Uncertainty relations in pre- and postselected systems

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(Received 25 August 2022; revised 22 May 2023; accepted 16 August 2023; published 7 September 2023)

In this work, we derive Robertson-Heisenberg–type uncertainty relation for two incompatible observables in a pre- and postselected (PPS) system. The newly defined standard deviation and the uncertainty relation in the PPS system have physical meanings which we present here. We demonstrate two unusual properties in the PPS system using our uncertainty relation. First, for commuting observables, the lower bound of the uncertainty relation in the PPS system does not become zero even if the initially prepared state, i.e., preselection, is the eigenstate of both the observables when specific postselections are considered. This implies that for such case, two commuting observables can disturb each other's measurement results which is in fully contrast with the Robertson-Heisenberg uncertainty relation. Second, unlike the standard quantum system, the PPS system makes it feasible to prepare sharply a quantum state (preselection) for noncommuting observables (to be detailed in the main text). Some applications of uncertainty and uncertainty relation in the PPS system are provided: (i) detection of mixedness of an unknown state, (ii) stronger uncertainty relation in the standard quantum system, (iii) "purely quantum uncertainty relation" that is, the uncertainty relation which is not affected (i.e., neither increasing nor decreasing) under the classical mixing of quantum states, (iv) state-dependent tighter uncertainty relation in the standard quantum system, and (v) tighter upper bound for the out-of-time-order correlation function.

DOI: 10.1103/PhysRevA.108.032206

I. INTRODUCTION

The uncertainty relation, which Heisenberg discovered, is one of the most well-known scientific findings [1,2]. It asserts that it is impossible to accurately measure the position and the momentum of a particle. In other words, measuring the position of a particle always affects the momentum of that particle and vice versa. Robertson developed the uncertainty relation known as "*Robertson-Heisenberg uncertainty relation*" (RHUR) [3] in the very later years to describe the difficulty of jointly sharp preparation of a quantum state (see Ref. [4] for the notion of *sharp preparation*) for incompatible observables. This relation not only limits the joint sharp preparation for noncommuting observables but also proved its usefulness: to formulate quantum mechanics [5,6], for entanglement detection [7,8], for the security analysis of quantum key distribution in quantum cryptography [9], as a fundamental building block for quantum mechanics and quantum gravity [10], etc.

On the one side, we have the standard quantum systems where the RHUR hold while pre- and postselected (PPS) systems, on the other side, are different kinds of quantum mechanical systems that were developed by Aharonov, Bergmann, and Lebowitz (ABL) [11–13] to address the issue of temporal asymmetry in quantum mechanics. Recently, in Refs. [14,15], the authors generalized the probabilities of obtaining the measurement results of an observable in a PPS system given by ABL [11].

In the later years, Aharonov, Albert, and Vaidman (AAV) [16] introduced the notion of "weak value" defined as

$$\langle A_w\rangle_\psi^\phi = \frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}, \tag{1}$$

in a pre- and postselected system when the observable A is measured weakly. Here, $|\psi\rangle$ and $|\phi\rangle$ are pre- and postselected states, respectively. Weak values have strange features for being complex and its real part can lie outside the max-min range of the eigenvalues of the operator of interest when the pre- and postselections are nearly orthogonal.

In order to obtain the real and imaginary parts of the weak value of A [17,18], first the system of interest and a pointer (ancilla) is prepared in the product state $|\psi\rangle\otimes|\xi\rangle$. Then the system pointer is evolved under the global unitary $U=\exp(-iHt)$, where $H=gA\otimes P_x$ is the von Nuemann Hamiltonian, A is the measurement operator of the system, P_x is the pointer's momentum observable, g is the coupling coefficient between system and pointer, and t is the interaction time. Now after the time evolution of the system-pointer, the system is projected to $|\phi\rangle$ and, as a result, the state of the pointer collapses to the unnormalized state $|\widetilde{\xi}_\phi\rangle\approx\langle\phi|\psi\rangle(1-igt\,\langle A_w\rangle_\psi^\phi P_x)\,|\xi\rangle$ in the limit $g\ll 1$, i.e., weak interaction. Now, it can be shown that the average position and momentum shifts of the pointer in state $|\xi_\phi\rangle=\frac{|\widetilde{\xi}_\phi\rangle}{\sqrt{\langle\widetilde{\xi}_\phi|\widetilde{\xi}_\phi\rangle}}$ are $\langle X\rangle_{\xi_\phi}=gt\,\mathrm{Re}(\langle A_w\rangle_\psi^\phi)$ and $\langle P_x\rangle_{\xi_\phi}=\frac{gt}{2\sigma^2}\,\mathrm{Im}(\langle A_w\rangle_\psi^\phi)$, respectively, with the Gaussian pointer $\langle x|\xi\rangle=(\frac{1}{\sqrt{2\pi}\sigma})^{1/2}e^{-x^2/4\sigma^2}$, σ is the root-

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mean-square (rms) width of the position distribution $|\langle x|\xi\rangle|^2$ of the pointer and thus providing the full knowledge of the weak value of A.

Recently, a lot of attention was paid to these aspects [19–40]. The measurements involving weak values are known as "weak measurements" or "weak PPS measurements." Since it depends on probabilistic postselection $|\phi\rangle$, a weak value can be thought of as conditional expectation value. Moreover, when the postselection is same as preselection, i.e., $|\phi\rangle = |\psi\rangle$, it becomes

$$\langle A\rangle_\psi = \langle\psi|A|\psi\rangle, \tag{2}$$

the expectation value in the standard quantum system. The PPS systems can therefore be thought of as being more general than the so-called standard quantum systems.

As pre- and postselected systems are already useful practically as well as fundamentally, then an immediate question can be asked whether there exists any uncertainty relation like the RHUR which can give the limitations on joint sharp preparation of the given pre- and postselected states when noncommuting observables are measured.

In this study, we demonstrate the existence of such uncertainty relations in PPS systems, which are expected as PPS systems are more generalized versions of standard ones. We first define the standard deviation of an observable in the PPS system for the given pre- and postselections with geometrical as well as physical interpretations. After that, we derive our main result of this paper "*uncertainty relations in pre- and postselected systems*" using the well known Cauchy-Schwarz inequality.

We provide the following physical applications of our results: (i) detection of the purity of an unknown state of any quantum systems (e.g., qubit, qutrit, two qubit, qutrit-qubit, etc.) using two different definitions of the uncertainty of an observable in the PPS system, (ii) stronger uncertainty relation in the standard quantum system (i.e., the uncertainty relation that can not be made trivial or the lower bound can not be made zero for almost all possible choices of initially prepared systems) using the uncertainty relation in the PPS system, (iii) purely quantum uncertainty relations, that is, the uncertainty relations which are not affected (i.e., neither increasing nor decreasing) under the classical mixing of quantum states using the uncertainty relations in PPS systems, (iv) state-dependent tighter uncertainty relation in the standard system by introducing the idea of postselection, and, finally (v) tighter upper bound for the out-of-time-order correlation function. Moreover, as the RHUR has plenty of applications, uncertainty relations in the PPS systems can also be applied in quantum optics, information, technologies, etc.

This paper is organized as follows. In Sec. II, we discuss uncertainty relations in standard quantum systems. In Sec. III, we derive our main results of this paper. Application of our results are given in Sec. IV and finally we conclude our work in Sec. V.

II. UNCERTAINTY RELATION IN STANDARD QUANTUM SYSTEM

In this section, we first interpret the standard deviation of an observable in standard quantum systems from geometrical as well as information-theoretic perspective. For establishing the standard deviation in a PPS system, we will introduce a similar interpretation. The RHUR's well-known interpretation is also provided here.

A. Standard deviation

We consider the system Hilbert space to be $\mathcal{H}$ and let $|\psi\rangle$ be a state vector in $\mathcal{H}$. Due to the probabilistic nature of the measurement outcomes of the observable A, the uncertainty in the measurement is defined as the standard deviation

$$\langle\Delta A\rangle_\psi = \sqrt{\langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2}. \tag{3}$$

Geometric interpretation. Standard deviation can be given a geometrical interpretation using the following proposition.

Proposition 1. If $|\psi\rangle \in \mathcal{H}$ is an initially prepared state of a standard quantum system and $A \in \mathcal{L}(\mathcal{H})$ is a Hermitian operator, then we can decompose $A|\psi\rangle \in \mathcal{H}$ as

$$A|\psi\rangle = \langle A\rangle_\psi |\psi\rangle + \langle\Delta A\rangle_\psi |\psi_A^\perp\rangle, \tag{4}$$

where $|\psi_A^\perp\rangle = \frac{1}{\langle\Delta A\rangle_\psi}(A - \langle A\rangle_\psi)|\psi\rangle$, and $\langle A\rangle_\psi = \langle\psi|A|\psi\rangle$. Equation (4) is sometimes known as the "Aharonov-Vaidman identity" [41].

Proof. Let $A|\psi\rangle$ and $|\psi\rangle$ are two nonorthogonal state vectors. Using Gram-Schmidt orthogonalization process, we find the unnormalized state vector $|\widetilde{\psi}_A^\perp\rangle \in \mathcal{H}$ orthogonal to $|\psi\rangle$ as

$$|\widetilde{\psi}_A^\perp\rangle = A|\psi\rangle - \frac{(\langle\psi|A)|\psi\rangle}{\langle\psi|\psi\rangle}|\psi\rangle = (A - \langle A\rangle_\psi)|\psi\rangle, \tag{5}$$

where $\langle\psi|\psi\rangle = 1$ and after normalization, Eq. (5) becomes

$$A|\psi\rangle = \langle A\rangle_\psi |\psi\rangle + \langle\Delta A\rangle_\psi |\psi_A^\perp\rangle, \tag{6}$$

where $|\psi_A^\perp\rangle = |\widetilde{\psi}_A^\perp\rangle/\sqrt{\langle\widetilde{\psi}_A^\perp|\widetilde{\psi}_A^\perp\rangle}$ and $\sqrt{\langle\widetilde{\psi}_A^\perp|\widetilde{\psi}_A^\perp\rangle} = \sqrt{\langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2} = \langle\Delta A\rangle_\psi$ and $\langle A\rangle_\psi = \langle\psi|A|\psi\rangle$. ■

So, Eq. (4) can be interpreted as the unnormalized state vector $A|\psi\rangle$ which has two components and these are $\langle A\rangle_\psi$ along $|\psi\rangle$ and $\langle\Delta A\rangle_\psi$ along $|\psi_A^\perp\rangle$. Here we interpret $\langle\Delta A\rangle_\psi$ as disturbance of the state vector due to the measurement of the operator A or as the measurement error (or standard deviation) of that operator when the system is prepared in the state $|\psi\rangle$. For instance, if we set up the system in one of the eigenstates of the observable A, then from Eq. (4), it can be seen that the standard deviation of A is zero.

Information-theoretic interpretation. From an information-theoretic approach, Eq. (3) can be written as

$$\langle\Delta A\rangle_\psi = \sqrt{\sum_{i=1}^{d-1}\left|\langle\psi_i^\perp|A|\psi\rangle\right|^2}, \tag{7}$$

where $\{|\psi\rangle, |\psi_1^\perp\rangle, |\psi_2^\perp\rangle, \ldots, |\psi_{d-1}^\perp\rangle\}$ forms an orthonormal basis such that $I = |\psi\rangle\langle\psi| + \sum_{i=1}^{d-1}|\psi_i^\perp\rangle\langle\psi_i^\perp|$ and "d" is the dimension of the system. So, the origin of the nonzero standard deviation $\langle\Delta A\rangle_\psi$ in the standard quantum system can also be thought of due to the nonzero contributions of the unnormalized fidelities $\{|\langle\psi_i^\perp|A|\psi\rangle|\}_{i=1}^{d-1}$ which can be viewed as the spread of the information of the observable A along $\{|\psi_i^\perp\rangle\}_{i=1}^{d-1}$ directions.

B. RHUR

The well-known RHUR for two noncommuting operators A and B on a Hilbert space $\mathcal{H}$ when the system is prepared in the state $|\psi\rangle$ is given by

$$\langle\Delta A\rangle_\psi^2 \langle\Delta B\rangle_\psi^2 \geqslant \left[\frac{1}{2i}\langle\psi|[A,B]|\psi\rangle\right]^2, \tag{8}$$

where $\langle\Delta A\rangle_\psi$ and $\langle\Delta B\rangle_\psi$ are the standard deviations of the operators A and B, respectively, and $[A,B]=AB-BA$ is the commutator of A and B. The derivation of Eq. (8) using the *Aharonov-Vaidman identity* can be found in [41,42]. The stronger version is obtained by adding the "Schrödinger's term" in Eq. (8) as

$$\langle\Delta A\rangle_\psi^2 \langle\Delta B\rangle_\psi^2 \geqslant \left[\frac{1}{2i}\langle\psi|[A,B]|\psi\rangle\right]^2 + \left[\frac{1}{2}\langle\psi|\{A,B\}|\psi\rangle - \langle A\rangle_\psi\langle B\rangle_\psi\right]^2. \tag{9}$$

The RHUR is usually interpreted as the following: it puts bound on the sharp preparation of a quantum state for two noncommuting observables. Hence, a quantum state in which the standard deviations of the two noncommuting observables are both zero cannot exist.

III. MAIN RESULTS

The idea of the standard deviation or information dispersion (see preceding section) is a crucial component of the theory in a preparation-measurement situation. Pre- and postselected systems are typical examples, therefore, we define the standard deviation (uncertainty) of an observable and show that for such systems, there exist RHUR-like uncertainty relations for two noncommuting observables.

A. Standard deviation in PPS system

Geometric definition. It is well known that when the preselection and the postselection are the same, the PPS system becomes the standard quantum system (see Introduction). The following proposition generalizes Eq. (4) for the PPS system.

Proposition 2. If a PPS system is in a preselected state $|\psi\rangle$ and postselected state $|\phi\rangle$, then for a Hermitian operator $A\in\mathcal{L}(\mathcal{H})$, we can decompose $A|\psi\rangle$ as

$$A|\psi\rangle = \langle\phi|A|\psi\rangle|\phi\rangle + \langle\Delta A\rangle_\psi^\phi|\phi_{A\psi}^\perp\rangle, \tag{10}$$

where

$$\langle\Delta A\rangle_\psi^\phi = \sqrt{\langle\psi|A^2|\psi\rangle - |\langle\phi|A|\psi\rangle|^2} \tag{11}$$

and $|\phi_{A\psi}^\perp\rangle = \frac{1}{\langle\Delta A\rangle_\psi^\phi}(A|\psi\rangle - \langle\phi|A|\psi\rangle|\phi\rangle)$, a normalized state vector which is orthogonal to $|\phi\rangle$.

Proof. We assume that $A|\psi\rangle$ and $|\phi\rangle$ are two nonorthogonal state vectors. The unnormalized state vector $|\widetilde{\phi}_{A\psi}^\perp\rangle\in\mathcal{H}$ which is orthogonal to $|\phi\rangle$ is obtained using Gram-Schmidt orthogonalization process as

$$\left|\widetilde{\phi}_{A\psi}^\perp\right\rangle = A|\psi\rangle - \frac{\langle\phi|(A|\psi\rangle)}{\langle\phi|\phi\rangle}|\phi\rangle = A|\psi\rangle - \langle\phi|A|\psi\rangle|\phi\rangle, \tag{12}$$

where $\langle\phi|\phi\rangle=1$ and after normalization, Eq. (12) becomes

$$A|\psi\rangle = \langle\phi|A|\psi\rangle|\phi\rangle + \langle\Delta A\rangle_\psi^\phi\left|\phi_{A\psi}^\perp\right\rangle,$$

where $|\phi_{A\psi}^\perp\rangle = |\widetilde{\phi}_{A\psi}^\perp\rangle/\sqrt{\langle\widetilde{\phi}_{A\psi}^\perp|\widetilde{\phi}_{A\psi}^\perp\rangle}$ and $\sqrt{\langle\widetilde{\phi}_{A\psi}^\perp|\widetilde{\phi}_{A\psi}^\perp\rangle} = \sqrt{\langle\psi|A^2|\psi\rangle - |\langle\phi|A|\psi\rangle|^2} = \langle\Delta A\rangle_\psi^\phi$. ■

To define the standard deviation of the observable A in the PPS system, we now present an argument which is similar to the one used to describe the standard deviation of an observable in a standard quantum system. So, Eq. (10) can be interpreted geometrically as the unnormalized state vector $A|\psi\rangle$ which has two components $\langle\phi|A|\psi\rangle$ along the postselection $|\phi\rangle$ and $\langle\Delta A\rangle_\psi^\phi$ along $|\phi_{A\psi}^\perp\rangle$. *Here we define* $\langle\Delta A\rangle_\psi^\phi$ *as the standard deviation of the observable A when the system is preselected in* $|\psi\rangle$ *and postselected in* $|\phi\rangle$.

The standard deviation $\langle\Delta A\rangle_\psi^\phi$ can be realized via the weak value of the observable A as

$$\langle\Delta A\rangle_\psi^\phi = \sqrt{\langle\psi|A^2|\psi\rangle - \left|\langle A_w\rangle_\psi^\phi\right|^2|\langle\phi|\psi\rangle|^2} \tag{13}$$

$$= \sqrt{\langle\Delta A\rangle_\psi^2 + \langle A\rangle_\psi^2 - \left|\langle A_w\rangle_\psi^\phi\right|^2|\langle\phi|\psi\rangle|^2}, \tag{14}$$

where $\langle A_w\rangle_\psi^\phi$ is the weak value of the observable A defined in Eq. (1) and we have used Eq. (3) to derive Eq. (14). $|\langle\phi|\psi\rangle|^2$ is the success probability of the postselection $|\phi\rangle$. Equation (13) is no longer a valid expression if pre- and postselected states are orthogonal to one another because in this situation, weak value is not defined. Then, go back to Eq. (11). It should be noted that Eq. (11) holds true whether the measurement is strong or weak.

Information-theoretic definition. Another expression of the standard deviation $\langle\Delta A\rangle_\psi^\phi$ in the PPS system can be derived by inserting an identity operator $I=|\phi\rangle\langle\phi| + \sum_{i=1}^{d-1}|\phi_i^\perp\rangle\langle\phi_i^\perp|$, where $\{|\phi\rangle, |\phi_1^\perp\rangle, |\phi_2^\perp\rangle, \ldots, |\phi_{d-1}^\perp\rangle\}$ forms an orthonormal basis in the first term of the right-hand side of Eq. (13) as

$$\langle\Delta A\rangle_\psi^\phi = \sqrt{\sum_{i=1}^{d-1}\left|\langle A_w\rangle_\psi^{\phi_i^\perp}\right|^2|\langle\phi_i^\perp|\psi\rangle|^2}. \tag{15}$$

From an information-theoretic perspective, Eq. (15) may now be understood as follows: nonzero standard deviation in the PPS system arises as a result of the nonzero contributions from the weak values $\{\langle A_w\rangle_\psi^{\phi_i^\perp}\}_{i=1}^{d-1}$ along the orthogonal postselections $\{|\phi_i^\perp\rangle\}_{i=1}^{d-1}$. Note that two consecutive measurements are taken into account in a PPS system: the operator of interest A and the projection operator $\Pi_\phi = |\phi\rangle\langle\phi|$ which corresponds to the postselection $|\phi\rangle$. As a result, it is hard to tell whether or not A has been measured when the weak value is zero. Because of this, it is crucial to have nonzero weak values which carry the information about the observable A. Null weak values have recently been given a useful interpretation [43]: if a successful postselection occurs with a null weak value, then the property represented by the observable A cannot be detected by the weakly coupled quantum pointer. In other words, the pointer state remains unchanged when the weak value is zero (see Introduction section). Thus, one should anticipate that the standard deviation in the PPS system should be zero if we obtain null weak values for the postselections $\{|\phi_i^\perp\rangle\}_{i=1}^{d-1}$, that means the information about the observable A is not dispersed throughout the postselections $\{|\phi_i^\perp\rangle\}_{i=1}^{d-1}$.

In addition to the standard deviation's geometrical and information-theoretical explanations [Eqs. (10) and (15), respectively] in the PPS system, we now study the minimum (zero) and maximum uncertainty (or standard deviation) which provide additional insights to understand the standard deviation.

Zero uncertainty. The uncertainty $\langle \Delta A\rangle_\psi^\phi$ defined in Eq. (11) in the PPS system is zero if and only if

$$A\,|\psi\rangle = \langle\phi_z|A|\psi\rangle\,|\phi_z\rangle\,, \tag{16}$$

or $|\phi_z\rangle \propto A\,|\psi\rangle$. We have used the notation $|\phi_z\rangle$ as the postselection for which uncertainty in PPS system becomes zero. The zero uncertainty in the PPS system can now be realized in the following way: the weak value $\langle A_w\rangle_\psi^{\phi_z}$ becomes nonzero i.e., $\frac{\langle\psi|A^2|\psi\rangle}{\langle\psi|A|\psi\rangle} \neq 0$ when we postselect the system to $|\phi_z\rangle$, and the weak values for all postselections $\{|\phi_{z_i}^\perp\rangle\}_{i=1}^{d-1}$ orthogonal to $|\phi_z\rangle$ are zero. As a result, the right side of Eq. (15) is reduced to zero. It should be noted that all postselections orthogonal to $|\phi_z\rangle$ are "legitimate postselections," meaning that their weak values are clearly specified. Equivalently, we can state that the information about the observable A is not dispersed along the postselections $\{|\phi_{z_i}^\perp\rangle\}_{i=1}^{d-1}$ as null weak values do not carry information about the observable A (according to the above information-theoretic definition). Hence, it is guaranteed that in a particular direction there will be one and only one nonzero weak value of A in a PPS system if and only if the condition (16) is met.

Usefulness of zero uncertainty state. In this paragraph we provide the following usefulness of the zero uncertainty postselected state $|\phi_z\rangle$.

(1) In a parameter estimation scenario, where the task is to obtain the precision limit in the estimation of interaction coefficient g in the interaction Hamiltonian $H = gA \otimes p$ (p is the pointer's momentum variable), Fisher information plays an important role whose maximum value is given by $F^{\max}(g) = 4\sigma^2\,\langle\psi|A^2|\psi\rangle$, where σ is the standard deviation of initial distribution of the pointer state and $|\psi\rangle$ is the initially prepared state of the system [44]. In an arbitrarily postselected state $|\phi\rangle$, Fisher information is given by $F_\phi(g) = 4\sigma^2|\langle\phi|A|\psi\rangle|^2 \leqslant F^{\max}(g)$ [44]. Recently, it was shown in Ref. [45] that the Fisher information can be expressed in terms of quasiprobability distribution of an arbitrary preselected state when the system is a PPS system. Although the quasiprobabilities are in general complex and can take negative values as well, the Fisher information is always positive and real. The Fisher information of the preselected state with negative quasiprobability distribution can surpass the usual quantum Fisher information (defined in standard quantum system). Violation of such limit implies that the error which occurs in estimating the unknown parameter can be reduced significantly using the Fisher information of the preselected state with negative quasiprobability distribution compared to the usual quantum Fisher information. One can immediately see using Eq. (11) that $F_\phi(g) = 4\sigma^2[\langle\psi|A^2|\psi\rangle - (\langle\Delta A\rangle_\psi^\phi)^2]$. Now it is obvious that for the zero-uncertainty postselected state $|\phi_z\rangle$ as appeared in Eq. (16), we have $F_{\phi_z}(g) = F^{\max}(g)$. Hence, to achieve the maximum Fisher information $F^{\max}(g)$ in the PPS system, one must postselect the system in $|\phi_z\rangle = A\,|\psi\rangle / \sqrt{\langle\psi|A^2|\psi\rangle}$ which corresponds to the zero uncertainty.

(2) The postselection $|\phi_z\rangle$ alone has the ability to provide the information (e.g., $\langle\Delta A\rangle_\psi$ and $\langle A\rangle_\psi$) about the observable A. Indeed by noting that $\langle\psi|A^2|\psi\rangle = p_z(\langle A_w\rangle_\psi^{\phi_z})^2$ and $\langle\psi|A|\psi\rangle = p_z\,\langle A_w\rangle_\psi^{\phi_z}$, we have $\langle\Delta A\rangle_\psi^2 = (1-p_z)p_z(\langle A_w\rangle_\psi^{\phi_z})^2$, where $p_z = |\langle\phi_z|\psi\rangle|^2$ is the probability of obtaining the postselection $|\phi_z\rangle = A\,|\psi\rangle / \sqrt{\langle\psi|A^2|\psi\rangle}$.

Maximum uncertainty. To achieve the maximum value of $\langle\Delta A\rangle_\psi^\phi$, the weak value $\langle A_w\rangle_\psi^\phi$ in Eq. (13) has to be zero, i.e., when the postselection $|\phi\rangle$ is orthogonal to $A\,|\psi\rangle$ and hence $\max(\langle\Delta A\rangle_\psi^\phi) = \sqrt{\langle\psi|A^2|\psi\rangle}$. Note that, in a preparation-measurement scenario, maximum measurement error is also found to be $\sqrt{\langle\psi|A^2|\psi\rangle}$ whether the measurement of the observable A is performed in standard system [see Eq. (3)] or while performing the best estimation the operator A from the measurement of another Hermitian operator [46].

B. Uncertainty relation in PPS system

After defining the standard deviation of an observable in a PPS system, interpreting it geometrically and informationally, and maintaining a parallel comparison and connection with the standard deviation in the standard system, we are now in a position to provide an uncertainty relation in a PPS system for two incompatible observables. One can formulate many different types of uncertainty relations in PPS systems (for example, [47]), but our interpretation of an uncertainty relation in a PPS system is based on the standard deviation defined in Eq. (11) or (13). Since the weak value of the observable A in the standard deviation (13) in the PPS system replaces the average value of the same observable A in the standard deviation (3) in standard quantum system, it is not surprising that the mathematical expression of the uncertainty relation in the PPS system is a modified version of the RHUR (8), where the average values of the incompatible observables A and B in Eq. (8) will be replaced by the weak values of the respective observables when the system is preselected in $|\psi\rangle$ and postselected in $|\phi\rangle$. The explicit form of the uncertainty relation in the PPS system is provided in the following theorem.

Theorem 1. Let $A, B \in \mathcal{L}(\mathcal{H})$ be two noncommuting Hermitian operators which are measured in the PPS system of our interest with $|\psi\rangle$ and $|\phi\rangle$ being pre- and postselected states, respectively, then the product of their standard deviations satisfies

$$\big(\langle\Delta A\rangle_\psi^\phi\big)^2\big(\langle\Delta B\rangle_\psi^\phi\big)^2 \geqslant \left[\frac{1}{2i}\,\langle\psi|[A,B]|\psi\rangle - \mathrm{Im}(W_{AB})\right]^2, \tag{17}$$

where $W_{AB}{=}\langle\psi|A|\phi\rangle\langle\phi|B|\psi\rangle{=}(\langle A_w\rangle_\psi^\phi)^*\,\langle B_w\rangle_\psi^\phi\,|\langle\phi|\psi\rangle|^2$ [using the definition of the weak value defined in Eq. (1)].

Proof. Cauchy-Schwarz inequality for two unnormalized state vectors $|\widetilde{\phi}_{A\psi}^\perp\rangle$ and $|\widetilde{\phi}_{B\psi}^\perp\rangle$ in $\mathcal{H}$ becomes

$$\langle\widetilde{\phi}_{A\psi}^\perp|\widetilde{\phi}_{A\psi}^\perp\rangle\langle\widetilde{\phi}_{B\psi}^\perp|\widetilde{\phi}_{B\psi}^\perp\rangle \geqslant |\langle\widetilde{\phi}_{A\psi}^\perp|\widetilde{\phi}_{B\psi}^\perp\rangle|^2. \tag{18}$$

Now, as $|\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle|^2 = [\mathrm{Re}(\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle)]^2 + [\mathrm{Im}(\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle)]^2$ and hence

$$|\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle|^2 \geqslant [\mathrm{Im}(\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle)]^2, \tag{19}$$

where $\mathrm{Im}(\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle) = \frac{1}{2i}(\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle - \langle\widetilde{\phi}^{\perp}_{B\psi}|\widetilde{\phi}^{\perp}_{A\psi}\rangle)$ and $\mathrm{Re}(\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle) = \frac{1}{2}(\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle + \langle\widetilde{\phi}^{\perp}_{B\psi}|\widetilde{\phi}^{\perp}_{A\psi}\rangle)$. Now put $|\widetilde{\phi}^{\perp}_{A\psi}\rangle = A\,|\psi\rangle - \langle\phi|A|\psi\rangle\,|\phi\rangle$ defined in Eq. (12) for operator A and similarly for operator B also, then we have

$$\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle = \langle\psi|AB|\psi\rangle - \langle\psi|A|\phi\rangle\,\langle\phi|B|\psi\rangle\,. \tag{20}$$

Note that $\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{A\psi}\rangle = (\,\langle\Delta A\rangle^{\phi}_{\psi}\,)^2$ is square of the standard deviation of the observable A in the PPS system defined in Eq. (11) and similarly $\langle\widetilde{\phi}^{\perp}_{B\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle = (\,\langle\Delta B\rangle^{\phi}_{\psi}\,)^2$ is square of the standard deviation of the observable B in the PPS system. Finally, putting these values and using Eqs. (19) and (20) in Eq. (18), it becomes Eq. (17). ■

Equation (17) is always true for any strong PPS systems [11–13] or weak PPS systems [16]. For weak PPS measurements [16], W_{AB} is expressed in terms of weak values of both the observables. If the pre- and postselected states are the same, i.e., $|\phi\rangle = |\psi\rangle$, then one gets back the RHUR (8) as argued before. Equation (17) with "Schrödinger's term" becomes

$$\left(\,\langle\Delta A\rangle^{\phi}_{\psi}\,\right)^2\left(\,\langle\Delta B\rangle^{\phi}_{\psi}\,\right)^2 \geqslant \left[\frac{1}{2i}\,\langle\psi|[A,B]|\psi\rangle - \mathrm{Im}(W_{AB})\right]^2 + \left[\frac{1}{2}\,\langle\psi|\{A,B\}|\psi\rangle - \mathrm{Re}(W_{AB})\right]^2. \tag{21}$$

The uncertainty relation (17) can be interpreted in the same way as we did for the RHUR (8). That is, it bounds the sharp preparation of the pair for pre- and postselections ($|\psi\rangle$, $|\phi\rangle$) for two noncommuting observables. The lower bound contains an additional term $\mathrm{Im}(W_{AB})$ compared to the RHUR (8). So even if $[A,B] \neq 0$, the bound on the right-hand side of Eq. (17) can become zero implying the possibility of both the standard deviations being zero implying further the possibility of sharp preparation of a pair of pre- and postselected states. Below, we provide the necessary and sufficient condition for such case (see *Observation 2*). Recently, the authors of [48,49] confirmed that in a PPS system using the ABL rule [11–13], it is possible to go beyond the standard lower bound in the RHUR for position and momentum observables. Not exactly, but a similar property, i.e., achieving arbitrary small lower bound (which depends on the pre- and postselections) of the product of standard deviations of two noncommuting observables in a PPS system is possible in the relations (17). We now explore two peculiar characteristics of the uncertainty relations (17) and (21) that cannot be observed in standard quantum systems.

Observation 1. If the lower bound in an uncertainty relation in any quantum system is nonzero, then we say that two incompatible observables disturb each others' measurement results. Now consider the following case. If $|\psi\rangle$ is a common eigenstate of both A and B, then $\langle\Delta A\rangle_{\psi} = 0$, $\langle\Delta B\rangle_{\psi} = 0$ implying that the measurement of one does not disturb the outcome of the other. Surprisingly, this property does not hold in the PPS system. Note that, even if $|\psi\rangle$ is a common eigenstate of both A and B, the lower bound of the relation (21) does not become zero for specific postselections which implies $\langle\Delta A\rangle^{\phi}_{\psi} \neq 0$, $\langle\Delta B\rangle^{\phi}_{\psi} \neq 0$. Hence, we can say that the measurement of A is invariably disturbed by the measurement of B or vice versa in a PPS system. In Ref. [50], Vaidman demonstrated the same property in a PPS system using the ABL rule.

Observation 2. With two noncommuting observables in the standard quantum system, sharp preparation of a quantum state is impossible. Or, equivalently, for an initially prepared state $|\psi\rangle$, it is impossible to have $\langle\Delta A\rangle_{\psi} = 0$, $\langle\Delta B\rangle_{\psi} = 0$ if $[A,B] \neq 0$. But in the PPS system, we can prepare *any* quantum state $|\psi\rangle$ which can give $\langle\Delta A\rangle^{\phi}_{\psi} = 0$, $\langle\Delta B\rangle^{\phi}_{\psi} = 0$ for a specific choice of postselection implying sharp preparation of $|\psi\rangle$ for noncommuting observables A and B. It is easy to show that both the uncertainties $\langle\Delta A\rangle^{\phi}_{\psi}$ and $\langle\Delta B\rangle^{\phi}_{\psi}$ are zero for the common postselection $|\phi_z\rangle$ if and only if

$$|\phi_z\rangle \propto A\,|\psi\rangle\,, \qquad |\phi_z\rangle \propto B\,|\psi\rangle\,.$$

After the normalization, we find the common postselection condition

$$|\phi_z\rangle = \frac{A\,|\psi\rangle}{\sqrt{\langle\psi|A^2|\psi\rangle}} = \frac{B\,|\psi\rangle}{\sqrt{\langle\psi|B^2|\psi\rangle}}, \tag{22}$$

up to some phase factors.

Example. Now, consider an example of two noncommuting observables $A = \frac{1}{\sqrt{2}}(I + \sigma_x)$ and $B = \frac{1}{\sqrt{2}}(\sigma_z + \sigma_x)$ with the initially prepared state $|0\rangle$. With these specific choices, it is possible to show that condition (22) is satisfied. Recall that in order to conduct the experiment using weak values, the average values of the observables must not be zero; for this reason, we did not take into account the Pauli observables σ_x and σ_y with initially prepared state $|0\rangle$. Nonetheless, if one does not adhere to weak values, this example is still true. So, the common postselection for this case is $|\phi_e\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and hence both the uncertainties $\langle\Delta A\rangle^{\phi_e}_{0}$ and $\langle\Delta B\rangle^{\phi_e}_{0}$ of the noncommuting observables A and B, respectively, are zero for the given initially prepared state $|0\rangle$ and the conditioned postselection $|\phi_e\rangle$ in Eq. (22). In the PPS system, it is now feasible to do the hitherto impossibly difficult task of jointly sharply preparing a quantum state for two noncommuting observables.

The aforementioned Observations 1 and 2 demonstrate that PPS systems are capable of being even stranger than their well-known unusual results, e.g., quantum Cheshire cats [24], measurement of a component of a spin-$\frac{1}{2}$ particle which can reach $100\hbar$ [16], etc.

Comments. The characteristics of the uncertainty relations (17) and (21) in PPS systems as compared to the RHUR (8) and Eq. (9) are substantially altered by the postselections. These uncertainty inequalities (17) and (21) will undoubtedly have applications like the RHUR for quantum foundations, information, and technologies. For instance, (i) they can be used for information extraction using commuting observables because the inequalities do not become trivial for particular choices of postselections, (ii) one can obtain a series of uncertainty inequalities by changing the postselections and that is advantageous for practical purposes (see stronger uncertainty relations in Sec. IV), (iii) existing applications of uncertainty relations (8) and (9) in standard systems, such as entanglement detection [7], quantum metrology [51,52], etc., can be revisited using uncertainty relations (17) and (21) in the PPS systems, (iv) PPS system based spin squeezing: spin-squeezed states are a class of states having squeezed spin variance along a certain direction, at the cost of antisqueezed variance along an orthogonal direction. This is done by using the RHUR (8) in the standard quantum system [53–56]. Such analysis can be reintroduced in the light of PPS systems. As there is no unique definition of spin squeezing in the standard quantum systems, it is, by means of Eq. (17), also possible to define the spin squeezing nonuniquely in the PPS system. A very careful analysis is required to see whether there exist some states in the PPS systems for which $\langle\Delta A\rangle_{\psi}^{\phi} = \langle\Delta B\rangle_{\psi}^{\phi}$ and inequality (17) is saturated similar to coherent spin states in the standard quantum systems.

Intelligent pre- and postselected states

In the standard quantum system, the states for which the equality condition holds in the RHUR (8) are known as intelligent states or minimum-uncertainty states [57–59]. Minimum-uncertainty states have been proposed to improve the accuracy of phase measurement in quantum interferometer [60]. Minimum-uncertainty states in the PPS systems can also be defined based on the following condition.

One can find the condition for which the inequality (17) saturates (see Appendix A) is given by

$$A\,|\psi\rangle - \langle\phi|A|\psi\rangle\,|\phi\rangle = \pm i\frac{\langle\Delta A\rangle_{\psi}^{\phi}}{\langle\Delta B\rangle_{\psi}^{\phi}}(B\,|\psi\rangle - \langle\phi|B|\psi\rangle\,|\phi\rangle). \tag{23}$$

If the sign of "i" on the right-hand side of Eq. (23) is taken to be positive (negative) when the observable A appears on the left-hand side of Eq. (23), then the sign of i on the right-hand side of Eq. (23) is taken to be negative (positive) when the observable B appears on the left-hand side of Eq. (23). So, the pre- and postselected states which satisfy the condition (23) can be referred as the "intelligent pre- and postselected states." For the given preselection and observables in Eq. (23), one can find the postselection which will make Eq. (17) the most tight, i.e., equality.

C. Uncertainty equality in PPS system

Recently, in Ref. [61], the authors have shown that there exist variance-based uncertainty equalities from which a series of uncertainty inequalities with hierarchical structure can be obtained. It was shown that stronger uncertainty relation given by Maccone and Pati [62] is a special case of these uncertainty inequalities. Here we show such uncertainty equalities in the PPS systems. We provide interpretation of the uncertainty inequalities derived from the uncertainty equalities. Further, in application Sec. IV, we use uncertainty equalities in PPS systems to obtain stronger uncertainty relations and state-dependent tighter uncertainty relations.

Theorem 2. The product of standard deviations of two noncommuting Hermitian operators $A, B \in \mathcal{L}(\mathcal{H})$ in a PPS system with pre- and postselected states $|\psi\rangle$ and $|\phi\rangle$, respectively, satisfies

$$\langle\Delta A\rangle_{\psi}^{\phi}\,\langle\Delta B\rangle_{\psi}^{\phi} = \frac{\mp\left(\frac{1}{2i}\,\langle\psi|[A,B]|\psi\rangle - \mathrm{Im}(W_{AB})\right)}{1-\frac{1}{2}\sum_{k=1}^{d-1}\left|\,\langle\psi|\frac{A}{\langle\Delta A\rangle_{\psi}^{\phi}} \pm i\frac{B}{\langle\Delta B\rangle_{\psi}^{\phi}}|\phi_k^{\perp}\rangle\,\right|^2}, \tag{24}$$

where we have assumed that $\langle\Delta A\rangle_{\psi}^{\phi}$ and $\langle\Delta B\rangle_{\psi}^{\phi}$ are nonzero, and the sign should be considered such that the numerator is always real and positive. Here $\{|\phi\rangle, |\phi_k^{\perp}\rangle_{k=1}^{d-1}\}$ is a complete orthonormal basis in the d-dimensional Hilbert space.

Proof. Consider an orthonormal complete basis $\{|\phi\rangle, |\phi_k^{\perp}\rangle_{k=1}^{d-1}\}$ in the d-dimensional Hilbert space $\mathcal{H}$. Now, define the projection operator $\Pi = I - |\phi\rangle\langle\phi|$ and the unnormalized state vector $|\xi^{\pm}\rangle = (\frac{A}{\langle\Delta A\rangle_{\psi}^{\phi}} \pm i\frac{B}{\langle\Delta B\rangle_{\psi}^{\phi}})\,|\psi\rangle$. Then we have the following identity:

$$\begin{aligned}
\langle\xi^{\mp}|\Pi|\xi^{\mp}\rangle &= \langle\xi^{\mp}|\xi^{\mp}\rangle - \langle\xi^{\mp}|\phi\rangle\,\langle\phi|\xi^{\mp}\rangle \\
&= \left\{\frac{\langle\psi|A^2|\psi\rangle}{\left(\langle\Delta A\rangle_{\psi}^{\phi}\right)^2} + \frac{\langle\psi|B^2|\psi\rangle}{\left(\langle\Delta B\rangle_{\psi}^{\phi}\right)^2} \mp \frac{i\,\langle\psi|[A,B]|\psi\rangle}{\langle\Delta A\rangle_{\psi}^{\phi}\,\langle\Delta B\rangle_{\psi}^{\phi}}\right\} \\
&\quad - \left\{\frac{|\langle\phi|A|\psi\rangle|^2}{\left(\langle\Delta A\rangle_{\psi}^{\phi}\right)^2} + \frac{|\langle\phi|B|\psi\rangle|^2}{\left(\langle\Delta B\rangle_{\psi}^{\phi}\right)^2} \pm \frac{2\,\mathrm{Im}(W_{AB})}{\langle\Delta A\rangle_{\psi}^{\phi}\,\langle\Delta B\rangle_{\psi}^{\phi}}\right\} \\
&= 2 \pm 2\frac{\left(\frac{1}{2i}\,\langle\psi|[A,B]|\psi\rangle - \mathrm{Im}(W_{AB})\right)}{\langle\Delta A\rangle_{\psi}^{\phi}\,\langle\Delta B\rangle_{\psi}^{\phi}},
\end{aligned} \tag{25}$$

where we have used Eq. (11) and $W_{AB} = \langle\psi|A|\phi\rangle\,\langle\phi|B|\psi\rangle$. Now, we use another expression of $\Pi = \sum_{k=1}^{d-1}|\phi_k^{\perp}\rangle\,\langle\phi_k^{\perp}|$ to calculate the same identity

$$\langle\xi^{\mp}|\Pi|\xi^{\mp}\rangle = \sum_{k=1}^{d-1}\left|\,\langle\psi|\frac{A}{\langle\Delta A\rangle_{\psi}^{\phi}} \pm i\frac{B}{\langle\Delta B\rangle_{\psi}^{\phi}}|\phi_k^{\perp}\rangle\right|^2. \tag{26}$$

So, from the Eqs. (25) and (26), we obtain the uncertainty equality (24) in the PPS system. ■

Theorem 3. The sum of the variances of two noncommuting Hermitian operators $A, B \in \mathcal{L}(\mathcal{H})$ in a PPS system with pre- and postselected states $|\psi\rangle$ and $|\phi\rangle$, respectively, satisfies

$$\begin{aligned}
\left(\langle\Delta A\rangle_{\psi}^{\phi}\right)^2 + \left(\langle\Delta B\rangle_{\psi}^{\phi}\right)^2 &= \pm(i\,\langle\psi|[A,B]|\psi\rangle - 2\,\mathrm{Im}(W_{AB})) \\
&\quad + \sum_{k=1}^{d-1}|\,\langle\phi_k^{\perp}|(A \mp iB)|\psi\rangle\,|^2.
\end{aligned} \tag{27}$$

Here, the "±" sign is taken suitably such that the first term in right side is always positive.

Proof. Consider an orthonormal complete basis $\{|\phi\rangle, |\phi_k^{\perp}\rangle_{k=1}^{d-1}\}$ in the d-dimensional Hilbert space $\mathcal{H}$ and hence $I - |\phi\rangle\langle\phi| = \sum_{k=1}^{d-1}|\phi_k^{\perp}\rangle\,\langle\phi_k^{\perp}|$. By equating the following two

$$\mathrm{Tr}((A \mp iB)|\psi\rangle\langle\psi|(A \pm iB)(I - |\phi\rangle\,\langle\phi|)),$$

$$\mathrm{Tr}\left[(A \mp iB)|\psi\rangle\langle\psi|(A \pm iB)\left(\sum_{k=1}^{d-1}|\phi_k^{\perp}\rangle\,\langle\phi_k^{\perp}|\right)\right],$$

we have Eq. (27). ■

An inequality can be obtained by discarding some of the terms in the summation corresponding to k or all the terms except one term in Eq. (24) or (27). It is also possible to obtain an arbitrarily tight inequality by discarding the minimum valued term inside the summation in the denominator of Eq. (24) for a particular value of k. Note that we have to optimize the minimum $|\langle\psi|\frac{A}{\langle\Delta A\rangle_\psi^\phi}\pm i\frac{B}{\langle\Delta B\rangle_\psi^\phi}|\phi_k^\perp\rangle|^2$ over all possible choice of basis $\{|\phi_k^\perp\rangle_{k=1}^{d-1}\}$ in the subspace orthogonal to $|\phi\rangle$.

In an experiment, let us assume that a few postselected states from $\{|\phi_k^\perp\rangle\}_{k=1}^{d-1}$ are not detected by the detector because of certain technical difficulties. Using such imprecise experimental data, one may still be able to obtain an uncertainty relation. In that case, the terms corresponding to the unregistered postselections in Eq. (24) or (27) are to be eliminated.

D. Uncertainty relation for mixed preselection in PPS system

So far, we have only considered the preselected state to be pure in a PPS system. Let us now generalize the definition of the standard deviation and derive the uncertainty relations for mixed preselected state in the PPS system. A direct generalization of the standard deviation defined in Eq. (11) is given by

$$\langle\Delta A\rangle_\rho^\phi=\sqrt{\mathrm{Tr}(A^2\rho)-\langle\phi|A\rho A|\phi\rangle}. \tag{28}$$

See Ref. [63] for the motivation for calling $\langle\Delta A\rangle_\rho^\phi$ as a standard deviation when the preselection is a mixed state. If $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$, where $\sum_i p_i=1$, then the variance is given by

$$\left(\langle\Delta A\rangle_\rho^\phi\right)^2=\sum_i p_i\left(\langle\Delta A\rangle_{\psi_i}^\phi\right)^2. \tag{29}$$

Equation (29) demonstrates the intriguing fact that the variance of A in PPS system, i.e., $\langle\mathrm{Var}A\rangle_\rho^\phi=\left(\langle\Delta A\rangle_\rho^\phi\right)^2$ *respects classical mixing of quantum states*. Mathematically, classical mixing of quantum states is represented by a density operator. By taking advantage of this property, one can obtain a purely quantum uncertainty relation when the preselection ρ is a mixed state (see Sec. IV). It may be noted here that in standard quantum systems, the variance $\mathrm{Var}A=\langle\Delta A\rangle_\rho^2$ increases, in general, under the classical mixing of quantum states.

To realize the standard deviation in PPS system via weak value for mixed preselected state, a generalization of the standard deviation $\langle\Delta A\rangle_\psi^\phi$ given in Eq. (13) can be *defined* as

$$\langle\Delta A_w\rangle_\rho^\phi=\sqrt{\mathrm{Tr}(A^2\rho)-|\langle A_w\rangle_\rho^\phi|^2\langle\phi|\rho|\phi\rangle}, \tag{30}$$

where $\langle A_w\rangle_\rho^\phi=\frac{\langle\phi|A\rho|\phi\rangle}{\langle\phi|\rho|\phi\rangle}$ is the weak value of the operator A when the pre- and postselections are ρ and $|\phi\rangle$, respectively. Now, $\langle\mathrm{Var}A_w\rangle_\rho^\phi:=(\langle\Delta A_w\rangle_\rho^\phi)^2$ can be viewed as a variance like quantity (henceforth called as *generalized variance*) of A involving weak value and, it is, in general, different from the variance $\langle\mathrm{Var}A\rangle_\rho^\phi=\left(\langle\Delta A\rangle_\rho^\phi\right)^2$, and $\langle\mathrm{Var}A_w\rangle_\rho^\phi$ is always nondecreasing under the classical mixing of quantum states. This property is certified by the following proposition.

Proposition 3. The generalized variance $\langle\mathrm{Var}A_w\rangle_\rho^\phi$ is lower bounded by the variance $\langle\mathrm{Var}A\rangle_\rho^\phi$, that is

$$\langle\mathrm{Var}A_w\rangle_\rho^\phi\geqslant\sum_i p_i\left(\langle\Delta A\rangle_{\psi_i}^\phi\right)^2=\langle\mathrm{Var}A\rangle_\rho^\phi, \tag{31}$$

where equality holds if the preselection ρ is a pure state.

Proof. Let $\rho=\sum_i p_i|\psi_i\rangle\langle\psi_i|$, then using Eq. (30) (after using the definition of the weak value for mixed preselection), we have

$$\begin{aligned}\langle\mathrm{Var}A_w\rangle_\rho^\phi&=\left(\langle\Delta A_w\rangle_\rho^\phi\right)^2\\&=\mathrm{Tr}(A^2\rho)-\frac{|\langle\phi|A\rho|\phi\rangle|^2}{\langle\phi|\rho|\phi\rangle}\\&=\sum_i p_i\langle\psi_i|A^2|\psi_i\rangle\\&\quad-\frac{|\sum_i\sqrt{p_i}\langle\phi|A|\psi_i\rangle\sqrt{p_i}\langle\psi_i|\phi\rangle|^2}{\langle\phi|\rho|\phi\rangle}\\&\geqslant\sum_i p_i\langle\psi_i|A^2|\psi_i\rangle\\&\quad-\frac{\left(\sum_i p_i|\langle\phi|A|\psi_i\rangle|^2\right)\left(\sum_i p_i\langle\phi|\psi_i\rangle\langle\psi_i|\phi\rangle\right)}{\langle\phi|\rho|\phi\rangle}\\&=\sum_i p_i\langle\psi_i|A^2|\psi_i\rangle-\sum_i p_i|\langle\phi|A|\psi_i\rangle|^2\\&=\sum_i p_i\left(\langle\Delta A\rangle_{\psi_i}^\phi\right)^2=\left(\langle\Delta A\rangle_\rho^\phi\right)^2=\langle\mathrm{Var}A\rangle_\rho^\phi,\end{aligned}$$

where we have used the Cauchy-Schwarz inequality for the complex numbers in the first inequality and Eq. (29) in the last line. When ρ is pure, equality holds automatically. ■

As $\langle\mathrm{Var}A\rangle_\rho^\phi$ does neither increase nor decrease under classical mixing of quantum states, the inequality $\langle\mathrm{Var}A_w\rangle_\rho^\phi\geqslant\langle\mathrm{Var}A\rangle_\rho^\phi$ clearly implies that under classical mixing of quantum states, the generalized variance $\langle\mathrm{Var}A_w\rangle_\rho^\phi$ is always nondecreasing. In fact, one can easily verify that $\langle\mathrm{Var}A_w\rangle_\rho^\phi$ is sum of the *quantum uncertainty* $\langle\mathrm{Var}A\rangle_\rho^\phi$ and the *classical uncertainty* $C(\rho,A,\phi):=\langle\phi|A\rho A|\phi\rangle-|\langle A_w\rangle_\rho^\phi|^2\langle\phi|\rho|\phi\rangle$, both of which will be discussed in detail in Sec. IV C.

It is important to note that, in general, the equality in Eq. (31) does not imply that the preselection ρ is pure. In Sec. IV A (see below), we show that only in the qubit system, equality of Eq. (31) implies that the preselection is a pure state. To make an equality in Eq. (31) in higher-dimensional systems, we need to put conditions on the observable and postselection (see below in Sec. IV A).

The uncertainty relation (17) or (21) can be generalized for mixed preselection ρ also which is given by

$$\left(\langle\Delta A\rangle_\rho^\phi\right)^2\left(\langle\Delta B\rangle_\rho^\phi\right)^2\geqslant\left[\frac{1}{2i}\langle[A,B]\rangle_\rho-\mathrm{Im}W_{AB}\right]^2, \tag{32}$$

where $W_{AB}=\langle\phi|B\rho A|\phi\rangle$. See the derivation of Eq. (32) in Appendix B. Equation (32) holds also when the definition of standard deviation defined in Eq. (30) is considered due the Proposition 3.

TABLE I. Comparison of different properties between standard quantum systems and PPS systems.

Properties	Standard quantum systems	Pre- and postselected systems
Standard deviation	$\langle\Delta A\rangle_\psi = (\langle\psi\|A^2\|\psi\rangle - \langle\psi\|A\|\psi\rangle^2)^{1/2}$	$\langle\Delta A\rangle_\psi^\phi = (\langle\psi\|A^2\|\psi\rangle - \|\langle A_w\rangle_\psi^\phi\|^2\|\langle\phi\|\psi\rangle\|^2)^{1/2}$
Zero standard deviation	Only if $\|\psi\rangle$ is an eigenstate of A, i.e., $\|\psi\rangle \propto A\|\psi\rangle$	Only if $\|\phi\rangle \propto A\|\psi\rangle$
Uncertainty relation	$\langle\Delta A\rangle_\psi^2 \langle\Delta B\rangle_\psi^2 \geqslant [\frac{1}{2i}\langle\psi\|[A,B]\|\psi\rangle]^2$	$\left(\langle\Delta A\rangle_\psi^\phi\right)^2\left(\langle\Delta B\rangle_\psi^\phi\right)^2 \geqslant \left[\frac{1}{2i}\langle\psi\|[A,B]\|\psi\rangle - \mathrm{Im}(W_{AB})\right]^2$
Joint sharp preparation	If $\|\psi\rangle$ is the eigenstate of both A and B	If $\|\phi\rangle = \frac{A\|\psi\rangle}{\sqrt{\langle\psi\|A^2\|\psi\rangle}} = \frac{B\|\psi\rangle}{\sqrt{\langle\psi\|B^2\|\psi\rangle}}$, up to some phase factors

See Table I for the comparison of different properties between standard quantum systems and PPS systems.

IV. APPLICATIONS

Suitably postselected systems can offer some essential information regarding quantum systems. Below, we provide a few applications of standard deviations and uncertainty relations in PPS systems.

A. Detection of mixedness of an unknown state

Practically, partial information about a quantum state is often of great help. For example, whether an interaction has taken place with the environment, one must verify the purity of the system's state. Quantum state tomography (QST) is the most resource intensive way to verify the purity of a quantum state but here we provide some results that can be used to detect purity of the quantum state using less resources compared to the QST.

We will use the inequality (31) in Proposition 3 to detect the mixedness of an unknown preselected state in a PPS system. The proofs of the following lemmas are given in Appendix C.

Lemma 1. Qubit system: In the case of a two-level quantum system (i.e., a qubit), equality in Eq. (31) holds if and only if the preselected state ρ is pure irrespective of choice of the observable A and the postselected state $|\phi\rangle$.

Lemma 2. Qutrit system: If for an observable A and a complete orthonormal basis $\{|\phi_k\rangle\}_{k=1}^3$ (to be used as postselected states) of any three-level quantum system (i.e., a qutrit), and the condition $\langle\phi_1|A|\phi_2\rangle = 0$ also holds good, then equality in Eq. (31) holds good if and only if the preselected state ρ is pure.

Lemma 3. Qubit-qubit system: Consider any two nonorthogonal postselections $|\phi_B\rangle$ and $|\phi'_B\rangle$ in the subsystem B. For any observable A, equality of $\langle\Delta(A\otimes I)_w\rangle_\rho^{\phi_{AB}}$ and $\langle\Delta(A\otimes I)\rangle_\rho^{\phi_{AB}}$ and separately of $\langle\Delta(A\otimes I)_w\rangle_\rho^{\phi'_{AB}}$ and $\langle\Delta(A\otimes I)\rangle_\rho^{\phi'_{AB}}$ hold only when the $2\otimes 2$ preselected state ρ is pure, where $|\phi_{AB}\rangle = |\phi_A\rangle|\phi_B\rangle$ and $|\phi'_{AB}\rangle = |\phi_A\rangle|\phi'_B\rangle$. Two nonorthogonal postselections $|\phi_B\rangle$ and $|\phi'_B\rangle$ in the subsystem B are required here due to the fact that there exists an unique $2\otimes 2$ mixed density matrix which satisfies the equality of Eq. (31).

Lemma 4. Qubit-qutrit system: If for an observable A and any complete orthonormal basis $\{|\phi_A^k\rangle\}_{k=1}^3$ (to be used as postselected states) for a qutrit, and the condition $\langle\phi_A^1|A|\phi_A^2\rangle = 0$ is considered, then equality of $\langle\Delta(A\otimes I)_w\rangle_\rho^{\phi_{AB}}$ and $\langle\Delta(A\otimes I)\rangle_\rho^{\phi_{AB}}$ and separately of $\langle\Delta(A\otimes I)_w\rangle_\rho^{\phi'_{AB}}$ and $\langle\Delta(A\otimes I)\rangle_\rho^{\phi'_{AB}}$ hold if and only if the $3\otimes 2$ preselected state ρ is pure.

Extension of this method for higher-dimensional systems will require more conditions to be imposed on the observable and postselections. So it might be difficult to apply our method for higher dimensions. To overcome this difficulties, Eq. (17) or (21) can be used to detect the mixedness of the initially prepared states. Note that Mal *et al.* have used the stronger version of the RHUR (9) to do so [64].

B. Stronger uncertainty relation

Motivation. If, for example, the initially prepared state of the system is an eigenstate of one of the two incompatible observables A and B, both the sides of the RHUR (8) become trivial (i.e., zero). For certain states, a trivial lower bound is always possible because the right side of the relation (8) contains the average of the commutator of incompatible observables. For such cases, the RHUR (8) does not capture the incompatibility of the noncommuting observables. One can think of adding *Schrödinger's term* in the RHUR but still this can be become trivial (e.g., when the prepared state is an eigenstate of either A or B). So, none of them are unquestionably appropriate to capture the incompatibility of the noncommuting observables.

It is Maccone and Pati [62] who considered a different uncertainty relation, based on the sum of the variances $\langle\Delta A\rangle_\psi^2 + \langle\Delta B\rangle_\psi^2$, that is guaranteed to be nontrivial (i.e., having nonzero lower bound) whenever the observables are incompatible on the given state $|\psi\rangle$. But there are shortcomings in the Maccone-Pati uncertainty relations (MPUR). It is easy to show that in two-dimensional Hilbert space [65] if, for example, the initial state $|\psi\rangle$ of the system is an eigenstate of the observable A, then one finds that the first inequality $\langle\Delta A\rangle_\psi^2 + \langle\Delta B\rangle_\psi^2 \geqslant \pm i\langle\psi|[A,B]|\psi\rangle + |\langle\psi|(A\pm iB)|\psi^\perp\rangle|^2$ in MPUR becomes $\langle\Delta B\rangle_\psi^2 \geqslant \langle\Delta B\rangle_\psi^2$, where $|\psi^\perp\rangle$ is arbitrary state orthogonal to $|\psi\rangle$. Similarly, it can be shown that the second inequality $\langle\Delta A\rangle_\psi^2 + \langle\Delta B\rangle_\psi^2 \geqslant \frac{1}{2}|\langle\psi_{A+B}^\perp|(A+B)|\psi\rangle|^2$ in MPUR becomes $\langle\Delta B\rangle_\psi^2 \geqslant \frac{1}{2}\langle\Delta B\rangle_\psi^2$, where $|\psi_{A+B}^\perp\rangle = [1/\langle\Delta(A+B)\rangle_\psi](A+B-\langle A+B\rangle_\psi)|\psi\rangle$ and $\langle\Delta(A+B)\rangle_\psi^2 = \langle(A+B)^2\rangle_\psi - \langle A+B\rangle_\psi^2$ for arbitrary dimensional Hilbert space if the initial state of the system is an eigenstate of the observable A [66]. It indicates that the first and second inequalities in MPUR for two and arbitrary dimensions, respectively, contain no information about the observable A and are therefore of no practical significance. In other words, we learn nothing new about the quantum system other than the trivial fact that $\langle\Delta B\rangle_\psi$ is non-negative. In addition, if the initially prepared state $|\psi\rangle$ is unknown, then $|\psi^\perp\rangle$ is likewise unknown in the MPUR inequalities and, so is the lower bound of MPUR. The first inequality in MPUR may be useful in a quantum system with Hilbert spaces of more than two dimensions.

Here, we demonstrate that relations (17) and (21) can be used to solve the triviality problem of the RHUR and the problem with MPUR that we have mentioned above, i.e., these uncertainty relations can provide nontrivial information about the observable A. Even if the initially prepared state (preselection) $|\psi\rangle$ is unknown, the lower bound of our stronger uncertainty relation can be calculated.

Consider the relation (17) which, using Eq. (14), becomes

$$\big(\langle\Delta A\rangle_\psi^2+\epsilon_A\big)\big(\langle\Delta B\rangle_\psi^2+\epsilon_B\big)\geqslant\left[\frac{1}{2i}\langle\psi|\,[A,B]\,|\psi\rangle-\mathrm{Im}(W_{AB})\right]^2, \tag{33}$$

where $\epsilon_X=\langle X\rangle_\psi^2-|\langle X_w\rangle_\psi^\phi|^2|\langle\phi|\psi\rangle|^2$, with $X=A$ or B. Now suppose $|\psi\rangle$ is an eigenstate of A then, Eq. (33) is nontrivial unless $|\phi\rangle=|\psi\rangle$, as, in the case when $|\phi\rangle\neq|\psi\rangle$, the inequality (33) becomes

$$\epsilon_A\big(\langle\Delta B\rangle_\psi^2+\epsilon_B\big)\geqslant[\mathrm{Im}(W_{AB})]^2. \tag{34}$$

Notice that, in the both sides of Eq. (34), there is a quantum state $|\phi\rangle$ which can be chosen independently in the standard quantum system. So, it is always possible to choose a suitable $|\phi\rangle$ such that the relation (33) is nontrivial. With "Schrödinger's term," the relations (33) and (34) become

$$\big(\langle\Delta A\rangle_\psi^2+\epsilon_A\big)\big(\langle\Delta B\rangle_\psi^2+\epsilon_B\big)\geqslant\left[\frac{1}{2i}\langle\psi|\,[A,B]\,|\psi\rangle-\mathrm{Im}(W_{AB})\right]^2+\left[\frac{1}{2}\langle\psi|\{A,B\}|\psi\rangle-\mathrm{Re}(W_{AB})\right]^2, \tag{35}$$

$$\epsilon_A\big(\langle\Delta B\rangle_\psi^2+\epsilon_B\big)\geqslant[\mathrm{Im}(W_{AB})]^2+\left[\frac{1}{2}\langle\psi|\{A,B\}|\psi\rangle-\mathrm{Re}(W_{AB})\right]^2, \tag{36}$$

respectively. As ϵ_A and ϵ_B can also be negative, the left-hand side of relation (33) can become lower than the left-hand side of relation (8). The same holds true for the right-hand side as well. So, for a fixed $|\psi\rangle$, we always want to have a nontrivial lower bound from the relations (8) and (33) which can be combined in a single uncertainty relation, i.e., the *stronger uncertainty relation*

$$\max\{\mathcal{L}_{\mathrm{RH}},\mathcal{L}_{\mathrm{PPS}}\}\geqslant\max\{\mathcal{R}_{\mathrm{RH}},\mathcal{R}_{\mathrm{PPS}}\},$$

where $\mathcal{L}_{\mathrm{RH}}=\langle\Delta A\rangle_\psi^2\,\langle\Delta B\rangle_\psi^2$, $\mathcal{L}_{\mathrm{PPS}}=(\langle\Delta A\rangle_\psi^2+\epsilon_A)(\langle\Delta B\rangle_\psi^2+\epsilon_B)$, $\mathcal{R}_{\mathrm{RH}}=[\frac{1}{2i}\langle\psi|[A,B]|\psi\rangle]^2$, and $\mathcal{R}_{\mathrm{PPS}}=[\frac{1}{2i}\langle\psi|\,[A,B]\,|\psi\rangle-\mathrm{Im}(W_{AB})]^2$.

In Fig. 1, comparison between the relations (8) and both (33) and (35) is shown. Equations (34) and (36) capture the information about the operator A when the initially prepared state $|\psi\rangle$ is one of the eigenstates of A, while MPUR fails to capture such information which we have already discussed.

Moreover, even if the initial state (i.e., preselection) is unknown, the lower bound of the uncertainty relation (33) can be calculated experimentally and in that case we need the average value of the Hermitian operator $\frac{1}{i}[A,B]$ and weak values of the operators A and B.

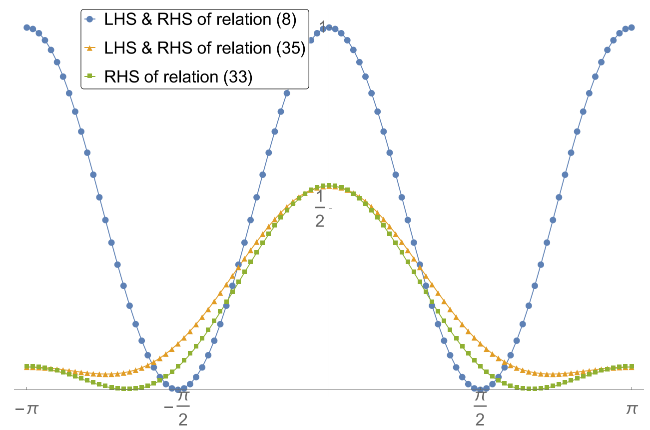

FIG. 1. Comparison between the RHURs (8) and (9), and the uncertainty relations (33) and (35). We choose $A=\sigma_x$, $B=\sigma_y$ for a spin-$\frac{1}{2}$ particle and $|\psi\rangle=\cos(\theta/2)\,|0\rangle+e^{i\xi}\sin(\theta/2)\,|1\rangle$, $|\phi\rangle=\cos(\omega/2)\,|0\rangle+e^{i\eta}\sin(\omega/2)\,|1\rangle$ with $\xi=0$, $\omega=\pi/3$, and $\eta=\pi/5$. The blue curve is the left-hand side of the RHUR and, for this particular case, it coincides with its lower bound, i.e., right-hand side of RHUR. The orange curve is the left-hand side of Eq. (35) and, for this particular case, it coincides with the right-hand side of Eq. (35). The green curve is the right-hand side of Eq. (33). Now, notice that, for $\theta=-\pi/2$ and $\pi/2$, the RHUR becomes trivial while for the same values of θ, the relation (33) as well as the relation (35) are nontrivial. For this particular choice of postselection, both the relations (33) and (35) are stronger than the RHUR (8). Note that the relation (35) is the strongest under this condition as it is nontrivial for all the values of θ. If, for the fixed values of θ and ξ, the relations (33) and (35) are trivial, then one should keep changing the values of ω and η (i.e., by choosing the postselection suitably) until they become nontrivial which is our main goal.

Sum uncertainty relation in the PPS system can also be used to obtain stronger uncertainty relation in the standard quantum system. One can easily show that

$$\big(\langle\Delta A\rangle_\psi^2+\epsilon_A\big)+\big(\langle\Delta B\rangle_\psi^2+\epsilon_B\big)\geqslant\pm(i\,\langle\psi|[A,B]|\psi\rangle-2\mathrm{Im}(W_{AB}))$$

holds in Eq. (27) in Theorem 3 by discarding the summation part which is always a positive number. This inequality remains strong against when $|\psi\rangle$ is one of the eigenstates of A by suitably choosing postselection $|\phi\rangle$.

C. Purely quantum uncertainty relation

Motivation. In practice, it is not always possible to carry out quantum mechanical tasks with pure states because of interactions with the environment. Because the mixed initial prepared state is a classical mixture of pure quantum states, any task or measurement involves a hybrid of classical and quantum parts. In modern technologies, it is considered that quantum advantage is more effective and superior to classical advantage. Hence, a hybrid of a quantum and classical component may be less advantageous than a quantum component alone. For example, the uncertainty of an observable A in standard quantum system increases in general under classical mixing of quantum states, i.e., $\langle \Delta A\rangle_\rho^2 \geqslant \sum_i p_i \langle \Delta A\rangle_{\psi_i}^2$ (where $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$) and this is disadvantageous in the sense that the knowledge about the observable is more uncertain than only when the average of the pure state uncertainties are considered. The uncertainty that one gets to see due to (classical) mixing of pure states is considered here as "classical uncertainty." The standard deviation $\langle \Delta A\rangle_\rho$ can be referred as the hybrid of classical and quantum uncertainties and hence the RHUR (8) can be considered as the hybrid uncertainty relation in the standard quantum systems.

Purely quantum uncertainty relation, a crucial component of the quantum world, may be very useful, but in order to obtain it, the classical uncertainty must be eliminated from the hybrid uncertainty relation. To do this, we first need to determine the purely quantum uncertainty of an observable, which can be done in a number of ways, such as by eliminating the classical component of the hybrid uncertainty or by specifying a purely quantum uncertainty straight away.

Any measure of purely quantum uncertainty should have at least the following intuitive and expected property (below "Φ" represents sometimes quantum observables, sometimes states, etc., for different types of quantum mechanical systems; for example, if the system is a PPS system, then Φ is the postselection $|\phi\rangle$ and if the system is a standard quantum system, then the term Φ disappears):

(i) Quantum uncertainty should not be affected (neither increasing nor decreasing) by the classical mixing of quantum states, i.e.,

$$\mathcal{Q}(\rho, A, \Phi) = \sum_i p_i \mathcal{Q}(\psi_i, A, \Phi), \text{ where } \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| .$$

Here $\mathcal{Q}(\rho, A, \Phi)$ is some measure of purely quantum uncertainty of the observable A for a given ρ. There might exist some other properties depending upon the nature of the system (e.g., standard systems, PPS systems, etc.) but we emphasize that the most important property of a purely quantum uncertainty should be (i).

It is seen that the variance of A in PPS system, i.e., $\langle \text{Var}A\rangle_\rho^\phi = (\langle\Delta A\rangle_\rho^\phi)^2$, is a purely quantum uncertainty which satisfies the property (i). Now the purely quantum mechanical uncertainty relation in this regard is Eq. (32).

As can be seen from Eq. (31) for mixed states, the second definition of the variance of A, i.e., $\langle \text{Var}A_w\rangle_\rho^\phi = (\langle \Delta A_w\rangle_\rho^\phi)^2$ in the PPS system defined in Eq. (30) is a hybrid uncertainty. Hence, the uncertainty in PPS system based on weak value has both classical and quantum parts. When measurement is carried out in the PPS system and weak values are involved, classical uncertainty may be crucial in determining how much classicality (in the form of classical uncertainty) the mixed state ρ possesses. Mixed states with less classicality should have more quantumness (in the form of quantum uncertainty), and vice versa. To distinguish classical uncertainty from the hybrid uncertainty $\langle \text{Var}A_w\rangle_\rho^\phi$, we subtract the quantum uncertainty $\langle \text{Var}A\rangle_\rho^\phi$ from it, i.e.,

$$C(\rho, A, \phi) = \left(\langle \Delta A_w\rangle_\rho^\phi\right)^2 - \left(\langle \Delta A\rangle_\rho^\phi\right)^2. \tag{37}$$

This is one of the good measures of classical uncertainty which should have some intuitive and expected properties:

(i) $\mathcal{C}(\rho, A, \Phi) \geqslant 0$ for a quantum state ρ,

(ii) $\mathcal{C}(\rho, A, \Phi) = 0$ when $\rho = |\psi\rangle\langle\psi|$ (absence of classical mixing),

(iii) total classical uncertainty of disjoint systems should be the sum of individual systems's classical uncertainties:

$$\mathcal{C}(\rho, A_1\otimes I + I\otimes A_2, \Phi) = \mathcal{C}(\rho, A_1\otimes I, \Phi) + \mathcal{C}(\rho, I\otimes A_2, \Phi),$$

when $\rho = \rho_1 \otimes \rho_2$.

One can show that all the properties (i)–(iii) of classical uncertainty are satisfied by $C(\rho, A, \phi)$ defined in Eq. (37). Particularly, property (iii) is satisfied by taking $\Phi = |\phi_1\rangle|\phi_2\rangle$. Here, $|\phi_1\rangle$ and $|\phi_2\rangle$ are postselections of the two disjoint systems, respectively.

There are some works by Luo and other authors regarding the purely quantum uncertainty relation. Initial attempt was made by Luo and Zhang [67] to obtain uncertainty relation by using skew information (introduced by Wigner and Yanase [68]) but it was found to be incorrect in general [69]. Later, another attempt was made by Luo himself [70], which is obtained by discarding the classical part from the hybrid uncertainty relation using skew information. But this uncertainty relation cannot be guaranteed to be an intrinsically quantum uncertainty relation [according to property (i)] as the uncertainty they claim to be a quantum uncertainty is a product of skew information (which is a convex function under the mixing of quantum states) and a concave function under the same mixing. After that, a series of successful and failed attempts was performed by modifying the works of Luo and other authors [71–74].

Instead, we have given a quantum uncertainty relation although it is based on pre- and postselections which is different from the standard quantum mechanics but a quantumness can be seen in the relation (32).

D. State-dependent tighter uncertainty relations in standard systems

The RHUR (8) or (9) is known not to be the tight one. Some existing tighter bounds are given in [61,62,75]. The drawback of these tighter uncertainty relations is that their lower bounds depend on the states perpendicular to the given state of the system. If the given state is unknown, then the lower bound of these uncertainty relations also remains unknown.

Here we show that by the use of arbitrary postselected state $|\phi\rangle$, the lower bound of the RHUR based on sum uncertainties can be made arbitrarily tight and even if the given state (i.e., preselection here) is unknown, the lower bound of our tighter uncertainty relation can be obtained in experiments.

Theorem 4. Let $\rho \in \mathcal{L}(\mathcal{H})$ be the density operator of the standard quantum system, then the sum of the standard deviations of two noncommuting observables $A, B \in \mathcal{L}(\mathcal{H})$ satisfies

$$\langle \Delta A\rangle_\rho^2 + \langle \Delta B\rangle_\rho^2 \geqslant \pm i\,\text{Tr}([A,B]\rho) + \langle\phi|C_\pm^\dagger \rho C_\pm|\phi\rangle , \tag{38}$$

where $C_\pm = A \pm iB - \langle A \pm iB\rangle_\rho I$ and the "$\pm$" sign is taken in such a way that the first term in the right-hand side is always positive.

Proof. Considering Eq. (27) for preselection $|\psi_j\rangle$ and multiply by p_j, and then after summing over j, we have

$$\sum_j p_j\big(\langle\Delta A\rangle^{\phi}_{\psi_j}\big)^2+\sum_j p_j\big(\langle\Delta B\rangle^{\phi}_{\psi_j}\big)^2$$
$$=\pm i\sum_j p_j\,\langle\psi_j|[A,B]|\psi_j\rangle\mp 2\,\mathrm{Im}\Bigg(\sum_j p_j\langle\phi|B|\psi_j\rangle\langle\psi_j|A|\phi\rangle\Bigg)$$
$$+\sum_{k=1}^{d-1}\sum_j p_j\,\langle\phi_k^{\perp}|(A\pm iB)|\psi_j\rangle\,\langle\psi_j|(A\mp iB)|\phi_k^{\perp}\rangle\,,\qquad(39)$$

where we have used $W_{AB}=\langle\phi|B|\psi_j\rangle\,\langle\psi_j|A|\phi\rangle$. By using Eq. (29) for A and B when $\rho=\sum_j p_j|\psi_j\rangle\langle\psi_j|$, we have

$$\big(\langle\Delta A\rangle^{\phi}_{\rho}\big)^2+\big(\langle\Delta B\rangle^{\phi}_{\rho}\big)^2$$
$$=\pm i\,\mathrm{Tr}([A,B]\rho)\mp 2\,\mathrm{Im}(\langle\phi|B\rho A|\phi\rangle)$$
$$+\sum_{k=1}^{d-1}\langle\phi_k^{\perp}|(A\pm iB)\rho(A\mp iB)|\phi_k^{\perp}\rangle$$
$$=\pm i\,\mathrm{Tr}([A,B]\rho)+\langle\phi|(A\pm iB)\rho(A\mp iB)|\phi\rangle$$
$$-\langle\phi|A\rho A|\phi\rangle-\langle\phi|B\rho B|\phi\rangle$$
$$+\sum_{k=1}^{d-1}\langle\phi_k^{\perp}|(A\pm iB)\rho(A\mp iB)|\phi_k^{\perp}\rangle\,,\qquad(40)$$

where $\mp 2\,\mathrm{Im}(\langle\phi|B\rho A|\phi\rangle)=\pm i(\langle\phi|B\rho A|\phi\rangle-\langle\phi|A\rho B|\phi\rangle)$ $=\langle\phi|(A\pm iB)\rho(A\mp iB)|\phi\rangle-\langle\phi|A\rho A|\phi\rangle-\langle\phi|B\rho B|\phi\rangle$ has been used. Now put $\big(\langle\Delta A\rangle^{\phi}_{\rho}\big)^2=\mathrm{Tr}(A^2\rho)-\langle\phi|A\rho A|\phi\rangle$ defined in Eq. (28) (similarly for B also) and after subtracting $\mathrm{Tr}(A\rho)^2+\mathrm{Tr}(B\rho)^2$ from both sides of Eq. (40) and using $|\phi\rangle\langle\phi|+\sum_{k=1}^{d-1}|\phi_k^{\perp}\rangle\langle\phi_k^{\perp}|=I$, we have

$$\langle\Delta A\rangle^2_{\rho}+\langle\Delta B\rangle^2_{\rho}$$
$$=\pm i\,\mathrm{Tr}([A,B]\rho)+\mathrm{Tr}[(A\pm iB)(A\mp iB)\rho]$$
$$-\mathrm{Tr}(A\rho)^2-\mathrm{Tr}(B\rho)^2$$
$$=\pm i\,\mathrm{Tr}([A,B]\rho)+\mathrm{Tr}[(A\pm iB)(A\mp iB)\rho]$$
$$-\mathrm{Tr}[(A\pm iB)\rho]\mathrm{Tr}[(A\mp iB)\rho]$$
$$=\pm i\,\mathrm{Tr}([A,B]\rho)+\mathrm{Tr}(M_{\mp}^{\dagger}M_{\mp}\rho)-|\mathrm{Tr}(M_{\mp}\rho)|^2$$
$$=\pm i\,\mathrm{Tr}([A,B]\rho)+\mathrm{Tr}[(M_{\mp}-\langle M_{\mp}\rangle_{\rho}I)^{\dagger}(M_{\mp}-\langle M_{\mp}\rangle_{\rho}I)\rho],\qquad(41)$$

where $M_{\mp}=A\mp iB$. Now let $C_{\pm}=M_{\pm}-\langle M_{\pm}\rangle_{\rho}I$ then Eq. (41) can be rewritten as

$$\langle\Delta A\rangle^2_{\rho}+\langle\Delta B\rangle^2_{\rho}=\pm i\,\mathrm{Tr}([A,B]\rho)+\langle\phi|C_{\pm}^{\dagger}\rho C_{\pm}|\phi\rangle$$
$$+\sum_{i}^{d-1}\langle\phi_i^{\perp}|C_{\pm}^{\dagger}\rho C_{\pm}|\phi_i^{\perp}\rangle\,,$$

where $\{|\phi\rangle,\{|\phi_i^{\perp}\rangle\}_{i=1}^{d-1}\}$ is an orthonormal basis in $\mathcal{H}$. By discarding the summation term which is always a positive number in the above equation, we obtain the inequality (38).■

Notice that the lower bound of Eq. (38) has different nonzero values depending on different choices of the postselections $|\phi\rangle$. The inequality (38) becomes an equality when $|\phi\rangle\propto(A\pm iB-\langle A\pm iB\rangle_{\rho}I)|\psi\rangle$, where $\rho=|\psi\rangle\langle\psi|$ is a pure state. In Refs. [61,62,75], the lower bound of the sum uncertainty relation depends on the state orthogonal to the initial pure state, and if the initial state is a mixed state, then the lower bound can not always be computed at least for the full-rank density matrix. The reason is that we cannot find a state which is orthogonal to all the eigenstates of a full-rank density matrix. Moreover, if the initial density matrix is unknown, then computing the lower bound will be hard. In contrast, Eq. (38) does not have such issues as the first and second terms in the right-hand side of Eq. (38) are the average values of the Hermitian operators $i[A,B]$ and $(A\pm iB-\langle A\pm iB\rangle_{\rho}I)|\phi\rangle\langle\phi|(A\mp iB-\langle A\mp iB\rangle_{\rho}I)$ in the state ρ, respectively, where $\langle A\pm iB\rangle_{\rho}=\langle A\rangle_{\rho}\pm i\,\langle B\rangle_{\rho}$. All of them can be obtained in experiments even if ρ is unknown.

E. Tighter upper bound for out-of-time-order correlators

Recently, Bong *et al.* [76] used the RHUR for unitary operators to give upper bound for out-of-time-order correlators (OTOC) which is defined by $F=\mathrm{Tr}[(W_t^{\dagger}V^{\dagger}W_tV)\rho]$, where V and W_t are fixed and time-dependent unitary operators, respectively. The OTOC diagnoses the spread of quantum information by measuring how quickly two commuting operators V and W fail to commute, which is quantified by $\langle|[W_t,V]|^2\rangle_{\rho}=2(1-\mathrm{Re}[F])$, where $|X|^2=X^{\dagger}X$. The OTOC has strong connection with chaos and information scrambling [77–79] and also with high-energy physics [80–83]. It is known that OTOC's upper bound is essential for limiting how quickly many-body entanglement can generate [80–82]. The standard upper bound for modulus of the OTOC given by Bong *et al.* [76] is $|F|\leqslant\cos(\theta_{VW_t}-\theta_{W_tV})$, where $\theta_{VW_t}=\cos^{-1}|\mathrm{Tr}(\rho VW_t)|$, $\theta_{W_tV}=\cos^{-1}|\mathrm{Tr}(\rho W_tV)|$.

Here, we show that uncertainty relation in PPS system for unitary operators can be used to derive tighter upper bound for the OTOC.

Theorem 5. Let $\rho\in\mathcal{L}(\mathcal{H})$ be the system's state and $|\phi\rangle$ be any arbitrary state, then modulus of the OTOC $F=\mathrm{Tr}[(W_t^{\dagger}V^{\dagger}W_tV)\rho]$ for fixed and time-dependent unitary operators $V,W_t\in\mathcal{L}(\mathcal{H})$, respectively, is upper bounded by

$$|F|=|\,\langle W_t^{\dagger}V^{\dagger}W_tV\rangle\,|\leqslant\cos\big(\theta^{\phi}_{VW_t}-\theta^{\phi}_{W_tV}\big),\qquad(42)$$

where $\theta^{\phi}_{VW_t}=\cos^{-1}||\sqrt{\rho}(VW_t)^{\dagger}\,|\phi\rangle\,||$ and $\theta^{\phi}_{W_tV}=\cos^{-1}||\sqrt{\rho}(W_tV)^{\dagger}\,|\phi\rangle\,||$. Here, $||\ldots||$ defines a vector norm.

Proof. For a given mixed state ρ and arbitrary state $|\phi\rangle$ which we consider to be pre- and postselections, respectively, the standard deviation $\langle\Delta X\rangle^{\phi}_{\rho}$ of any operator X in the PPS system is defined as $(\langle\Delta X\rangle^{\phi}_{\rho})^2=\mathrm{Tr}(XX^{\dagger}\rho)-\langle\phi|X^{\dagger}\rho X|\phi\rangle=\mathrm{Tr}((\sqrt{\rho}X_0^{\phi})^{\dagger}\sqrt{\rho}X_0^{\phi})=||\sqrt{\rho}X_0^{\phi}||_F^2$, where $X_0^{\phi}=X-X\,|\phi\rangle\,\langle\phi|$ and $||A||_F=\sqrt{\mathrm{Tr}(A^{\dagger}A)}$ denotes the Frobenius norm of the operator A. When X is a Hermitian operator, $\langle\Delta X\rangle^{\phi}_{\rho}$ becomes the standard deviation of X defined in Eq. (28). Now consider X to be unitary operators U and V. So, we can derive uncertainty relation for two unitary operators U and V using the Cauchy-Schwarz inequality for operators with Frobenius norm when the system is in pre- and

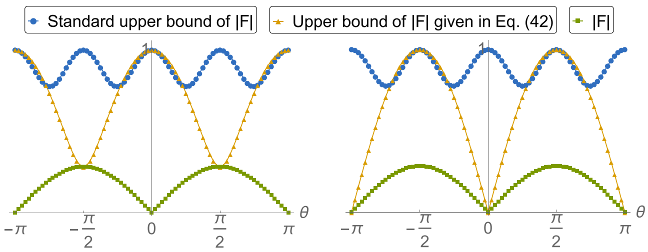

FIG. 2. For both the figures, the blue curve is the standard upper bound for $|F|$ given by Bong *et al.* [76] and the green curve is $|F|$. We have considered $V = \sigma_z$ and $W_t = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$ for a fixed time. Initially prepared state is $|\psi\rangle = \cos(\theta/2)\,|0\rangle + e^{i\pi/11}\sin(\theta/2)\,|1\rangle$. Now in the left figure, the orange curve is the upper bound of $|F|$ given in Eq. (42) when the postselection is $|\phi_1\rangle = \cos(\pi/2)\,|0\rangle + e^{i\pi/2}\sin(\pi/2)\,|1\rangle$. In the right figure, the orange curve is the upper bound of $|F|$ given in Eq. (42) when the postselection is $|\phi_2\rangle = \cos(\pi/4)\,|0\rangle + e^{i\pi/2}\sin(\pi/4)\,|1\rangle$. Here for two (or more) different postselections, it is clearly seen that the upper bound given in Eq. (42) is tighter than the standard upper bound given by Bong *et al.* [76].

postselections ρ and $|\phi\rangle$, respectively, as

$$\langle\Delta U\rangle_\rho^\phi \,\langle\Delta V\rangle_\rho^\phi \geqslant \left|\mathrm{Tr}\left[\left(\sqrt{\rho}U_0^\phi\right)^\dagger \sqrt{\rho}V_0^\phi\right]\right| = |\mathrm{Tr}(VU^\dagger\rho) - \langle\phi|U^\dagger\rho V|\phi\rangle|, \tag{43}$$

where $\langle\Delta U\rangle_\rho^\phi = \sqrt{1 - \langle\phi|U^\dagger\rho U|\phi\rangle}$ and similarly for V also. Now, by replacing $U \to V^\dagger W_t^\dagger$ and $V \to W_t^\dagger V^\dagger$, (43) becomes

$$\begin{aligned} &|\mathrm{Tr}(W_t^\dagger V^\dagger W_t V\rho)| \\ &\quad\leqslant |\langle\phi|(V^\dagger W_t^\dagger)^\dagger \rho W_t^\dagger V^\dagger|\phi\rangle| + \langle\Delta(V^\dagger W_t^\dagger)\rangle_\rho^\phi \,\langle\Delta(W_t^\dagger V^\dagger)\rangle_\rho^\phi \\ &\quad\leqslant ||\sqrt{\rho}V^\dagger W_t^\dagger\,|\phi\rangle\,||\,||\sqrt{\rho}W_t^\dagger V^\dagger\,|\phi\rangle\,|| \\ &\qquad + \sqrt{1 - ||\sqrt{\rho}V^\dagger W_t^\dagger\,|\phi\rangle\,||^2}\sqrt{1 - ||\sqrt{\rho}W_t^\dagger V^\dagger\,|\phi\rangle\,||^2}, \end{aligned} \tag{44}$$

where we used the Cauchy-Schwarz inequality for vectors and $\langle\Delta(V^\dagger W_t^\dagger)\rangle_\rho^\phi = \sqrt{1 - ||\sqrt{\rho}V^\dagger W_t^\dagger\,|\phi\rangle\,||^2}$ and $\langle\Delta(W_t^\dagger V^\dagger)\rangle_\rho^\phi = \sqrt{1 - ||\sqrt{\rho}W_t^\dagger V^\dagger\,|\phi\rangle\,||^2}$, where $||\,|\chi\rangle\,|| = \sqrt{\langle\chi|\chi\rangle}$ denotes vector norm.

Now, by setting $||\sqrt{\rho}(VW_t)^\dagger\,|\phi\rangle\,|| = \cos\theta_{VW_t}^\phi$ and $||\sqrt{\rho}(W_tV)^\dagger\,|\phi\rangle\,|| = \cos\theta_{W_tV}^\phi$ in (44), the inequality (42) is proved. ■

In Fig. 2, it is shown that by suitably choosing $|\phi\rangle$, the upper bound of $|F|$ in Eq. (42) can be made tighter than the standard upper bound given by Bong *et al.* [76]. Hence, we conclude that the tighter upper bound for the modulus of the OTOC is

$$|F| \leqslant \min\left\{\min_\phi\left\{\cos\left(\theta_{VW_t}^\phi - \theta_{W_tV}^\phi\right)\right\}, \cos(\theta_{VW_t} - \theta_{W_tV})\right\}.$$

V. CONCLUSION

We have defined standard deviation of an observable in a PPS system, interpreted it geometrically as well as informationally from the perspective of weak PPS measurements, and subsequently derived the Robertson-Heisenberg–type uncertainty relation for two noncommuting observables. Such uncertainty relations in PPS system impose limitations on the joint sharp preparation of pre- and postselected states for two incompatible observables. We provided the necessary and sufficient condition for zero uncertainty of an observable and show its usefulness in achieving optimized Fisher information in quantum metrology. We have derived both product and sum uncertainty equalities from which a series of uncertainty inequalities can be obtained. The generalization of uncertainty relation for mixed preselection in PPS system has also been discussed. We have demonstrated that the PPS system can exhibit more bizarre behaviors than the usual ones. For instance, it is possible in PPS system that measurement of two compatible observables can disturb each other's measurement results. i.e., the lower bound in the uncertainty relation can be made nonzero by suitably choosing postselections. A similar property in PPS system was first shown by Vaidman [50]. It is also possible that a quantum state (preselection) can be prepared in a PPS system for which both of the standard deviations of incompatible observables are zero although this is not possible in a standard quantum system (see Sec. III B).

The standard deviation and uncertainty relation in the PPS system have been used to provide physical applications. (i) We have used two different definitions of the standard deviations in the PPS system to detect purity of an unknown state. (ii) The uncertainty relation in the PPS system is used to derive the stronger uncertainty relation (i.e., nontrivial for all possible choices of initially prepared states) in the standard quantum system. For two-dimensional quantum system, the stronger uncertainty relation by Maccone-Pati [62] fails to provide the information about the incompatible observables when the system state is an eigenstate of either observable. We have shown that our stronger uncertainty relation overcomes this shortcoming of Maccone-Pati uncertainty relation. (iii) Since the variance in the PPS system remains unaffected (i.e., neither increases nor decreases) by the classical mixing of quantum states, we have concluded that the uncertainty relation in the PPS system is a purely quantum uncertainty relation. In contrast, variance in the standard system increases in general under the classical mixing of quantum states. Following this observation we have provided a measure of classical uncertainty whose less value implies more purely quantum uncertainty. (iv) Tighter sum uncertainty relation in the standard quantum system has been derived where the tightness depends on the postselection. (v) Uncertainty relation in PPS system for two unitary operators has been used to provide tighter upper bound for out-of-time-order correlators.

Future directions: (i) It will be interesting if the global minimum for sum of uncertainties of noncommuting observables in the PPS system exists because that can be used to detect entanglement by suitably choosing postselections, similar to the work by Hofmann and Takeuchi [7]. (ii) Applications and implications of the ideas like "zero uncertainty" and "joint sharp preparation of a quantum state for noncommuting observables" need more attention. (iii) This is a matter of further study if the uncertainty relation (17) in PPS system has applications similar to the RHUR (8), such as quantum metrology, spin squeezing, improving the accuracy of phase measurement in quantum interferometers, etc. (iv) We have derived the condition for the "intelligent pre- and postselected states" to achieve the minimum bound of the uncertainty relation in the PPS system and intelligent pre- and postselected states can be exploited to get highly precise phase measurements because many theoretical and experimental efforts have been made in recent years involving the minimum uncertainty states (for which the RHUR saturates) and the spin-squeezing states in the standard quantum systems (see, for example, [52,56,60]) for precise phase measurements.

ACKNOWLEDGMENTS

We would like to thank K. Mølmer and D. R. M. Arvidsson-Shukur for bringing Refs. [14,15,48,49] and [45], respectively, to our attention. We acknowledge Debmalya Das for valuable comments on this work.

APPENDIX A

Here we derive the condition for which the inequality (17) saturates. In the Cauchy-Schwarz inequality (18), the remainder and the real term to be vanished for the equality condition of Eq. (17), i.e.,

$$|\widetilde{\phi}^{\perp}_{A\psi}\rangle - \frac{\langle\widetilde{\phi}^{\perp}_{B\psi}|\widetilde{\phi}^{\perp}_{A\psi}\rangle}{\langle\widetilde{\phi}^{\perp}_{B\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle}|\widetilde{\phi}^{\perp}_{B\psi}\rangle = 0, \tag{A1}$$

$$\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle + \langle\widetilde{\phi}^{\perp}_{B\psi}|\widetilde{\phi}^{\perp}_{A\psi}\rangle = 0. \tag{A2}$$

Now take the inner product between $\langle\phi^{\perp}_{A\psi}|$ and Eq. (A1), and use the condition (A2), then we have

$$\langle\widetilde{\phi}^{\perp}_{A\psi}|\widetilde{\phi}^{\perp}_{A\psi}\rangle + \frac{(\langle\widetilde{\phi}^{\perp}_{B\psi}|\widetilde{\phi}^{\perp}_{A\psi}\rangle)^2}{\langle\widetilde{\phi}^{\perp}_{B\psi}|\widetilde{\phi}^{\perp}_{B\psi}\rangle} = 0. \tag{A3}$$

Now using $\langle\widetilde{\phi}^{\perp}_{X\psi}|\widetilde{\phi}^{\perp}_{X\psi}\rangle = (\langle\Delta X\rangle^{\phi}_{\psi})^2$, where $X = \{A, B\}$; Eq. (A3) becomes

$$\langle\widetilde{\phi}^{\perp}_{B\psi}|\widetilde{\phi}^{\perp}_{A\psi}\rangle = \pm i\,\langle\Delta A\rangle^{\phi}_{\psi}\,\langle\Delta B\rangle^{\phi}_{\psi}\,. \tag{A4}$$

Finally, use Eqs. (A4) and (12) in Eq. (A1) to obtain the condition (23).

APPENDIX B

To show that the uncertainty relation (17) or (21) is also valid for mixed preselected state ρ, we consider the following operator:

$$T = A^{\phi}_0 + (\gamma + i\epsilon)B^{\phi}_0, \tag{B1}$$

where $A^{\phi}_0 = A - A\,|\phi\rangle\,\langle\phi|$ and $B^{\phi}_0 = B - B\,|\phi\rangle\,\langle\phi|$, and γ, ϵ are some real parameters. Now for any operator T, the inequality

$$\mathrm{Tr}(\rho T T^{\dagger}) \geqslant 0 \tag{B2}$$

holds. Using Eq. (B1), we have

$$\begin{aligned}\mathrm{Tr}(\rho T T^{\dagger}) =& \big(\langle\Delta A\rangle^{\phi}_{\rho}\big)^2 + (\gamma^2+\epsilon^2)\big(\langle\Delta B\rangle^{\phi}_{\rho}\big)^2 \\ &+ \gamma(\langle\{A, B\}\rangle_{\rho} - 2\,\mathrm{Re}W_{AB}) \\ &- i\epsilon(\langle[A, B]\rangle_{\rho} - 2\,\mathrm{Im}W_{AB}) \geqslant 0,\end{aligned} \tag{B3}$$

where $(\langle\Delta A\rangle^{\phi}_{\rho})^2 = \mathrm{Tr}(\rho A^{\phi}_0 A^{\phi\dagger}_0)$ is defined in Eq. (28), $\langle[A, B]\rangle_{\rho} = \mathrm{Tr}(\rho[A, B])$, $\langle\{A, B\}\rangle_{\rho} = \mathrm{Tr}(\rho\{A, B\})$, and $W_{AB} = \mathrm{Tr}(\Pi_{\phi}B\rho A)$. Now one finds the quantity $\mathrm{Tr}(\rho T T^{\dagger})$ is minimum for $\gamma = -\frac{\langle\{A,B\}\rangle_{\rho} - 2\,\mathrm{Re}W_{AB}}{2(\langle\Delta B\rangle^{\phi}_{\rho})^2}$ and $\epsilon = \frac{i(\langle[A,B]\rangle_{\rho} - 2i\,\mathrm{Im}W_{AB})}{2(\langle\Delta B\rangle^{\phi}_{\rho})^2}$. Hence, $\min_{\gamma,\epsilon}\mathrm{Tr}(\rho T T^{\dagger}) \geqslant 0$ becomes

$$\begin{aligned}\big(\langle\Delta A\rangle^{\phi}_{\rho}\big)^2\big(\langle\Delta B\rangle^{\phi}_{\rho}\big)^2 \geqslant& \left[\frac{1}{2i}\,\langle[A, B]\rangle_{\rho} - \mathrm{Im}W_{AB}\right]^2 \\ &+ \left[\frac{1}{2}\,\langle\{A, B\}\rangle_{\rho} - \mathrm{Re}W_{AB}\right]^2.\end{aligned} \tag{B4}$$

By discarding the second term which is a positive number in the right-hand side of Eq. (B4), the uncertainty relation (32) is achieved.

APPENDIX C

Here we show the proofs of all the Lemmas to detect mixedness of an unknown state in qubit, qutrit, qubit-qubit, and qubit-qutrit systems. Let us recall the mathematical expression of the statement of Proposition 3 which is given by

$$\langle\Delta A_w\rangle^{\phi}_{\rho} \geqslant \langle\Delta A\rangle^{\phi}_{\rho}\,. \tag{C1}$$

In the following, we will use Eq. (C1) to prove all the Lemmas. The general form of a mixed state is $\rho = \sum_i p_i\,|\psi_i\rangle\,\langle\psi_i|$ and the condition for which the equality of Eq. (C1) holds is (see proof of the Proposition 3)

$$\langle\phi|A|\psi_i\rangle = \lambda\,\langle\phi|\psi_i\rangle\,, \tag{C2}$$

where λ is some constant which depends on the index of $|\phi\rangle$ (e.g., for $|\phi_k\rangle$, it is λ_k).

1. The proof of Lemma 1

Proof. We first assume that each $|\psi_i\rangle$ is distinct and hence from Eq. (C2), we have a set of equations

$$\langle\phi|\,(A - \lambda I)\,|\psi_i\rangle = 0, \tag{C3}$$

for each $|\psi_i\rangle$. Denote the unnormalized state vector $|\widetilde{\phi}^{\lambda}_{A}\rangle = (A - \lambda I)\,|\phi\rangle$. As $|\widetilde{\phi}^{\lambda}_{A}\rangle$ is a unnormalized state vector different from $|\phi\rangle$ and the $|\psi_i\rangle$, $\forall\ i$ are orthogonal to $|\widetilde{\phi}^{\lambda}_{A}\rangle$, it implies that $\{|\psi_i\rangle\}_{i=1}$ are confined in one-dimensional Hilbert space. Hence, each $|\psi_i\rangle$ is the same initially prepared state that is ρ is a pure state in a qubit system. ■

2. The proof of Lemma 2

Proof. The qubit argument can not be generalized for the higher-dimensional systems. The reason is simply because in three-dimensional Hilbert space (for example) all the $|\psi_i\rangle$ can be confined in a two-dimensional subspace of the Hilbert space which is orthogonal to $|\widetilde{\phi}^{\lambda}_{A}\rangle$. To make an "if and only if" condition, we consider the orthogonal basis $\{|\phi_k\rangle\}^{3}_{k=1}$ as valid postselections. Here, valid postselections are those postselections for which weak values are defined.

As there are three postselections in three-dimensional Hilbert space, we have three sets of equations like (C2) for the equality of the inequality (C1):

$$\{\langle\phi_1|\,(A-\lambda_1 I)\,|\psi_i\rangle = 0\}_{i=1}, \tag{C4}$$

$$\{\langle\phi_2|\,(A-\lambda_2 I)\,|\psi_i\rangle = 0\}_{i=1}, \tag{C5}$$

$$\{\langle\phi_3|\,(A-\lambda_3 I)\,|\psi_i\rangle = 0\}_{i=1}. \tag{C6}$$

Now, there are three possibilities which are implied by (C4), (C5), and (C6): (i) The state vectors $\{|\widetilde{\phi}_{kA}^{\lambda_k}\rangle = (A-\lambda_k I)\,|\phi_k\rangle\}_{k=1}^{3}$ span the whole three-dimensional Hilbert space $\mathcal{H}$, (ii) $\{|\widetilde{\phi}_{kA}^{\lambda_k}\rangle\}_{k=1}^{3}$ span a two-dimensional Hilbert space $\mathcal{H}$, (iii) $\{|\widetilde{\phi}_{kA}^{\lambda_k}\rangle\}_{k=1}^{3}$ span a one-dimensional Hilbert space $\mathcal{H}$.

Below, we will show that possibility (i) is discarded naturally whereas to discard possibility (iii), we need a condition on observable A and postselection $|\phi\rangle$. Then, possibility (ii) will automatically indicate that all the $\{|\psi_i\rangle\}_{i=1}$ are the same, i.e., ρ is pure.

To start with possibility (i), let us assume that possibility (i) is true, then $\{|\psi_i\rangle\}_{i=1}$ has to be orthogonal to $\{|\widetilde{\phi}_{kA}^{\lambda_k}\rangle\}_{k=1}^{3}$ according to (C4), (C5), and (C6) implying $|\psi_i\rangle = 0\ \forall\ i$, i.e., $\rho = 0$. So we discard this possibility.

Possibility (iii) implies

$$\mathcal{N}_1(A-\lambda_1 I)\,|\phi_1\rangle = \mathcal{N}_2(A-\lambda_2 I)\,|\phi_2\rangle = \mathcal{N}_3(A-\lambda_3 I)\,|\phi_3\rangle \tag{C7}$$

along the z axis (for example) and hence $\{|\psi_i\rangle\}_{i=1}$ span two-dimensional xy plane. Here $\mathcal{N}_k$ are normalization constants. Now the inner product of (C7) with $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$, respectively gives

$$\mathcal{N}_1(\langle\phi_1|A|\phi_1\rangle - \lambda_1) = \mathcal{N}_2\,\langle\phi_1|A|\phi_2\rangle = \mathcal{N}_3\,\langle\phi_1|A|\phi_3\rangle\,, \tag{C8}$$

$$\mathcal{N}_1\,\langle\phi_2|A|\phi_1\rangle = \mathcal{N}_2(\langle\phi_2|A|\phi_2\rangle - \lambda_2) = \mathcal{N}_3\,\langle\phi_2|A|\phi_3\rangle\,, \tag{C9}$$

$$\mathcal{N}_1\,\langle\phi_3|A|\phi_1\rangle = \mathcal{N}_2\,\langle\phi_3|A|\phi_2\rangle = \mathcal{N}_3(\langle\phi_3|A|\phi_3\rangle - \lambda_3). \tag{C10}$$

Now, the particular choice

$$\langle\phi_1|A|\phi_2\rangle = 0 \tag{C11}$$

implies that Eqs. (C8) and (C9) do not hold if $\langle\phi_1|A|\phi_3\rangle \neq 0$ and $\langle\phi_2|A|\phi_3\rangle \neq 0$, respectively. If either of Eq. (C8) and (C9) does not hold then possibility (iii) is discarded. But, if $\langle\phi_1|A|\phi_3\rangle = 0$ and $\langle\phi_2|A|\phi_3\rangle = 0$, then we have to proceed further. Note that, by setting $\langle\phi_1|A|\phi_2\rangle = 0$ from Eq. (C11), $\langle\phi_1|A|\phi_3\rangle = 0$ and $\langle\phi_2|A|\phi_3\rangle = 0$ in Eqs. (C8), (C9), and (C10), we have

$$\lambda_k = \langle\phi_k|A|\phi_k\rangle \quad \text{for} \quad k = 1, 2, 3. \tag{C12}$$

Now, it is easy to see that with the values of λ_k from Eq. (C12), $\{\langle\phi_k|\widetilde{\phi}_{kA}^{\lambda_k}\rangle = 0\}_{k=1}^{3}$ holds. This implies that $\{|\widetilde{\phi}_{kA}^{\lambda_k}\rangle\}_{k=1}^{3}$ cannot be confined in one-dimensional Hilbert space, i.e., along a particular axis and in our assumption it is the z axis. But according to Eq. (C7), $\{|\widetilde{\phi}_{kA}^{\lambda_k}\rangle\}_{k=1}^{3}$ are along the z axis. Hence, it shows the contradiction and we discard the possibility (iii) when the condition $\langle\phi_1|A|\phi_2\rangle = 0$ is considered.

Finally, the possibility (ii) implies that $\{|\psi_i\rangle\}_{i=1}$ must be spanned in one-dimensional Hilbert space $\mathcal{H}$ that is, each $|\psi_i\rangle$ is the same initially prepared state which is a pure state.

So, we conclude that if for an observable A and a complete orthonormal basis $\{|\phi_k\rangle\}_{k=1}^{3}$ (to be used as postselected states) of any three-level quantum system (i.e., a qutrit), the condition $\langle\phi_1|A|\phi_2\rangle = 0$ is considered, then the equality in Eq. (C1) holds good if and only if the preselected state ρ is pure. ■

3. The proof of Lemma 3

Proof. For this bipartite system, we consider the observable and the postselection to be $A\otimes I$ and $|\phi_{AB}\rangle = |\phi_A\rangle\,|\phi_B\rangle$, respectively. The standard deviations defined in Eqs. (28) and (30) for the given bipartite state ρ become

$$\begin{aligned}\left(\langle\Delta(A\otimes I)_w\rangle_\rho^{\phi_{AB}}\right)^2 &= \mathrm{Tr}[(A\otimes I)^2\rho] - \frac{|\,\langle\phi_{AB}|(A\otimes I)\rho|\phi_{AB}\rangle\,|^2}{\langle\phi_{AB}|\rho|\phi_{AB}\rangle}\\ &= \mathrm{Tr}[A^2\rho_A] - \frac{|\,\langle\phi_A|A\rho_A^{\phi_B}|\phi_A\rangle\,|^2}{\langle\phi_A|\rho_A^{\phi_B}|\phi_A\rangle},\end{aligned} \tag{C13}$$

$$\begin{aligned}\left(\langle\Delta(A\otimes I)\rangle_\rho^{\phi_{AB}}\right)^2 &= \mathrm{Tr}[(A\otimes I)^2\rho] - \langle\phi_{AB}|(A\otimes I)\rho(A\otimes I)|\phi_{AB}\rangle\\ &= \mathrm{Tr}[A^2\rho_A] - \langle\phi_A|A\rho_A^{\phi_B}A|\phi_A\rangle\,,\end{aligned} \tag{C14}$$

respectively, where $\rho_A^{\phi_B} = \langle\phi_B|\rho|\phi_B\rangle$ is the collapsed density operator of the subsystem A when a projection operator $\Pi_{\phi_B} = |\phi_B\rangle\,\langle\phi_B|$ is measured in the subsystem B. In a qubit-qubit system, the subsystem A is two dimensional and hence $\langle\Delta(A\otimes I)_w\rangle_\rho^{\phi_{AB}}$ from Eq. (C13) and $\langle\Delta(A\otimes I)\rangle_\rho^{\phi_{AB}}$ from Eq. (C14) are equal "if and only if" $\rho_A^{\phi_B}$ is pure. Now, $\rho_A^{\phi_B}$ being pure can be from ρ being both pure and mixed. If ρ is pure, then $\rho_A^{\phi_B}$ is always pure but if ρ mixed, then it is easy to see that $\rho_A^{\phi_B}$ is pure only when $\rho = \sum_{i=1}^{2} p_i\,|\psi_A^i\rangle\,\langle\psi_A^i| \otimes |\phi_B^i\rangle\,\langle\phi_B^i|$, where $|\phi_B^1\rangle = |\phi_B\rangle$ and $\sum_{i=1}^{2} |\phi_B^i\rangle\,\langle\phi_B^i| = I$. So, let us consider another postselection $|\phi_B'\rangle$ (which is not orthogonal to $\{|\phi_B^i\rangle\}_{i=1}^{2}$) and if we find $\rho_A^{\phi_B'}$ to be pure which is equivalent to the equality of $\langle\Delta(A\otimes I)_w\rangle_\rho^{\phi_{AB}'}$ and $\langle\Delta(A\otimes I)\rangle_\rho^{\phi_{AB}'}$, then we are sure that the bipartite state ρ is a pure state (due to the virtue of qubit system discussed above), where $|\phi_{AB}'\rangle = |\phi_A\phi_B'\rangle$.

So, here is the conclusion: Consider any two nonorthogonal postselections $|\phi_B\rangle$ and $|\phi_B'\rangle$ in the subsystem B. For any observable A, equality of $\langle\Delta(A\otimes I)_w\rangle_\rho^{\phi_{AB}}$ and $\langle\Delta(A\otimes I)\rangle_\rho^{\phi_{AB}}$ and separately of $\langle\Delta(A\otimes I)_w\rangle_\rho^{\phi_{AB}'}$ and $\langle\Delta(A\otimes I)\rangle_\rho^{\phi_{AB}'}$ hold *only* when the $2\otimes 2$ preselected state ρ is pure. ■

4. The proof of Lemma 4

Proof. The treatment above with the condition of the qutrit system, we have the following conclusion: if for an observable A and any complete orthonormal basis $\{|\phi_A^k\rangle\}_{k=1}^{3}$ (to be used as postselected states) for a qutrit, the condition $\langle\phi_A^3|A|\phi_A^1\rangle = 0$ is considered, then equality of $\langle\Delta(A\otimes I)_w\rangle_\rho^{\phi_{AB}}$ and $\langle\Delta(A\otimes I)\rangle_\rho^{\phi_{AB}}$ and separately of $\langle\Delta(A\otimes I)_w\rangle_\rho^{\phi_{AB}'}$ and $\langle\Delta(A\otimes I)\rangle_\rho^{\phi_{AB}'}$ hold if and only if the $3\otimes 2$ preselected state ρ is pure, where $|\phi_{AB}'\rangle = |\phi_A\phi_B'\rangle$. ■

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[66] The second inequality in MPUR [62] is given by $\langle\Delta A\rangle^2_\psi + \langle\Delta B\rangle^2_\psi \geqslant \frac{1}{2}|\langle\psi^{\perp}_{A+B}|(A+B)|\psi\rangle|^2$, where $|\psi^{\perp}_{A+B}\rangle = [1/\langle\Delta(A+B)\rangle_\psi](A+B-\langle A+B\rangle_\psi)|\psi\rangle$ and $\langle\Delta(A+B)\rangle^2_\psi = \langle(A+B)^2\rangle_\psi - \langle A+B\rangle^2_\psi$. If $|\psi\rangle$ is the eigenstate of the observable A, then $\langle\Delta(A+B)\rangle_\psi = \langle\Delta B\rangle_\psi$ and using the expression of $|\psi^{\perp}_{A+B}\rangle$ in the above inequality, we have $\langle\Delta B\rangle^2_\psi \geqslant \frac{1}{2}\langle\Delta B\rangle^2_\psi$.

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