mathcal{B}}_\pm(\vec{p})$ are, respectively, eigenvectors of $\tilde{\mathbf{H}}_0(\vec{p})$ with eigenvalues $-\varpi(\vec{p})$ and $\varpi(\vec{p})$.

Eqs. (30) and (38) imply

$$\mathbf{A}_\pm + \mathbf{B}_\pm = \mathbf{C}_\pm, \tag{41}$$

$$\boldsymbol{\mathcal{A}}_- = \mathbf{A}_-, \quad \boldsymbol{\mathcal{B}}_+ = \mathbf{B}_+, \tag{42}$$

and we can use (35), (36), and (37) to show that

$$\tilde{\boldsymbol{\Phi}}(\cdot, z) \to e^{iz\widehat{\varpi}}\mathbf{A}_\pm + e^{-iz\widehat{\varpi}}\mathbf{B}_\pm = e^{-iz\widehat{\tilde{\mathbf{H}}}_0}\mathbf{C}_\pm \quad \text{for} \quad z \to \pm\infty. \tag{43}$$

In particular, the four-component function defined by $\boldsymbol{\Psi}(\cdot, z) := e^{iz\widehat{\tilde{\mathbf{H}}}_0}\tilde{\boldsymbol{\Phi}}(\cdot, z)$ satisfies

$$\boldsymbol{\Psi}(\cdot, z) \to \mathbf{C}_\pm \quad \text{for} \quad z \to \pm\infty. \tag{44}$$

Because $\widehat{\tilde{\mathbf{H}}}_0$ describes the propagation of waves in the absence of interactions, $\boldsymbol{\Psi}(\cdot, z)$ plays the role of the interaction-picture momentum-space wave functions [35]. Expressing $\tilde{\boldsymbol{\Phi}}(\vec{p}, z)$ in terms of $\boldsymbol{\Psi}(\vec{p}, z)$ and substituting the result in (17), we find

$$i\partial_z\boldsymbol{\Psi}(\cdot, z) = \widehat{\boldsymbol{\mathcal{H}}}\boldsymbol{\Psi}(\cdot, z) + \boldsymbol{\mathfrak{J}}(\cdot, z), \tag{45}$$

where

$$\widehat{\boldsymbol{\mathcal{H}}}(z) := e^{iz\widehat{\tilde{\mathbf{H}}}_0}(\widehat{\tilde{\mathbf{H}}} - \widehat{\tilde{\mathbf{H}}}_0)e^{-iz\widehat{\tilde{\mathbf{H}}}_0}, \quad \boldsymbol{\mathfrak{J}}(\cdot, z) := e^{iz\widehat{\tilde{\mathbf{H}}}_0}\tilde{\mathbf{J}}(\cdot, z). \tag{46}$$

We can express the general solution of the nonhomogeneous Schrödinger equation (45) in terms of the evolution operator for the corresponding homogeneous equation, namely, the operator $\widehat{\boldsymbol{\mathcal{U}}}(z, z_0)$ satisfying $i\partial_z\widehat{\boldsymbol{\mathcal{U}}}(z, z_0) = \widehat{\boldsymbol{\mathcal{H}}}(z)\widehat{\boldsymbol{\mathcal{U}}}(z, z_0)$ and $\widehat{\boldsymbol{\mathcal{U}}}(z_0, z_0) = \widehat{\mathbf{I}}$ for all $z, z_0 \in \mathbb{R}$. This gives

$$\boldsymbol{\Psi}(\cdot, z) = \widehat{\boldsymbol{\mathcal{U}}}(z, z_0)\left[\boldsymbol{\Psi}(\cdot, z_0) - i\int_{z_0}^{z} dz'\,\widehat{\boldsymbol{\mathcal{U}}}(z_0, z')\boldsymbol{\mathfrak{J}}(\cdot, z')\right], \tag{47}$$

where we have made use of $\widehat{\boldsymbol{\mathcal{U}}}(z', z_0)^{-1} = \widehat{\boldsymbol{\mathcal{U}}}(z_0, z')$. We also recall that $\widehat{\boldsymbol{\mathcal{U}}}(z, z_0)$ admits the Dyson series expansion [35]:

$$\widehat{\boldsymbol{\mathcal{U}}}(z, z_0) = \widehat{\mathbf{I}} + \sum_{\ell=1}^{\infty}(-i)^\ell\int_{z_0}^{z} dz_\ell\int_{z_0}^{z_\ell} dz_{\ell-1}\cdots$$
$$\times\int_{z_0}^{z_2} dz_1\,\widehat{\boldsymbol{\mathcal{H}}}(z_\ell)\widehat{\boldsymbol{\mathcal{H}}}(z_{\ell-1})\cdots\widehat{\boldsymbol{\mathcal{H}}}(z_1). \tag{48}$$

Following Ref. [27], we define the fundamental transfer matrix of the medium according to

$$\widehat{\mathbf{M}} := \widehat{\pi}_k\,\widehat{\boldsymbol{\mathcal{U}}}(+\infty, -\infty)\,\widehat{\pi}_k. \tag{49}$$

This is a linear operator acting in $\mathscr{F}_k^4 := \{\mathbf{F} \in \mathscr{F}^4 \,|\, \mathbf{F}(\vec{p}) = \mathbf{0} \text{ for } |\vec{p}| \geqslant k\}$. In view of (39), (44), (47), and (49),

$$\mathbf{C}_+ = \widehat{\mathbf{M}}\,\mathbf{C}_- + \mathbf{D}, \tag{50}$$

where

$$\mathbf{D} := -i\widehat{\pi}_k\int_{-\infty}^{\infty} dz'\,\widehat{\boldsymbol{\mathcal{U}}}(+\infty, z')\boldsymbol{\mathfrak{J}}(\cdot, z'), \tag{51}$$

and we have benefited from the identity $\widehat{\boldsymbol{\mathcal{U}}}(+\infty, -\infty)\,\widehat{\boldsymbol{\mathcal{U}}}(-\infty, z') = \widehat{\boldsymbol{\mathcal{U}}}(+\infty, z')$.

III. RADIATION BY AN OSCILLATING SOURCE IN A LINEAR MEDIUM

If the source is localized or more generally confined to the region bounded by the planes $z = a_\pm$, the emitted wave satisfies the outgoing asymptotic boundary condition. To quantify this condition, we examine the behavior of the four-component fields $\mathbf{\Phi}$ for $z \to \pm\infty$. Performing the inverse Fourier transform of both sides of (43), we find

$$\mathbf{\Phi}(\vec{r}, z) \to \frac{1}{4\pi^2} \int_{\mathscr{D}_k} d^2p\, e^{i\vec{p}\cdot\vec{r}}\big[\mathbf{A}_\pm(\vec{p})e^{i\varpi(\vec{p})z} + \mathbf{B}_\pm(\vec{p})e^{-i\varpi(\vec{p})z}\big] \text{ for } z \to \pm\infty, \tag{52}$$

where $\mathscr{D}_k := \{\vec{p} \in \mathbb{R}^2 \,|\, |\vec{p}| < k\}$. This identifies $\mathbf{A}_\pm$ and $\mathbf{B}_\pm$ with the Fourier coefficients of the right- and left-going waves along the z axis, respectively. Therefore, the outgoing asymptotic boundary condition corresponds to $\mathbf{A}_- = \mathbf{B}_+ = \mathbf{0}$. In view of (41), we can also write it in the form $\mathbf{C}_- = \mathbf{B}_-$ and $\mathbf{C}_+ = \mathbf{A}_+$. Substituting these equations in (40), we have

$$\widehat{\mathbf{\Pi}}_2\mathbf{C}_- = \mathbf{C}_-, \quad \widehat{\mathbf{\Pi}}_1\mathbf{C}_- = \widehat{\mathbf{\Pi}}_2\mathbf{C}_+ = \mathbf{0}, \quad \widehat{\mathbf{\Pi}}_1\mathbf{C}_+ = \mathbf{C}_+. \tag{53}$$

Next, we apply $\widehat{\mathbf{\Pi}}_j$ to both sides of (50) and use (53) to establish

$$\widehat{\mathbf{\Pi}}_2\widehat{\mathbf{M}}\,\mathbf{C}_- = -\widehat{\mathbf{\Pi}}_2\mathbf{D}, \tag{54}$$

$$\mathbf{C}_+ = \widehat{\mathbf{\Pi}}_1(\widehat{\mathbf{M}} - \widehat{\mathbf{I}})\mathbf{C}_- + \widehat{\mathbf{\Pi}}_1\mathbf{D}. \tag{55}$$

These are linear equations for $\mathbf{C}_\pm$ which are to be solved in $\mathscr{F}_k^4$.

Equations (54) and (55) have the same structure as Eqs. (57) and (58) of Ref. [27] for the four-component fields $\mathbf{T}^l_\pm$ which determine the asymptotic expression for the scattered waves. The only difference is that in the scattering setup considered in [27], $\mathbf{D}$ is determined by the incident wave whose source resides at $z = -\infty$. This suggests that we can pursue the approach of Refs. [27,36] to express the scaled electric field $\boldsymbol{\mathcal{E}}$ for the electromagnetic wave reaching the detectors placed at $z = \pm\infty$ as follows:

$$\boldsymbol{\mathcal{E}}(\mathbf{r}) = \frac{k|\cos\vartheta|e^{ikr}}{2\pi i r}\,\mathbf{\Xi}(\vartheta,\varphi)^T\mathbf{C}_\pm(\vec{k}) \quad \text{for } r\cos\vartheta \to \pm\infty, \tag{56}$$

where r, ϑ, and φ are, respectively, the radial, polar, and azimuthal spherical coordinates of the position $\mathbf{r}$ of the detector,

$$\mathbf{\Xi}(\vartheta,\varphi)^T := \big[\mathbf{e}_x \quad \mathbf{e}_y \quad \sin\vartheta\sin\varphi\,\mathbf{e}_z \quad -\sin\vartheta\cos\varphi\,\mathbf{e}_z\big], \tag{57}$$

and $\vec{k}$ is the projection of the wave vector $\mathbf{k} := k\,\hat{\mathbf{r}}$ onto the x-y plane, i.e.,

$$\vec{k} := \frac{k\,\vec{r}}{r} = k\,(\sin\vartheta\cos\varphi\,\mathbf{e}_x + \sin\vartheta\sin\varphi\,\mathbf{e}_y). \tag{58}$$

With the help (57), we can show that

$$\mathbf{\Xi}(\vartheta,\varphi)^T\mathbf{C}_\pm(\vec{k}) = \mathbf{c}^+_\pm + [(\mathbf{c}^-_\pm \times \hat{\mathbf{r}})\cdot\mathbf{e}_z]\mathbf{e}_z \quad \text{for} \quad \pm\cos\vartheta > 0, \tag{59}$$

where

$$\mathbf{c}^+_\pm := C_{\pm1}(\vec{k})\,\mathbf{e}_x + C_{\pm2}(\vec{k})\,\mathbf{e}_y, \quad \mathbf{c}^-_\pm := C_{\pm3}(\vec{k})\,\mathbf{e}_x + C_{\pm4}(\vec{k})\,\mathbf{e}_y, \tag{60}$$

and $C_{\pm i}$ are the entries of $\mathbf{C}_\pm$, so that $\mathbf{C}^T_\pm = [C_{\pm1}\ C_{\pm2}\ C_{\pm3}\ C_{\pm4}]$.

Equation (56) reduces the solution of the radiation problem for oscillating sources to the determination of the four-component functions $\mathbf{C}_\pm$. The entries of these are, however, not independent. To see this, we introduce the two-component functions

$$\vec{C}^+_\pm := \begin{bmatrix} C_{\pm1} \\ C_{\pm2} \end{bmatrix}, \quad \vec{C}^-_\pm := \begin{bmatrix} C_{\pm3} \\ C_{\pm4} \end{bmatrix}, \tag{61}$$

so that

$$\mathbf{C}_\pm = \begin{bmatrix} \vec{C}^+_\pm \\ \vec{C}^-_\pm \end{bmatrix}. \tag{62}$$

Employing (27), (53), and (62), we then find

$$\vec{C}^+_\pm(\vec{p}) = \mp\frac{1}{\varpi(\vec{p})}\,\tilde{\mathbf{L}}_0(\vec{p})\vec{C}^-_\pm(\vec{p}). \tag{63}$$

Next, we use (19), (58), (63), and $\varpi(\vec{k}) = |k_z|$ to show that

$$\vec{C}^+_\pm(\vec{k}) = -\frac{k}{2|k_z|}\begin{bmatrix} 1 - \sin^2\vartheta\cos^2\varphi & -\sin^2\vartheta\sin\varphi\cos\varphi \\ -\sin^2\vartheta\sin\varphi\cos\varphi & 1 - \sin^2\vartheta\sin^2\varphi \end{bmatrix}\vec{G}_\pm(\vec{k}), \tag{64}$$

where

$$\vec{G}_\pm(\vec{p}) := \mp 2i\,\boldsymbol{\sigma}_2\vec{C}^-_\pm(\vec{p}). \tag{65}$$

Substituting (64) in (61) to determine $C_{\pm i}$ and using the result in (60), we have

$$\mathbf{c}^+_\pm = -\frac{k}{2|k_z|}[\mathbf{g}_\pm - (\hat{\mathbf{r}}\cdot\mathbf{g}_\pm)\hat{\mathbf{r}} + (\hat{\mathbf{r}}\cdot\mathbf{g}_\pm)(\mathbf{e}_z\cdot\hat{\mathbf{r}})\mathbf{e}_z], \tag{66}$$

where $\mathbf{g}_\pm$ is the vector lying in the x-y plane whose x and y components respectively coincide with the first and second entries of $\vec{G}_\pm(\vec{k})$, i.e.,

$$\mathbf{g}_\pm := \big\{[1 \quad 0]\vec{G}_\pm(\vec{k})\big\}\,\mathbf{e}_x + \big\{[0 \quad 1]\vec{G}_\pm(\vec{k})\big\}\,\mathbf{e}_y \tag{67}$$

$$:= \pm 2\,\mathbf{e}_z \times \mathbf{c}^-_\pm. \tag{68}$$

The latter equation follows from (60), (61), and (65). Solving it for $\mathbf{c}^-_\pm$, we find

$$\mathbf{c}^-_\pm = \pm\tfrac{1}{2}\,\mathbf{g}_\pm \times \mathbf{e}_z. \tag{69}$$

We can express $\mathbf{c}^+_\pm$ by substituting (69) in (66). More interesting is the identity

$$\mathbf{c}^+_\pm + [(\mathbf{c}^-_\pm \times \hat{\mathbf{r}})\cdot\mathbf{e}_z]\mathbf{e}_z = \frac{k}{2|k_z|}\,\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{g}_\pm) \quad \text{for} \quad \pm\cos\vartheta > 0,$$

which in view of (2), (56), and (59) leads us to the following remarkably simple equation for the electric field of the wave reaching the detectors:

$$\mathbf{E}(\mathbf{r},t)=\frac{k\,e^{i(kr-\omega t)}}{4\pi i\sqrt{\varepsilon_0}\,r}\,\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{g}_\pm)\quad\text{for}\quad r\cos\vartheta\to\pm\infty. \tag{70}$$

According to this equation, the information about the radiation of the source is contained in a pair of vectors lying in the x-y plane, namely, $\mathbf{g}_\pm$.[7]

Next, we explore the utility of (70) in solving the textbook problem of the radiation of an oscillating source placed in vacuum [1].

In the absence of scatterers, $\hat{\boldsymbol{\varepsilon}}=\hat{\boldsymbol{\mu}}=\mathbf{I}$,

$$\widehat{\tilde{\mathbf{H}}}=\tilde{\mathbf{H}}_0,\quad \widehat{\boldsymbol{\mathcal{H}}}(z)=\widehat{\mathbf{0}},\quad \widehat{\boldsymbol{\mathcal{U}}}(z,z')=\widehat{\mathbf{I}},\quad \widehat{\boldsymbol{\mathcal{M}}}=\widehat{\mathbf{I}},\quad \widehat{\mathbf{M}}=\widehat{\pi}_k,$$

and Eqs. (51), (54), and (55) respectively take the form

$$\mathbf{D}=-i\widehat{\pi}_k\int_{-\infty}^{\infty}dz\,\mathfrak{J}(\cdot,z),\quad \mathbf{C}_-=-\widehat{\boldsymbol{\Pi}}_2\mathbf{D},\quad \mathbf{C}_+=\widehat{\boldsymbol{\Pi}}_1\mathbf{D}. \tag{71}$$

In view of (28), (33), (34), (46), and (71),

$$\begin{aligned}\mathbf{C}_-(\vec{p})&=i\chi_k(\vec{p})\int_{-\infty}^{\infty}dz\,e^{iz\varpi(\vec{p})}\boldsymbol{\Pi}_2(\vec{p})\tilde{\mathbf{J}}(\vec{p},z)\\&=i\chi_k(\vec{p})\boldsymbol{\Pi}_2(\vec{p})\,\tilde{\tilde{\mathbf{J}}}\,(\vec{p},-\varpi(\vec{p})),\end{aligned} \tag{72}$$

$$\begin{aligned}\mathbf{C}_+(\vec{p})&=-i\chi_k(\vec{p})\int_{-\infty}^{\infty}dz\,e^{-iz\varpi(\vec{p})}\boldsymbol{\Pi}_1(\vec{p})\tilde{\mathbf{J}}(\vec{p},z)\\&=-i\chi_k(\vec{p})\boldsymbol{\Pi}_1(\vec{p})\,\tilde{\tilde{\mathbf{J}}}\,(\vec{p},\varpi(\vec{p})),\end{aligned} \tag{73}$$

where

$$\chi_k(\vec{p}):=\begin{cases}1 & \text{for}\quad |\vec{p}|<k,\\ 0 & \text{for}\quad |\vec{p}|\geqslant k,\end{cases}$$

$\tilde{\tilde{\mathbf{J}}}$ stands for the three-dimensional Fourier transform of $\mathbf{J}$, i.e.,

$$\tilde{\tilde{\mathbf{J}}}\,(\vec{p},p_z):=\int_{-\infty}^{\infty}dz\,e^{-ip_zz}\tilde{\mathbf{J}}(\vec{p})=\int_{\mathbb{R}^3}d^3\mathbf{r}\,e^{-i\mathbf{p}\cdot\mathbf{r}}\mathbf{J}(\mathbf{r}), \tag{74}$$

and $\mathbf{p}$ denotes $\vec{p}+p_z\mathbf{e}_z$, which we also express as $(\vec{p},p_z)$. Recall that the wave vector $\mathbf{k}$ for a detected wave is given by $\mathbf{k}:=k\hat{\mathbf{r}}$ and that the detectors lie on the planes $z=\pm\infty$. In particular, $\mathbf{k}\in\mathbb{R}^3$, $|\mathbf{k}|=k>0$, and $k_z=k\cos\vartheta\neq 0$. These in turn imply $\chi_k(\vec{k})=1$ and $\mathbf{k}=\vec{k}\pm\varpi(\vec{k})\mathbf{e}_z$ for $\pm\cos\vartheta>0$. Using these relations together with (72) and (73), we obtain

$$\mathbf{C}_-(\vec{k})=i\boldsymbol{\Pi}_2(\vec{k})\,\tilde{\tilde{\mathbf{J}}}\,(\mathbf{k})\quad\text{for}\quad\cos\vartheta<0, \tag{75}$$

$$\mathbf{C}_+(\vec{k})=-i\boldsymbol{\Pi}_1(\vec{k})\,\tilde{\tilde{\mathbf{J}}}\,(\mathbf{k})\quad\text{for}\quad\cos\vartheta>0. \tag{76}$$

Next, we introduce $\vec{P}:=[p_x\quad p_y]^T$ and use (12) and (19) to show that

$$\tilde{\mathbf{J}}(\vec{p},z)=\frac{i}{k}\begin{bmatrix}\tilde{\mathcal{J}}_z(\vec{p},z)\vec{P}\\ ik\boldsymbol{\sigma}_2\tilde{\vec{\mathcal{J}}}(\vec{p},z)\end{bmatrix}, \tag{77}$$

$$ik^{-1}\boldsymbol{\sigma}_2\tilde{\mathbf{L}}_0(\vec{p})\vec{P}=\vec{P}. \tag{78}$$

These equations together with (21) and (27) imply

$$\boldsymbol{\Pi}_1(\vec{p})\,\tilde{\tilde{\mathbf{J}}}\,(\vec{p},p_z)=\frac{1}{2\varpi(\vec{p})}\begin{bmatrix}\tilde{\mathbf{L}}_0(\vec{p})\,\boldsymbol{\sigma}_2\,\vec{\mathscr{G}}_-(\vec{p},p_z)\\ -\varpi(\vec{p})\,\boldsymbol{\sigma}_2\,\vec{\mathscr{G}}_-(\vec{p},p_z)\end{bmatrix}, \tag{79}$$

$$\boldsymbol{\Pi}_2(\vec{p})\,\tilde{\tilde{\mathbf{J}}}\,(\vec{p},p_z)=-\frac{1}{2\varpi(\vec{p})}\begin{bmatrix}\tilde{\mathbf{L}}_0(\vec{p})\,\boldsymbol{\sigma}_2\,\vec{\mathscr{G}}_+(\vec{p},p_z)\\ \varpi(\vec{p})\,\boldsymbol{\sigma}_2\,\vec{\mathscr{G}}_+(\vec{p},p_z)\end{bmatrix}, \tag{80}$$

where

$$\vec{\mathscr{G}}_\pm(\vec{p},p_z):=\tilde{\tilde{\vec{\mathcal{J}}}}\,(\vec{p},p_z)\pm\varpi(\vec{p})^{-1}\,\tilde{\tilde{\mathcal{J}}}_z\,(\vec{p},p_z)\vec{P}. \tag{81}$$

In particular, for $\mathbf{p}=\mathbf{k}$, we have $\vec{p}=\vec{k}$ and $\vec{P}=\vec{\mathfrak{K}}:=[k_x\quad k_y]^T=[k\sin\vartheta\cos\varphi\quad k\sin\vartheta\sin\varphi]^T$, and (75), (76), and (79)–(81) yield

$$\mathbf{C}_\pm(\vec{k})=\frac{i}{2|k_z|}\begin{bmatrix}-\tilde{\mathbf{L}}_0(\vec{k})\,\boldsymbol{\sigma}_2\,\vec{G}_0(\mathbf{k})\\ k_z\,\boldsymbol{\sigma}_2\,\vec{G}_0(\mathbf{k})\end{bmatrix}\quad\text{for}\quad\pm\cos\vartheta>0, \tag{82}$$

where

$$\vec{G}_0(\mathbf{k}):=\tilde{\tilde{\vec{\mathcal{J}}}}\,(\mathbf{k})-k_z^{-1}\,\tilde{\tilde{\mathcal{J}}}_z\,(\mathbf{k})\vec{\mathfrak{K}}=\frac{k}{k_z}\begin{bmatrix}\mathbf{e}_y\cdot[\hat{\mathbf{r}}\times\tilde{\tilde{\vec{\mathcal{J}}}}\,(\mathbf{k})]\\ -\mathbf{e}_x\cdot[\hat{\mathbf{r}}\times\tilde{\tilde{\vec{\mathcal{J}}}}\,(\mathbf{k})]\end{bmatrix}, \tag{83}$$

and we have also benefited from the fact that

$$k_z=\pm\varpi(\vec{k})\quad\text{for}\quad\pm\cos\vartheta>0. \tag{84}$$

In view of (62), (82), and (84), $\vec{C}^-_\pm(\vec{k})=\pm\frac{i}{2}\boldsymbol{\sigma}_2\vec{G}_0(\mathbf{k})$ for $\pm\cos\vartheta>0$. Using this relation in (65), we find that for the system we consider,

$$\vec{G}_\pm(\vec{k})=\vec{G}_0(\mathbf{k})\quad\text{for}\quad\pm\cos\vartheta>0.$$

This equation together with (67) and (83) imply

$$\mathbf{g}_\pm=\mathbf{g}_0\quad\text{for}\quad\pm\cos\vartheta>0, \tag{85}$$

where

$$\mathbf{g}_0:=-\frac{k}{k_z}\,\mathbf{e}_z\times[\hat{\mathbf{r}}\times\tilde{\tilde{\vec{\mathcal{J}}}}\,(\mathbf{k})]=\tilde{\tilde{\vec{\mathcal{J}}}}\,(\mathbf{k})-\sec\vartheta\,\tilde{\tilde{\mathcal{J}}}_z\,(\mathbf{k})\,\hat{\mathbf{r}}. \tag{86}$$

Substituting (85) in (70) and making use of (1), (86), and the fact that $\mathbf{E}$ is a continuous function of $\hat{\mathbf{r}}$, we have

$$\begin{aligned}\mathbf{E}(\mathbf{r},t)&=\frac{k\,e^{i(kr-\omega t)}}{4\pi i\sqrt{\varepsilon_0}\,r}\,\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times\tilde{\tilde{\vec{\mathcal{J}}}}\,(\mathbf{k})]\quad\text{for}\quad r\to\infty\\&=\frac{kZ_0}{4\pi i}\,\frac{e^{i(kr-\omega t)}}{r}\,\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times\tilde{\tilde{\mathbf{J}}}\,(k\hat{\mathbf{r}},0)]\quad\text{for}\quad r\to\infty,\end{aligned} \tag{87}$$

where $Z_0:=\sqrt{\mu_0/\varepsilon_0}$ is the vacuum impedance and $\tilde{\tilde{\mathbf{J}}}\,(k\hat{\mathbf{r}},t):=\int_{\mathbb{R}^3}d^3r\,e^{-i\mathbf{k}\cdot\mathbf{r}}\mathbf{J}(\mathbf{r},t)$.

[7]This holds also for the scattering of electromagnetic waves where the roles of $\mathbf{C}_\pm$ and $\mathbf{c}^\pm_\pm$ are, respectively, played by $\mathbf{T}^{l/r}_\pm$ and $\mathbf{t}^\pm_\pm$ of Ref. [27]. In particular, we can express the electric field of the scattered wave arriving at the detectors in terms of the vectors $\mathbf{g}_\pm$ defined by (67) with $\mathbf{c}^\pm_\pm$ changed to $\mathbf{t}^\pm_\pm$.

Equation (87) coincides with the outcome of the standard treatment of the radiation of an oscillating source in vacuum [1]; Eqs. (9.4), (9.5), and (9.8) of Ref. [1] also imply (87). Note, however, that our derivation of this equation is manifestly gauge invariant; unlike its textbook derivation, it does not involve the retarded vector potential and the Lorentz gauge condition.

Although the derivation of (87) rests on the assumption that the source of the radiation resides in empty space, we can also use it to describe the radiation of the source in a general linear medium provided that we characterize the electromagnetic response of the medium not in terms of its permittivity and permeability tensors but in terms of its bound charge density and bound and polarization current densities [37]. This requires replacing the free current density $\mathbf{J}$ appearing in (87) with the sum of the free, bound, and polarization current densities: $\mathbf{J} \to \mathbf{J}' := \mathbf{J} + \mathbf{J}_{\rm b} + \mathbf{J}_{\rm p}$, where $\mathbf{J}_{\rm b}$ and $\mathbf{J}_{\rm p}$ respectively stand for the bound and polarization current densities. Because we do not know the explicit form of $\mathbf{J}_{\rm b}$ and $\mathbf{J}_{\rm p}$, the substitution of $\mathbf{J}'$ for $\mathbf{J}$ in (87) gives a formula for the electric field of the detected wave which is of little practical value. Nevertheless, using the description of the medium in terms of the effective current density $\mathbf{J}'$, we can establish the general validity of Eq. (85) provided that we let $\mathbf{J}'$ play the role of $\mathbf{J}$ in the definition of $\mathbf{g}_0$. In other words, we have

$$\mathbf{g}_\pm = \mathbf{g} \quad \text{for} \quad \pm\cos\theta > 0, \tag{88}$$

where

$$\mathbf{g} := \tilde{\tilde{\mathcal{J}}}'(\mathbf{k}) - \sec\vartheta\; \tilde{\tilde{\mathcal{J}}}'_z(\mathbf{k})\,\hat{\mathbf{r}}, \tag{89}$$

$\tilde{\tilde{\mathcal{J}}}' := \tilde{\tilde{\mathcal{J}}} + \tilde{\tilde{\mathcal{J}}}_{\rm b} + \tilde{\tilde{\mathcal{J}}}_{\rm p}$, $\mathcal{J}_{\rm b}$ and $\mathcal{J}_{\rm p}$ are the vector-valued functions satisfying $\mathbf{J}_{\rm p}(\mathbf{r},t) = \mu_0^{-1/2} e^{-i\omega t}\mathcal{J}_{\rm p}(\mathbf{r})$ and $\mathbf{J}_{\rm b}(\mathbf{r},t) = \mu_0^{-1/2} e^{-i\omega t}\mathcal{J}_{\rm b}(\mathbf{r})$, and $\tilde{\tilde{\mathcal{J}}}'_z$ is the z component of $\tilde{\tilde{\mathcal{J}}}'$.

Substituting (88) in (70) and making use of the fact that the electric field is a continuous function of ϑ, we find the following expression for the electric field of the detected wave:

$$\mathbf{E}(\mathbf{r},t) = \frac{k\, e^{i(kr-\omega t)}}{4\pi i\sqrt{\varepsilon_0}\, r}\; \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{g}) \quad \text{for} \quad r \to \infty. \tag{90}$$

Because $\mathbf{g}$ is orthogonal to the z axis, the right-hand side of this relation is not manifestly covariant. The definition of $\mathbf{g}$, however, shows that $\hat{\mathbf{r}} \times \mathbf{g} = \hat{\mathbf{r}} \times \tilde{\tilde{\mathcal{J}}}'$. This confirms the covariance of $\hat{\mathbf{r}} \times \mathbf{g}$ and, consequently, that of the right-hand side of (90).

We would like to emphasize that because we do not know the explicit form of $\mathcal{J}'$, we cannot use (89) to compute $\mathbf{g}$. We can instead employ our transfer-matrix approach and the knowledge of the permittivity and permeability tensors of the medium to determine $\mathbf{g}$.

This completes our general discussion of the radiation problem for an oscillating source in a linear medium. It leads to the following prescription for computing the electric field of the wave arriving at the detectors.

(1) Find the evolution operator $\widehat{\mathcal{U}}(+\infty, z)$, the four-component field $\mathbf{D}$, and the transfer matrix $\widehat{\mathbf{M}}$.

(2) Solve (54) for $\mathbf{C}_-$.

(3) Read off the expressions for $\vec{C}_-^-$, $\vec{G}_-$, and $\mathbf{g}_-$ using (62), (65), and (67).

(4) Substitute $\mathbf{g}_-$ for the $\mathbf{g}$ in (90).

IV. RADIATION OF AN OSCILLATING SOURCE IN THE PRESENCE OF POINT SCATTERERS

Suppose that an oscillating source is placed next to a finite collection of nonmagnetic point scatterers lying in the x-y plane with otherwise arbitrary positions, so that the relative permittivity and permeability tensors of the system take the form

$$\hat{\boldsymbol{\varepsilon}}(\vec{r}, z) = \mathbf{I} + \delta(z)\sum_{a=1}^{N} \mathfrak{Z}_a\, \delta(\vec{r} - \vec{r}_a), \qquad \hat{\boldsymbol{\mu}}\,(\vec{r}, z) = \mathbf{I}, \tag{91}$$

where N is the number of point scatterers, $\vec{r}_a = (x_a, y_a)$ signify their positions in the x-y plane, and $\mathfrak{Z}_a$ are 3×3 complex matrices with entries $\mathfrak{z}_{a,ij}$.[8] Then, as we show in Ref. [27], for the generic cases where $\mathfrak{z}_{a,33} \neq 0$, we find the following expression for the interaction-picture Hamiltonian (46):

$$\widehat{\mathcal{H}}(z) = i\delta(z)\widehat{\mathbf{V}}, \tag{92}$$

where

$$\widehat{\mathbf{V}} := k\sum_{a=1}^{N}\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathscr{Z}_a & \mathbf{0}\end{bmatrix}\widehat{v}_a; \tag{93}$$

$\mathscr{Z}_a$ are the 2×2 matrices given by

$$\mathscr{Z}_a := \frac{1}{\mathfrak{z}_{a,33}}\begin{bmatrix} \mathfrak{z}_{a,22}\mathfrak{z}_{a,33} - \mathfrak{z}_{a,23}\mathfrak{z}_{a,32} & \mathfrak{z}_{a,23}\mathfrak{z}_{a,31} - \mathfrak{z}_{a,21}\mathfrak{z}_{a,33} \\ \mathfrak{z}_{a,13}\mathfrak{z}_{a,32} - \mathfrak{z}_{a,12}\mathfrak{z}_{a,33} & \mathfrak{z}_{a,11}\mathfrak{z}_{a,33} - \mathfrak{z}_{a,13}\mathfrak{z}_{a,31}\end{bmatrix}\boldsymbol{\sigma}_2 = \frac{1}{\mathfrak{z}_{a,33}}\begin{bmatrix} \mathcal{Z}_{a,11} & -\mathcal{Z}_{a,12} \\ -\mathcal{Z}_{a,21} & \mathcal{Z}_{a,22}\end{bmatrix}\boldsymbol{\sigma}_2 = \frac{i}{\mathfrak{z}_{a,33}}\begin{bmatrix} -\mathcal{Z}_{a,12} & -\mathcal{Z}_{a,11} \\ \mathcal{Z}_{a,22} & \mathcal{Z}_{a,21}\end{bmatrix}; \tag{94}$$

$\mathcal{Z}_{a,ij}$ stands for the minor of the $\mathfrak{z}_{a,ij}$ entry of $\mathfrak{Z}_a$, i.e., $\mathcal{Z}_{a,ij}$ is the determinant of the 2×2 matrix obtained by deleting the ith row and jth column of $\mathfrak{Z}_a$, for every positive integer m; $\widehat{v}_a : \mathscr{F}^m \to \mathscr{F}^m$ is the linear operator defined by[9]

$$\widehat{v}_a\mathbf{F}(\vec{p}) := \check{\mathbf{F}}(\vec{r}_a)\, e^{-i\vec{r}_a\cdot\vec{p}}; \tag{95}$$

$\mathbf{F} \in \mathscr{F}^m$ is arbitrary; and $\check{\mathbf{F}} := \mathcal{F}^{-1}\mathbf{F}$ is the two-dimensional inverse Fourier transform of $\mathbf{F}$, i.e., $\check{\mathbf{F}}(\vec{r}) := \frac{1}{4\pi^2}\int_{\mathbb{R}^2} d^2\vec{p}\,'\, e^{i\vec{r}_a\cdot\vec{p}\,'}\mathbf{F}(\vec{p}\,')$.

According to (92), $\widehat{\mathcal{H}}(z_1)\widehat{\mathcal{H}}(z_2) = \mathbf{0}$ for all $z_1, z_2 \in \mathbb{R}$. Therefore, the Dyson series (48) terminates, and we find

$$\widehat{\mathcal{U}}(+\infty, z) = \widehat{\mathbf{I}} + \widehat{\mathbf{V}}\int_z^\infty \delta(z')dz' = \widehat{\mathbf{I}} + \begin{cases}\widehat{\mathbf{V}} & \text{for} \quad z < 0, \\ \widehat{\mathbf{0}} & \text{for} \quad z > 0.\end{cases} \tag{96}$$

[8] $\vec{r}_a \neq \vec{r}_b$ for $a \neq b$.

[9] Reference [27] uses $\tilde{\delta}(i\vec{\nabla}_p - \vec{r}_a)$ for what we call $\widehat{v}_a$.

Notice that $\widehat{\mathcal{U}}(+\infty, 0)$ involves $\int_0^\infty \delta(z')dz'$, which is ill defined. This causes no problem in computing the fundamental transfer matrix (49) because the latter requires the knowledge of $\widehat{\mathcal{U}}(+\infty, -\infty)$. According to (49) and (96),

$$\widehat{\mathbf{M}} = \widehat{\pi}_k + \widehat{\pi}_k \widehat{\mathbf{V}} \widehat{\pi}_k. \tag{97}$$

In view of Eq. (51), the problem with $\int_0^\infty \delta(z')dz'$ does not affect the calculation of the four-component function $\mathbf{D}$ either if $\tilde{\mathbf{J}}(\vec{p}, z)$ and, consequently, $\mathfrak{J}(\vec{p}, z)$ are continuous functions of z at $z = 0$. Under this condition, we can use (51) and (96) to infer

$$\mathbf{D}(\vec{p}) = -i\chi_k(\vec{p})\bigg[\int_{-\infty}^{\infty} dz\, \mathfrak{J}(\vec{p}, z) + \int_{-\infty}^{0} dz\, \widehat{\mathbf{V}}\mathfrak{J}(\vec{p}, z)\bigg]. \tag{98}$$

To obtain a more explicit expression for the right-hand side of this equation, we examine the structure of the four-component functions $\mathfrak{J}$ and $\mathbf{J}$ in the presence of the point scatterers given by (91).

According to (12), $\mathbf{J}$ involves $\hat{\varepsilon}_{33}^{-1}\mathcal{J}_z$ and $\hat{\varepsilon}_{33}^{-1}\mathcal{J}_z\vec{K}_{\mathcal{E}}$. To deal with the fact that $\hat{\varepsilon}_{33}$ and $\vec{K}_{\mathcal{E}}$ have δ-function singularities, we employ the following distributional identity which we prove in Appendix A of Ref. [27]:

$$\frac{\delta(\vec{r} - \vec{r}_a)}{1 + \sum_{c=1}^N \mathfrak{a}_c\, \delta(\vec{r} - \vec{r}_c)} = 0,$$

where $\mathfrak{a}_c$ are nonzero numbers. This together with (6) allows us to show that whenever $\mathcal{J}_z$ is a continuous function on the x-y plane, $\hat{\varepsilon}_{33}^{-1}\mathcal{J}_z = \mathcal{J}_z$ and $\hat{\varepsilon}_{33}^{-1}\mathcal{J}_z\vec{K}_{\mathcal{E}} = \mathbf{0}$. Substituting these in (12) and taking note of (46), we see that the presence of point scatterers does not affect $\mathbf{J}$ or $\mathfrak{J}$. In particular, (77), (79), and (80) hold. In view of these equations and (31) and (46), we have

$$\begin{aligned}\chi_k(\vec{p}) &\int_{-\infty}^{\infty} dz\, \mathfrak{J}(\vec{p}, z)\\ &= \chi_k(\vec{p})[\boldsymbol{\Pi}_1(\vec{p})\, \tilde{\tilde{\mathbf{J}}}\,(\vec{p}, \varpi(\vec{p})) + \boldsymbol{\Pi}_2(\vec{p})\, \tilde{\tilde{\mathbf{J}}}\,(\vec{p}, -\varpi(\vec{p}))]\\ &= \frac{\chi_k(\vec{p})}{2\varpi(\vec{p})}\begin{bmatrix} \tilde{\mathbf{L}}_0(\vec{p})\,\boldsymbol{\sigma}_2[\vec{\mathscr{G}}_-(\vec{p}, \varpi(\vec{p})) - \vec{\mathscr{G}}_+(\vec{p}, -\varpi(\vec{p}))] \\ -\varpi(\vec{p})\,\boldsymbol{\sigma}_2[\vec{\mathscr{G}}_-(\vec{p}, \varpi(\vec{p})) + \vec{\mathscr{G}}_+(\vec{p}, -\varpi(\vec{p}))]\end{bmatrix},\end{aligned} \tag{99}$$

where $\vec{\mathscr{G}}_\pm$ are given by (81).

Next, we use (19), (24), (46), (77), (93), and (95) to show that

$$\begin{aligned}\chi_k(\vec{p}) &\int_{-\infty}^{0} dz\, \widehat{\mathbf{V}}\mathfrak{J}(\vec{p}, z)\\ &= \begin{bmatrix} \mathbf{0} \\ \chi_k(\vec{p}) \sum_{a=1}^N e^{-i\vec{r}_a\cdot\vec{p}}\, \mathscr{Z}_a(\vec{\mathfrak{R}}_a + \vec{\mathfrak{S}}_a)\end{bmatrix},\end{aligned} \tag{100}$$

where

$$\vec{\mathfrak{R}}_a := k\int_{-\infty}^{0} dz\, \vec{\mathcal{R}}(\vec{r}_a, z), \quad \vec{\mathfrak{S}}_a := k\int_{-\infty}^{0} dz\, \vec{\mathcal{S}}(\vec{r}_a, z), \tag{101}$$

$$\vec{\mathcal{R}}(\vec{r}, z) := \frac{i}{4\pi^2 k}\int_{\mathscr{D}_k} d^2p\, e^{i\vec{p}\cdot\vec{r}} \cos[z\varpi(\vec{p})]\, \tilde{\mathcal{J}}_z(\vec{p}, z)\, \vec{P}, \tag{102}$$

$$\vec{\mathcal{S}}(\vec{r}, z) := -\frac{i}{4\pi^2}\int_{\mathscr{D}_k} d^2p\, \frac{e^{i\vec{p}\cdot\vec{r}}\sin[z\varpi(\vec{p})]}{\varpi(\vec{p})}\, \tilde{\mathbf{L}}_0(\vec{p})\boldsymbol{\sigma}_2\, \tilde{\vec{\mathcal{J}}}(\vec{p}, z). \tag{103}$$

Substituting (99) and (100) in (98), we find the explicit form of the four-component function $\mathbf{D}$.

Let us recall that to determine the four-component functions $\mathbf{C}_-$, we need to solve (54). First, we evaluate both sides of (54) at some $\vec{p} \in \mathbb{R}^2$ and use (97) to write the resulting equation in the form

$$\mathbf{C}_-(\vec{p}) = -\boldsymbol{\Pi}_2(\vec{p})[\chi_k(\vec{p})\widehat{\mathbf{V}}\mathbf{C}_-(\vec{p}) + \mathbf{D}(\vec{p})]. \tag{104}$$

We also note that (62), (63), (93), and (95) imply

$$\chi_k(\vec{p})\widehat{\mathbf{V}}\mathbf{C}_-(\vec{p}) = k\begin{bmatrix}\mathbf{0} \\ \vec{X}(\vec{p})\end{bmatrix}, \tag{105}$$

where

$$\vec{X}(\vec{p}) := \chi_k(\vec{p})\sum_{b=1}^N e^{-i\vec{r}_b\cdot\vec{p}}\, \mathscr{Z}_b\vec{X}_b, \tag{106}$$

$$\begin{aligned}\vec{X}_b &:= \check{\vec{C}}{}^+_-(\vec{r}_b) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} d^2p\, e^{i\vec{p}\cdot\vec{r}_b}\vec{C}^+_-(\vec{p})\\ &= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} d^2p\, \frac{e^{i\vec{p}\cdot\vec{r}_b}}{\varpi(\vec{p})}\tilde{\mathbf{L}}_0(\vec{p})\vec{C}^-_-(\vec{p}).\end{aligned} \tag{107}$$

Next, we use the entries of $\mathbf{D}$, which we denote by D_i, to introduce the two-component functions, $\vec{D}^+ := [D_1 \quad D_2]^T$ and $\vec{D}^- := [D_3 \quad D_4]^T$, so that

$$\mathbf{D} := \begin{bmatrix}\vec{D}^+ \\ \vec{D}^-\end{bmatrix}. \tag{108}$$

Substituting (105) in (104) and using (27), (62), (106), and (108), we obtain

$$\vec{C}^-_-(\vec{p}) = -\frac{k\chi_k(\vec{p})}{2}\sum_{a=1}^N e^{-i\vec{r}_a\cdot\vec{p}}\mathscr{Z}_a\vec{X}_a + \vec{\Delta}(\vec{p}), \tag{109}$$

where

$$\begin{aligned}\vec{\Delta}(\vec{p}) &:= \frac{1}{2}\bigg[\frac{1}{\varpi(\vec{p})}\tilde{\mathbf{L}}_0(\vec{p})\vec{D}^+(\vec{p}) - \vec{D}^-(\vec{p})\bigg]\\ &= \frac{i\chi_k(\vec{p})}{2}\bigg[\sum_{a=1}^N e^{-i\vec{r}_a\cdot\vec{p}}\, \mathscr{Z}_a(\vec{\mathfrak{R}}_a + \vec{\mathfrak{S}}_a)\\ &\quad - \boldsymbol{\sigma}_2\bigg\{\tilde{\tilde{\vec{\mathcal{J}}}}\,(\vec{p}, -\varpi(\vec{p})) + \frac{\tilde{\tilde{\mathcal{J}}}_z\,(\vec{p}, -\varpi(\vec{p}))}{\varpi(\vec{p})}\,\vec{P}\bigg\}\bigg].\end{aligned} \tag{110}$$

and we have also employed (21), (81), and (98)–(100).

Plugging (110) in (109), setting $\vec{p} = \vec{k}$, and using (65), we find

$$\vec{G}_-(\vec{k}) := \vec{G}_0(\mathbf{k}) + \vec{G}_s(\mathbf{k}) \quad \text{for} \quad \cos\vartheta < 0, \tag{111}$$

where $\vec{G}_0(\mathbf{k})$ is given by (83) and

$$\vec{G}_s(\mathbf{k}) := -\boldsymbol{\sigma}_2\sum_{a=1}^N e^{-i\vec{r}_a\cdot\vec{k}}\mathscr{Z}_a(\vec{\mathfrak{R}}_a + \vec{\mathfrak{S}}_a + ik\vec{X}_a). \tag{112}$$

In view of (67), (83), (86), (88), and (111), $\mathbf{g} = \mathbf{g}_- = \mathbf{g}_0 + \mathbf{g}_s$, where[10]

$$\mathbf{g}_s := \{[1 \quad 0]\,\vec{G}_s(\mathbf{k})\}\mathbf{e}_x + \{[0 \quad 1]\,\vec{G}_s(\mathbf{k})\}\mathbf{e}_y. \tag{113}$$

Substituting $\mathbf{g} = \mathbf{g}_0 + \mathbf{g}_s$ in (90) and using (86), we have

$$\mathbf{E}(\mathbf{r},t) = \frac{k\,e^{i(kr-\omega t)}}{4\pi i\sqrt{\varepsilon_0}\,r}\{\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times\tilde{\tilde{\mathcal{J}}}\,(k\hat{\mathbf{r}})] + \hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{g}_s)\}$$
$$\text{for} \quad r\to\infty. \tag{114}$$

Equations (112)–(114) reduce the solution of the radiation problem we are considering to the determination of $\vec{X}_a$. To calculate the latter, first, we use (63) and (109) to derive

$$\vec{C}_-^{+}(\vec{p}) = -\frac{k\chi_k(\vec{p})}{2}\sum_{a=1}^{N}\frac{e^{-i\vec{r}_a\cdot\vec{p}}}{\varpi(\vec{p})}\,\tilde{\mathbf{L}}_0(\vec{p})\mathscr{Z}_a\vec{X}_a + \vec{\Gamma}(\vec{p}), \tag{115}$$

where

$$\vec{\Gamma}(\vec{p}) := \varpi(\vec{p})^{-1}\tilde{\mathbf{L}}_0(\vec{p})\vec{\Delta}(\vec{p}). \tag{116}$$

Performing the inverse Fourier transform of both sides of (115), we then find

$$\check{\vec{C}}_-^{+}(\vec{r}) = -\sum_{b=1}^{N}\boldsymbol{\mathcal{L}}(\vec{r}-\vec{r}_b)\mathscr{Z}_b\vec{X}_b + \check{\vec{\Gamma}}(\vec{r}), \tag{117}$$

where

$$\boldsymbol{\mathcal{L}}(\vec{r}) := \frac{k}{8\pi^2}\int_{\mathscr{D}_k} d^2p\,\frac{e^{i\vec{p}\cdot\vec{r}}}{\varpi(\vec{p})}\,\tilde{\mathbf{L}}_0(\vec{p}). \tag{118}$$

For $\vec{r} = \vec{r}_a$, (117) gives the following system of equations for $\vec{X}_b$:

$$\sum_{b=1}^{N}\mathbf{A}_{ab}\vec{X}_b = \check{\vec{\Gamma}}(\vec{r}_a), \tag{119}$$

where for all $a, b \in \{1, 2, \ldots, N\}$,

$$\mathbf{A}_{ab} := \delta_{ab}\mathbf{I} + \boldsymbol{\mathcal{L}}_{ab}\mathscr{Z}_b, \tag{120}$$

$$\boldsymbol{\mathcal{L}}_{ab} := \boldsymbol{\mathcal{L}}(\vec{r}_a - \vec{r}_b). \tag{121}$$

Equation (119) has a unique solution if and only if, for all $a, c \in \{1, 2, \ldots, N\}$, there are 2×2 matrices $\mathbf{B}_{ac}$ such that

$$\sum_{b=1}^{N}\mathbf{B}_{ab}\mathbf{A}_{bc} = \delta_{ac}\mathbf{I}. \tag{122}$$

Multiplying both sides of (119) by $\mathbf{B}_{ca}$ from the left, summing over a, and making use of (122), we have

$$\vec{X}_c = \sum_{b=1}^{N}\mathbf{B}_{cb}\check{\vec{\Gamma}}(\vec{r}_b). \tag{123}$$

Substituting this equation in (112) and using the result in (114), we obtain the electric field of the wave reaching the detectors.

[10]We have also computed $\mathbf{C}_+$, $\vec{G}_+$, and $\mathbf{g}_+$ and checked that, indeed, $\mathbf{g}_+ = \mathbf{g}_-$.

V. RADIATION OF A PERFECT DIPOLE IN THE PRESENCE OF POINT SCATTERERS

By definition, the electric dipole moment of a charge distribution with charge density ρ is given by $\mathfrak{p}(t) := \int_{\mathbb{R}^3} d^3r\,\mathbf{r}\rho(\mathbf{r},t)$. For a charge distribution corresponding to a localized oscillating source of angular frequency ω, we can use the continuity equation $\nabla\cdot\mathbf{J} = i\omega\rho$, the identity $\int_{\mathbb{R}^3} d^3r(\nabla\cdot\mathbf{J})\mathbf{r} = -\int_{\mathbb{R}^3} d^3r\,\mathbf{J}$, and Eq. (1) to show that [1]

$$\mathfrak{p}(t) = i\omega^{-1}\int_{\mathbb{R}^3} d^3r\,\mathbf{J}(\mathbf{r},t) = \frac{i\sqrt{\varepsilon_0}\,e^{-i\omega t}}{k}\int_{\mathbb{R}^3} d^3r\,\mathcal{J}(\mathbf{r}). \tag{124}$$

This suggests that we can model a perfect electric dipole by a scaled current density of the form

$$\mathcal{J}(\mathbf{r}) = \mathbf{j}\,\delta(\mathbf{r}-\mathbf{a}), \tag{125}$$

where $\mathbf{j} := \frac{-ik}{\sqrt{\varepsilon_0}}\mathfrak{p}(0)$, $\mathfrak{p}(0)$ is the electric dipole moment of the dipole at $t = 0$, and $\mathbf{a}$ is its position.

According to (125),

$$\tilde{\mathcal{J}}(\vec{p},z) = \mathbf{j}\,\delta(z-a_z)e^{-i\vec{a}\cdot\vec{p}}, \qquad \tilde{\tilde{\mathcal{J}}}\,(\mathbf{p}) = \mathbf{j}\,e^{-i\mathbf{a}\cdot\mathbf{p}}, \tag{126}$$

where $\vec{a} := (a_x, a_y)$ and a_x, a_y, and a_z are Cartesian components of $\mathbf{a}$, so that $\mathbf{a} = (a_x, a_y, a_z) = (\vec{a}, a_z)$. The first relation in (126) shows that $\tilde{\mathcal{J}}(\vec{p},z)$ is a continuous function of z at $z = 0$, and we can safely employ the results of the preceding section provided that $a_z \neq 0$, i.e., the dipole does not lie on the x-y plane. In the following we assume that this condition holds.

Using the second relation in (126), we can write (114) in the form

$$\mathbf{E}(\mathbf{r},t) = \frac{k\,e^{i(kr-\omega t)}}{4\pi i\sqrt{\varepsilon_0}\,r}\,[e^{-ik\mathbf{a}\cdot\hat{r}}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{j}) + \hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{g}_s)]$$
$$\text{for} \quad r\to\infty. \tag{127}$$

Therefore, to determine the electric field reaching the detectors we need to calculate $\mathbf{g}_s$. To do this, first we use (101)–(103), (110), and (126) to calculate $\vec{\mathfrak{R}}_b$, $\vec{\mathfrak{S}}_b$, and $\vec{\Delta}$. This gives

$$\vec{\mathfrak{R}}_b = k\,j_z\theta(-a_z)\vec{R}_b(a_z), \qquad \vec{\mathfrak{S}}_b = k\,\theta(-a_z)\mathbf{S}_b(a_z)\vec{J}, \tag{128}$$

$$\vec{\Delta}(\vec{p}) = \frac{i\chi_k(\vec{p})}{2}\Bigg\{k\theta(-a_z)\sum_{b=1}^{N}e^{-i\vec{r}_b\cdot\vec{p}}\mathscr{Z}_b[j_z\vec{R}_b(a_z) + \mathbf{S}_b(a_z)\vec{J}]$$
$$- e^{-i\vec{a}\cdot\vec{p}}e^{ia_z\varpi(\vec{p})}\boldsymbol{\sigma}_2\left[\vec{J} + \frac{j_z}{\varpi(\vec{p})}\vec{P}\right]\Bigg\}, \tag{129}$$

where

$$\theta(x) := \begin{cases}1 & \text{for } x \geqslant 0,\\ 0 & \text{for } x < 0,\end{cases} \qquad \vec{J} := \begin{bmatrix} j_x \\ j_y\end{bmatrix}, \tag{130}$$

$$\vec{R}_b(a_z) := \vec{R}(\vec{r}_b - \vec{a}, a_z), \qquad \mathbf{S}_b(a_z) := \mathbf{S}(\vec{r}_b - \vec{a}, a_z), \tag{131}$$

$$\vec{R}(\vec{r},z) := \frac{i}{4\pi^2 k}\int_{\mathscr{D}_k} d^2p\,e^{i\vec{r}\cdot\vec{p}}\cos[z\varpi(\vec{p})]\vec{P}, \tag{132}$$

$$\mathbf{S}(\vec{r},z) := -\frac{i}{4\pi^2}\int_{\mathscr{D}_k} d^2p\,\frac{e^{i\vec{r}\cdot\vec{p}}\sin[z\varpi(\vec{p})]}{\varpi(\vec{p})}\,\tilde{\mathbf{L}}_0(\vec{p})\boldsymbol{\sigma}_2, \tag{133}$$

and j_x, j_y, and j_z are components of $\mathbf{j}$.

Next, we substitute (129) in (116) to find $\vec{\Gamma}(\vec{p})$; take the inverse Fourier transform of the resulting equation; and use (21), (78), (123), and the identity $\sum_{c=1}^{N}\mathbf{B}_{ac}\boldsymbol{\mathcal{L}}_{cb}\mathscr{Z}_b=\delta_{ab}\mathbf{I}-\mathbf{B}_{ab}$, which follows from (120) and (122), to show that

$$\vec{X}_a=\frac{i}{2}\sum_{b=1}^{N}\mathbf{B}_{ab}[j_z\vec{\mathscr{R}}_b(a_z)+\mathscr{S}_b(a_z)\vec{J}]+i\theta(-a_z)\left\{\sum_{b=1}^{N}-\mathbf{B}_{ab}[j_z\vec{R}_b(a_z)+\mathbf{S}_b(a_z)\vec{J}]+j_z\vec{R}_a(a_z)+\mathbf{S}_a(a_z)\vec{J}\right\},\tag{134}$$

where

$$\vec{\mathscr{R}}_b(a_z):=\vec{\mathscr{R}}(\vec{r}_b-\vec{a},a_z),\quad \mathscr{S}_b(a_z):=\mathscr{S}(\vec{r}_b-\vec{a},a_z),\tag{135}$$

$$\vec{\mathscr{R}}(\vec{r},z):=\frac{i}{4\pi^2k}\int_{\mathscr{D}_k}d^2p\;e^{i\vec{r}\cdot\vec{p}}e^{iz\varpi(\vec{p})}\vec{P},\tag{136}$$

$$\mathscr{S}(\vec{r},z):=-\frac{1}{4\pi^2}\int_{\mathscr{D}_k}d^2p\;\frac{e^{i\vec{r}\cdot\vec{p}}e^{iz\varpi(\vec{p})}}{\varpi(\vec{p})}\tilde{\mathbf{L}}_0(\vec{p})\boldsymbol{\sigma}_2.\tag{137}$$

Notice that changing the term $e^{i\vec{r}\cdot\vec{p}}$ on the right-hand sides of (132), (133), (136), and (137) to $e^{-i\vec{r}\cdot\vec{p}}$ multiplies the left-hand sides of (132) and (136) by a minus sign while not affecting the left-hand sides of (133) and (137). This shows that we can replace $e^{i\vec{r}\cdot\vec{p}}$ in (132) and (136) by $i\sin(\vec{r}\cdot\vec{p})$ and in (133) and (137) by $\cos(\vec{r}\cdot\vec{p})$. Doing this, we find

$$\vec{R}(\vec{r},z)=\mathrm{Re}[\vec{\mathscr{R}}(\vec{r},z)],\quad \mathbf{S}(\vec{r},z)=\mathrm{Re}[\mathscr{S}(\vec{r},z)],\tag{138}$$

where Re stands for the real part of its argument. It is also worth mentioning that we can express $\vec{\mathscr{R}}(\vec{r},z)$ and $\mathscr{S}(\vec{r},z)$ in terms of the function

$$h(\vec{r},z):=\frac{1}{4\pi^2k}\int_{\mathscr{D}_k}d^2p\;\frac{e^{i\vec{r}\cdot\vec{p}}e^{iz\sqrt{k^2-p^2}}}{\sqrt{k^2-p^2}}=\frac{1}{2\pi}\int_0^1du\;\frac{u\,e^{ikz\sqrt{1-u^2}}J_0(k|\vec{r}|u)}{\sqrt{1-u^2}},\tag{139}$$

where J_0 stands for the zero-order Bessel function of the first kind. It is easy to check that

$$\vec{\mathscr{R}}(\vec{r},z)=-i\vec{\partial}\,\partial_z h(\vec{r},z),\tag{140}$$

$$\mathscr{S}(\vec{r},z)=i\begin{bmatrix}\partial_x^2+k^2 & \partial_x\partial_y\\ \partial_x\partial_y & \partial_y^2+k^2\end{bmatrix}h(\vec{r},z).\tag{141}$$

According to (139), the real and imaginary parts of $h(\vec{r},z)$ are, respectively, even and odd functions of z. In view of (140) and (141), this implies that the real part of $\vec{\mathscr{R}}(\vec{r},z)$ and the imaginary part of $\mathscr{S}(\vec{r},z)$ are even functions of z, while the imaginary part of $\vec{\mathscr{R}}(\vec{r},z)$ and the real part of $\mathscr{S}(\vec{r},z)$ are odd functions of z. We can use these observations together with (131), (135), and (138) to establish the following identities:

$$\vec{R}_b(-a_z)=\vec{R}_b(a_z),\quad \mathbf{S}_b(-a_z)=-\mathbf{S}_b(a_z),\tag{142}$$

$$\mathrm{Im}[\vec{\mathscr{R}}_b(-a_z)]=-\mathrm{Im}[\vec{\mathscr{R}}_b(a_z)],\quad \mathrm{Im}[\mathscr{S}_b(-a_z)]=\mathrm{Im}[\mathscr{S}_b(a_z)],\tag{143}$$

where Im stands for the imaginary part of its argument.

Having calculated $\vec{X}_a$, we can use (112) and (128) to establish

$$\begin{aligned}\vec{G}_s(\mathbf{k})&=\frac{k}{2}\boldsymbol{\sigma}_2\sum_{a,b=1}^{N}e^{-i\vec{r}_a\cdot\vec{k}}\mathscr{Z}_a\mathbf{B}_{ab}\{j_z\vec{\mathscr{R}}_b(a_z)+\mathscr{S}_b(a_z)\vec{J}-2\theta(-a_z)[j_z\vec{R}_b(a_z)+\mathbf{S}_b(a_z)\vec{J}]\}\\&=\frac{k}{2}\boldsymbol{\sigma}_2\sum_{a,b=1}^{N}e^{-i\vec{r}_a\cdot\vec{k}}\mathscr{Z}_a\mathbf{B}_{ab}(j_z\{\mathrm{sgn}(a_z)\vec{R}_b(a_z)+i\mathrm{Im}[\vec{\mathscr{R}}_b(a_z)]\}+\{\mathrm{sgn}(a_z)\mathbf{S}_b(a_z)+i\mathrm{Im}[\mathscr{S}_b(a_z)]\}\vec{J})\\&=\frac{k}{2}\boldsymbol{\sigma}_2\sum_{a,b=1}^{N}e^{-i\vec{r}_a\cdot\vec{k}}\mathscr{Z}_a\mathbf{B}_{ab}[\mathrm{sgn}(a_z)j_z\vec{\mathscr{R}}_b(|a_z|)+\mathscr{S}_b(|a_z|)\vec{J}],\end{aligned}\tag{144}$$

where $\mathrm{sgn}(x):=x/|x|$ stands for the sign of x and we have also made use of (138), (142), and (143). Recalling that $\mathbf{g}_s$ is the vector having the entries of $\vec{G}_s(\mathbf{k})$ as its components, we can read off the latter from (144) and substitute the result in (127) to obtain the electric field of the wave reaching the detectors.

The appearance of $\mathrm{sgn}(a_z)$ on the right-hand side of (144) might give the impression that it depends on our choice of the direction of the z axis. This is unacceptable because $\mathbf{g}_s$ enters the expression for the electric field, which must not depend on our choice of the coordinate system. As a consistency check on the validity of (144), we examine the behavior of its right-hand side under the change of coordinates $\mathfrak{T}$ that flips the sign of the z component of all vectors. First, we use (91) to infer that under this coordinate transformation, $\mathfrak{Z}_{a,ij}$ is left invariant unless one and only one of i and j is 3, in which case it changes sign. In light of (94), (120), and (122), this implies that $\mathscr{Z}_a$, $\mathbf{A}_{ab}$, and, consequently, $\mathbf{B}_{ab}$ are left invariant under $\mathfrak{T}$. Equations (135), (136), and (137) show that the same applies to $\vec{\mathscr{R}}_b(|a_z|)$ and $\mathscr{S}_b(|a_z|)$. It is also clear that $\mathfrak{T}$ implies $\mathrm{sgn}(a_z)\to-\mathrm{sgn}(a_z)$, $j_z\to-j_z$, and $\vec{J}\to\vec{J}$. These observations prove the invariance of the right-hand side of (144) under $\mathfrak{T}$.

In the remainder of this section we explore the consequences of our findings for the simplest special case, namely, an oscillating perfect dipole in the presence of a single point scatterer ($N=1$). Without loss of generality we choose our coordinate system in such a way that the point scatterer lies at the origin while the dipole is on the z axis. Then, $\vec{r}_1 = \vec{a} = \vec{0}$, and (113), (118), (120)–(122), (135)–(138), and (144) imply

$$\boldsymbol{\mathcal{L}}_{11} = \boldsymbol{\mathcal{L}}(\vec{0}) = -\frac{ik^3}{6\pi}\boldsymbol{\sigma}_2, \tag{145}$$

$$\mathbf{B}_{11} = \mathbf{A}_{11}^{-1} = \mathfrak{B} := \left(\mathbf{I} - \frac{ik^3}{6\pi}\boldsymbol{\sigma}_2\mathscr{Z}_1\right)^{-1}, \tag{146}$$

$$\vec{\mathscr{R}}_1(z) = [0 \quad 0]^T, \quad \mathscr{S}_1(z) = \frac{ik^2}{3\pi}\mathfrak{s}(kz)\mathbf{I}, \tag{147}$$

$$\vec{G}_s(\mathbf{k}) = \mathfrak{s}(|a_z|k)(\mathfrak{B} - \mathbf{I})\vec{J}, \tag{148}$$

$$\mathbf{g}_s = \mathfrak{s}(|a_z|k)[\{(\mathfrak{B}_{11} - 1)j_x + \mathfrak{B}_{12}j_y\}\mathbf{e}_x + \{\mathfrak{B}_{21}j_x + (\mathfrak{B}_{22} - 1)j_y\}\mathbf{e}_y], \tag{149}$$

where

$$\mathfrak{s}(x) := \frac{-3i\left[(x^2 + ix - 1)e^{ix} - \frac{x^2}{2} + 1\right]}{2x^3}. \tag{150}$$

The following are consequences of Eqs. (91), (94), (114), (146), (149), and (150).

(1) $\mathbf{g}_s$ and, consequently, the electric field of the detected radiation do not depend on j_z and blow up when $\det(\mathbf{I} - \frac{ik^3}{6\pi}\boldsymbol{\sigma}_2\mathscr{Z}_1) = 0$. This condition marks a spectral singularity of the system which can exist if the point scatterer is made of gain material [38,39]. In the absence of a source, the spectral singularity corresponds to a configuration where the scatterer begins amplifying the background noise and emits coherent radiation [40,41]. This is the basic mechanism that applies to every laser. In the presence of a source, tuning the parameters of the system to approach a spectral singularity so that $\det(\mathbf{I} - \frac{ik^3}{6\pi}\boldsymbol{\sigma}_2\mathscr{Z}_1) \approx 0$ causes the point scatterer to function as an amplifier for the radiated wave.[11]

(2) The term $\mathfrak{s}(|a_z|k)$ in (149) determines the dependence of the electric field of the emitted wave on the distance $|a_z|$ between the dipole and the point scatterer. For $|a_z| \to 0$, $\mathfrak{s}(|a_z|k) \to 1$, and $\mathbf{g}_s$ tends to a nonzero constant value. For $|a_z|k \to \infty$, $\mathfrak{s}(|a_z|k) \to 0$, and $\mathbf{g}_s$ tends to zero. Therefore, as expected, the presence of a distant point scatterer ($|a_z| \gg k^{-1}$) does not have a noticeable effect on the radiation of the dipole.

(3) The z component of $\mathbf{j}$ does not affect the response of the point scatterer to the radiation emitted by the dipole. This has to do with the fact that $\vec{r}_1 - \vec{a} = \vec{0}$, which causes $\vec{\mathscr{R}}_1(z)$ to vanish for all z.

4. If the principal axes of the point scatterer are aligned along the coordinate axes, there are $\mathfrak{z}_x, \mathfrak{z}_y, \mathfrak{z}_z \in \mathbb{C}$ such that

$$\boldsymbol{\mathfrak{Z}}_1 = \begin{bmatrix} \mathfrak{z}_x & 0 & 0 \\ 0 & \mathfrak{z}_y & 0 \\ 0 & 0 & \mathfrak{z}_z \end{bmatrix}, \quad \boldsymbol{\sigma}_2\mathscr{Z}_1 = \begin{bmatrix} \mathfrak{z}_x & 0 \\ 0 & \mathfrak{z}_y \end{bmatrix},$$

$$\mathfrak{B} = \begin{bmatrix} \beta(\mathfrak{z}_x) & 0 \\ 0 & \beta(\mathfrak{z}_y) \end{bmatrix},$$

where

$$\beta(\mathfrak{z}) := \left(1 - \frac{i\mathfrak{z}k^3}{6\pi}\right)^{-1} \tag{151}$$

and (149) gives

$$\mathbf{g}_s = \mathfrak{s}(|a_z|k)[\{\beta(\mathfrak{z}_x) - 1\}j_x\,\mathbf{e}_x + \{\beta(\mathfrak{z}_y) - 1\}j_y\,\mathbf{e}_y].$$

In particular, if $\mathfrak{z}_x = \mathfrak{z}_y$, so that the point scatterer is uniaxial or isotropic, we have

$$\mathbf{g}_s = \mathfrak{s}(|a_z|k)[\beta(\mathfrak{z}_x) - 1]\,\vec{j}, \tag{152}$$

where $\vec{j} := j_x\mathbf{e}_x + j_y\mathbf{e}_y$.

Let $\langle u(\mathbf{r})\rangle$ and $\langle u_0(\mathbf{r})\rangle$ denote the time-averaged energy density of the detected wave at $\mathbf{r}$ in the presence and absence of point scatterers, respectively. According to (127), the ratio of these quantities, which equals the ratio of the time-averaged intensities of these waves [37], is given by

$$\frac{\langle u(\mathbf{r})\rangle}{\langle u_0(\mathbf{r})\rangle} = \langle\hat{u}(\hat{\mathbf{r}})\rangle := \frac{|\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times (e^{-ik\mathbf{a}\cdot\hat{r}}\mathbf{j} + \mathbf{g}_s)|^2}{|\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{j})|^2}. \tag{153}$$

Substituting (150), (151), and (152) in this equation, we find $\langle\hat{u}(\hat{\mathbf{r}})\rangle$ for $\mathfrak{z}_x = \mathfrak{z}_y$. This in particular implies

$$\langle\hat{u}(\hat{\mathbf{r}})\rangle = \begin{cases} |e^{ika_z}\mathfrak{s}(|a_z|k)[\beta(\mathfrak{z}_x) - 1] + 1|^2 & \text{for } \hat{\mathbf{r}} = \mathbf{e}_z, \\ |e^{ika_z\cos\vartheta}\mathfrak{s}(|a_z|k)[\beta(\mathfrak{z}_x) - 1] + 1|^2 & \text{for } j_z = 0. \end{cases} \tag{154}$$

Figure 2 shows the graphs of the normalized time-averaged intensity (154) measured by detectors located at $z = +\infty$ as a function of a_zk. As expected, $\langle\hat{u}(\mathbf{e}_z)\rangle$ tends to 1 as $|a_zk|$ grows, and its deviation from 1 is more pronounced for larger values of $|\mathfrak{z}_x|k^3$. For $\mathfrak{z}_xk^3 = 1$, $|a_zk| \lesssim 0.9$, and $0 \leqslant \vartheta \lesssim 55^\circ$, $\langle\hat{u}(\hat{\mathbf{r}})\rangle$ turns out to be a one-to-one function of a_zk. This shows that one can, in principle, use the value of the normalized time-averaged intensity for sufficiently low energy waves to determine the relative position of the source with respect to the point scatterer or vice versa if one can identify the line joining them (i.e., the z axis).[12] This simple example suggests the possibility of using the exact solution of the radiation problem in the presence of point scatterers for the purpose of addressing the inverse problem of locating the scatterers using the data on their response to the incident radiation, which is a problem of great practical importance.

VI. CONCLUSION

Multidimensional generalizations of the transfer matrix of scattering theory in one dimension have been developed and utilized since the 1980s basically for the purpose of numerical investigation of wave propagation in stratified media [28]. The basic idea behind these developments is to dissect the medium into a large number of thin layers along a propagation axis, discretize the transverse degrees of freedom in each layer, assign a transfer matrix to each layer (which is a matrix relating the amplitude of the wave at the points representing

[11] Making $|\det(\mathbf{I} - \frac{ik^3}{6\pi}\boldsymbol{\sigma}_2\mathscr{Z}_1)|$ too small leads to the emergence of nonlinear effects which render our analysis inapplicable.

[12] Plotting the graph of $\langle\hat{u}(\hat{\mathbf{r}})\rangle$ for different real and complex values of $\mathfrak{z}_xk^3$, we have checked that this feature is not sensitive to the value of $\mathfrak{z}_xk^3$.

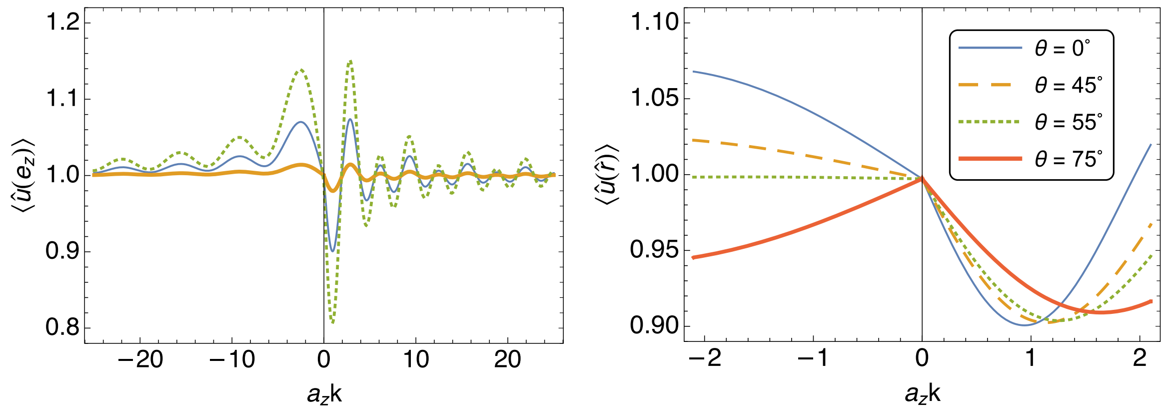

FIG. 2. Plots of the normalized time-averaged intensity as a function of a_zk for $\mathfrak{z}_x = \mathfrak{z}_y$. The left panel shows the plots of $\langle \hat{u}(\mathbf{e}_z)\rangle$ for $\mathfrak{z}_x k^3 = 0.2$ (thick orange curve), 1.0 (thin blue curve), and 2.0 (dotted green curve). The right panel shows the plots of $\langle \hat{u}(\hat{\mathbf{r}})\rangle$ for $j_z = 0$, $\mathfrak{z}_x k^3 = 1$, and $\vartheta = 0^\circ, 45^\circ, 55^\circ$, and 75°. For $|a_zk| \lesssim 0.9$ and $0 \leqslant \vartheta \lesssim 55^\circ$, the value of $\langle \hat{u}(\hat{\mathbf{r}})\rangle$ determines a_zk uniquely.

one of the two large boundaries of the layer to the other), and multiply them according to a particular composition rule to obtain the transfer matrix for the bulk. Recently, we introduced a fundamental notion of the transfer matrix for scalar [25] and electromagnetic [27] waves whose definition does not require the slicing or discretization of the medium. This notion forms the basis of a dynamical formulation of the stationary scattering that allows for analytic calculations and is particularly effective in dealing with point interactions.

In the present article, we explored the utility of the fundamental transfer matrix in the study of the problem of radiation in a general linear scattering medium. This led to a general method for solving this problem. Using this method we showed that the electric field of the wave emitted by an oscillating source has the form

$$\mathbf{E}(\mathbf{r},t) = \frac{k\, e^{i(kr-\omega t)}}{4\pi i\sqrt{\varepsilon_0}\, r}\, \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{g}) \quad \text{for} \quad r \to \infty,$$

where $\mathbf{g}$ is a vector belonging to the x-y plane that stores all the information about the current density characterizing the source and the permittivity and permeability tensors of the medium. We provided the following procedure for the calculation of $\mathbf{g}$.

(1) Determine the evolution operator $\widehat{\mathcal{U}}(+\infty, z)$ given by (48) which specifies the dynamics of the nonunitary effective quantum system corresponding to the interaction-picture Hamiltonian $\widehat{\boldsymbol{\mathcal{H}}}(z)$.

(2) Calculate the fundamental transfer matrix $\widehat{\mathbf{M}}$ and the four-component field $\mathbf{D}$, which are respectively given by (49) and (51).

(3) Solve the integral equation (54) for the four-component field $\mathbf{C}_-$.

4. Read off the expressions for $\vec{C}_-^-$, $\vec{G}_-$, and $\mathbf{g}_-$ using (62), (65), and (67), and identify $\mathbf{g}$ with $\mathbf{g}_-$.

We successfully applied our method to describe the radiation of an oscillating source placed in a medium consisting of a regular or irregular planar array of nonmagnetic, possibly anisotropic and active or lossy point scatterers. For this system, which is relevant to the study of nanoparticles with extremely large refractive indices [42], the determination of $\mathbf{g}$ requires the solution of a linear system of $2N$ algebraic equations (119), where N is the number of point scatterers, and the evaluation of the integral in (139) which seems not to admit an explicit expression in terms of the known functions. This is clearly not a major problem, for we can compute it numerically. We can also find the numerical solution of (119) for large arrays consisting of as many as hundreds of point scatterers.

A distinctive feature of our treatment of point scatterers is that it avoids the singularities of their traditional treatments [23,24]. This is among the main difficulties in dealing with the radiation problem in the presence of point scatterers which we have been able to circumvent.

ACKNOWLEDGMENTS

This work was supported by the Scientific and Technological Research Council of Türkiye (TÜBİTAK) in the framework of Project No. 120F061 and by the Turkish Academy of Sciences (TÜBA).

[1] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999).

[2] R. G. Newton, *Scattering Theory of Waves and Particles*, 2nd ed. (Dover, New York, 2013).

[3] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics* (American Mathematical Society, Providence, RI, 2005).

[4] R. de L. Kronig and W. G. Penney, Quantum mechanics of electrons in crystal lattices, Proc. R. Soc. London Ser. A **130**, 499 (1931).

[5] E. Fermi, Sul moto dei neutroni nelle sostanze idrogebare, Ric. Sci. **7**, 13 (1936) [On the motion of neutrons in hydrogenous substances, in *E. Fermi Collected Papers*, Vol. 1, *Italy, 1921–1938* (University of Chicago Press, Chicago, 1962), pp. 980–1016].

[6] L. L. Foldy, The multiple scattering of waves. I. General theory of isotropic scattering by randomly distributed scatterers, Phys. Rev. **67**, 107 (1945).

[7] E. H. Lieb and Q. Liniger, Exact analysis of an interacting Bose gas. I. The general solution and the ground state, Phys. Rev. **130**, 1605 (1963).

[8] E. H. Lieb and Q. Liniger, Exact analysis of an interacting Bose gas. II. The excitation spectrum, Phys. Rev. **130**, 1616 (1963).

[9] J.-P. Antoine, P. Exner, and P. Seba, A mathematical model of heavy-quarkonia mesonic decays, Ann. Phys. (NY) **233**, 1 (1994).

[10] J. M. Cerveró and R. Rodríguez, Infinite chain of N different deltas: A simple model for a quantum wire, Eur. Phys. J. B **30**, 239 (2002).

[11] M. T. Batchelor, X. W. Guan, and A. Kundu, One-dimensional anyons with competing δ-function and derivative δ-function potentials, J. Phys. A **41**, 352002 (2008).

[12] H. Ghaemi-Dizicheh, A. Mostafazadeh, and M. Sarisaman, Spectral singularities and tunable slab lasers with 2D material coating, J. Opt. Soc. Am. B **37**, 2128 (2020).

[13] S. Flügge, *Practial Quantum Mechanics I* (Springer, New York, 1971).

[14] A. Mostafazadeh, Transfer matrix in scattering theory: A survey of basic properties and recent developments, Turk. J. Phys. **44**, 472 (2020).

[15] C. Thorn, Quark confinement in the infinite-momentum frame, Phys. Rev. D **19**, 639 (1979).

[16] R. Jackiw, Delta-function potentials in two- and three-dimensional quantum mechanics, in *M. A. B. Beg Memorial Volume*, edited by A. All and P. Hoodbhoy (World Scientific, Singapore, 1991), pp. 25–42.

[17] L. R. Mead and J. Godines, An analytical example of renormalization in two-dimensional quantum mechanics, Am. J. Phys. **59**, 935 (1991).

[18] C. Manuel and R. Tarrach, Perturbative renormalization in quantum mechanics, Phys. Lett. B **328**, 113 (1994).

[19] S. K. Adhikari and T. Frederico, Renormalization Group in Potential Scattering, Phys. Rev. Lett. **74**, 4572 (1995).

[20] S. Adhikari, T. Frederico, and R. M. Marinho, Lattice discretization in quantum scattering, J. Phys. A **29**, 7157 (1996).

[21] I. Mitra, A. DasGupta, and B. Dutta-Roy, Regularization and renormalization in scattering from Dirac delta potentials, Am. J. Phys. **66**, 1101 (1998).

[22] H. Bui and A. Mostafazadeh, Geometric scattering of a scalar particle moving on a curved surface in the presence of point defects, Ann. Phys. (NY) **407**, 228 (2019).

[23] P. de Vries, D. V. van Coevorden, and A. Lagendijk, Point scatterers for classical waves, Rev. Mod. Phys. **70**, 447 (1998).

[24] D. P. Challa, G. Hu, and M. Sini, Multiple scattering of electromagnetic waves by finitely many point-like obstacles, Math. Models Methods Appl. Sci. **24**, 863 (2014).

[25] F. Loran and A. Mostafazadeh, Fundamental transfer matrix and dynamical formulation of stationary scattering in two and three dimensions, Phys. Rev. A **104**, 032222 (2021).

[26] F. Loran and A. Mostafazadeh, Renormalization of multi-delta-function point scatterers in two and three dimensions, the coincidence-limit problem, and its resolution, Ann. Phys. (NY) **443**, 168966 (2022).

[27] F. Loran and A. Mostafazadeh, Fundamental transfer matrix for electromagnetic waves, scattering by a planar collection of point scatterers, and anti-$\mathcal{PT}$-symmetry, Phys. Rev. A **107**, 012203 (2023).

[28] J. B. Pendry, A transfer matrix approach to localisation in 3D, J. Phys. C **17**, 5317 (1984).

[29] J. B. Pendry, Transfer matrices and conductivity in two- and three-dimensional systems. I. Formalism, J. Phys.: Condens. Matter **2**, 3273 (1990).

[30] J. B. Pendry and P. M. Bell, Transfer matrix techniques for electromagnetic waves, in *Photonic Band Gap Materials*, edited by C. M. Soukoulis, NATO Advanced Studies Institute, Series E, Applied Sciences (Springer, Dordrecht, 1996), Vol. 315, pp. 203–228.

[31] F. Loran and A. Mostafazadeh, Perfect broad-band invisibility in isotropic media with gain and loss, Opt. Lett. **42**, 5250 (2017).

[32] F. Loran and A. Mostafazadeh, Exactness of the Born approximation and broadband unidirectional invisibility in two dimensions, Phys. Rev. A **100**, 053846 (2019).

[33] F. Loran and A. Mostafazadeh, Class of exactly solvable scattering potentials in two dimensions, entangled-state pair generation, and a grazing-angle resonance effect, Phys. Rev. A **96**, 063837 (2017).

[34] D. W. Berreman, Optics in stratified and anisotropic media: 4×4-matrix formulation, J. Opt. Soc. Am. **62**, 502 (1972).

[35] J. J. Sakurai, *Modern Quantum Mechanics* (Addison-Wesley, New York, 1994).

[36] F. Loran and A. Mostafazadeh, Transfer-matrix formulation of the scattering of electromagnetic waves and broadband invisibility in three dimensions, J. Phys. A **53**, 165302 (2020).

[37] D. J. Griffiths, *Introduction to Electrodynamics* (Pearson, Essex, 2014).

[38] A. Mostafazadeh, Spectral Singularities of Complex Scattering Potentials and Infinite Reflection and Transmission Coefficients at Real Energies, Phys. Rev. Lett. **102**, 220402 (2009).

[39] S. Longhi, Spectral singularities and Bragg scattering in complex crystals, Phys. Rev. A **81**, 022102 (2010).

[40] A. Mostafazadeh, Optical spectral singularities as threshold resonances, Phys. Rev. A **83**, 045801 (2011).

[41] H. Ghaemi-Dizicheh, A. Mostafazadeh, and M. Sarısaman, Nonlinear spectral singularities and laser output intensity, J. Opt. **19**, 105601 (2017).

[42] X. Cao, A. Ghandriche, and M. Sini, The electromagnetic waves generated by a cluster of nanoparticles with high refractive indices, arXiv:2209.02413.