Leggett-Garg inequalities in the quantum field theory of neutrino oscillations

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We investigate Leggett-Garg inequalities for neutrino oscillations in the quantum field theoretical setting. We derive an exact flavor-mass uncertainty relation and prove that this uncertainty product yields an upper bound to the violation of the inequalities. The relation between temporal nonclassicality and quantum uncertainty realizes the Lüders upper bound to the violation of the Leggett-Garg inequalities in quantum field theory, analogous to the Tsirelson upper bound to the violation of the Bell inequalities. By studying the problem both in the exact field-theoretical setting and in the limiting quantum mechanical approximation, we show that on average the inequalities are violated more often and more strongly in quantum field theory than in quantum mechanics.

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I. INTRODUCTION

Neutrino physics covers a very broad spectrum of applications in different fields of scientific investigation [1–4]. In particular, the mixed structure of elementary particles has significant implications in the framework of quantum information and quantum resource theory. Indeed, flavor mixing is closely associated with the entanglement of single-particle states and other nonlocal features, both bipartite and multipartite [5–11].

In the framework of quantum field theory, flavor mixing features important aspects which differentiates it from the usual quantum mechanical scheme [3,4,12–17]. For example, in the flavor Fock space approach we employ here, one can consider the unitary inequivalence between the flavor and mass Fock spaces which can be built from the respective vacuum states [18–23], and the possible dynamical origin of mixing, arising from *D*-branes or chirally symmetric models [24,25]. Other important developments concern the relation between neutrino physics and gravity. These include the study of neutrinos in curved backgrounds, in astrophysical regimes as well as in connection with the cornerstone principles of general relativity, including general covariance and the equivalence principle. For a comprehensive review of these topics, see Ref. [26].

Following a seminal work on Leggett-Garg inequalities in particle mixing [27], the quantum nature of neutrino oscillations has been probed with the MINOS experiment data by observing the violation of a reduced Leggett-Garg type inequality derived in the single-particle quantum mechanical approximation of quantum field theory [28]. Leggett-Garg inequalities are a central object in fundamental quantum physics and quantum information science, as they can be viewed as the temporal analog of the Bell inequalities quantifying spatial quantum nonlocality [29–33].

Among other possible applications [34–36], Leggett-Garg type inequalities have been proposed as a tool to discriminate between Dirac and Majorana fermions by studying neutrino oscillations in matter within phenomenological models of dissipative environments [37]. Moreover, different forms of the inequalities have been discussed in the study of three-flavor neutrino oscillations [38,39] and in the investigation of possible nonstandard neutrino interactions beyond the standard model [40].

Despite the fundamental insight into temporal nonclassicality that they could provide at the microscopic scale, Leggett-Garg inequalities in elementary particle physics have been mostly studied in the quantum mechanical approximation rather than in the exact framework of quantum field theory. In this paper, we address Leggett-Garg inequalities in the full generality of the quantum field theory of neutrino mixing and oscillations [18–23], an ideal playground for the analysis of temporal nonclassicality.

The first main achievement of the present work is that Leggett-Garg temporal inequalities in neutrino physics are intimately related to quantum uncertainty. We obtain this result by observing that in flavor oscillation processes there exists a conserved charge besides the one associated with total lepton number conservation, and that this quantity is the mass-charge operator. From the noncommutativity of lepton and mass charges and using the Robertson-Schrödinger prescription, we obtain a flavor-mass uncertainty relation that mirrors

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the Leggett-Garg inequality in Wigner form [41]. Indeed, the Wigner form of the inequalities turns out to be more robust than the standard ones under unsharp measurements [39,41], and more suitable for the study of neutrino oscillations, being entirely expressed in terms of the observable oscillation probabilities [39].

The second main achievement, a direct consequence of the first one, is that the flavor-mass uncertainty product is an upper bound to the violation of the Leggett-Garg inequality in neutrino oscillations. This represents an upper limit to the violation of the temporal inequalities, the so-called Lüders bound, yielding the temporal analog of the Tsirelson upper bound to the violation of the spatial Bell inequalities in the quantum field theoretical framework of neutrino mixing. Finally, by comparing the different expressions, we show that the Leggett-Garg inequalities are typically violated more generically and more frequently in quantum field theory than in quantum mechanics.

The paper is organized as follows: In Sec. II we introduce the mass-charge operator and the flavor-mass uncertainty relations. In Sec. III we compute WLGI for neutrino oscillations and we compare the results in quantum field theory and quantum mechanics. Finally in Sec. IV we present the conclusions. In the Appendix, we give a brief introduction to neutrino mixing and oscillations in QFT.

II. NEUTRINO MIXING AND FLAVOR-MASS UNCERTAINTY

A. Flavor charges and mass-charge operator

Consider the Lagrangian density for two flavor Dirac neutrino fields:

$$\mathcal{L}(x) = (\bar{\nu}_e(x) \quad \bar{\nu}_\mu(x)) \left(i\partial\!\!\!/ - \begin{pmatrix} m_e & m_{e\mu} \\ m_\mu & m_{e\mu} \end{pmatrix} \right) \begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \end{pmatrix}, \quad (1)$$

which is the free part of the weak interaction Lagrangian in the *flavor* basis [42]. This expression can be diagonalized in the *mass* basis [1]

$$\begin{pmatrix} \nu_e(x) \\ \nu_\mu(x) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1(x) \\ \nu_2(x) \end{pmatrix}, \quad (2)$$

where $\tan 2\theta = 2m_{e\mu}/(m_\mu - m_e)$. The mass and flavor representations are unitarily inequivalent representations of the fermionic anticommutation relations [18–23], and in the following we will work in the flavor basis.

One can verify that the Lagrangian (1) is invariant under the action of the group $U(1)\times U(1)$ [24]. Hence, the ensuing conserved charges are

$$Q = \sum_{\sigma=e,\mu} \int d^3\mathbf{x} : \nu_\sigma^\dagger \, \nu_\sigma :, \quad (3)$$

$$Q_M = \int d^3\mathbf{x} : \left(\nu_e^\dagger \quad \nu_\mu^\dagger\right) \begin{pmatrix} m_e & m_{e\mu} \\ m_\mu & m_{e\mu} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} : . \quad (4)$$

Here, Q is the generator of the global phase transformation $\nu_\sigma \to e^{i\alpha}\nu_\sigma$ associated with the *total lepton number* (indeed, it commutes with the neutrino production-detection vertex [2,42]), while Q_M is a *mass-charge operator*. The lepton charge can be decomposed as

$$Q = Q_e(t) + Q_\mu(t), \quad (5)$$

where

$$Q_\sigma(t) = \int d^3\mathbf{x} : \nu_\sigma^\dagger \, \nu_\sigma :, \quad \sigma = e, \mu, \quad (6)$$

are the flavor lepton charges [2,43].

B. Flavor-mass uncertainty relations

Based on the above, we can introduce an uncertainty relation stemming from the noncommutativity of the flavor charges Q_σ with Q_M. Consider the Robertson-Schrödinger uncertainty relation for Q_σ and Q_M:

$$\sigma_Q^2 \, \sigma_M^2 \geqslant \tfrac{1}{4} \, |\langle [Q_\sigma(t), \, Q_M] \rangle_\sigma|^2. \quad (7)$$

The above inequality defines a *flavor-mass uncertainty relation*.

Using Eq. (A16) one can expand the charges as

$$Q_\rho(t) = \sum_r \int d^3\mathbf{k} \left[\alpha^{r\dagger}_{\mathbf{k},\rho}(t)\, \alpha^r_{\mathbf{k},\rho}(t) - \beta^{r\dagger}_{\mathbf{k},\rho}(t)\, \beta^r_{\mathbf{k},\rho}(t)\right], \quad (8)$$

$$\begin{aligned} Q_M = {} & \sum_{r,\rho} m_\rho \int d^3\mathbf{k} \left[\alpha^{r\dagger}_{\mathbf{k},\rho}(t)\, \alpha^r_{\mathbf{k},\rho}(t) - \beta^{r\dagger}_{\mathbf{k},\rho}(t)\, \beta^r_{\mathbf{k},\rho}(t)\right] \\ & + m_{e\mu} \sum_{r,\sigma\neq\rho} \int d^3\mathbf{k} \, \big\{ |U_\mathbf{k}| \big[\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\, \alpha^r_{\mathbf{k},\rho}(t) \\ & - \beta^{r\dagger}_{\mathbf{k},\sigma}(t)\, \beta^r_{\mathbf{k},\rho}(t)\big] + \epsilon^r |V_\mathbf{k}| \left[\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\, \beta^{r\dagger}_{-\mathbf{k},\rho}(t) \right. \\ & \left. - \beta^r_{-\mathbf{k},\sigma}(t)\, \alpha^r_{\mathbf{k},\rho}\right] \big\}, \end{aligned} \quad (9)$$

where $\epsilon^r \equiv (-1)^r$, and the coefficients $|U_\mathbf{k}|$, $|V_\mathbf{k}|$ are given by the expressions (A8) and (A20).

Furthermore, it is worth noting that ${}_\sigma\langle|Q_M|\rangle_\sigma = m_\sigma$, where $\langle\cdots\rangle_\sigma = \langle\nu^r_{\mathbf{k},\sigma}|\cdots|\nu^r_{\mathbf{k},\sigma}\rangle$. Bearing this in mind, computing the variances, one finds

$$\sigma_Q^2 = \langle Q_\sigma^2(t)\rangle_\sigma - \langle Q_\sigma(t)\rangle_\sigma^2 = \mathcal{Q}_{\sigma\to\sigma}(t)[1 - \mathcal{Q}_{\sigma\to\sigma}(t)], \quad (10)$$

$$\sigma_M^2 = \langle Q_M^2\rangle_\sigma - \langle Q_M\rangle_\sigma^2 = m_{e\mu}^2. \quad (11)$$

The field-theoretical flavor oscillation probability is the expectation value of the flavor charges with respect to a reference flavor state [43], and its explicit form is reported in Eq. (A24). The evaluation of the right-hand side of Eq. (7) yields

$$\begin{aligned} |\langle [Q_\sigma(t), \, Q_M]\rangle_\sigma| &= m_{e\mu}\, \mathcal{C}(t) \\ &\equiv m_{e\mu} \sin(2\theta) |[|U_\mathbf{k}|^2 \sin(2\omega^-_\mathbf{k} t) \\ &\quad + |V_\mathbf{k}|^2 \sin(2\omega^+_\mathbf{k} t)]|, \end{aligned} \quad (12)$$

and thus the quantum field-theoretical flavor-mass uncertainty relation

$$\mathcal{F}_{\text{QFT}} \equiv \mathcal{Q}_{\sigma\to\sigma}(t)[1 - \mathcal{Q}_{\sigma\to\sigma}(t)] - \tfrac{1}{4}\mathcal{C}^2(t) \geqslant 0. \quad (13)$$

The ultrarelativistic limit is achieved when $m_i/|\mathbf{k}| \ll 1$ while keeping $\delta m^2 = m_2^2 - m_1^2 \neq 0$; under these conditions, one has $|U_\mathbf{k}| \approx 1$ and $|V_\mathbf{k}| \approx 0$, thus implying that the exact field-theoretical expression $\mathcal{F}_{\text{QFT}}$ reduces to the quantum mechanical approximation $\mathcal{F}_{\text{QM}}$:

$$\mathcal{F}_{\text{QM}} \equiv \mathcal{P}_{\sigma\to\sigma}(t)[1 - \mathcal{P}_{\sigma\to\sigma}(t)] - \tfrac{1}{4}\mathcal{P}_{\sigma\to\rho}(2t) \geqslant 0. \quad (14)$$

Within such approximation, neutrino flavor states (A21) are just linear superpositions of (single-particle) definite-mass states [1] [see Eq. (A5)], and the quantum mechanical oscillation probability (A12) is recovered. Therefore, in this regime, Eq. (7) becomes

$$\mathcal{P}_{\sigma\to\rho}(t)\left[1-\mathcal{P}_{\sigma\to\rho}(t)\right] \geqslant \tfrac{1}{4}\mathcal{P}_{\sigma\to\rho}(2t), \quad \sigma \neq \rho. \tag{15}$$

The flavor-mass uncertainty relation yields that the lower bound on the transition probability product at a generic time t is set by the flavor oscillation probability at time $2t$.

In the following, we will show that this fact unambiguously establishes the existence of time correlations for flavor neutrinos in terms of Leggett-Garg temporal inequalities and their relation with the flavor-mass uncertainty principle derived above.

III. LEGGETT-GARG INEQUALITIES IN QUANTUM FIELD THEORY

Consider an observer performing measurements on a dichotomic variable having outputs ± 1 at different times t_0, t_1, t_2. Let $O(t)$ be an operator which quantifies such observable ($O|\pm\rangle = m|\pm\rangle$, with $m = \pm 1$). One can then introduce temporal relations, analogs of the Bell inequalities, that are known as the *Leggett-Garg* temporal inequalities [29]. Various forms of the latter can be obtained, by working with the joint probabilities $P(m_i, m_j)$ on the measurements performed at three different times t_0, t_1, t_2 [33,41].

In what follows, we investigate temporal inequalities in the field theory framework for neutrino oscillations. We will establish that temporal inequalities are violated more frequently and more generically in quantum field theory than in the quantum mechanical approximation, thus showing that quantum field theory is more nonlocal than quantum mechanics in the time regime. Moreover, the flavor-mass uncertainty product provides an upper bound to the violation of the Leggett-Garg inequalities, thus realizing the Lüders temporal analog of the Tsirelson upper limit on the violation of the Bell inequalities [44–47]. Indeed, the quantum mechanical upper bound to the standard Leggett-Garg inequality, typically referred to as the Lüders bound [29,32,48–50], holds also for the Wigner form of the Leggett-Garg inequality [41], with the temporal correlation functions belonging to the standard set of inequalities replaced by a probabilistic representation of macrorealism.

First, let us focus on $O(t) = Q_3(t) \equiv [Q_e(t) - Q_\mu(t)]$ [i.e., the flavor charges defined in Eq. (6)]. One can verify that $Q_3(t)|\nu^r_{\mathbf{k},e}(t)\rangle = |\nu^r_{\mathbf{k},e}(t)\rangle$ and $Q_3(t)|\nu^r_{\mathbf{k},\mu}(t)\rangle = -|\nu^r_{\mathbf{k},\mu}(t)\rangle$. Hence, this operator is a dichotomic variable quantifying the neutrino flavor. Without loss of generality, let us assume that a muonic neutrino is produced at time $t_0 = 0$. After that, two measurements are performed at $t_1 = t$ and $t_2 = 2t$.

The Leggett-Garg temporal inequality in Wigner form [39] reads

$$P(m_1, m_2) - P(m_0, m_1) - P(-m_0, m_2) \leqslant 0, \tag{16}$$

where $m_0 = m_1 = m_2 = 1$. After some algebra, in the exact quantum field-theoretical formulation we obtain

$$\mathcal{W}_{\mathrm{QFT}} \equiv \mathcal{Q}_{e\to e}(t)\,\mathcal{Q}_{\mu\to e}(t) - \mathcal{Q}_{\mu\to e}(2t) \leqslant 0. \tag{17}$$

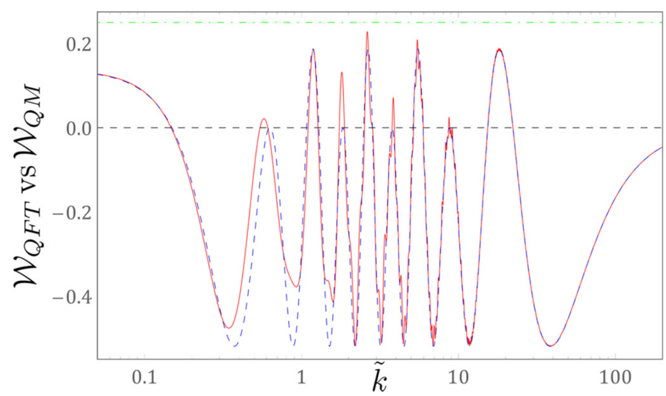

FIG. 1. Violation of the Leggett-Garg inequality in Wigner form, as a function of the dimensionless parameter $\tilde{k} \equiv |\mathbf{k}|/\sqrt{m_1 m_2}$, in quantum field theory (red solid line) and in quantum mechanics (blue dashed line), for sample values $m_1 = 2$, $m_2 = 30$, $t = 1$, and $\theta = \pi/6$. Horizontal dot-dashed green line: Lüders bound $\mathcal{W}^{\max}_{\mathrm{QFT}} = \mathcal{W}^{\max}_{\mathrm{QM}} = 0.25$. All plotted quantities are dimensionless.

In the ultrarelativistic regime, the limiting quantum mechanical approximation to Eq. (17) reads

$$\mathcal{W}_{\mathrm{QM}} \equiv \mathcal{P}_{e\to e}(t)\,\mathcal{P}_{\mu\to e}(t) - \mathcal{P}_{\mu\to e}(2t) \leqslant 0, \tag{18}$$

which coincides with the standard form used for the investigation of temporal correlations in neutrino physics [39].

In Fig. 1, we report $\mathcal{W}_{\mathrm{QFT}}$ and $\mathcal{W}_{\mathrm{QM}}$ as functions of $|\mathbf{k}|$, for given values of the masses and of the time t (which here plays the rôle of the baseline). As expected, $\mathcal{W}_{\mathrm{QFT}}$ and $\mathcal{W}_{\mathrm{QM}}$ coincide in the ultraviolet and in the infrared limits, with a significant deviation occurring in the intermediate regime $|\mathbf{k}| \approx \sqrt{m_1 m_2}$. The pronounced nonclassical behavior of quantum field theory is consistent with the general result of Refs. [51,52] obtained for the Bell inequalities in quantum field theory. The quantum field theoretical violation of the Leggett-Garg inequality appears more *generic* than the quantum mechanical one, meaning that it is realized for a wider class of states and a larger set of physical parameters. For instance, from Fig. 1 one can observe that there are many more momentum intervals over which the violation occurs at the field theory level (17), but does not occur at the quantum mechanical level (18).

Next, we investigate the relation between the temporal inequalities and the flavor-mass uncertainty (15). By using Eq. (A12), one can immediately observe that the inequality in Eq. (15) is saturated for $\theta = 0$, $\theta = \pi/2$ (no mixing) and $\theta = \pi/4$ (maximal mixing). In Fig. 2 we report the behavior of $\mathcal{W}_{\mathrm{QFT}}$, $\mathcal{F}_{\mathrm{QFT}}$, $\mathcal{W}_{\mathrm{QM}}$, and $\mathcal{F}_{\mathrm{QM}}$. In all cases, the uncertainty product is an upper bound to the violation of the Leggett-Garg inequality, both in quantum field theory and in quantum mechanics. In the quantum mechanical regime the bound is a trivial consequence of the fact that, denoting $\mathcal{F} - \mathcal{W} \equiv \Delta$, we have

$$\Delta_{\mathrm{QM}} = \tfrac{3}{4}\,\mathcal{P}_{\sigma\to\rho}(2t) \geqslant 0. \tag{19}$$

For $\theta = \pi/4$, $\mathcal{F}_{\mathrm{QM}}$ is identically vanishing and the temporal inequalities are never violated, since $\mathcal{W}_{\mathrm{QM}} \leqslant 0$. Indeed, for this value of the mixing angle neutrino states are

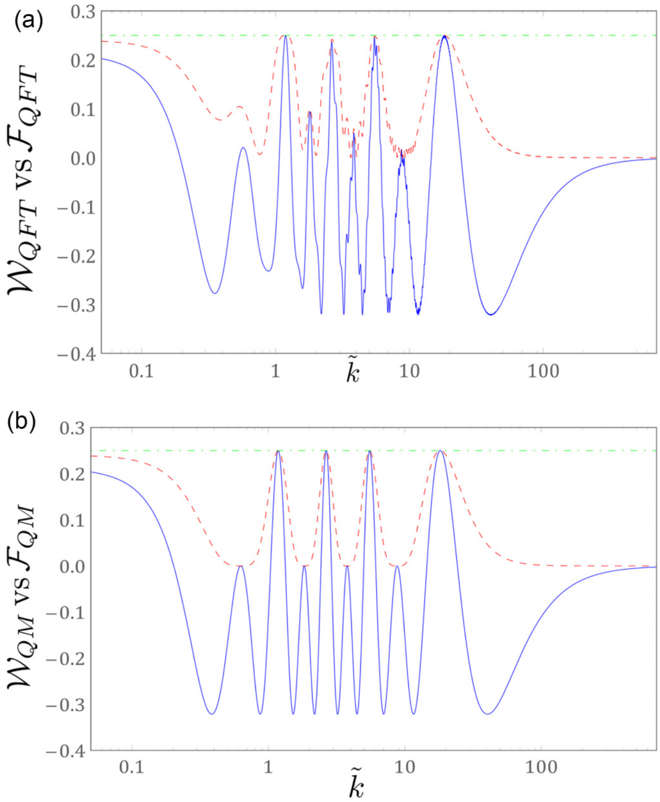

FIG. 2. (a) Behavior of $\mathcal{W}_{\mathrm{QFT}}$ (blue solid line) and of $\mathcal{F}_{\mathrm{QFT}}$ (red dashed line) as functions of the dimensionless parameter $\tilde{k} \equiv |\mathbf{k}|/\sqrt{m_1 m_2}$, for sample values $m_1 = 2$, $m_2 = 30$, $t = 1$, and $\theta = \pi/8$. (b) Behavior of $\mathcal{W}_{\mathrm{QM}}$ (blue solid line) and of $\mathcal{F}_{\mathrm{QM}}$ (red dashed line) for the same sample values. In both panels the horizontal dot-dashed green line is the Lüders bound $\mathcal{W}^{\max}_{\mathrm{QFT}} = \mathcal{W}^{\max}_{\mathrm{QM}} = 0.25$. All plotted quantities are dimensionless.

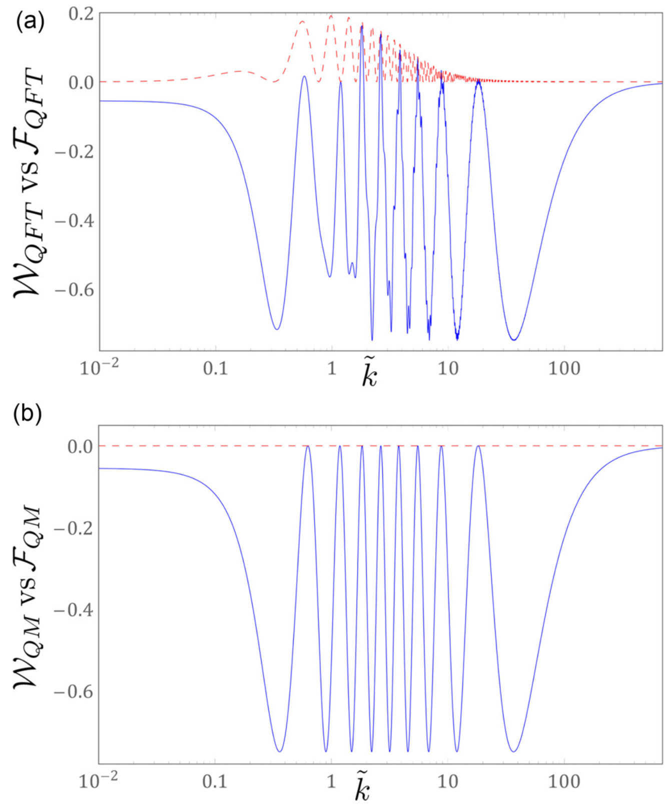

FIG. 3. (a) Behavior of $\mathcal{W}_{\mathrm{QFT}}$ (blue solid line) and of $\mathcal{F}_{\mathrm{QFT}}$ (red dashed line) as functions of the dimensionless parameter $\tilde{k} \equiv |\mathbf{k}|/\sqrt{m_1 m_2}$, for sample values $m_1 = 2$, $m_2 = 30$, $t = 1$ and $\theta = \pi/4$. (b) Behavior of $\mathcal{W}_{\mathrm{QM}}$ (blue solid line) and of $\mathcal{F}_{\mathrm{QM}}$ (red dashed line) for the same sample values. All plotted quantities are dimensionless.

near-*classical* coherent states that minimize the flavor-mass uncertainty product.

Turning to quantum field theory, consider the quantity

$$\begin{aligned} 4\,\Delta_{\mathrm{QFT}} = \sin^2(2\theta)[&\sin^2(\omega^-_{\mathbf{k}}t)\,|U_{\mathbf{k}}|^2(4-|U_{\mathbf{k}}|^2) \\ &+ \sin^2(\omega^+_{\mathbf{k}}t)\,|V_{\mathbf{k}}|^2(4-|V_{\mathbf{k}}|^2) \\ &- 2|V_{\mathbf{k}}|^2\,|U_{\mathbf{k}}|^2\,\sin(\omega^+_{\mathbf{k}}t)\,\sin(\omega^-_{\mathbf{k}}t)]. \end{aligned} \tag{20}$$

Recalling that $|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$ for any $\mathbf{k}$, we obtain

$$\begin{aligned} \sin^2(2\theta)\{&|V_{\mathbf{k}}|^2\,|U_{\mathbf{k}}|^2[\sin(\omega^-_{\mathbf{k}}t) - \sin(\omega^+_{\mathbf{k}}t)]^2 \\ &+ 3|V_{\mathbf{k}}|^2\,\sin^2(\omega^+_{\mathbf{k}}t) + 3\,|U_{\mathbf{k}}|^2\,\sin^2(\omega^-_{\mathbf{k}}t)\} \geqslant 0. \end{aligned} \tag{21}$$

Therefore, $\mathcal{F}_{\mathrm{QFT}}$ is always an upper bound to $\mathcal{W}_{\mathrm{QFT}}$.

It is straightforward to verify that both $\mathcal{F}_{\mathrm{QFT}}$ and $\mathcal{F}_{\mathrm{QM}}$ feature a global maximum equal to $1/4$ in correspondence of the mixing angle $\theta = \pi/8$. Such value realizes the Lüders bound both in the quantum mechanical and in the quantum field theoretical setting, fixing the maximum allowed quantum violation of the Leggett-Garg inequality $\mathcal{W}^{\max}_{\mathrm{QFT}} = \mathcal{W}^{\max}_{\mathrm{QM}} = 0.25$. This result has been achieved by means of the flavor-mass uncertainty relations, which provide an upper bound for the Wigner form (16) in both the relativistic and the nonrelativistic regimes. From Fig. 3 we observe that strong violations of the Leggett-Garg inequality in Wigner form occur in quantum field theory even when, as in the case of maximal flavor mixing $\theta = \pi/4$, there is no violation in the quantum mechanical limit, thereby certifying the higher degree of temporal nonclassicality of quantum field theory with respect to quantum mechanics.

Such stronger violation of the Leggett-Garg inequalities in quantum field theory with respect to quantum mechanics perfectly mirrors the case of the spatial Bell inequalities; for the latter case, Summers and Werner proved, in the full generality of algebraic quantum field theory [51,52], that Bell inequalities are always maximally violated in quantum field theory, even for the vacuum state, thus showing that quantum fluctuations of the vacuum cannot be reproduced with local hidden variable theories.

In neutrino oscillations, the flavor vacuum $|0\rangle_{e\mu}$ [18] is a nontrivial entangled state of neutrino-antineutrino pairs, and this feature provides the source of both the pronounced nonlocal behavior of flavor mixing in the temporal regime and the discrepancy between $\mathcal{W}_{\mathrm{QFT}}$ and $\mathcal{W}_{\mathrm{QM}}$ for $|\mathbf{k}| \approx \sqrt{m_1 m_2}$.

IV. DISCUSSION AND OUTLOOK

We have investigated the Leggett-Garg inequalities in the quantum field theory of neutrino oscillations. In analogy and in agreement with the general result holding for the Bell inequalities, we have found that the temporal inequalities exhibit a violation that is more nonclassical in quantum field theory than in quantum mechanics.

We have derived a flavor-mass uncertainty relation out of the mass-charge operator and the lepton charges. Such an uncertainty relation provides an upper bound to the violation of the Leggett-Garg inequalities at all energy scales, establishing a temporal analog of the Tsirelson bound for Bell inequalities in the case of neutrino oscillations.

It would be interesting to consider extensions to three-flavor mixing with *CP* violation and to nonstandard neutrino interactions [40], in order to probe signatures of new physics beyond the standard model phenomenology and to assess the possibility of applying the present formalism in different settings. For instance, we may think of the axion-photon mixing [53–57] and apply the same considerations developed in the present work to quantum optical frameworks.

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APPENDIX: NEUTRINO OSCILLATIONS: QUANTUM MECHANICS VERSUS QUANTUM FIELD THEORY

In a quantum field-theoretical setting, the starting point in the investigation of oscillation phenomena is provided by the mixing transformation

$$\nu_\sigma(x) = \sum_j U_{\sigma\, j}\, \nu_j(x), \tag{A1}$$

where ν_σ are the *flavor fields* (i.e., the ones involved in the weak interaction), ν_j are the mass fields (which describe neutrinos with definite masses), and U is the unitary mixing matrix.

It is commonly believed that a flavor Fock space can be built in the ultrarelativistic limit $m_j/|\mathbf{k}| \to 0$, where m_j are the neutrino masses [16]. Under these circumstances, annihilation operators are defined as

$$\tilde{\alpha}^r_{\mathbf{k},\sigma} = \sum_j U^*_{\sigma\, j}\, \alpha^r_{\mathbf{k},j}, \tag{A2}$$

where $\alpha^r_{\mathbf{k},j}$ are the annihilation operator of fields with definite mass

$$\nu_j(x) = \sum_r \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \left[u^r_{\mathbf{k},j}(t)\, \alpha^r_{\mathbf{k},j} + v^r_{-\mathbf{k},j}(t)\, \beta^{r\dagger}_{-\mathbf{k},j}\right] e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{A3}$$

with $u^r_{\mathbf{k},j}(t) = u^r_{\mathbf{k},j} e^{-i\omega_{\mathbf{k},j}}$, $v^r_{-\mathbf{k},j}(t) = v^r_{-\mathbf{k},j} e^{i\omega_{\mathbf{k},j}}$, $\omega_{\mathbf{k},j} = \sqrt{|\mathbf{k}|^2 + m_j^2}$. Similar relations hold for $\beta^r_{\mathbf{k},j}$. Flavor states can thus be constructed as

$$\left|\nu^r_{\mathbf{k},\sigma}\right\rangle_P \equiv \tilde{\alpha}^{r\dagger}_{\mathbf{k},\sigma}|0\rangle, \tag{A4}$$

where $|0\rangle$ is the vacuum state, which is annihilated by $\alpha^r_{\mathbf{k},j}$ and $\beta^r_{\mathbf{k},j}$ (*mass vacuum*). These states are none other than the flavor states originally introduced by Pontecorvo and collaborators [1],

$$\left|\nu^r_{\mathbf{k},\sigma}\right\rangle_P = \sum_j U^*_{\sigma\, j}\left|\nu^r_{\mathbf{k},j}\right\rangle. \tag{A5}$$

In the relativistic limit, these are eigenstates of flavor charges (6) at fixed time,

$$\lim_{m_i/|\mathbf{k}|\to 0} Q_\sigma(0)\left|\nu^r_{\mathbf{k},\sigma}\right\rangle_P = \left|\nu^r_{\mathbf{k},\sigma}\right\rangle_P. \tag{A6}$$

However, this is not true at all energy scales. To see this, let us explicitly consider the two-flavor case; the mixing transformation takes the form (2). We can then evaluate the oscillation formula as the expectation value of the flavor charge on a reference neutrino state [43]

$$\begin{aligned}\widetilde{\mathcal{P}}_{e\to\mu}(t) &\equiv {}_P\!\left\langle\nu^r_{\mathbf{k},e}\right|Q_\mu(t)\left|\nu^r_{\mathbf{k},e}\right\rangle_P\\ &= \frac{\sin^2(2\theta)}{2}\{1 - |U_{\mathbf{k}}|\cos[(\omega_{\mathbf{k},1} - \omega_{\mathbf{k},2})t]\},\end{aligned} \tag{A7}$$

where $|U_{\mathbf{k}}| \equiv u^{r\dagger}_{\mathbf{k},1} u^r_{\mathbf{k},2}$:

$$|U_{\mathbf{k}}| = \mathcal{A}_{\mathbf{k}}\left(1 + \frac{|\mathbf{k}|^2}{(\omega_{\mathbf{k},1} + m_1)(\omega_{\mathbf{k},2} + m_2)}\right), \tag{A8}$$

$$\mathcal{A}_{\mathbf{k}} = \sqrt{\left(\frac{\omega_{\mathbf{k},1} + m_1}{2\omega_{\mathbf{k},1}}\right)\left(\frac{\omega_{\mathbf{k},2} + m_2}{2\omega_{\mathbf{k},2}}\right)}. \tag{A9}$$

In the ultrarelativistic limit, $|U_{\mathbf{k}}| \to 1$, and we recover the standard oscillation formula

$$\mathcal{P}_{e\to\mu}(t) = \sin^2(2\theta)\, \sin^2\left(\frac{\omega_{\mathbf{k},1} - \omega_{\mathbf{k},2}}{2}t\right). \tag{A10}$$

In particular, we observe that

$$\widetilde{\mathcal{P}}_{e\to\mu}(0) = \frac{\sin^2(2\theta)}{2}(1 - |U_{\mathbf{k}}|), \tag{A11}$$

which is physically inconsistent, because it entails that flavor is undefined even at $t = 0$. Note that, in terms of Pontecorvo states, the description of neutrino oscillations in the ultra-relativistic limit is equivalent to a quantum mechanical approach with the Hamiltonian given by

$$H = \sum_{j,\mathbf{k},r} \omega_{\mathbf{k},j}\left|\nu^r_{\mathbf{k},j}\right\rangle\!\left\langle\nu^r_{\mathbf{k},j}\right|, \tag{A12}$$

with which the quantum mechanical oscillation probability can be computed, yielding

$$P_{\sigma\to\rho}(t) = \left|{}_P\!\left\langle\nu^r_{\mathbf{k},\rho}\right|e^{iHt}\left|\nu^r_{\mathbf{k},\sigma}\right\rangle_P\right|^2, \tag{A13}$$

thus recovering the result (A10).

In order to solve such inconsistencies, notice that Eq. (2) can be equivalently rewritten as [18]

$$\nu_\sigma(x) = G^{-1}_\theta(t)\, \nu_j(x)\, G_\theta(t), \tag{A14}$$

with $(\sigma, j) = (e, 1), (\mu, 2)$, and $G_\theta(t)$ given by

$$G_\theta(t) = \exp\left\{\theta \int d^3\mathbf{x}\,[\nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x)]\right\}. \quad (A15)$$

From (A3) and (A14) it follows that flavor fields can be expanded as

$$\nu_\sigma(x) = \sum_r \int \frac{d^3\mathbf{k}}{(2\pi)^{\frac{3}{2}}}\left[u^r_{\mathbf{k},j}\,\alpha^r_{\mathbf{k},\sigma}(t) + v^r_{-\mathbf{k},j}\,\beta^{r\dagger}_{-\mathbf{k},\sigma}(t)\right]e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (A16)$$

with $(\sigma, j) = (e, 1), (\mu, 2)$, and where the flavor ladder operators are given by

$$\begin{pmatrix} \alpha^r_{\mathbf{k},\sigma}(t) \\ \beta^r_{-\mathbf{k},\sigma}(t) \end{pmatrix} = G_\theta^{-1}(t)\begin{pmatrix} \alpha^r_{\mathbf{k},j}(t) \\ \beta^r_{-\mathbf{k},j}(t) \end{pmatrix} G_\theta(t). \quad (A17)$$

In the Heisenberg picture, the *flavor vacuum* is

$$|0\rangle_{e,\mu} = G_\theta^{-1}(0)\;|0\rangle_{1,2}, \quad (A18)$$

where $|0\rangle_{1,2}$ denotes the mass vacuum [cf. Eq. (A4)] for the two-flavor case. One can easily verify that $|0\rangle_{e,\mu}$ is annihilated by the flavor operators defined in Eq. (A17). Moreover, one can prove that

$$\lim_{V\to\infty}\,{}_{1,2}\langle 0|0\rangle_{e,\mu} = \lim_{V\to\infty} e^{\frac{V}{(2\pi)^3}\int d^3\mathbf{k}\,\ln(1-\sin^2\theta|V_\mathbf{k}|^2)^2} = 0, \quad (A19)$$

where

$$|V_\mathbf{k}| = \mathcal{A}_\mathbf{k}\left(\frac{|\mathbf{k}|}{\omega_{\mathbf{k},1}+m_1} - \frac{|\mathbf{k}|}{\omega_{\mathbf{k},2}+m_2}\right), \quad (A20)$$

meaning that flavor and massive fields belong to unitarily inequivalent representations of the anticommutation relations.

Now, exact flavor eigenstates can be explicitly constructed as

$$\left|\nu^r_{\mathbf{k},\sigma}\right\rangle = \alpha^{r\dagger}_{\mathbf{k},\sigma}\,|0\rangle_{e,\mu}, \quad (A21)$$

where flavor operators are taken at reference time $t = 0$. One can prove that

$$Q_\sigma(0)\left|\nu^r_{\mathbf{k},\sigma}\right\rangle = \left|\nu^r_{\mathbf{k},\sigma}\right\rangle. \quad (A22)$$

The corresponding oscillation formula can be found by taking the expectation value of the flavor charges [43]

$$\mathcal{Q}_{\sigma\to\rho}(t) = \langle Q_\rho(t)\rangle_\sigma. \quad (A23)$$

Explicitly

$$\begin{aligned}\mathcal{Q}_{\sigma\to\rho}(t) &= \sin^2(2\theta)[|U_\mathbf{k}|^2\sin^2(\omega^-_\mathbf{k}t) + |V_\mathbf{k}|^2\sin^2(\omega^+_\mathbf{k}t)],\\ \mathcal{Q}_{\sigma\to\sigma}(t) &= 1 - \mathcal{Q}_{\sigma\to\rho}(t), \quad \sigma\neq\rho,\end{aligned} \quad (A24)$$

where $\omega^\pm_\mathbf{k} \equiv (\omega_{\mathbf{k},2} \pm \omega_{\mathbf{k},1})/2$. Note that

$$\mathcal{Q}_{\sigma\to\rho}(t) \approx \mathcal{P}_{\sigma\to\rho}(t) \quad \text{when } m_i/|\mathbf{k}| \to 0,\ \omega^-_\mathbf{k} \neq 0. \quad (A25)$$

We thus recover the usual phenomenological results in the ultrarelativistic limit.

[1] S. M. Bilenky and B. Pontecorvo, Phys. Rep. **41**, 225 (1978).
[2] S. M. Bilenky and S. T. Petcov, Rev. Mod. Phys. **59**, 671 (1987); **61**, 169(E) (1989); **60**, 575 (1988).
[3] M. Beuthe, Phys. Rep. **375**, 105 (2003).
[4] C. Giunti and C. Kim, *Fundamentals of Neutrino Physics and Astrophysics* (Oxford University Press, Oxford, 2007).
[5] M. Blasone, F. Dell'Anno, S. De Siena, and F. Illuminati, Europhys. Lett. **85**, 50002 (2009).
[6] M. Blasone, F. Dell'Anno, S. De Siena, M. Di Mauro, and F. Illuminati, Phys. Rev. D **77**, 096002 (2008).
[7] M. Blasone, F. Dell'Anno, S. De Siena, and F. Illuminati, Adv. High Energy Phys. **2014**, 359168 (2014).
[8] A. K. Alok, S. Banerjee, and S. U. Sankar, Nucl. Phys. B **909**, 65 (2016).
[9] K. Dixit, J. Naikoo, S. Banerjee, and A. Kumar Alok, Eur. Phys. J. C **79**, 96 (2019).
[10] K. Dixit, J. Naikoo, B. Mukhopadhyay, and S. Banerjee, Phys. Rev. D **100**, 055021 (2019).
[11] M. Blasone, F. Illuminati, G. G. Luciano, and L. Petruzziello, Phys. Rev. A **103**, 032434 (2021).
[12] A. Ioannisian and A. Pilaftsis, Phys. Rev. D **59**, 053003 (1999).
[13] E. K. Akhmedov and A. Y. Smirnov, Found. Phys. **41**, 1279 (2011).
[14] W. Grimus, J. Phys. G **47**, 085004 (2020).
[15] D. V. Naumov and V. A. Naumov, Phys. Part. Nuclei **51**, 1 (2020).
[16] G. Fantini, A. Gallo Rosso, F. Vissani, and V. Zema, Adv. Ser. Direct. High Energy Phys. **28**, 37 (2018).
[17] L. Smaldone and G. Vitiello, Universe **7**, 504 (2021).
[18] M. Blasone and G. Vitiello, Ann. Phys. (NY) **244**, 283 (1995); **249**, 363(E) (1996).
[19] K. C. Hannabuss and D. C. Latimer, J. Phys. A **33**, 1369 (2000).
[20] K. Fujii, C. Habe, and T. Yabuki, Phys. Rev. D **64**, 013011 (2001).
[21] C. R. Ji and Y. Mishchenko, Phys. Rev. D **65**, 096015 (2002).
[22] N. E. Mavromatos, S. Sarkar, and W. Tarantino, Mod. Phys. Lett. A **28**, 1350045 (2013).
[23] C. Y. Lee, Mod. Phys. Lett. A **35**, 2030015 (2020).
[24] M. Blasone, P. Jizba, N. E. Mavromatos, and L. Smaldone, Phys. Rev. D **100**, 045027 (2019).
[25] M. Blasone, P. Jizba, N. E. Mavromatos, and L. Smaldone, Phys. Rev. D **102**, 025021 (2020).
[26] G. G. Luciano and L. Petruzziello, Int. J. Mod. Phys. D **29**, 2043002 (2020).
[27] D. Gangopadhyay, D. Home, and A. S. Roy, Phys. Rev. A **88**, 022115 (2013).
[28] J. A. Formaggio, D. I. Kaiser, M. M. Murskyj, and T. E. Weiss, Phys. Rev. Lett. **117**, 050402 (2016).
[29] A. J. Leggett and A. Garg, Phys. Rev. Lett. **54**, 857 (1985).
[30] J. Kofler and C. Brukner, Phys. Rev. Lett. **99**, 180403 (2007).
[31] J. Kofler and C. Brukner, Phys. Rev. Lett. **101**, 090403 (2008).
[32] C. Emary, N. Lambert, and F. Nori, Rep. Prog. Phys. **77**, 016001 (2014).
[33] S. Kumari and A. K. Pan, Europhys. Lett. **118**, 50002 (2017).
[34] J. Naikoo, A. K. Alok, S. Banerjee, and S. U. Sankar, Phys. Rev. D **99**, 095001 (2019).
[35] S. Shafaq and P. Mehta, J. Phys. G **48**, 085002 (2021).
[36] X.-Z. Wang and B.-Q. Ma, Eur. Phys. J. C **82**, 133 (2022).

[37] M. Richter, B. Dziewit, and J. Dajka, Phys. Rev. D **96**, 076008 (2017).
[38] D. Gangopadhyay and A. S. Roy, Eur. Phys. J. C **77**, 260 (2017).
[39] J. Naikoo, S. Kumari, S. Banerjee, and A. Pan, J. Phys. G **47**, 095004 (2020).
[40] K. Dixit and A. K. Alok, Eur. Phys. J. Plus **136**, 334 (2021).
[41] D. Saha, S. Mal, P. K. Panigrahi, and D. Home, Phys. Rev. A **91**, 032117 (2015).
[42] M. Blasone, P. Jizba, and L. Smaldone, Phys. Rev. D **99**, 016014 (2019).
[43] M. Blasone, P. A. Henning, and G. Vitiello, Phys. Lett. B **451**, 140 (1999).
[44] B. S. Tsirelson, Lett. Math. Phys. **4**, 93 (1980).
[45] S. Wehner, Phys. Rev. A **73**, 022110 (2006).
[46] N. Linden, S. Popescu, A. J. Short, and A. Winter, Phys. Rev. Lett. **99**, 180502 (2007).
[47] M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter, and M. Żukowski, Nature (London) **461**, 1101 (2009).
[48] C. Budroni and C. Emary, Phys. Rev. Lett. **113**, 050401 (2014).
[49] A. Kumari, M. Qutubuddin, and A. K. Pan, Phys. Rev. A **98**, 042135 (2018).
[50] A. Kumari and A. K. Pan, Ann. Phys. (NY) **534**, 2100401 (2022).
[51] S. J. Summers and R. Werner, Commun. Math. Phys. **110**, 247 (1987).
[52] S. J. Summers and R. Werner, J. Math. Phys. **28**, 2440 (1987).
[53] G. G. Raffelt and D. S. P. Dearborn, Phys. Rev. D **36**, 2211 (1987).
[54] M. S. Turner, Phys. Rep. **197**, 67 (1990).
[55] M. Christensson and M. Fairbairn, Phys. Lett. B **565**, 10 (2003).
[56] A. Friedland, M. Giannotti, and M. Wise, Phys. Rev. Lett. **110**, 061101 (2013).
[57] A. Capolupo, I. De Martino, G. Lambiase, and A. Stabile, Phys. Lett. B **790**, 427 (2019).