Heisenberg-Langevin approach to driven superradiance

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We present an analytical approach for the study of driven Dicke superradiance based on a Heisenberg-Langevin formulation. We calculate the steady-state fluctuations of both the atomic-spin and light-field operators. While the atoms become entangled below a critical drive, exhibiting spin squeezing, we show that the radiated light is in a classical-like coherent state whose amplitude and spectrum are identical to those of the incident driving field. Therefore, the nonlinear atomic system scatters light as a linear classical scatterer. Our results are consistent with the recent theory of coherently radiating spin states. The presented Heisenberg-Langevin approach should be simple to generalize for treating superradiance beyond the permutation-symmetric Dicke model.

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I. INTRODUCTION

Superradiance describes the cooperative radiation of an ensemble of quantum emitters into common photonic modes. A conceptually simple case that captures the essence of cooperative radiation is that of Dicke superradiance, where all the constituents of an ensemble of two-level atoms are coupled to the common photonic modes in an identical manner, thus forming an effective "collective-spin" dipole [1–3]. Superradiance was observed in both atoms [4–8] and artificial emitters [9] and plays a role in various quantum phenomena and technologies, ranging from phase transitions [8,10–13] to narrowband superradiant lasers [14–17].

The situation wherein the atoms are additionally driven by a resonant laser (cooperative resonance fluorescence) can be studied by a driven-dissipative master equation of the Dicke model. Mean-field theory yields a second-order phase transition of the steady-state atomic population, or "magnetization," as a function of the drive [18–23]. Spin squeezing was recently found in the steady state by a numerical solution of the master equation [23–26], with a supporting analytical result obtained in [25]. For the radiated light, intensity correlations $g^{(2)}$ were calculated and found to exhibit bunching correlations above the phase-transition point but no correlations below it [20]. More recently, it was found that the appearance of so-called coherently radiating spin states (CRSSs) as the steady state of driven Dicke superradiance underlies these results [27].

Here we present a simple analytical approach for driven superradiance based on Heisenberg-Langevin (HL) equations. While this HL approach is, in principle, equivalent to the master equation used previously, the HL equations are natural for the direct analytical treatment of both spin and field fluctuations via their operator-form solution. In particular, we account for spin and field fluctuations around the mean field using the Holstein-Primakoff approximation. For the spin fluctuations the operator-valued solutions are in a Bogoliubov transformation form, implying quantum correlations, as verified by the subsequent calculation of spin squeezing. For the field operator, we find that the fluctuations are proportional to the vacuum field, thus proving that the radiated field below the phase-transition point is in a coherent state. We also calculate the two-time correlation of the field, finding that the spectrum is δ peaked at the incident-drive frequency. Surprisingly, the light is thus scattered from the many-atom system as if the latter is a linear system, although the atomic system is highly nonlinear, which is evident from its phase transition. We discuss the consistency and relation of these results with the predictions of CRSS theory [27].

This paper is organized as follows. In Sec. II we derive the HL equations of the driven Dicke model, focusing on a relevant cavity-scheme realization. After recalling the mean-field solution in Sec. III, we treat spin fluctuations and squeezing in Sec. IV. Section V is devoted to the analysis of the radiated light. Finally, our conclusions are presented in Sec. VI.

II. MODEL

We begin with the derivation of the HL equations of motion that describe the driven Dicke model, considering a system of atoms in a damped cavity as realized in most typical experiments [4–6,9]. Realizations of Dicke physics also exist in other systems in which many atoms are coupled to a common photon bath, e.g., in waveguide quantum electrodynamics (QED) [24] or even in an elongated atomic ensemble in free space [3,8]; however, the cavity case considered here is conceptually the most straightforward one as it directly emphasizes a single common photonic mode.

A. System and Hamiltonian

We consider the system displayed in Fig. 1: N two-level atoms are trapped inside an optical cavity driven by external

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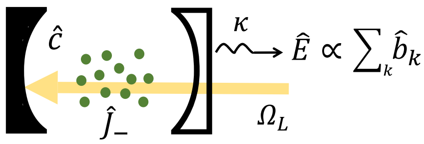

FIG. 1. Cavity realization of driven superradiance. An atomic ensemble is trapped inside a cavity, wherein all atoms (green dots) are identically coupled to a cavity mode (lowering operator $\hat{c}$) and hence described by a collective-spin dipole (lowering operator $\hat{J}_-$). The cavity field is damped through its mirrors at rate κ to the outside propagating modes $\hat{b}_k$, which form the radiated field $\hat{E}$, and is driven by a laser with Rabi-field amplitude Ω_L. Here a single-sided cavity scheme is presented, with one out-coupling mirror (right-hand side).

laser light thorough the cavity mirrors. The atomic positions are such that all atoms are identically coupled to the cavity mode (i.e., well within the cavity mode waist and at longitudinal positions that are multiples of cavity wavelength apart). The Hamiltonian of the atoms and the cavity is given by

$$\hat{H}_S = \hbar\omega_a\hat{J}_z + \hbar\omega_c\hat{c}^\dagger\hat{c} + \hbar[\hat{c}^\dagger(g^*\hat{J}_- + \Omega_L e^{-i\omega_a t}) + \text{H.c.}]. \tag{1}$$

Here $\hat{c}$ is the boson lowering operator of the cavity mode of frequency ω_c, whereas $\hat{J}_\alpha = (1/2)\sum_{n=1}^{N}\hat{\sigma}_n^\alpha$ ($\alpha \in \{x, y, z\}$) are the collective-spin operators of the atomic ensemble, with $\hat{\sigma}_n^\alpha$ being the Pauli operator of a two-level atom $n \in \{1, \dots, N\}$ with resonant frequency ω_a. The external laser, resonant with the atoms and with amplitude Ω_L, drives the cavity via its mirrors. The cavity mode is also coupled to the atoms via the dipole coupling g, which is identical for all atoms, where $\hat{J}_- = \hat{J}_x - i\hat{J}_y = \sum_{n=1}^{N}\hat{\sigma}_n^- = \hat{J}_+^\dagger$ is the collective-spin lowering operator of the atoms and $\hat{\sigma}_n^- = (\hat{\sigma}_n^+)^\dagger$ is the Pauli lowering operator of atom n.

In addition, the cavity mode is coupled through its mirrors to a one-dimensional (1D) continuum of propagating photon modes characterized by the wave number k and corresponding boson modes $\hat{b}_k$ and frequencies vk (v is the speed of light). The Hamiltonians describing this 1D photon reservoir and its coupling to the system are given by (here for a one-sided cavity; Fig. 1)

$$\hat{H}_R = \sum_{k>0}\hbar vk\hat{b}_k^\dagger\hat{b}_k,$$
$$\hat{H}_{SR} = \hbar\sum_{k>0}(\eta\hat{b}_k^\dagger\hat{c} + \text{H.c.}), \quad \eta \equiv \sqrt{\frac{v}{L}\kappa}, \tag{2}$$

respectively, where the coupling constant η is taken to be k independent (consistent with the Markov approximation) and L is the quantization length of the 1D continuum. The total Hamiltonian is given by $\hat{H} = \hat{H}_S + \hat{H}_R + \hat{H}_{SR}$. We note that we neglect here the direct spontaneous emission from atoms to photon modes in transverse directions outside the cavity. For a dilute ensemble this is an individual-atom process that is typically much slower than the relevant Dicke dynamics discussed here.

B. Heisenberg-Langevin equations

We begin with eliminating the reservoir modes $\hat{b}_k$ by inserting the solution of their Heisenberg equations into the equation for $\hat{c}$, obtaining within the usual Markov approximation [28]

$$\dot{\tilde{c}} = \left(i\delta_c - \frac{\kappa}{2}\right)\tilde{c} - ig^*\tilde{J}_- - i\Omega_L + \hat{E}_0(t), \quad \delta_c = \omega_a - \omega_c. \tag{3}$$

Here the system operators are already written in a rotated frame, $\tilde{c} = \hat{c}e^{i\omega_L t}$ and $\tilde{J}_- = \hat{J}_- e^{i\omega_L t}$, whereas the Langevin vacuum noise of the reservoir is given by $\hat{E}_0(t) = -i\sum_k \eta^* e^{-i(vk-\omega_L)t}\hat{b}_k(0)$, satisfying (assuming an initial vacuum state)

$$\langle\hat{E}_0(t)\hat{E}_0^\dagger(t')\rangle = \kappa\delta(t - t'). \tag{4}$$

Next, we eliminate the cavity mode by assuming that its damping rate κ in the 1D continuum is much faster than the typical timescale of variations in $\tilde{J}_-$, i.e., $\kappa \gg |\dot{\tilde{J}}_-/\tilde{J}_-|$. Within this coarse-grained dynamical picture and for times t much longer than $1/\kappa$, the elimination of $\tilde{c}$ is equivalent to setting $\dot{\tilde{c}} = 0$ in Eq. (3) and inserting the solution for $\tilde{c}$ into the Heisenberg equations for atomic variables such as $\tilde{J}_-$ and $\hat{J}_z$. Finally, we obtain (denoting $\tilde{J}_\mp \to \hat{J}_\mp$ for simplicity)

$$\dot{\hat{J}}_- = (\gamma - i2\Delta)\hat{J}_z\hat{J}_- - i2\hat{J}_z[\Omega + \hat{f}(t)],$$
$$\dot{\hat{J}}_z = -\gamma\hat{J}_+\hat{J}_- + i\hat{J}_+[\Omega + \hat{f}(t)] - i[\Omega^* + \hat{f}^\dagger(t)]\hat{J}_-, \tag{5}$$

with the coefficients

$$\gamma = \frac{|g|^2\kappa}{\delta_c^2 + (\kappa/2)^2}, \quad \Delta = \frac{-|g|^2\delta_c}{\delta_c^2 + (\kappa/2)^2}, \quad \Omega = \frac{-2g\Omega_L}{2\delta_c + i\kappa} \tag{6}$$

and the effective Langevin input-vacuum noise (filtered by the cavity), $\hat{f}(t) \approx [2g/(\kappa - i2\delta_c)]\hat{E}_0(t)$, satisfying

$$\langle\hat{f}(t)\hat{f}^\dagger(t')\rangle = \gamma\delta(t - t'). \tag{7}$$

Equations (5) form the HL equations of the driven Dicke model, with an effective emission rate γ of an atom to the outside modes via the cavity and an effective laser drive with Rabi frequency Ω. The collective shift Δ describes the resonant dipole-dipole interactions between pairs of atoms [29], corresponding to an effective Hamiltonian $\hat{H}_{\text{dd}} = -\hbar\sum_n\sum_m \Delta_{nm}\hat{\sigma}_n^+\hat{\sigma}_m^-$. Here the dipole-dipole kernel $\Delta_{nm} = \Delta$ is uniform for all atom pairs n and m since all atoms are coupled identically to the mediating cavity photon mode. In treatments of superradiance in free space, such coherent dipole-dipole effects are often ignored [3], whereas they vanish in a waveguide QED superradiance scheme [24]. In the cavity setting, they exist, however, if one allows for atom-cavity detuning δ_c as seen in Eq. (6) for Δ [23,26].

We note that while this specific derivation was performed starting from the damped-cavity model, equivalent HL equations (5) can be derived by considering other models of photon continua to which all atoms are identically coupled. Here the cavity mode effectively becomes a continuum due to its fast damping rate κ.

C. Equivalent master equation

The HL equations (5) are equivalent to the following master equation for the density matrix of the atoms:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_{\text{eff}}, \hat{\rho}] + \gamma\left[\hat{J}_-\hat{\rho}\hat{J}_+ - \frac{1}{2}(\hat{J}_+\hat{J}_-\hat{\rho} + \hat{\rho}\hat{J}_+\hat{J}_-)\right],$$
$$\hat{H}_{\text{eff}} = -\hbar\Delta\hat{J}_+\hat{J}_- - \hbar(\Omega\hat{J}_+ + \Omega^*\hat{J}_-). \quad (8)$$

This master equation with $\Delta = 0$ is a typical starting point for the analysis of driven Dicke superradiance presented in previous works [18–22,24,25,30], whereas the additional dipole-dipole term Δ is considered in Refs. [23,26]. Here instead we will use the HL formulation of Eq. (5) in order to derive analytical results for fluctuations and correlations of atomic and photonic degrees of freedom. We will use the master equation as a numerical verification of the one-time correlation functions. Since the total spin $\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = j(j+1)$ is conserved under the dynamics of Eqs. (5) and (8), the initial state sets the SU(2) spin representation j. Assuming an initial ground state for the N atoms, we have $j = N/2$, and the Hilbert space that spans Eqs. (8) is of size $2j + 1 = N + 1$ and can be easily solved numerically for reasonable N.

III. MEAN-FIELD SOLUTION

As a first step of our analytical approach, we begin with the mean-field solution of the model in the steady state. This solution will provide the basis for the subsequent study of quantum fluctuations using the HL operator approach. To obtain the mean-field equations, we take the average over the HL equations (5) such that the Langevin vacuum-noise terms vanish and perform the factorization of operator products $\langle\hat{J}_\alpha\hat{J}_\beta\rangle \approx \langle\hat{J}_\alpha\rangle\langle\hat{J}_\beta\rangle$ (with $\alpha, \beta \in \{x, y, z\}$). This factorization is justified for $N \to \infty$ under the mean-field assumption that fluctuations of observables are much smaller than their mean. It is important to note that such a factorization does not mean that there are no correlations between the atoms that comprise the collective spin $\hat{J}_\alpha$ [18]: in fact, we see below that the atoms are entangled [23–26]. Considering the conservation of the total spin $\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = j(j+1)$, with $j = N/2 \gg 1$, the solution to the mean-field equations becomes (see also [23])

$$\langle\hat{J}_z\rangle = -\frac{N}{2}\sqrt{1 - \frac{|\Omega|^2}{\Omega_c^2}}, \quad \langle\hat{J}_-\rangle = -\frac{\Omega}{\Delta + i\gamma/2}, \quad (9)$$

with the critical driving field defined by

$$\Omega_c = \Omega_c(\Delta) = \frac{N}{4}\sqrt{\gamma^2 + 4\Delta^2}. \quad (10)$$

The steady-state population inversion (or magnetization) $\langle\hat{J}_z\rangle$ thus exhibits a second-order phase transition as a function of the drive Ω, where it vanishes at the critical value Ω_c. The latter increases with the strength of the dipole-dipole shift Δ, as seen in Eq. (10). For $|\Omega| > \Omega_c$ oscillatory solutions of the mean-field equations [18] exist which, nevertheless, appear to decay to zero at long timescales upon consideration of the full quantum problem [23]. Figure 2 displays $\langle\hat{J}_z\rangle$ obtained with the exact numerical solutions of the master equation (8) for $N = 50$ and different values of Δ. Very good agreement with the mean-field expression (9) is exhibited when $|\Omega|$ is not too close to the critical point Ω_c. In particular, calculations with different values of Δ all collapse to the same curve when Ω is scaled to the corresponding $\Omega_c(\Delta)$ from Eq. (10). Disagreement between mean-field and numerical results is observed around Ω_c due to the fact that the mean value of $\langle\hat{J}_z\rangle$ near Ω_c becomes increasingly small while fluctuations grow, in contradiction to the mean-field assumption. The second-order transition predicted by the mean-field solution in the thermodynamic limit $N \to \infty$ then becomes smoother at finite N.

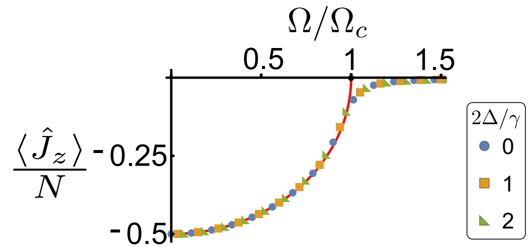

FIG. 2. Population inversion $\langle\hat{J}_z\rangle$ of the collective atomic system as a function of the driving field Ω. Results obtained from the numerical solution of Eq. (8) with $N = 50$ atoms and for different values of the dipole-dipole shift $2\Delta/\gamma = 0, 1, 2$ all collapse to the same curve when Ω is scaled by the critical field $\Omega_c(\Delta)$ from Eq. (10). The red line represents the analytical mean-field solution from Eq. (9), exhibiting a second-order phase transition. The exact numerical solutions agree with the mean-field result until they diverge away a bit before the transition point due to the finite value of N.

The mean-field solution (9) can also be written as a mean of the spin vector $\hat{\mathbf{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)$ in a Bloch sphere,

$$\langle\hat{\mathbf{J}}\rangle = \begin{pmatrix}\langle\hat{J}_x\rangle\\ \langle\hat{J}_y\rangle\\ \langle\hat{J}_z\rangle\end{pmatrix} = -\frac{N}{2}\begin{pmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{pmatrix}, \quad (11)$$

with the angles in spherical coordinates given by

$$\sin\theta = \frac{|\Omega|}{\Omega_c}, \quad \phi = \arg(\Delta + i\gamma/2) - \arg(\Omega). \quad (12)$$

For later purposes, it is instructive to introduce a rotated coordinate system in which the mean spin vector is directed to the south pole of the Bloch sphere and hence appears as a ground state in this rotated system. Spin operators in the rotated system, described by the vector $\hat{\mathbf{J}}' = (\hat{J}'_x, \hat{J}'_y, \hat{J}'_z)$, are related to the original spin operators $\hat{\mathbf{J}} = (\hat{J}_x, \hat{J}_y, \hat{J}_z)$ via the rotation matrix $\mathcal{R}$ as

$$\hat{\mathbf{J}}' = \mathcal{R}^{-1}\hat{\mathbf{J}}, \quad \langle\hat{\mathbf{J}}'\rangle = -\frac{N}{2}\begin{pmatrix}0\\0\\1\end{pmatrix},$$
$$\mathcal{R} = \begin{pmatrix}\cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi\\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi\\ -\sin\theta & 0 & \cos\theta\end{pmatrix}. \quad (13)$$

As required, in the rotated system the mean spin vector $\langle\hat{\mathbf{J}}'\rangle$ points to the south pole, defining the $-z'$ axis as the mean spin direction.

IV. SPIN FLUCTUATIONS AND SQUEEZING

We now turn to the analysis of small fluctuations of spin variables around the mean-field solution (Sec. IV A). This will allow us to estimate atomic correlations such as spin squeezing (Sec. IV B) and, later on, also the fluctuations in the scattered field (Sec. V).

A. Collective spin fluctuations in the Holstein-Primakoff approximation

We recall that within its representation in the rotated system (13), the mean spin vector $\hat{\mathbf{J}}' = (\hat{J}'_x, \hat{J}'_y, \hat{J}'_z)$ is directed towards the axis z' and vanishes along the x' and y' axes. In order to analyze fluctuations around this mean, we first define the spin lowering operator in the rotated basis, $\hat{J}'_- = \hat{J}'_x - i\hat{J}'_y$, and rewrite the HL equations (5) in terms of the rotated-spin operators $\hat{J}'_-$ and $\hat{J}'_z$ using the transformation $\mathcal{R}$ from (13). As in the original basis, the HL equations in the rotated basis are also nonlinear in their relevant variables, $\hat{J}'_-$ and $\hat{J}'_z$; however, the linearization of the equations for small fluctuations around the mean field is simpler in this rotated basis. To this end, we use the Holstein-Primakoff transformation, which is an exact representation of SU(2) spin operators (here of spin $j = N/2$) in terms of a bosonic operator $\hat{a}$ (satisfying $[\hat{a}, \hat{a}^\dagger] = 1$) [31],

$$\hat{J}'_- = \sqrt{N - \hat{a}^\dagger\hat{a}}\,\hat{a}, \quad \hat{J}'_z = \hat{a}^\dagger\hat{a} - \frac{N}{2}. \tag{14}$$

We see that the limit $\hat{a} \to 0$ restores the mean-field solution $\hat{\mathbf{J}}' \to \langle \hat{\mathbf{J}}' \rangle = O(N)$ from (13), where quantum fluctuations are neglected. As a first correction to the mean-field assumption, we then consider small spin fluctuations, $|\hat{a}| \sim O(1) \ll \sqrt{N}$, and expand the nonlinear HL equation for $\hat{J}'_-$ to leading orders in the small parameter $1/\sqrt{N}$. This is achieved using the approximation

$$\hat{J}'_- \approx \sqrt{N}\,\hat{a}, \quad \hat{J}'_z \approx -\frac{N}{2}, \tag{15}$$

and the subsequent linearization of the HL equation to the first orders of $\hat{a}$ and the noise $\hat{f}$. Finally, we obtain the HL equation for the spin fluctuations $\hat{a}$,

$$\dot{\hat{a}} = -\left(N\frac{\gamma}{2}\cos\theta - iN\Delta\frac{1+\cos^2\theta}{2}\right)\hat{a} - iN\Delta\frac{\sin^2\theta}{2}\hat{a}^\dagger + i\sqrt{N}\left[\frac{1+\cos\theta}{2}e^{i\phi}\hat{f}(t) - \frac{1-\cos\theta}{2}e^{-i\phi}\hat{f}^\dagger(t)\right]. \tag{16}$$

This yields coupled linear equations for $\hat{a}$ and $\hat{a}^\dagger$ whose solution in the steady state for times $t \gg (N\gamma\cos\theta/2)^{-1}$ is

$$\hat{a}(t) = \frac{1}{\sqrt{\cos\theta}}\left[\frac{1+\cos\theta}{2}e^{i\phi}\hat{b}(t) + \frac{1-\cos\theta}{2}e^{-i\phi}\hat{b}^\dagger(t)\right],$$
$$\hat{b}(t) = i\sqrt{N\cos\theta}\int_0^t dt' e^{-N\cos\theta[\frac{\gamma}{2}-i\Delta](t-t')}\hat{f}(t'). \tag{17}$$

B. Bogoliubov transformation form

The operator-form solution for the spin fluctuation (17), along with the correlation function (7) of the Langevin vacuum noise $\hat{f}(t)$, now allows to evaluate correlations of the collective spin. However, even without performing specific calculations, the operator solution itself is already very insightful. Noting that the operator $\hat{b}$ satisfies the bosonic commutation relations $[\hat{b}, \hat{b}^\dagger] = 1$ [by using Eq. (7) for $\hat{f}$], we identify that the lowering operator of the spin fluctuation $\hat{a}$ exhibits a Bogoliubov transformation form $\hat{a} = u\hat{b} + v\hat{b}^\dagger$ with the Bogoliubov coefficients

$$u = \frac{1+\cos\theta}{2\sqrt{\cos\theta}}e^{i\phi}, \quad v = \frac{1-\cos\theta}{2\sqrt{\cos\theta}}e^{-i\phi}. \tag{18}$$

From Eq. (17) we observe that $\hat{b}$ is proportional to the integrated vacuum noise $\hat{f} \propto \hat{E}_0$. Recalling that a Bogoliubov-transformed vacuum noise is known to represent quantum-correlated noise in the form of a squeezed vacuum [32,33], we then expect the spin fluctuations $\hat{a}$ to exhibit quantum squeezing. In particular, the bosonic squeezing parameter can be read off directly from the Bogoliubov transformation and is generally given by (see [32] or Appendix A)

$$\xi_B^2 \equiv \min_\varphi \mathrm{Var}[\hat{a}e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi}] = (|u| - |v|)^2 = \cos\theta, \tag{19}$$

where in the last equality we used the Bogoliubov coefficients for our specific case, Eqs. (18). When $\xi_B^2 < 1$, quantum squeezing exists, which is the case whenever v does not vanish, i.e., when $\hat{a}$ includes the raising operator of the vacuum field, $\hat{b}^\dagger \propto \int dt\hat{f}^\dagger(t) \propto \int dt\hat{E}_0^\dagger(t)$.

In the following we will see that this simple analysis exactly predicts the quantum correlations of spin fluctuations and that the above bosonic squeezing ξ_B^2 in fact becomes identical to the so-called spin-squeezing parameter. Therefore, the analytical operator-form solution provided by the HL approach allows for a direct estimate of quantum spin correlations: they are expected to grow with the driving field $|\Omega|/\Omega_c = \sin\theta > 0$, as indicated by decreasing values of $\xi_B^2 = \cos\theta < 1$. This is further discussed below.

C. Spin squeezing

A particularly relevant characterization of collective-spin fluctuations is provided by the spin-squeezing parameter [34,35]. Spin squeezing quantifies fluctuations of the spin vector perpendicular to its mean direction and is linked to the sensitivity of quantum-enhanced metrology with collections of spins [36–38] and their underlying pairwise entanglement [39,40]. Within the rotated spin representation from (13), where the mean is directed to $-z'$ so that $|\langle\hat{\mathbf{J}}'\rangle| = |\langle\hat{J}'_z\rangle|$, the spin-squeezing parameter is given by [34,36,37]

$$\xi^2 = \min_\varphi \frac{\mathrm{Var}[\hat{J}'_\varphi]N}{|\langle\hat{J}'_z\rangle|^2}, \quad \hat{J}'_\varphi = \cos\varphi\hat{J}'_x + \sin\varphi\hat{J}'_y; \tag{20}$$

that is, it is proportional to the minimal variance of the fluctuations along the plane perpendicular to the main spin direction ($x'y'$ plane). Spin squeezing exists for $\xi^2 < 1$, implying that the collective spin has improved phase sensitivity to rotations compared to the standard quantum limit $\xi^2 = 1$ of an uncorrelated coherent spin state.

Within our mean-field and small-fluctuations assumption, we use $|\langle\hat{J}'_z\rangle| \approx N/2$ and the bosonic approximation (15) for $\hat{J}'_\mp = \hat{J}'_x \mp i\hat{J}'_y$ to obtain the spin-squeezing parameter in terms

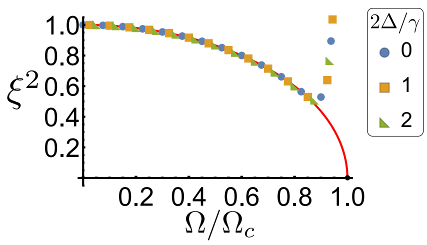

FIG. 3. Spin squeezing ξ^2 as a function of the driving field Ω. Results obtained from the numerical solution of Eq. (8) with $N = 50$ atoms and for different values of the dipole-dipole shift $2\Delta/\gamma = 0, 1, 2$ all collapse to the same curve when Ω is scaled by the critical field $\Omega_c(\Delta)$ from Eq. (10). The red line represents the analytical solution from Eq. (22). The exact numerical solutions agree with the analytical result until they diverge away close to the transition point, where ξ^2 begins degrading (growing) with Ω (see text).

of the bosonic operators $\hat{a}$:

$$\xi^2 = \min_{\varphi} \mathrm{Var}[\hat{a}e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi}]. \tag{21}$$

This expression is identical to the definition of ξ_B^2 from Eq. (19), so the spin-squeezing parameter becomes

$$\xi^2 = \cos\theta = \sqrt{1 - \frac{|\Omega|^2}{\Omega_c^2(\Delta)}}. \tag{22}$$

We observe that the spin squeezing is determined by the ratio between the driving field and the critical field, $|\Omega|/\Omega_c$. It depends on the dipole-dipole interaction Δ through the critical field $\Omega_c(\Delta)$ from Eq. (10). This generalizes the analytical result of Ref. [25], obtained for the case with $\Delta = 0$ using a master-equation approach. When the drive is weak $|\Omega|/\Omega_c \to 0$, no spin squeezing exists, $\xi^2 = 1$, since the system is in a coherent spin state wherein all atoms are in the ground state. As the drive increases, population in the atoms is created, such that collective emission is possible, building entanglement and spin-squeezing correlations between the atoms, $\xi^2 < 1$. At the critical point $|\Omega|/\Omega_c = 1$ the spin-squeezing parameter vanishes: this result is valid only at the limit $N \to \infty$, where it does not contradict the Heisenberg limit $\xi^2 \geqslant 1/N$ [34]. For finite N, our mean-field assumption of small fluctuations breaks down as we approach the critical point, where fluctuations become increasingly large (e.g., $\max_\varphi \mathrm{Var}[\hat{J}'_\varphi] \propto 1/\cos\theta$ diverges near the critical point).

It is instructive to compare the analytical result (22) to that obtained by an exact numerical solution of the master equation for a finite N, as explained above. In Fig. 3 we observe excellent agreement between the analytical and numerical solutions up to a driving field somewhat below the critical point $|\Omega| < \Omega_c(\Delta)$, above which the two solutions diverge. As in Fig. 2, the dependence on Δ is captured by plotting the numerical solutions for different values of Δ, which all collapse to the curve as a function of the driving field Ω (e.g., taken to be real) scaled to the corresponding critical field $\Omega_c(\Delta)$, as anticipated analytically in Eq. (22). We observe that the exact solution obtains its optimal (minimal) value for the squeezing ξ^2 close to the point where it begins to diverge away from the analytical result. Therefore, this optimal value for ξ^2 should improve (become smaller) with increasing N [25]. The scaling of the optimal ξ^2, being a finite-size effect, cannot be accounted for by the above mean-field based results (valid for $N \to \infty$). This scaling can be obtained analytically using CRSS theory, yielding $\xi^2 \sim N^{-1/3}$ [27].

V. RADIATED LIGHT

So far we have treated the field degrees of freedom as a reservoir that generates driven-dissipative dynamics of the atoms. However, superradiance is essentially a scattering problem of an input coherent-state field off a collective dipole $\hat{J}_-$ formed by the atoms. As such, the total field exhibits the general form

$$\hat{E}(t) = \hat{E}_{\text{free}}(t) + G\hat{J}_-. \tag{23}$$

The first term is the freely propagating coherent-state field in the absence of atoms, consisting of an average field and vacuum fluctuations. It may include the influence of linear optical elements such as the cavity mirrors in the cavity realization of Fig. 1. The second term is the field component scattered by the atomic dipole $\hat{J}_-$, with a coupling coefficient G (describing field propagation from the atoms to the detector). While the first term exhibits noncorrelated coherent-state statistics of the input field, the second term may exhibit correlations generated by the nonlinearity of the atoms [41]. In superradiance, the considered atomic system is clearly nonlinear, as we have already seen that the population inversion $\langle\hat{J}_z\rangle$ is a nonlinear function of the driving field Ω [see Eq. (9)]. Nevertheless, we show in the following that, surprisingly, the scattered component of the field is also a coherent-state field, linear in the input field. This holds for any driving field Ω smaller than the critical field Ω_c.

Although our result is valid for any realization of superradiance, we focus for concreteness on the cavity realization considered above. We define the total observable field as the field propagating out of the cavity (in the rotated frame ω_L)

$$\hat{E}(t) = -i\sum_{k>0} \eta^* \hat{b}_k(t) e^{i\omega_L t} - i\Omega_L, \tag{24}$$

where $-i\Omega_L$ is the average component of the input coherent field. Using the same HL approach as in Sec. II B, we solve for $\hat{E}(t)$ within the coarse-grained dynamics at $t \gg 1/\kappa$, obtaining Eq. (23) with (see Appendix B)

$$\hat{E}_{\text{free}}(t) = (1+\chi)[\hat{E}_0(t) - i\Omega_L], \quad \chi = \frac{\kappa}{i\delta_c - \kappa/2},$$
$$G = -ig^*\chi. \tag{25}$$

Here χ describes the linear response of the cavity to the input field $\hat{E}_0(t) - i\Omega_L$ (vacuum + coherent drive), which interferes with the input, yielding the factor $1 + \chi$. Therefore, the atom-free field indeed has the form of a coherent-state field composed of vacuum + average components. In the following we will show that this turns out to be the case also for the total field.

A. Average field

Taking the average of Eq. (23), the vacuum term $\hat{E}_0$ does not contribute, so the free-field component from Eq. (25) gives $-i\Omega_L(1+\chi)$. For the scattered part we use $G = -ig^*\chi$ from

Eq. (25) and $\langle\hat{J}_-\rangle$ from Eq. (9), obtaining $G\langle\hat{J}_-\rangle = i\chi\Omega_L$. The total average field then becomes

$$\langle\hat{E}\rangle = -i\Omega_L, \tag{26}$$

equal to the incident average field. So the average radiated field in superradiance is linear in the incident-field amplitude even though the atomic system is nonlinear.

B. Field fluctuations

The HL approach allows us to gain direct access to field operators which contain information on the quantum statistics of the field. We will use it here to show that the fluctuating part of the radiated field is proportional to vacuum fluctuations. To this end, we first focus on the scattered component of the field $G\hat{J}_-$. Using the transformation (13), we write $\hat{J}_-$ in terms of the rotated-system spin operators as

$$\begin{aligned}\hat{J}_- &= e^{-i\phi}\left(\frac{\cos\theta+1}{2}\hat{J}'_- + \frac{\cos\theta-1}{2}\hat{J}'_+ + \sin\theta\hat{J}'_z\right)\\ &\approx e^{-i\phi}\left(\frac{\cos\theta+1}{2}\sqrt{N}\hat{a} + \frac{\cos\theta-1}{2}\sqrt{N}\hat{a}^\dagger - \sin\theta\frac{N}{2}\right).\end{aligned} \tag{27}$$

In the second line we use the Holstein-Primakoff linearization, Eq. (15). Plugging in the solution for $\hat{a}$ from Eq. (17), we then obtain for the fluctuating part of the field $\hat{E}$ from (23) and (25):

$$\hat{\mathcal{E}}(t) \equiv \hat{E} - \langle\hat{E}\rangle = (1+\chi)\hat{E}_0(t) + G\sqrt{N\cos\theta}\hat{b}(t). \tag{28}$$

The first term describes the vacuum fluctuations $\propto \hat{E}_0$ of the coherent free-field component from Eq. (25). The second term originates from the fluctuating part of the scattered field $G\hat{J}_-$ from Eq. (27) and is also essentially proportional to integrated vacuum fluctuations $\hat{E}_0$ [recalling that $\hat{b}$ in Eq. (17) is an integral of $\hat{f} \propto \hat{E}_0$].

Therefore, unlike the spin fluctuations $\hat{a} = u\hat{b} + v\hat{b}^\dagger$ from Eqs. (17) and (18), which contain both lowering and raising operators of the input-field fluctuations, it turns out that the radiated field fluctuations contain only the former. This is because the transformation coefficients from $\hat{a}$ and $\hat{a}^\dagger$ to $\hat{J}_- \sim \hat{\mathcal{E}}$ in Eq. (27), together with the Bogoliubov coefficients u and v of $\hat{a}$, all depend on $\cos\theta$ in such a way that leads to an exact cancellation of the coefficient for $\hat{b}^\dagger$ in $\hat{\mathcal{E}}$, as seen in Eq. (28). As we further discuss below, this leads to vanishing squeezing for light at any drive strength $|\Omega|/\Omega_c = \sin\theta < 1$, while spin squeezing does exist. This result is equivalent to the geometrical interpretation given by the so-called dipole-projected squeezing [27].

C. Coherent-state radiation

The fact that the field fluctuations $\hat{\mathcal{E}}(t)$ are essentially proportional to the vacuum field $\hat{E}_0(t)$ implies that the total radiated field is in a coherent state. This can be formally shown as follows. Considering the initial state of the total system $|\psi_0\rangle$, we have $\hat{E}_0(t)|\psi_0\rangle = 0$ (since we assumed a vacuum state for the photon reservoir), leading to $\hat{\mathcal{E}}(t)|\psi_0\rangle = 0$. Now, consider the state of the total system at some steady-state time t (in the Schrödinger picture), $|\psi(t)\rangle = \hat{U}(t)|\psi_0\rangle$, where $\hat{U}(t)$ is the evolution operator of the total system. In order to prove that this state describes a coherent state of the radiated field, we have to show it is an eigenstate of the Schrödinger-picture field operator $\hat{E}(0)$. Using the relation between the Heisenberg- and Schrödinger-picture operators, $\hat{E}(t) = \hat{U}^\dagger(t)\hat{E}(0)\hat{U}(t)$, we have

$$\begin{aligned}\hat{E}(0)|\psi(t)\rangle &= \hat{U}(t)\hat{E}(t)|\psi_0\rangle = \hat{U}(t)[\langle\hat{E}\rangle + \hat{\mathcal{E}}(t)]|\psi_0\rangle\\ &= \langle\hat{E}\rangle|\psi(t)\rangle,\end{aligned} \tag{29}$$

where in the last equality $\hat{\mathcal{E}}(t)|\psi_0\rangle = 0$ is used. So the radiated field is a coherent state with amplitude $\langle\hat{E}\rangle = -i\Omega_L$ from (26), identical to the incident field.

D. Light squeezing

The above prediction of a classical-like coherent-state radiation implies that the radiated field does not exhibit any quantum correlations. This prediction should be valid in the limit $N \to \infty$, recalling that our analytical HL approach relied on linearizing quantum fluctuations around the mean-field solution. For finite N, we therefore expect to approach this result with increasing N. We now demonstrate this by showing that the quantum-squeezing correlations of the radiated field decay exponentially with N.

Defining the quadrature operator of the radiated field, $\hat{X}_\varphi = e^{i\varphi}\hat{E} + e^{-i\varphi}\hat{E}^\dagger$, the bosonic squeezing parameter of the radiated field is given by

$$\xi_E^2 = \min_\varphi \frac{\mathrm{Var}[\hat{X}_\varphi]}{V_0} = 1 + \frac{2}{V_0}(\langle\hat{\mathcal{E}}^\dagger\hat{\mathcal{E}}\rangle - |\langle\hat{\mathcal{E}}^2|), \tag{30}$$

with $V_0 \equiv [\hat{E}, \hat{E}^\dagger] = [\hat{E}_0, \hat{E}_0^\dagger] = \kappa\delta(t=0)$ being the vacuum-noise level. Quantum-squeezing correlations exist if the quadrature noise can become lower than that of the vacuum, i.e., for $\xi_E^2 < 1$. This requires the existence of the phase-dependent correlator $\langle\hat{\mathcal{E}}^2\rangle$, which, however, vanishes for a coherent-state field whose fluctuations $\hat{\mathcal{E}}$ are equivalent to a vacuum field, as we predict analytically for $N \to \infty$. For finite N we can use the numerical solution of the master equation (8) to calculate ξ_E^2 by first expressing it in terms of collective-spin variables. From Eq. (23) for the total field and (25) for its free component, we obtain $\hat{\mathcal{E}} = (1+\chi)\hat{E}_0 + G(\hat{J}_- - \langle\hat{J}_-\rangle)$, so that

$$\begin{aligned}\xi_E^2 &= 1 + \frac{2|G|^2}{V_0}Q,\\ Q &= (\langle\hat{J}_+\hat{J}_-\rangle - \langle\hat{J}_+\rangle\langle\hat{J}_-\rangle) - |\langle\hat{J}_-^2\rangle - \langle\hat{J}_-\rangle^2|.\end{aligned} \tag{31}$$

For a coherent-state field we expect $\xi_E^2 = 1$ (no squeezing) and hence $Q = 0$. Since Q is expressed via spin correlators, we calculate it exactly from the numerical steady-state solution of the master equation (8) as described above. In Fig. 4 we observe, for different values of $\Omega/\Omega_c = \sin\theta < 1$, that $|Q|$ decreases exponentially for large atom numbers N, consistent with the prediction of a coherent-state radiated field for $N \gg 1$.

We conclude that the field collectively scattered by the many-atom system, as described by the driven Dicke problem, is a coherent state and thus exhibits no quantum-squeezing correlations. Moreover, the fact that this result is valid in the strong-field regime $\Omega \lesssim \Omega_c$, where the atomic population is

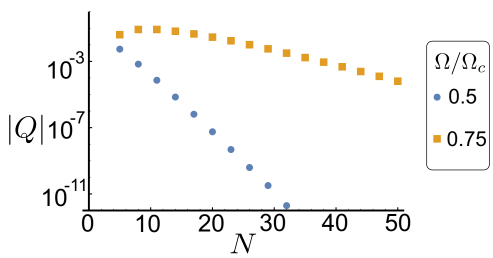

FIG. 4. Quantum-squeezing correlations of the radiated light, Eq. (31), as a function of atom number N (semilog plot). Exact numerical results, obtained by solving Eq. (8) for different values of $\Omega/\Omega_c < 1$ and $\Delta = 0$, show that field correlations vanish exponentially with $N \gg 1$ ($|Q| \to 0$). This agrees with the analytical prediction of coherent-state radiation.

a highly nonlinear function of the field, is in stark contrast to the common situation where a nonlinear scatterer effectively generates photon correlations [41]. In particular, if we consider the analogous case of independently driven atoms, it is well known that quantum squeezing can, in principle, be generated [42,43]. This is because the nonlinearity of each atom generates squeezed radiation, and if the radiation from all the statistically independent atoms is coherently added, the total radiated field is squeezed. In collective radiation, we find, however, that the opposite situation occurs. While the atoms are quantum correlated (spin squeezed), these correlations remarkably lead to the exact cancellation of quantum correlations in the radiated light. In the HL approach, this stems from the exact cancellation of the Bogoliubov coefficient of the raising-operator component of the scattered field, a discussed below Eq. (28). A complementary interpretation is given by the emergence of a CRSS in the steady state [27].

E. Spectrum

Having access to the field operator $\hat{E}(t)$, the HL approach also allows us to directly calculate two-time correlations and spectra. The spectrum of the radiated field in a steady-state time t is given as usual by the Fourier transform on the time difference τ of the two-time correlation, $\langle \hat{E}^\dagger(t)\hat{E}(t+\tau)\rangle$. Since this is a normal-ordered correlator, the fluctuating part of $\hat{E}$ in Eq. (28) drops, as it is proportional to the lowering operator $\hat{E}_0$. This trivially yields $\langle \hat{E}^\dagger(t)\hat{E}(t+\tau)\rangle = |\langle\hat{E}\rangle|^2 = |\Omega_L|^2$. The spectrum of the radiated field is then a single δ peak at the incident frequency ω_L (recalling that we work in the laser-rotated frame). This again shows that the collective atomic dipole scatters light as a linear optical element even though the atomic population exhibits a strongly nonlinear dependence on the drive Ω_L.

VI. CONCLUSIONS

In this work we presented a HL approach to driven Dicke superradiance in the steady state. The analytical results for steady-state spin squeezing agree with and generalize those obtained in Refs. [23–26]. Furthermore, our finding that the radiated field is in a coherent state underlies previous results on uncorrelated photon statistics below the transition [20]. These HL-based results are consistent with the formation of a CRSS as described in [27]. The HL approach is thus complementary to the CRSS description of superradiance. On the one hand, it is based on the approximate analysis of small fluctuations around the mean field for $N \to \infty$ and did not yield finite-size scalings with N or the full atomic state as in CRSS. On the other hand, it is simpler to generalize for treating superradiance beyond the permutation-symmetric Dicke case, e.g., by performing the Holstein-Primakoff approximation for each individual atom separately, while allowing for direct estimation of atom and field correlations.

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APPENDIX A: SQUEEZING IN TERMS OF BOGOLIUBOV COEFFICIENTS

Here we review the relation $\xi_B^2 = (|u| - |v|)^2$ [32] for the squeezing of a Bogoliubov-transformed vacuum. For a boson mode $\hat{a}$, one defines the quadrature operator $\hat{x}_\varphi = \hat{a}e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi}$ and the squeezing parameter as in Eq. (19),

$$\xi_B^2 = \min_\varphi \mathrm{Var}[\hat{x}_\varphi] = 1 + 2\langle \hat{a}^\dagger \hat{a}\rangle - 2|\langle \hat{a}^2\rangle|, \tag{A1}$$

where in the last equality we already assumed $\langle \hat{a}\rangle = 0$ and used $[\hat{a}, \hat{a}^\dagger] = 1$. Consider $\hat{a}$ in a Bogoliubov form, $\hat{a} = u\hat{b} + v\hat{b}^\dagger$, with the boson mode $\hat{b}$ satisfying $[\hat{b}^\dagger, \hat{b}] = 1$. Taking the state of the system as the vacuum of mode $\hat{b}$, we have $\langle \hat{b}^\dagger\hat{b}\rangle, \langle \hat{b}^2\rangle = 0$, obtaining $\xi_B^2 = 1 + 2|v|^2 - 2|u||v|$. Using the relation $|u|^2 - |v|^2 = 1$, enforced by the requirement $[\hat{a}, \hat{a}^\dagger] = 1$, we finally obtain $\xi_B^2 = (|u| - |v|)^2$.

APPENDIX B: OUTPUT FIELD

Here we elaborate on the derivation of the general expression for the output field, Eqs. (23) and (25), in the one-sided cavity scheme in Fig. 1. We begin by defining the outside propagating field

$$\hat{E}(x,t) = -i\sum_{k>0}\eta^*\hat{b}_k(t)e^{ikx}e^{i\omega_L t} - i\Omega_L e^{i\frac{\omega_L}{v}x}. \tag{B1}$$

Here x is the propagation axis: in the one-sided scheme, $x = 0$ denotes the position of the out-coupling mirror (right-hand-side mirror in Fig. 1), so that $x < 0$ denotes incoming left-propagating fields, whereas $x > 0$ denotes outgoing right-propagating fields. The radiated field from Eq. (24) is then defined by taking $x = 0^+ > 0$. As in the derivation of the HL equations in Sec. II B, we first formally solve the Heisenberg equations for $\hat{b}_k(t)$, obtaining in the Markov approximation

$$\hat{E}(x,t) = \hat{E}_0(x,t) - i\Omega_L e^{i\frac{\omega_L}{v}x} - e^{i\frac{\omega_L}{v}x}\kappa\int_0^t dt'\tilde{c}(t')\delta(t - x/c - t'). \tag{B2}$$

Here $\hat{E}_0(x,t)$ is the vacuum field from Eq. (4) with the exponentials e^{ikx} in the mode expansion $\hat{b}_k(0)$. For $x = 0^- < 0$, the

Dirac δ function does not contribute, and we indeed obtain the input field $\hat{E}_0(t) - i\Omega_L$. For $x = 0^+ > 0$ we obtain

$$\hat{E}(t) \equiv \hat{E}_0(0^+, t) = \hat{E}_0(t) - i\Omega_L - \kappa\tilde{c}(t). \quad \text{(B3)}$$

Finally, inserting the coarse-grained solution for $\tilde{c}(t)$ [obtained for simplicity by setting $\dot{\tilde{c}} = 0$ in Eq. (3)], we arrive at Eqs. (23) and (25).

[1] R. H. Dicke, Coherence in spontaneous radiation processes, Phys. Rev. **93**, 99 (1954).

[2] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995).

[3] M. Gross and S. Haroche, Superradiance: An essay on the theory of collective spontaneous emission, Phys. Rep. **93**, 301 (1982).

[4] Y. Kaluzny, P. Goy, M. Gross, J. M. Raimond, and S. Haroche, Observation of Self-Induced Rabi Oscillations in Two-Level Atoms Excited Inside a Resonant Cavity: The Ringing Regime of Superradiance, Phys. Rev. Lett. **51**, 1175 (1983).

[5] M. A. Norcia, M. N. Winchester, J. R. Cline, and J. K. Thompson, Superradiance on the millihertz linewidth strontium clock transition, Sci. Adv. **2**, e1601231 (2016).

[6] M. A. Norcia, J. R. K. Cline, J. A. Muniz, J. M. Robinson, R. B. Hutson, A. Goban, G. E. Marti, J. Ye, and J. K. Thompson, Frequency Measurements of Superradiance from the Strontium Clock Transition, Phys. Rev. X **8**, 021036 (2018).

[7] D. D. Grimes, S. L. Coy, T. J. Barnum, Y. Zhou, S. F. Yelin, and R. W. Field, Direct single-shot observation of millimeter-wave superradiance in Rydberg-Rydberg transitions, Phys. Rev. A **95**, 043818 (2017).

[8] G. Ferioli, A. Glicenstein, I. Ferrier-Barbut, and A. Browaeys, Observation of a non-equilibrium superradiant phase transition in free space, Nat. Phys. (2023), doi: 10.1038/s41567-023-02064-w.

[9] A. Angerer, K. Streltsov, T. Astner, S. Putz, H. Sumiya, S. Onoda, J. Isoya, W. J. Munro, K. Nemoto, J. Schmiedmayer, and J. Majer, Superradiant emission from colour centres in diamond, Nat. Phys. **14**, 1168 (2018).

[10] E. M. Kessler, G. Giedke, A. Imamoglu, S. F. Yelin, M. D. Lukin, and J. I. Cirac, Dissipative phase transition in a central spin system, Phys. Rev. A **86**, 012116 (2012).

[11] P. Kirton, M. M. Roses, J. Keeling, and E. G. Dalla Torre, Introduction to the Dicke model: From equilibrium to nonequilibrium, and vice versa, Adv. Quantum Technol. **2**, 1800043 (2019).

[12] K. Hepp and E. H. Lieb, On the superradiant phase transition for molecules in a quantized radiation field: The Dicke maser model, Ann. Phys. (NY) **76**, 360 (1973).

[13] K. Hepp and E. H. Lieb, Equilibrium statistical mechanics of matter interacting with the quantized radiation field, Phys. Rev. A **8**, 2517 (1973).

[14] F. Haake, M. I. Kolobov, C. Seeger, C. Fabre, E. Giacobino, and S. Reynaud, Quantum noise reduction in stationary superradiance, Phys. Rev. A **54**, 1625 (1996).

[15] D. Meiser, J. Ye, D. R. Carlson, and M. J. Holland, Prospects for a Millihertz-Linewidth Laser, Phys. Rev. Lett. **102**, 163601 (2009).

[16] J. G. Bohnet, Z. Chen, J. M. Weiner, D. Meiser, M. J. Holland, and J. K. Thompson, A steady-state superradiant laser with less than one intracavity photon, Nature (London) **484**, 78 (2012).

[17] K. Debnath, Y. Zhang, and K. Mølmer, Lasing in the superradiant crossover regime, Phys. Rev. A **98**, 063837 (2018).

[18] P. Drummond and H. Carmichael, Volterra cycles and the cooperative fluorescence critical point, Opt. Commun. **27**, 160 (1978).

[19] P. D. Drummond, Observables and moments of cooperative resonance fluorescence, Phys. Rev. A **22**, 1179 (1980).

[20] H. Carmichael, Analytical and numerical results for the steady state in cooperative resonance fluorescence, J. Phys. B **13**, 3551 (1980).

[21] R. Puri and S. Lawande, Exact steady-state density operator for a collective atomic system in an external field, Phys. Lett. A **72**, 200 (1979).

[22] J. Hannukainen and J. Larson, Dissipation-driven quantum phase transitions and symmetry breaking, Phys. Rev. A **98**, 042113 (2018).

[23] D. Barberena, R. J. Lewis-Swan, J. K. Thompson, and A. M. Rey, Driven-dissipative quantum dynamics in ultra-long-lived dipoles in an optical cavity, Phys. Rev. A **99**, 053411 (2019).

[24] A. González-Tudela and D. Porras, Mesoscopic Entanglement Induced by Spontaneous Emission in Solid-State Quantum Optics, Phys. Rev. Lett. **110**, 080502 (2013).

[25] T. E. Lee, C.-K. Chan, and S. F. Yelin, Dissipative phase transitions: Independent versus collective decay and spin squeezing, Phys. Rev. A **90**, 052109 (2014).

[26] K. Tucker, D. Barberena, R. J. Lewis-Swan, J. K. Thompson, J. G. Restrepo, and A. M. Rey, Facilitating spin squeezing generated by collective dynamics with single-particle decoherence, Phys. Rev. A **102**, 051701(R) (2020).

[27] O. Somech and E. Shahmoon, Quantum entangled states of a classically radiating macroscopic spin, arXiv:2204.05455.

[28] M. O. Scully and M. S. Zubairy, *Quantum Optics* (Cambridge University Press, Cambridge, 1999).

[29] R. Lehmberg, Radiation from an N-atom system. I. General formalism, Phys. Rev. A **2**, 883 (1970).

[30] J. Huber, P. Kirton, and P. Rabl, Phase-space methods for simulating the dissipative many-body dynamics of collective spin systems, SciPost Phys. **10**, 045 (2021).

[31] A. Auerbach, *Interacting Electrons and Quantum Magnetism* (Springer, New York, 2012).

[32] P. Drummond and Z. Ficek (eds.), *Quantum Squeezing* (Springer-Verlag, Berlin, Heidelberg, 2004).

[33] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, Heidelberg, 2007).

[34] J. Ma, X. Wang, C. Sun, and F. Nori, Quantum spin squeezing, Phys. Rep. **509**, 89 (2011).

[35] M. Kitagawa and M. Ueda, Squeezed spin states, Phys. Rev. A **47**, 5138 (1993).

[36] D. J. Wineland, J. J. Bollinger, W. M. Itano, F. L. Moore, and D. J. Heinzen, Spin squeezing and reduced quantum noise in spectroscopy, Phys. Rev. A **46**, R6797 (1992).

[37] D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, Squeezed atomic states and projection noise in spectroscopy, Phys. Rev. A **50**, 67 (1994).

[38] C. L. Degen, F. Reinhard, and P. Cappellaro, Quantum sensing, Rev. Mod. Phys. **89**, 035002 (2017).

[39] J. K. Korbicz, J. I. Cirac, and M. Lewenstein, Spin Squeezing Inequalities and Entanglement of *N* Qubit States, Phys. Rev. Lett. **95**, 120502 (2005).

[40] A. Sørensen, L.-M. Duan, J. I. Cirac, and P. Zoller, Many-particle entanglement with Bose–Einstein condensates, Nature (London) **409**, 63 (2001).

[41] D. E. Chang, V. Vuletić, and M. D. Lukin, Quantum nonlinear optics—Photon by photon, Nat. Photonics **8**, 685 (2014).

[42] D. F. Walls and P. Zoller, Reduced Quantum Fluctuations in Resonance Fluorescence, Phys. Rev. Lett. **47**, 709 (1981).

[43] H. T. Dung, L. Knöll, and D.-G. Welsch, Resonant dipole-dipole interaction in the presence of dispersing and absorbing surroundings, Phys. Rev. A **66**, 063810 (2002).