Entanglement statistics of randomly interacting spins

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We investigate the entanglement in the ground state of systems comprising two and three qubits with random interactions. Since the Hamiltonians also contain deterministic one-body terms, by varying the interaction strength, one can continuously interpolate between deterministic separable eigenstates and fully random entangled eigenstates, with nontrivial intermediate behavior. Entanglement strongly depends on the underlying topology of the interaction among the qubits. Since W states correspond to a zero-measure set as compared to the set of Greenberger-Horne-Zeilinger (GHZ) states, in all investigated cases the ground states are of the latter type. However, for a certain class of interactions (nonseparable collective potential) high GHZ entanglement is produced, while for fully separable pairwise interactions the marginal GHZ ground states concentrate in the vicinity of W states.

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I. INTRODUCTION

Randomness and entanglement are two fundamental features of quantum mechanics. While the former can ultimately be traced back to Born's postulate (see Ref. [1] for a deeper discussion), the latter arises from the tensor structure of composite Hilbert spaces and the superposition principle [2], in the standard framework of quantum theory. From the perspective of quantum information science, both are valuable resources for executing communication and information processing tasks.

Apart from randomness with purely quantum origins, stochasticity may appear as a consequence of the complexity of interactions among the several parts of a physical system, as, for instance, in atomic nuclei [3]. This may be modeled by random Hamiltonians, which typically also generate entanglement. A direct approach to the problem is to bypass the analysis of Hamiltonians and study random states directly, by considering random rotations of a reference vector, for example. This sort of state has been studied analytically and numerically from various perspectives, mostly focusing on bipartite entanglement [4–11]. Recently, the entanglement properties of three-qubit random states have also been studied [12].

However, this kind of *completely* random state does not naturally occur in all physical situations involving randomness. A perhaps more natural problem is to study states associated with Hamiltonians which are partially random but retain some kind of structure, either through their eigenvectors or via dynamics. There have been many investigations in this direction, particularly in connection with the problem of thermalization [13–20].

In this work, we study entanglement in ground states of low-dimensional qubit systems in which interactions are fully random, but the Hamiltonians also contain deterministic one-body terms. The situation can be thought of as a spin lattice with random interactions, where each spin is subjected to a constant magnetic field. The relative intensity between one-body terms and interaction can be varied continuously, tuning the eigenstates from deterministic to fully random, with interesting intermediate phenomena.

We show that the statistical properties of bipartite and tripartite entanglement heavily depends on the topological nature of the interaction potentials.

In the next section we provide the necessary background on random matrices and entanglement quantification. Section III addresses the simple case of two qubits. Sections IV and V consider systems comprising three qubits. We provide closing remarks in Sec. VI.

II. PRELIMINARY CONCEPTS

A. Random matrices

Complex Hamiltonians have long been modeled as random matrices in a variety of systems, from nuclear physics to quantum dots, from microwave billiards to disordered media

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[21–23]. The simplest model is to enforce hermiticity but draw the samples according to a Gaussian measure,

$$P(\mathcal{H}) \propto e^{-\mathrm{Tr}(\mathcal{H}^2)/\sigma^2}, \tag{1}$$

where σ is the standard deviation of the distribution. This has two important consequences: first, the off-diagonal matrix elements are independent, identically distributed random variables; second, the ensemble is rotation invariant, in the sense that $P(U\mathcal{H}U^\dagger) = P(\mathcal{H})$ for any unitary transformation U. In the absence of any specific symmetries, the matrix is complex Hermitian without any further constraint. If it has dimension N, this is called the $GUE(N)$—Gaussian unitary ensemble [24].

Modeling the Hamiltonian as a random matrix means giving up the ambition of obtaining results that describe any specific system and, instead, focusing on properties that may be of universal validity, representative of systems that are typical in some sense, or set a null hypothesis against which any results or conjectures supposed to hold for a given system may be compared.

The eigenstates of a matrix from $GUE(N)$ may be arranged as a unitary matrix, and this is distributed uniformly in the unitary group. We will refer to this by the standard terminology of *Haar measure*. Basically, normalized eigenstates are uniformly distributed as points in the complex sphere [21].

The parameter σ controls the variance of the interactions, such that for small σ the matrix $\mathcal{H}$ becomes vanishingly small, while for large σ the matrix $\mathcal{H}$ may have very large elements, in the sense that

$$\langle \mathcal{H}_{ij}\rangle = 0, \quad \langle |\mathcal{H}_{ij}|^2\rangle = \sigma^2. \tag{2}$$

For a single spin, for example, with the total Hamiltonian given by $\hat{s}_z + V$, where $\hat{s}_z$ is the usual Pauli matrix and V is taken at random from the $GUE(2)$, the situation will be as follows: for small σ, the eigenvalues will be close to $\pm\frac{1}{2}$ and the eigenvectors will be close to $|0\rangle$ and $|1\rangle$; for large σ, the eigenvalues will be a pair of correlated random variables and the eigenstates will be random vectors with Haar measure.

B. Entanglement involving two and three qubits

Entanglement [2] quantification is a difficult problem and, in general, the degree of nonseparability embodied by a quantum state is not uniquely captured by a single figure of merit. Different quantifiers may lead to distinct orderings. However, when it comes to two or even three qubits, things simplify considerably. In the first case we employ the concurrence [25] as the entanglement quantifier, whereas in the second, more complex case we characterize nonseparability with both the concurrence (of bipartitions and reduced states) and the three-tangle [26].

Given an arbitrary two-qubit state represented by a density matrix ρ, its concurrence is determined by the eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, with $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4$, of the matrix $\rho\tilde{\rho}$, where $\tilde{\rho} = (\hat{s}_y \otimes \hat{s}_y)\rho^*(\hat{s}_y \otimes \hat{s}_y)$, and given by

$$C = \max\{\sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}, 0\}.$$

This reduces to $C = 2|ad - bc|$, for the general pure state $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$. The squared concurrence is often referred to in the literature as the two-tangle.

For a pure state $|\Psi\rangle$ of three qubits, the three-tangle, or residual entanglement, τ, is a quantifier of genuine three-partite entanglement, which is nonzero only for the class of Greenberger-Horne-Zeilinger (GHZ) states, vanishing for fully separable, biseparable, and W states [27]. It is given by

$$\tau = C_{1|23}^2 - C_{12}^2 - C_{13}^2, \tag{3}$$

where 1,2,3 are labels for the three qubits; 1|23 refers to the bipartition where 1 is a subsystem and 23 is the other subsystem, while 12 and 13 denote reduced systems, where 3 and 2 have been traced out, respectively. The concurrence of 1|23 is $C_{1|23} = 2\sqrt{\det\rho_1}$ [26], where $\rho_1 = \mathrm{Tr}_{23}(|\Psi\rangle\langle\Psi|)$.

Although not evident from the definition, the three-tangle is invariant under any permutation of the subsystems, being a feature of the whole system.

We will also refer to the total concurrence $C_t = C_{12} + C_{13} + C_{23}$. For any pure state of three qubits we have $C_t \leqslant 4/3$ (the equality holds if and only if $|\Psi\rangle$ is a maximally entangled W state) [27].

III. TWO-QUBIT RANDOM EIGENSTATES

We begin our study by considering a four-dimensional Hilbert space, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, with $\mathcal{H}_i = \mathbb{C}^{\otimes 2}$ being the Hilbert space of a spin, the total Hamiltonian being given by

$$H = H_1 + H_2 + V, \tag{4}$$

with

$$H_1 = \hat{s}_{z,1} \otimes \mathbb{1}_2, \quad H_2 = \mathbb{1}_1 \otimes \hat{s}_{z,2}, \tag{5}$$

where $\hat{s}_{z,i}$ stands for the Pauli operator in the z direction, acting on $\mathcal{H}_i$.

The interaction term V is random in one of the two following ways. Either (i) $V = V_1 \otimes V_2$, where V_i are taken from $GUE(2)$, or else (ii) $V = V_{12}$ is directly taken from $GUE(4)$.

To make meaningful comparisons between situations (i) and (ii), we must guarantee that the interaction term V has the same average magnitude for a given σ. That is to say, when we take $\sigma_{12} = \sigma$ as the standard deviation of the Gaussian ensemble related to V_{12}, we choose $\sigma_1 = \sigma_2 = \sqrt{\sigma}$ for V_1 and V_2, respectively. With this we get

$$\langle |(V_{12})_{ij}|^2\rangle = \langle |(V_1 \otimes V_2)_{ij}|^2\rangle \propto \sigma^2.$$

Considering $\sigma_1 = \sigma_2$ incurs no essential loss of generality, since one can easily show that the results depend only on the product $\sigma_1\sigma_2$ (which equals σ_{12}). So, in the remainder of this section we will consider a single deviation parameter σ, as described in the previous paragraph. Both kinds of interactions V vanish when $\sigma = 0$, leading to a direct sum total Hamiltonian $H_1 + H_2$ with a separable ground state.

In Fig. 1(a) we plot the average ground-state concurrence, computed from 10^5 realizations, as a function of the parameter σ.

In the regime $\sigma \ll 1$, we found that $\langle C\rangle \sim \sigma$, for both kinds of interactions. This asymptotics can easily be derived from perturbation theory, as follows. The ground state of $H = H_1 + H_2$ ($\sigma = 0$) is simply $|g\rangle = |00\rangle$, with eigenvalue $E_{\mathrm{ground}} = -2$. Since $\langle |(V_1 \otimes V_2)_{ij}|^2\rangle \propto \sigma^2$ ($\Rightarrow V_{ij} \propto \sigma$) and the first-order correction from nondegenerate perturbation theory is of the form $\sum_j (V_{gj}/E_{jg})|\psi_j^\perp\rangle$, with $E_{jg} = E_j -$

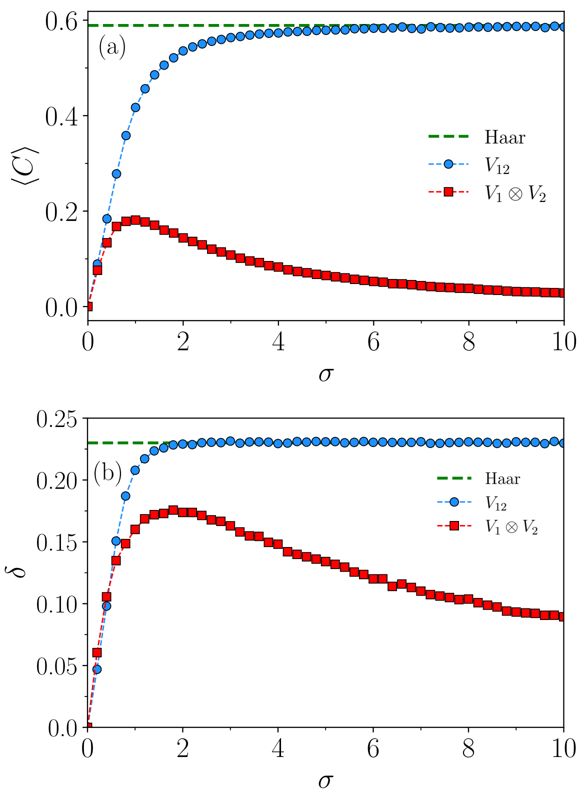

FIG. 1. (a) Average concurrence $\langle C \rangle$ as a function of σ with $N = 5 \times 10^4$ samples for the two types of interaction potential: V_{12} (blue circles) and $V_1 \otimes V_2$ (red squares). (b) Standard deviations of C as a function of σ. Horizontal lines indicate the results for Haar-random states.

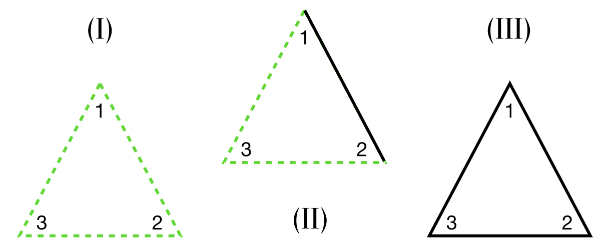

FIG. 2. Diagrams representing the three types of interactions considered in this section. In the triangles, qubits are the corners and interactions are the sides. Bold segments represent nonseparable interactions and dashed segments represent separable interactions.

E_{ground} and $\langle 00|\psi_j^{\perp}\rangle = 0$, the new ground state can be written as $|g'\rangle \propto |00\rangle + \sigma|g^{\perp}\rangle$ for small σ, where $|g^{\perp}\rangle$ is a ket in the subspace spanned by $\{|01\rangle, |10\rangle, |11\rangle\}$. Calculating the concurrence of this pure state leads to $C \propto |\sigma + O(\sigma^2)| \approx \sigma$ for sufficiently small σ. This justifies the linear behavior observed in the previous plots.

More interestingly, in case (i) the average concurrence attains a maximum for $\sigma = 1$ and vanishes asymptotically, whereas in situation (ii) it grows monotonically with σ, tending to the corresponding value for a Haar random state, which is given by [5]

$$\langle C \rangle_{\text{Haar}} = \frac{3\pi}{16} \approx 0.589. \quad (6)$$

These two distinct behaviors for different kinds of interaction can be understood as follows. When $\sigma \to \infty$, the terms H_1 and H_2 in the Hamiltonian become negligible compared to V, i.e., $H \approx V$. In case (i) this leads to a Hamiltonian with separable eigenstates, whereas in case (ii) the Hamiltonian remains nonfactorable. In the opposite limit of $\sigma = 0$, for both cases, the potential terms vanish and the Hamiltonian becomes $H_1 + H_2$, which gives rise to separable fundamental states. Therefore, in case (i), there must be at least one maximum. We do not have a rigorous proof of the fact that the maximum is located at $\sigma = 1$. However, there is a simple reasoning to justify it. In case (i), the entanglement is exclusively produced by the combined effect of both terms (single-body and two-body). Any of these terms alone leads to zero ground-state entanglement. It is thus expected that the entanglement attains a maximum when the two terms have equal weights, which corresponds to $\sigma = 1$.

In Fig. 1(b) we plot the concurrence standard deviation $\delta = \sqrt{\langle C^2\rangle - \langle C\rangle^2}$ [in the current context of two qubits we can see that the curves $\langle C\rangle(\sigma)$ and $\delta(\sigma)$ are very similar, but this is not always the case for three qubits].

As expected, under interaction V_{12} the quantity δ rapidly converges to the corresponding value of Haar random states (these have $\langle C^2\rangle_{\text{Haar}} = 2/5$, which leads to $\delta_{\text{Haar}} \approx 0.23$ [5,6]).

We note that, whereas in case (ii) the average concurrence is about twice as large as the corresponding standard deviation, in case (i) both $\langle C\rangle$ and δ have similar magnitudes for a given σ.

IV. THREE QUBITS, COLLECTIVE INTERACTION

Now we investigate the degree of entanglement of three-qubit ground states, thus, in an eight-dimensional Hilbert space, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, with $\mathcal{H}_i = \mathbb{C}^{\otimes 2}$, $i = 1, 2, 3$. The considered random Hamiltonians are given by

$$H = H_1 + H_2 + H_3 + V, \quad (7)$$

where $H_1 = \hat{s}_{z,1} \otimes \mathbb{1}_2 \otimes \mathbb{1}_3$, $H_2 = \mathbb{1}_1 \otimes \hat{s}_{z,2} \otimes \mathbb{1}_3$, and $H_3 = \mathbb{1}_1 \otimes \mathbb{1}_2 \otimes \hat{s}_{z,3}$.

In this case, a larger variety of nonequivalent potentials exist. Initially we study the following interactions:

$$V_{\text{I}} = V_1 \otimes V_2 \otimes V_3, \quad (8)$$

with each V_i in $GUE(2)$, fully separable;

$$V_{\text{II}} = V_{12} \otimes V_3, \quad (9)$$

where V_{12} and V_3 are in $GUE(4)$ and $GUE(2)$, respectively, partially separable; and

$$V_{\text{III}} = V_{123}, \quad (10)$$

in $GUE(8)$, fully nonseparable. See Fig. 2 for a schematic depiction of these interactions, which are all such that the three qubits interact collectively. In the next section we study pairwise interactions.

In order to characterize the ground state genuine three-partite entanglement we use the three-tangle (residual entanglement); the pairwise entanglement, via the concurrence of

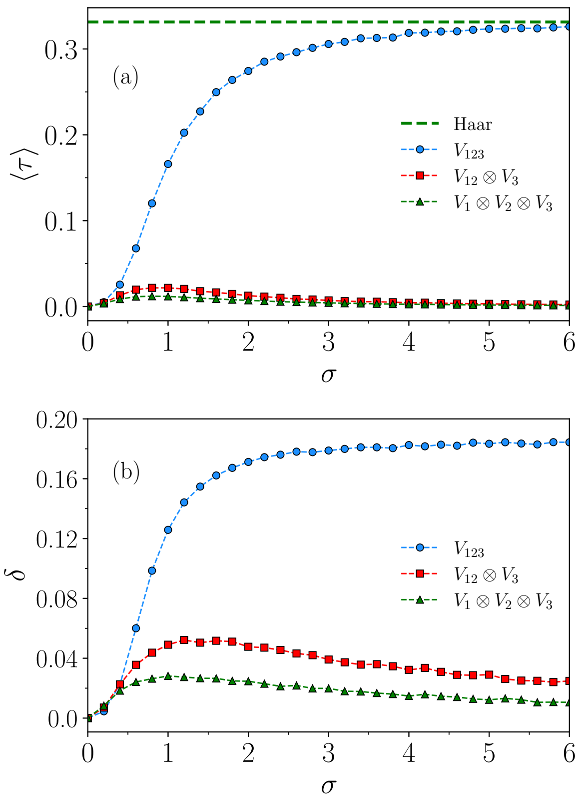

FIG. 3. (a) Average three-tangle $\langle\tau\rangle$ with 5×10^4 samples for different interactions. (b) Corresponding standard deviation δ.

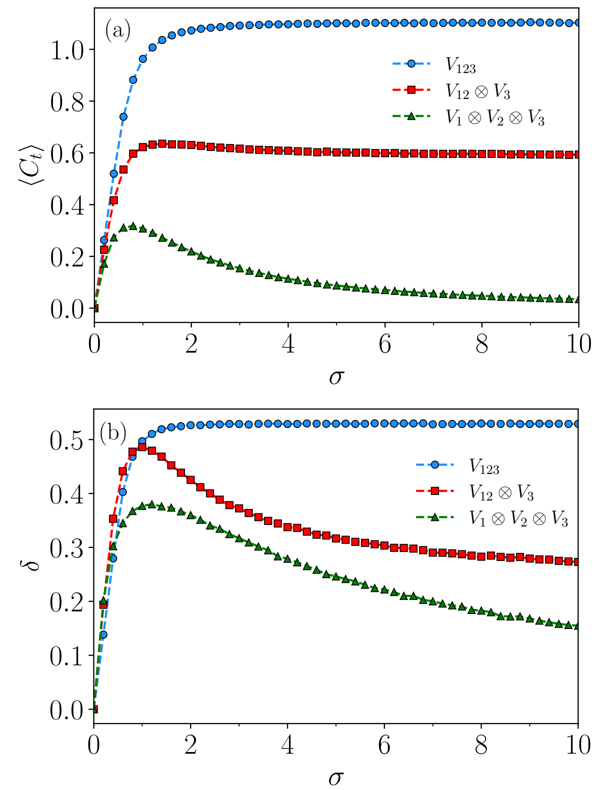

FIG. 4. (a) Total concurrence $\langle C_t\rangle$ with ρ_{12} and 15×10^4 samples for different interactions. (b) Corresponding standard deviation δ.

the reduced systems (12, 13, and 23); and the entanglement of the bipartitions, through the concurrence of the states related to the bipartitions (1|23, 12|3, and 13|2).

If we take the variance of case (III) to be $\sigma_{123} = \sigma$, then we should have $\sigma_{12}\,\sigma_3 = \sigma$ and $\sigma_1\,\sigma_2\,\sigma_3 = \sigma$, such that

$$\langle |(V_{\mathrm{I}})_{ij}|^2\rangle = \langle |(V_{\mathrm{II}})_{ij}|^2\rangle = \langle |(V_{\mathrm{III}})_{ij}|^2\rangle \propto \sigma^2.$$

The results only depend on the product of the involved deviations, so for definiteness we adopt the most symmetric choice: $\sigma_{12} = \sigma^{2/3}$ and $\sigma_3 = \sigma^{1/3}$ for case (II) and $\sigma_1 = \sigma_2 = \sigma_3 = \sigma^{1/3}$ for case (I).

In Fig. 3(a) we show the averaged three-tangle for the potentials (I), (II), and (III). The first feature that stands out is the much higher ability of the totally nonseparable interactions of type (III) to generate sizable values of τ, with $\langle\tau\rangle$ being more than an order of magnitude larger than that coming from cases (I) and (II).

In these latter cases, $\langle\tau\rangle$ has a maximum at $\sigma \approx 1$ and decays as $\sigma \to \infty$. This can be understood with the same reasoning as in the previous section: the one-body terms in (7) become negligible as $\sigma \to \infty$, and the ground states become either separable [case (I)] or biseparable [case (II)]. The three-tangle, as a genuine three-partite entanglement quantifier, becomes zero in both cases. In the opposite limit, $\sigma = 0$, the interactions vanish and the Hamiltonian becomes $H_1 + H_2 + H_3$, leading to nonentangled fundamental states. Again we do not have a closed analytical justification for the occurrence of the maximum at $\sigma \approx 1$, but the reasoning of balancedness between the one-body and three-body terms may also apply.

In case (III), $\langle\tau\rangle$ grows monotonically with σ and converges to the average three-tangle of pure, Haar-random three-qubit states, $\langle\tau\rangle_{\mathrm{Haar}} = 1/3$ [28].

The fact that $\langle\tau\rangle$ is very small for cases (I) and (II), at least two orders of magnitude smaller than 1 (the value of τ for a maximally entangled GHZ state), indicates that the generated ground states are either close to W states or to separable states. We shall return to this point later.

In Fig. 3(b) we show the corresponding standard deviations, which display qualitative behaviors similar to those of the average.

As a complementary characterization, in Fig. 4 we plot the average and standard deviation for the total concurrence $C_t = C_{12} + C_{13} + C_{23}$. Again, it is interaction of type (III) that leads to the highest values. Average total concurrence only decays with σ for the totally separable interaction of type (I), as expected. For sizable values of σ, the average is twice the standard deviation for types (III) and (II), but half that value for type (I).

V. THREE QUBITS, PAIRWISE INTERACTIONS

Here we study random interactions with a pairwise structure which are physically relevant, as, for instance,

$$V_a = \tfrac{1}{2}(\mathbb{1}_1 \otimes V_{23} + V_{12} \otimes \mathbb{1}_3), \tag{11}$$

$$V_b = \tfrac{1}{2}(\mathbb{1}_1 \otimes V_2 \otimes V_3 + V_1 \otimes V_2 \otimes \mathbb{1}_3), \tag{12}$$

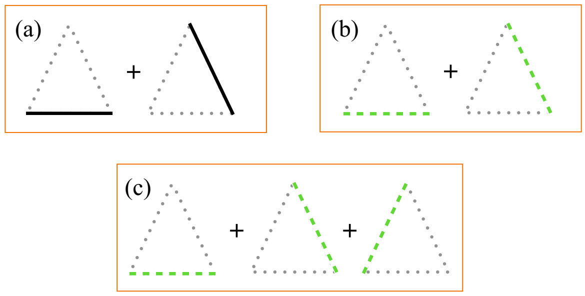

FIG. 5. Diagrams representing the three types of interactions considered in this section. We use the same convention as in Fig 2. Dotted gray segments denote absence of interaction.

such that no interaction between qubits 1 and 3 exists. In addition, we consider the more symmetric interaction

$$V_c = \tfrac{1}{3}(\mathbb{1}_1 \otimes V_2 \otimes V_3 + V_1 \otimes \mathbb{1}_2 \otimes V_3 + V_1 \otimes V_2 \otimes \mathbb{1}_3), \tag{13}$$

see Fig. 5 for a schematic depiction.

The multiplicative factors are there to make the average intensity of the full interaction term the same in all cases. For the same reason, we set $\sigma_{23} = \sigma_{12} = \sigma$ in V_a and V_b, and $\sigma_1 = \sigma_2 = \sigma_3 = \sigma^{1/2}$ in V_c.

In Fig. 6 we show the averaged three-tangle of the ground states coming from these interactions, along with the corresponding standard deviations. Here, for all investigated cases there is a very low degree of genuine three-partite entanglement of GHZ type, as compared to the potential $V_{\rm III}$, for which $\langle\tau\rangle$ is typically larger by one order of magnitude [see Fig. 4(a)].

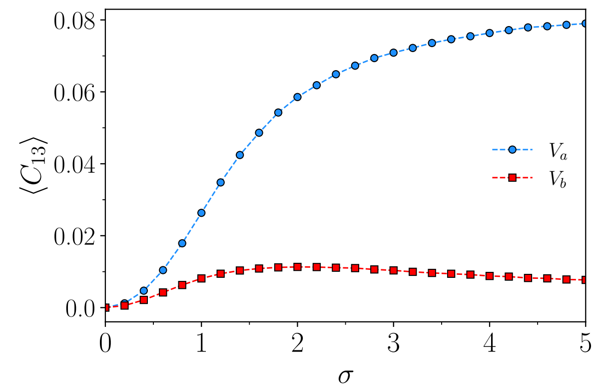

FIG. 7. Average concurrence $\langle C_{13}\rangle$ from 5×10^4 samples for the interactions V_a and V_b.

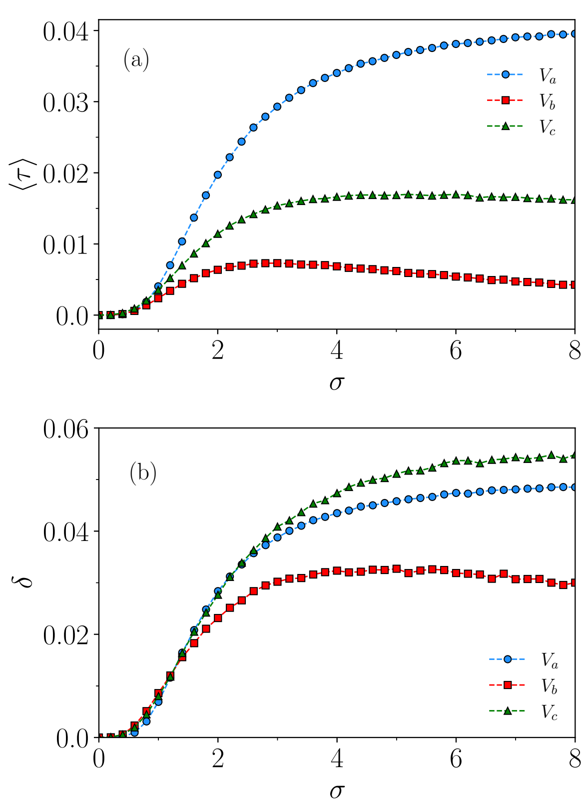

FIG. 6. (a) Average three-tangle $\langle\tau\rangle$ from 6×10^5 samples for the interactions V_a, V_b, and V_c. We find $\langle\tau\rangle \sim \sigma^3$ for small σ. (b) Corresponding standard deviation δ.

A curious feature in Fig. 6(b) is that, although the larger values of $\langle\tau\rangle$ are those produced by V_a, the deviations attached to V_c are consistently the larger ones. In all cases the averages and the deviations have the same order of magnitude.

Notice that in Fig. 6(a), the three-tangle remains close to zero for $0 < \sigma < 0.5$. In fact, we will show, based on numerical results, that $\langle\tau\rangle \sim \sigma^3$ for small σ. This will be relevant for future discussions.

A. Hamiltonian swapping

The two standard ways to produce entanglement are either to make the parties interact or to carry out a swapping operation (based on entangled measurements) [29,30]. If, however, we consider the Hamiltonian (7) with either potential V_a or V_b, no interaction occurs between qubits 1 and 3. Notwithstanding, the average ground state entanglement between these qubits is finite (although small).

This may be considered as an instance of entanglement swapping without measurement—see, for instance, Ref. [31]—since the correlation appears neither from a direct interaction nor from an entangling measurement (standard swapping). However, as can be seen in Fig. 7, the average concurrence between 1 and 3 is quite low, saturating around $\langle C_{13}\rangle \sim 0.001$ and $\langle C_{13}\rangle \sim 0.08$ for interactions V_a and V_b, respectively. Therefore, the states are close to being biseparable.

B. Concentration near W states

As we saw in the previous section, $V_{\rm III}$ gives rise to states with the highest average three-tangle among the investigated interactions. That is to say, this form of V generates GHZ states which are, on average, relatively far from the boundary with W states. The question arises whether there is a form of the random interaction that produces the latter states. The strict answer is negative because that would require fine tuning in order to ensure $\tau = 0$. In other words, in the space of parameters of three-qubit pure states (χ), the dimension of the subspace of GHZ states is the same as dim χ, while on the other hand, the subspace defined by $\tau = 0$ has lower

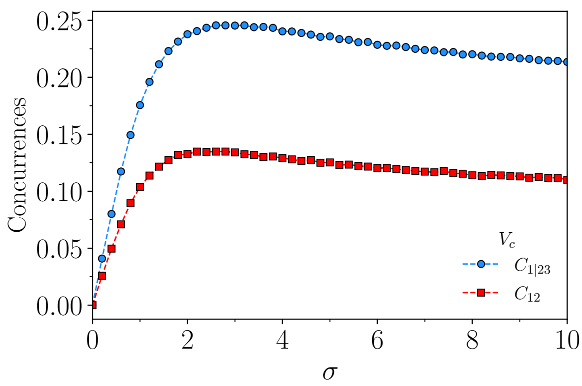

FIG. 8. $\langle C_{1|23}\rangle$ and $\langle C_{13}\rangle$ with 5×10^4 samples for the interaction potential V_c. Both quantities present a steep linear growth for small σ.

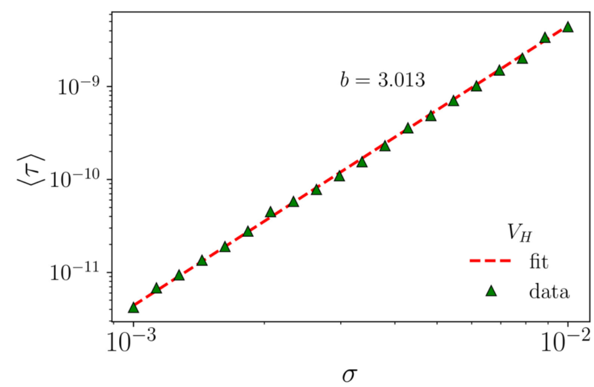

FIG. 9. Log-log plot of $\langle\tau\rangle$ against σ. We observe a power law $\sim\sigma^b$, with $b = 3.01$.

dimension. Therefore, the probability to generate a W state as an eigenvector of a random Hamiltonian is zero.

This can be understood in a more precise way by employing the optimal parametrization reported in Ref. [32]. For an arbitrary pure three-qubit state, $|\Psi\rangle$, there is always a basis for which one can write

$$|\Psi\rangle = a_0|000\rangle + a_1e^{i\varphi}|100\rangle + a_2|101\rangle + a_3|110\rangle + a_4|111\rangle,$$

where the coefficients a_j are non-negative real numbers, and $0 \leqslant \varphi \leqslant \pi$. It is easy to show that the corresponding three-tangle is given simply by $\tau = (2a_0a_4)^2$, so that $\tau = 0$ would require either $a_0 = 0$ or $a_4 = 0$ (or both). If $a_0 = 0$ and $a_4 \neq 0$, we typically get biseparable states. If $a_0 \neq 0$ and $a_4 = 0$, we typically obtain W states. Of course, these situations correspond to zero-measure sets, as compared to the set of GHZ states.

It is important, however, to note that the Haar measure does not correspond to a uniform distribution of the coefficients a_j. In particular, it is not correct to state that W and separable states are equally likely. Indeed, for the investigated Hamiltonians, the concurrences concerning any reduced density matrices and bipartitions are typically not zero, even for very low values of three-tangle.

Consider, for instance, the symmetric interaction potential V_c. It leads to ground states with $\tau < 0.05$ for any value of σ, as one can see in Fig. 6(a). On the other hand, the concurrences of bipartitions and reduced systems are sizable, as can be seen in Fig. 8, where $C_{12} = C_{13} = C_{23}$ and $C_{1|23} = C_{2|13} = C_{3|12}$, due to the mentioned symmetry. This indicates a possible close proximity to W states.

For small values of σ we found $C_{12} \sim \sigma$ and $C_{1|23} \sim \sigma$, from perturbation theory. Therefore, from Eq. (3), one might expect $\tau \sim \sigma^2$. However, the second-order term is dominated by the next-order term $\sim\sigma^3$. That is to say, in this region we may have very small three-tangle and not-so-small concurrences.

Let us carry out a more quantitative analysis. The linear regimes of the concurrences in Fig. 8, in the range $\sigma \in [0.01, 0.5]$, are found to be given by $\langle C_{1|23}\rangle \approx 0.198\,\sigma$ and $\langle C_{13}\rangle \approx 0.121\,\sigma$. By considering the symmetry of the potential, one can write the three-tangle as $\langle\tau\rangle = \langle C_{1|23}\rangle^2 - 2\langle C_{13}\rangle^2 \approx a\sigma^2 + O(\sigma^3)$, with $a \leqslant 10^{-3}$. Therefore, the cubic term dominates (see Fig. 9), and this explains the slow growth of $\langle\tau\rangle$ in this range, in contrast with the faster growth of the concurrences. The only possible conclusion is that the GHZ ground states are marginal, in the sense that they concentrate near the defining set of W states.

To support this conclusion, let us consider the fundamental states of V_c with $\sigma = 0.5$. We computed 10^6 ground states $|\Psi\rangle$, and calculated the overlap of each with the nearest state having $\tau = 0$ (numerically $\tau < 10^{-6}$), which we denote $|\Phi\rangle$. We get 99.89% of ground states with $p = |\langle\Psi|\Phi\rangle|^2 > 0.98$, see Fig. 10. For these states, $C_{12} < 0.4$, with about 14% of the states with $C_{12} > 0.1$. In addition, we found that $\tau < \min\{C_{12}^2, C_{13}^2, C_{23}^2\}$ for more than 99.99% of the states $|\Phi\rangle$. Usually, the three-tangle is several orders of magnitude smaller than the smallest two-tangle.

We conclude that the ground states are very close to being W states, and not so close to being separable. So, in practice, the GHZ states so generated are almost indistinguishable from the nearest W state. Although the formal difference between these two classes of states is well defined, we may find actual situations for which the numerical and practical distinction become difficult.

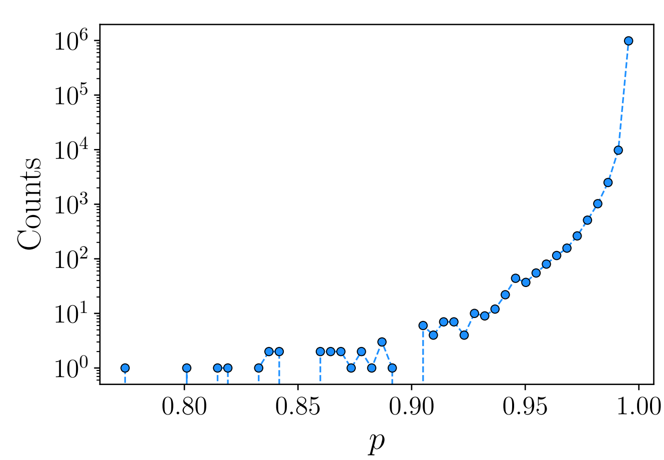

FIG. 10. Overlap $p = |\langle\Psi|\Phi\rangle|^2$ between the Hamiltonian's ground state and the nearest $\tau = 0$ state, for $\sigma = 0.5$, from 10^6 samples.

VI. CLOSING REMARKS

In this work we investigated the entanglement in the ground state of Hamiltonians containing deterministic one-body terms (spins subjected to a specified magnetic field) and random interaction terms, for two and three qubits. By varying the relative intensity of these contributions one can interpolate between separable and fully random (Haar distributed) states. It is clear that, although the characterization of Haar random states is relevant, restricting attention to them leaves a wealth of physically relevant situations unaddressed.

We found that the amount and nature of the resulting entanglement strongly depends on the underlying topology of the interaction terms. In the case of three qubits, we considered three types of collective interaction and three types of pairwise interaction, all differing in their degrees of separability. We found strong GHZ entanglement with a fully nonseparable collective interaction [a random matrix from $GUE(8)$], and the production of week GHZ entanglement, with ground states concentrating near the set of W states, for the opposite case of a fully separable pairwise interaction.

An interesting perspective is to increase the number of qubits in order to investigate more deeply the role of interaction topology in entanglement production. Where should we generically expect to find more multipartite entanglement, in the ground state of a fully connected network of spins or in a system with many pairwise interactions? The answer probably depends on the type of entanglement that is required.

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