Vacuum torque, propulsive forces, and anomalous tangential forces: Effects of nonreciprocal media out of thermal equilibrium

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From the generalized fluctuation-dissipation theorem, it is known that a body at rest made of nonreciprocal material may experience a torque, even in vacuum, if it is not in thermal equilibrium with its environment. However, it does not experience self-propulsion in such circumstances, except in higher order. Nevertheless, such a body may experience both a normal torque and a lateral force when adjacent to an ordinary surface with transverse translational symmetry. We explore how these phenomena arise, discuss what terminal velocities might be achieved, and point out some of the limitations of applying our results to observations, including the Lorenz-Lorentz correction and the cooling due to radiation. In spite of these limitations, the effects discussed would seem to be observable.

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I. INTRODUCTION

There is a long history of theoretical predictions of quantum or Casimir friction, where a particle or extended body that moves parallel to a surface experiences a force opposing its motion. The subject seems to have originated with Teodorovich [1] and Levitov [2]. (For a selected bibliography on this subject before 2016, see Ref. [3].) The friction is typically conceived to arise because of dissipation in the surface. (For a subset of papers on this subject, the reader is referred to Refs. [4–9]. For a readable overview, see Ref. [10].) Quantum friction can also result if the body itself is made of dissipative material. However, much earlier, it was recognized that, even in vacuum, a moving body or an atom without dissipation will experience friction due to the surrounding radiation field—this is the famous Einstein-Hopf effect [11].

In previous papers, we have considered quantum vacuum friction due to field and dipole fluctuations [12,13]. For low velocities, the condition that the atom or particle not gain or lose energy, the nonequilibrium steady-state condition [14], implies equal temperatures of the body and the environment, while relativistic velocities typically imply that the body be substantially hotter than the environment. (For other earlier work on nonequilibrium friction, see Refs. [15–17] for example.) The forces we considered there are true frictions, in that they always oppose the motion, and they vanish at zero velocity and at zero temperature.

Here we consider forces and torques that arise in the vacuum, or near other bodies, when the relative velocity is zero. This requires not only that the system be out of thermal equilibrium, so that the temperature of the body is different from that of its environment, but also that the electrical properties that characterize the material constituting the bodies be exotic, nonreciprocal, at least in lowest order. Nonreciprocity seems not to be possible for an isolated body; the typical way it can be achieved is through the introduction of an external magnetic field or some other appropriate external influence. Thus, it is something of an oxymoron to discuss nonreciprocal vacuum torque or friction.

Of course, there is much earlier work on such nonequilibrium phenomena, involving heat transfer, torque, and nonreciprocal surface forces [18–22]. (Further references will be provided as our discussion continues.) We also note that similar behavior arises in quantum thermal rotators, where a quantum torque is generated when the system is out of thermal equilibrium and exhibits broken spatial symmetry (see, for example, Ref. [23]).

The outline of this paper is as follows. In Sec. II we discuss how the fluctuation-dissipation theorem is modified for a nonreciprocal susceptibility. We then display in Sec. III a simple model of such a nonreciprocal material as a simplification of that given in Ref. [24]. The corresponding quantum vacuum torque, first found in Refs. [25,26], is derived in Sec. IV. In Sec. V we rederive the modified torque if the body is slowly rotating, which was first worked out in Ref. [24]. If the body is hotter or not too much colder than the environment, the ordinary quantum frictional torque acts as a drag and the body acquires a terminal angular velocity which should be readily observable, provided this temperature difference can be maintained. In Sec. VI the effect on the torque of an underlying plate, be it a dielectric slab or a perfect conductor, is investigated. Then we turn to the force, which can of course be

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inferred from the torque. The quantum vacuum force is shown to vanish, in the weak-susceptibility approximation that we use (Sec. VII), while, if an underlying surface is present, there is a component of the force parallel to the surface, as shown in examples of an imperfectly and a perfectly conducting plate (Sec. VIII). Again, if the nonequilibrium temperature difference could be maintained, a substantial terminal velocity could be achieved. In Sec. IX we calculate the time it would take for a body at rest to reach thermal equilibrium with the environment, unless some mechanism were supplied to keep it hotter or colder than the background. Possible suppression effects due to the Lorenz-Lorentz (Clausius-Mossotti) correction are discussed in Sec. X, although the resulting torques and forces should still be experimentally measurable. A summary and conclusions are provided in Sec. XI.

In this paper we concentrate on first-order effects in the susceptibility. Second-order effects, particularly important for quantum vacuum forces, are left for future work.

Throughout, we use Heaviside-Lorentz (HL) electromagnetic units. We also set $\hbar = c = 1$, except when numerical values are given. It should be noted that many authors (including some of us) often use Gaussian (G) units, for which the polarizability differs by a factor of 4π.

II. GENERALIZED FLUCTUATION-DISSIPATION THEOREM

Let $\mathbf{x}(t)$ be some dynamical variable (such as an electric dipole moment). In terms of its frequency Fourier transform, the fluctuation-dissipation theorem (FDT) tells us the expectation value of the symmetrized quadratic product of the frequency-transformed variables,[1]

$$\langle S\mathbf{x}(\omega)\mathbf{x}(\nu)\rangle = 2\pi\delta(\omega+\nu)\Im\boldsymbol{\chi}(\omega)\coth\frac{\beta\omega}{2}, \tag{2.1}$$

where $\beta = 1/T$ is the inverse temperature of the system and $\boldsymbol{\chi}$ is the generalized susceptibility. The latter is defined as the linear response function relating the dynamical variable to a driving force $\mathbf{f}$:

$$\mathbf{x}(t) = \int dt'\,\boldsymbol{\chi}(t-t')\cdot\mathbf{f}(t'). \tag{2.2a}$$

Thus, for the relevant example of the electric dipole moment driven by an electric field, through the polarizability,

$$\mathbf{d}(t) = \int dt'\,\boldsymbol{\alpha}(t-t')\cdot\mathbf{E}(t'). \tag{2.2b}$$

We assume causality, so $\boldsymbol{\chi}(t-t')$ is nonzero only for $t-t' \geqslant 0$. Alternatively, the fluctuating quantity might be the electric field, driven by the electric polarization, and the susceptibility would be the retarded electric Green's function.

Typically, we regard the susceptibility tensor as diagonal, or at least symmetric. More generally, the "imaginary part" that occurs in the FDT is actually the anti-Hermitian part

$$\Im\boldsymbol{\chi} = \frac{1}{2i}(\boldsymbol{\chi}-\boldsymbol{\chi}^\dagger), \tag{2.3}$$

that is,

$$(\Im\boldsymbol{\chi})_{ij}(\omega) = \frac{1}{2i}[\chi_{ij}(\omega)-\chi^*_{ji}(\omega)] = \frac{1}{2i}[\chi_{ij}(\omega)-\chi_{ji}(-\omega)], \tag{2.4}$$

which uses the fact that $\chi_{ij}(\omega)$ is the Fourier transform of a real response function. Unusual properties emerge from this if $\boldsymbol{\chi}$ is not symmetric: This means that $\Im\boldsymbol{\chi}$ has both real and imaginary parts in the conventional sense. The real part is

$$\mathrm{Re}(\Im\boldsymbol{\chi})_{ij}(\omega) = \tfrac{1}{2}[\mathrm{Im}\chi_{ij}(\omega)+\mathrm{Im}\chi_{ji}(\omega)], \tag{2.5a}$$

which is symmetric in the indices but odd in ω. These are the components that give rise to the quantum friction force and torque. The imaginary part of $\Im\boldsymbol{\chi}$ is

$$\mathrm{Im}(\Im\boldsymbol{\chi})_{ij}(\omega) = -\tfrac{1}{2}[\mathrm{Re}\chi_{ij}(\omega)-\mathrm{Re}\chi_{ji}(\omega)], \tag{2.5b}$$

which is antisymmetric in the indices but even in ω.

This property, as we will see, leads to unusual phenomena for nonreciprocal bodies, spontaneous quantum torque, and quantum propulsion. The term "nonreciprocal" seems to have a variety of meanings in the literature; in this paper we will take it to mean that Eq. (2.5b) is nonzero.

For an ordinary material, $\chi_{ij}(\omega)$ is symmetric in the indices, which means that the anti-Hermitian part coincides with the usual imaginary part

$$\chi_{ij}(\omega) = \chi_{ji}(\omega) \Rightarrow (\Im\boldsymbol{\chi})_{ij}(\omega) = \mathrm{Im}\chi_{ij}(\omega). \tag{2.6}$$

Where a susceptibility depends on continuous coordinates as well, such as the Green's dyadic that describes the electric field, reciprocity means invariance under interchange of discrete indices and continuous coordinates:

$$\Gamma_{ij}(\mathbf{r},\mathbf{r}';\omega) = \Gamma_{ji}(\mathbf{r}',\mathbf{r};\omega), \tag{2.7a}$$

which implies

$$(\Im\boldsymbol{\Gamma})_{ij}(\mathbf{r},\mathbf{r}';\omega) = \mathrm{Im}\Gamma_{ij}(\mathbf{r},\mathbf{r}';\omega). \tag{2.7b}$$

It is easy to check that this is satisfied by the Green's dyadic for a dielectric half space, for example, which has off-diagonal elements, symmetric in the tensor indices. When the latter is expressed as a two-dimensional Fourier transform, as is convenient when the environment consists of a dielectric slab perpendicular to the z axis,

$$\Gamma_{ij}(\mathbf{r},\mathbf{r}';\omega) = \int\frac{(d\mathbf{k}_\perp)}{(2\pi)^2}e^{i\mathbf{k}_\perp\cdot(\mathbf{r}-\mathbf{r}')_\perp}g_{ij}(z,z';\mathbf{k}_\perp,\omega), \tag{2.8}$$

the FDT is expressed in terms of

$$(\Im\mathbf{g})_{ij}(z,z';\mathbf{k}_\perp,\omega) = \frac{1}{2i}[g_{ij}(z,z';\mathbf{k}_\perp,\omega) - g_{ji}(z',z;-\mathbf{k}_\perp,-\omega)]. \tag{2.9}$$

III. MODEL FOR NONRECIPROCAL MATERIAL

In order to create a nonreciprocal response, one needs an external influence, which is supplied by a magnetic field. Let us suppose an oscillator with damping η is driven by both an electric field and a constant magnetic field

$$m\frac{d^2\mathbf{r}}{dt^2} + m\eta\frac{d\mathbf{r}}{dt} + m\omega_0^2\mathbf{r} = e\left(\mathbf{E}+\frac{d\mathbf{r}}{dt}\times\mathbf{B}\right) \tag{3.1a}$$

[1] $\Im\chi$ signifies the anti-Hermitian part, defined in Eq. (2.3).

or, equivalently, in the frequency domain

$$\big(-\omega^2 - i\omega\eta + \omega_0^2\big)\mathbf{r}(\omega) = \frac{e}{m}[\mathbf{E}(\omega) - i\omega\mathbf{r}(\omega)\times\mathbf{B}]. \quad (3.1b)$$

We can solve this equation for the components of $\mathbf{r}$ in terms of the electric field and recognize that $\mathbf{B}$ can be incorporated into the cyclotron frequency $\boldsymbol{\omega}_c = \frac{e}{m}\mathbf{B}$. We express the result in terms of the electric susceptibility, defined by $\mathbf{P}(\omega) = ne\mathbf{r}(\omega) = \boldsymbol{\chi}(\omega)\cdot\mathbf{E}(\omega)$, with n the number density of charges. If the magnetic field lies in the z direction, this immediately yields an electric susceptibility that is nonsymmetric and nonreciprocal,

$$\boldsymbol{\chi} = \omega_p^2 \begin{pmatrix} \frac{\omega_0^2-\omega^2-i\omega\eta}{(\omega_0^2-\omega^2-i\omega\eta)^2-\omega^2\omega_c^2} & \frac{-i\omega\omega_c}{(\omega_0^2-\omega^2-i\omega\eta)^2-\omega^2\omega_c^2} & 0 \\ \frac{i\omega\omega_c}{(\omega_0^2-\omega^2-i\omega\eta)^2-\omega^2\omega_c^2} & \frac{\omega_0^2-\omega^2-i\omega\eta}{(\omega_0^2-\omega^2-i\omega\eta)^2-\omega^2\omega_c^2} & 0 \\ 0 & 0 & \frac{1}{\omega_0^2-\omega^2-i\omega\eta} \end{pmatrix}, \quad (3.2)$$

in terms of the plasma frequency $\omega_p^2 = ne^2/m$. For a metal, we would set the restoring force to zero, so $\omega_0 = 0$, and we exactly recover the form given by Guo and Fan [24]. In particular, this provides us with a model for the anti-Hermitian part of the susceptibility,

$$\chi_{xy} = -\chi_{yx} = -i\frac{\omega_p^2\omega_c/\omega}{(\omega+i\eta)^2 - \omega_c^2}. \quad (3.3)$$

Numerically, for the charge and mass of the electron,

$$\omega_c = \frac{eB}{m} = m\frac{B}{B_c}, \quad B_c = \frac{m^2}{e} = 4.41\times 10^9\ \mathrm{T}, \quad (3.4)$$

so for a magnetic field of strength 1 T, $\omega_c \sim 10^{-4}$ eV, far smaller than the damping parameter for gold, for example, $\eta \approx 3.5\times 10^{-2}$ eV. Thus, to a good approximation, we can use for a metal

$$\chi_{xy} - \chi_{yx} \approx -2i\frac{\omega_c\omega_p^2}{\omega}\frac{1}{(\omega+i\eta)^2}. \quad (3.5)$$

IV. QUANTUM VACUUM TORQUE

The vacuum torque on a stationary body was discussed in general terms in Ref. [25] and subsequently in Ref. [26]. Somewhat earlier, Ref. [20] showed that a topologically insulating film in a magnetic field, out of thermal equilibrium, experiences a torque, which seems to be an instance of this same phenomenon. A torque on a body at rest requires that it be composed of nonreciprocal material, which is characterized by having a real part of the susceptibility which is nonsymmetric, and that the temperature of the body, T', be different from that of the environment, T.

The torque on an arbitrary body, described by electric susceptibility $\boldsymbol{\chi}(\mathbf{r};\omega)$, is, in terms of the polarization $\mathbf{P}$,

$$\begin{aligned}\boldsymbol{\tau}(t) &= \int(d\mathbf{r})\,\mathbf{r}\times[\rho_{\rm eff}(\mathbf{r},t)\mathbf{E}(\mathbf{r},t) + \mathbf{j}_{\rm eff}(\mathbf{r},t)\times\mathbf{B}(\mathbf{r},t)] \\ &= \int(d\mathbf{r})\,\mathbf{r}\times\left(-\boldsymbol{\nabla}\cdot\mathbf{P}(\mathbf{r},t)\mathbf{E}(\mathbf{r},t) + \frac{\partial\mathbf{P}(\mathbf{r},t)}{\partial t}\times\mathbf{B}(\mathbf{r},t)\right).\end{aligned} \quad (4.1)$$

Writing this in terms of Fourier transforms and eliminating $\mathbf{B}$ in favor of $\mathbf{E}$, we have

$$\begin{aligned}\boldsymbol{\tau} &= \int(d\mathbf{r})\,\mathbf{r}\times\int\frac{d\omega}{2\pi}\frac{d\nu}{2\pi}e^{-i(\omega+\nu)t}\left(-\boldsymbol{\nabla}\cdot\mathbf{P}(\mathbf{r};\omega)\mathbf{E}(\mathbf{r};\nu) - \frac{\omega}{\nu}\mathbf{P}(\mathbf{r};\omega)\times[\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r};\nu)]\right) \\ &= \int(d\mathbf{r})\frac{d\omega}{2\pi}\frac{d\nu}{2\pi}e^{-i(\omega+\nu)t}\,\mathbf{r}\times\left[\frac{\omega}{\nu}\{\boldsymbol{\nabla}\cdot[\mathbf{P}(\mathbf{r};\omega)\mathbf{E}(\mathbf{r};\nu)] - \mathbf{P}(\mathbf{r};\omega)\cdot\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{r};\nu)\} - \left(1+\frac{\omega}{\nu}\right)[\boldsymbol{\nabla}\cdot\mathbf{P}(\mathbf{r};\omega)]\mathbf{E}(\mathbf{r};\nu)\right].\end{aligned} \quad (4.2a)$$

In terms of components, the latter means

$$\tau_i = \int(d\mathbf{r})\frac{d\omega}{2\pi}\frac{d\nu}{2\pi}e^{-i(\omega+\nu)t}\epsilon_{ijk}x_j\left[\frac{\omega}{\nu}\{\nabla_l[P_l(\mathbf{r};\omega)E_k(\mathbf{r};\nu)] - P_l(\mathbf{r};\omega)\nabla_k E_l(\mathbf{r};\nu)\} - \left(1+\frac{\omega}{\nu}\right)E_k(\mathbf{r};\nu)\nabla_l P_l(\mathbf{r};\omega)\right]. \quad (4.2b)$$

Here the source of the electric field is the electric polarization

$$\mathbf{E}(\mathbf{r};\nu) = \int(d\mathbf{r}')\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}';\nu)\cdot\mathbf{P}(\mathbf{r}';\nu), \quad (4.3a)$$

where $\boldsymbol{\Gamma}$ is the retarded electromagnetic Green's dyadic, while the polarization is linearly related to the electric field

$$\begin{aligned}\mathbf{P}(\mathbf{r};\omega) &= \int(d\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\boldsymbol{\chi}(\mathbf{r};\omega)\cdot\mathbf{E}(\mathbf{r}';\omega) \\ &= \boldsymbol{\chi}(\mathbf{r};\omega)\cdot\mathbf{E}(\mathbf{r};\omega),\end{aligned} \quad (4.3b)$$

where we assume that the electric susceptibility is local in space. This means that there are two origins for the quantum torque: field fluctuations and dipole fluctuations. We evaluate the two contributions to the torque by use of the FDT,

$$\begin{aligned}&\langle SE_i(\mathbf{r};\omega)E_j(\mathbf{r}';\nu)\rangle \\ &\quad = 2\pi\delta(\omega+\nu)(\Im\boldsymbol{\Gamma})_{ij}(\mathbf{r},\mathbf{r}';\omega)\coth\frac{\beta\omega}{2},\end{aligned} \quad (4.4a)$$

$$\begin{aligned}&\langle SP_i(\mathbf{r};\omega)P_j(\mathbf{r}';\nu)\rangle \\ &\quad = 2\pi\delta(\omega+\nu)\delta(\mathbf{r}-\mathbf{r}')(\Im\boldsymbol{\chi})_{ij}(\mathbf{r};\omega)\coth\frac{\beta'\omega}{2},\end{aligned} \quad (4.4b)$$

where $\beta = 1/T$, $\beta' = 1/T'$, and S indicates that the symmetrized expectation values are used. Therefore, the last term in Eq. (4.2) vanishes, because the sum of the two frequencies is zero, leaving us with[2]

$$\boldsymbol{\tau} = \int (d\mathbf{r}) \frac{d\omega}{2\pi}\frac{d\nu}{2\pi} e^{-i(\omega+\nu)t}[\mathbf{P}(\mathbf{r};\omega)\times\mathbf{E}(\mathbf{r};\nu) + \mathbf{P}(\mathbf{r};\omega)\cdot(\mathbf{r}\times\nabla)\cdot\mathbf{E}(\mathbf{r};\nu)]. \tag{4.5}$$

Here the notation in the last term signifies that the free vector index is on the angular momentum operator; the **P** and **E** are dotted together.

Using Eqs. (4.3b) and (4.4a), we find for the EE contribution to the torque

$$\tau_i^{\rm EE} = \int (d\mathbf{r}) \frac{d\omega}{2\pi} \coth\frac{\beta\omega}{2} \epsilon_{ijk}\Big[\chi_{jl}(\mathbf{r};\omega)(\Im\boldsymbol{\Gamma})_{lk}(\mathbf{r},\mathbf{r};\omega) + \chi_{lm}(\mathbf{r};\omega)x_j\nabla'_k(\Im\boldsymbol{\Gamma})_{ml}(\mathbf{r},\mathbf{r}';\omega)\Big|_{\mathbf{r}-\mathbf{r}'\equiv\mathbf{R}\to\mathbf{0}}\Big]. \tag{4.6}$$

Here $\boldsymbol{\Gamma}$ is taken to be the usual vacuum retarded Green's dyadic $\boldsymbol{\Gamma}^0$, the divergenceless part of which can be written as[3]

$$\boldsymbol{\Gamma}^{0\prime}(\mathbf{r},\mathbf{r}';\omega) = (\nabla\nabla - \mathbf{1}\nabla^2)\frac{e^{i\omega R}}{4\pi R} = \Big[\hat{\mathbf{R}}\hat{\mathbf{R}}(3-3i\omega R-\omega^2R^2) - \mathbf{1}(1-i\omega R-\omega^2R^2)\Big]\frac{e^{i\omega R}}{4\pi R^3}, \tag{4.7}$$

where $R=|\mathbf{r}-\mathbf{r}'|$ and $\hat{\mathbf{R}}=\mathbf{R}/R$. It is evident that the second term in Eq. (4.6) vanishes here because ${\rm Im}\,e^{i\omega R}/R$ is an even function of R. When (4.7) is rotationally averaged in the coincidence limit ($\mathbf{R}\to\mathbf{0}$), we obtain

$$\boldsymbol{\Gamma}^{0\prime}(\mathbf{r},\mathbf{r}';\omega)\to\mathbf{1}\left(\frac{\omega^2}{6\pi R}+i\frac{\omega^3}{6\pi}+O(R)\right). \tag{4.8}$$

Therefore, we are left with only a single term for the torque,

$$\tau_i^{\rm EE} = \int \frac{d\omega}{2\pi}\coth\frac{\beta\omega}{2}\epsilon_{ijk}{\rm Re}\,\alpha_{jk}(\omega)\frac{\omega^3}{6\pi}, \tag{4.9}$$

where the mean polarizability[4] of the body[5] is given by

$$\alpha_{jk}(\omega) = \int (d\mathbf{r})\,\chi_{jk}(\mathbf{r};\omega), \tag{4.10}$$

the real part of which is picked out by the necessity of the integrand being even in ω. Thus, nonreciprocity is necessary for a vacuum torque in first order.

For the PP fluctuation part, Eq. (4.5) is written as

$$\boldsymbol{\tau}^{\rm PP} = \int (d\mathbf{r})\,(d\mathbf{r}')\frac{d\omega}{2\pi}\frac{d\nu}{2\pi}e^{-i(\omega+\nu)t}[\mathbf{P}(\mathbf{r};\omega)\times\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}';\nu)\cdot\mathbf{P}(\mathbf{r}';\nu) + \mathbf{P}(\mathbf{r};\omega)\cdot(\mathbf{r}\times\nabla)\cdot\boldsymbol{\Gamma}(\mathbf{r},\mathbf{r}';\nu)\cdot\mathbf{P}(\mathbf{r}';\nu)], \tag{4.11}$$

[2]This is made up of the internal and external torques, as given in Ref. [27], Eqs. (4.47) and (4.46).

[3]The omitted term is proportional to $\delta(\mathbf{r}-\mathbf{r}')$, which does not contribute in the point-split limit.

[4]The nonlinear effects occurring in the Lorenz-Lorentz law relate the polarizability to the permittivity; the polarizability is implied thereby to be linear (see Ref. [28] and Sec. X).

[5]Note that there is no requirement that the body be spherical, so any rotation would be observable.

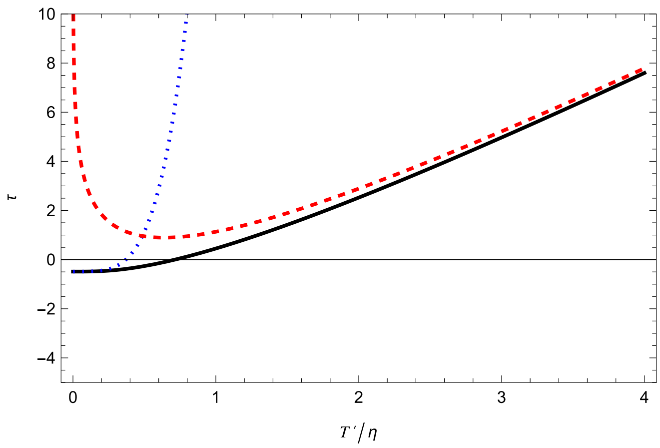

FIG. 1. Apart from the prefactor $\eta\omega_c\omega_p^2V/3\pi^2$, the quantum vacuum torque in Eq. (4.14) or (4.15) is shown as a function of T'/η by the solid line for $T/\eta=0.714$, appropriate for an environment at room temperature, 300 K, and a gold body. The red dashed line is the high-temperature approximation, which utilizes (A8b), while the blue dotted line is the low-temperature approximation, which utilizes (A10b). In both approximations, the dependence on the environmental temperature T is treated exactly.

to which the FDT (4.4b) is to be applied. Again, for vacuum, the coincidence limit of the second term is zero, and because of the diagonal form of the limit of the Green's dyadic, only the antisymmetric part of the susceptibility survives upon use of Eq. (4.4b). Specifically, the ith component of the quantity in square brackets in Eq. (4.11) becomes

$$\delta(\mathbf{r}-\mathbf{r}')2\pi\delta(\omega+\nu)\epsilon_{ijk}(\Im\boldsymbol{\chi})_{jk}(\mathbf{r};\omega)\left(-i\frac{\omega^3}{6\pi}\right)\coth\frac{\beta'\omega}{2}. \tag{4.12}$$

Here we have recognized that the antisymmetric part of $\Im\boldsymbol{\chi}$ is even in ω, according to Eq. (2.5b), and therefore only the odd part of the vacuum Green's dyadic survives. The resulting torque is thus of the same form as for the EE contribution, except for the sign, and the replacement $\beta\to\beta'$.

The combination of the two contributions thus yields the torque on a nonreciprocal body in vacuum, when the temperature of the body, T', differs from that of the blackbody radiation, T, due to PP and EE fluctuations:

$$\tau_i = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\omega^3}{6\pi}\epsilon_{ijk}{\rm Re}\,\alpha_{jk}(\omega)\left(\coth\frac{\beta\omega}{2}-\coth\frac{\beta'\omega}{2}\right). \tag{4.13}$$

This result exactly agrees with that of Guo and Fan [25], for zero rotational velocity, and with that of Strekha *et al.* [26]. However, there is no quantum vacuum force in this static situation, at least in first order, which we will demonstrate in Sec. VII. (Actually, this is already evident from the vanishing of the external torque contribution.)

Let us use the model in Eq. (3.5) to give an estimate of the size of the torque. Inserting this into Eq. (4.13) and letting $\omega=\eta x$ gives

$$\tau_z = \frac{\eta\omega_c\omega_p^2V}{3\pi^2}\int_{-\infty}^{\infty}dx\frac{x^3}{(x^2+1)^2}\left(\coth\frac{\beta'\eta x}{2}-\coth\frac{\beta\eta x}{2}\right) = \frac{4\eta\omega_c\omega_p^2V}{3\pi^2}[I_2(\beta'\eta)-I_2(\beta\eta)], \tag{4.14}$$

where V is the volume of the body and the integrals are defined by Eq. (A5) of Appendix A. As expected, this is positive if $T' > T$. These integrals are readily evaluated in Eq. (A6):

$$\tau_z = \frac{\eta\omega_c\omega_p^2 V}{3\pi^2}\left[\frac{\pi}{\eta}(T-T') + 2\ln\frac{T}{T'} + 2\psi\left(\frac{\eta}{2\pi T}\right) - 2\psi\left(\frac{\eta}{2\pi T'}\right) + \frac{\eta}{2\pi T}\psi'\left(\frac{\eta}{2\pi T}\right) - \frac{\eta}{2\pi T'}\psi'\left(\frac{\eta}{2\pi T'}\right)\right]. \tag{4.15}$$

In Fig. 1 we show the torque together with the low- and high-temperature approximations that result from the corresponding approximations for $I_2(\beta'\eta)$, presented in Appendix A. For a gold ($\omega_p = 9\,\text{eV}$ and $\eta = 0.035\,\text{eV}$) nanosphere of radius 100 nm, with $\omega_c = 10^{-4}\,\text{eV}$, the prefactor in Eq. (4.15) is 8×10^{-25} N m.

V. TORQUE ON A ROTATING BODY

Of course, a torque on a body will cause it to rotate. So, what is the torque on a rotating body? Naturally, there should be a vacuum torque on a rotating body made of ordinary (reciprocal) material, just as there is quantum vacuum friction on a linearly moving body [12,13]. The nonreciprocal aspect of this torque was first treated in Ref. [25].

We consider a body rotating about the z axis passing through its center of mass with angular velocity Ω. The formula (4.5) should still apply, with the external torque (the second term there) still not contributing if the background is vacuum. However, the polarization and electric fields should now refer to the body (rotating) frame, denoted by a prime subsequently. For low velocities, these are related to those in the blackbody (unprimed) frame by a rotation

$$E'_x(\mathbf{r}',t) = E_x(\mathbf{r},t)\cos\Omega t + E_y(\mathbf{r},t)\sin\Omega t, \tag{5.1a}$$

$$E'_y(\mathbf{r}',t) = E_y(\mathbf{r},t)\cos\Omega t - E_x(\mathbf{r},t)\sin\Omega t, \tag{5.1b}$$

which means for the frequency transforms

$$E'_x(\mathbf{r}';\omega) = \frac{1}{2}[E_x(\mathbf{r};\omega_+) + E_x(\mathbf{r};\omega_-)] + \frac{1}{2i}[E_y(\mathbf{r};\omega_+) - E_y(\mathbf{r};\omega_-)], \tag{5.2a}$$

$$E'_y(\mathbf{r}';\omega) = \frac{1}{2}[E_y(\mathbf{r};\omega_+) + E_y(\mathbf{r};\omega_-)] - \frac{1}{2i}[E_x(\mathbf{r};\omega_+) - E_x(\mathbf{r};\omega_-)], \tag{5.2b}$$

where $\omega_\pm = \omega \pm \Omega$. The $\mathbf{P}$ transforms in the same way.

The strategy followed to calculate the quantum rotational friction is the same as that used for quantum rectilinear friction [13]. There are two contributions: field fluctuations and dipole fluctuations. For the former we use Eq. (4.3b) to replace the polarization by the electric field but now understood in the body frame. Then we have to transform both electric fields to the blackbody frame. For vacuum friction we can use Eq. (4.8) for the Green's dyadic that appears when the FDT is employed for the fields. We also only keep half the terms: Only those proportional to $\delta(\omega_\pm + \nu_\mp)$ do not average to zero in time. The result of a straightforward calculation is

$$\tau_z^{\rm EE} = \int \frac{d\omega}{2\pi}\frac{\omega^3}{6\pi}[\mathrm{Im}(\alpha_{xx}+\alpha_{yy})(\omega_-) + \mathrm{Re}(\alpha_{xy}-\alpha_{yx})(\omega_-)]\coth\frac{\beta\omega}{2}, \tag{5.3}$$

where we have also noticed that under $\omega \to -\omega$, $\omega_+ \to -\omega_-$.

The procedure for the PP fluctuations is similar, except now we replace $\mathbf{E}$ by $\mathbf{P}$ according to Eq. (4.3a). This holds in the blackbody frame, so $\mathbf{P}$ must be transformed back to the body (rotating) frame. Simplifications occur as before for the vacuum case and we find after a bit of algebra

$$\tau_z^{\rm PP} = -\int \frac{d\omega}{2\pi}\frac{\omega^3}{6\pi}[\mathrm{Im}(\alpha_{xx}+\alpha_{yy})(\omega_-) + \mathrm{Re}(\alpha_{xy}-\alpha_{yx})(\omega_-)]\coth\frac{\beta'\omega_-}{2}. \tag{5.4}$$

Thus, when the two contributions are added, we find for the torque on a (slowly) rotating body:

$$\tau_z = \frac{1}{12\pi^2}\int_{-\infty}^{\infty} d\omega\,\omega_+^3[\mathrm{Im}(\alpha_{xx}+\alpha_{yy})(\omega) + \mathrm{Re}(\alpha_{xy}-\alpha_{yx})(\omega)]\left(\coth\frac{\beta\omega_+}{2} - \coth\frac{\beta'\omega}{2}\right). \tag{5.5}$$

This is precisely the torque found in Ref. [25] (recall that $4\pi\alpha^{\rm G} = \alpha^{\rm HL}$). This result for an isotropic (reciprocal) particle was given in Ref. [29], which further considered the effect of a magnetic field.[6] Note that if $\Omega = 0$ the first term involving the diagonal polarizabilities vanishes because the integrand is odd and the second term reproduces Eq. (4.13).

It is illuminating to expand this expression to leading order in the rotational velocity Ω (this is the adiabatic approximation):

$$\tau_z = \frac{1}{12\pi^2}\int_{-\infty}^{\infty} d\omega\,\omega^3\left[\mathrm{Re}(\alpha_{xy}-\alpha_{yx})(\omega)\left(\coth\frac{\beta\omega}{2} - \coth\frac{\beta'\omega}{2}\right) - \Omega\frac{3}{\omega}\mathrm{Im}(\alpha_{xx}+\alpha_{yy})(\omega)\left(\coth\frac{\beta'\omega}{2} - \coth\frac{\beta\omega}{2}\right) - \Omega\frac{\beta}{2}\mathrm{Im}(\alpha_{xx}+\alpha_{yy})(\omega)\mathrm{csch}^2\frac{\beta\omega}{2}\right]. \tag{5.6}$$

The first term here is the (nonreciprocal) quantum vacuum torque (4.13), the second is the nonequilibrium contribution to the ordinary (reciprocal) quantum vacuum frictional torque, and the third term is the analog of the Einstein-Hopf quantum vacuum friction. The sum of the two frictional terms is a drag if $T' > T$. If $T' < T$, the angular velocity changes sign, so initially the friction remains a drag, but for sufficiently low temperatures T' the second term in Eq. (5.6) will dominate and exponential growth of the angular velocity will ensue, insofar as the low-velocity approximation remains valid.

[6] Earlier related works on forces and torques on bodies with various kinds of asymmetries include Refs. [19–21,30].

Note that if the last two terms constitute a drag, the nonreciprocal torque found here will lead to the body rotating with a constant terminal angular velocity. Writing Eq. (5.6) in the abbreviated form[7]

$$\tau_z = I\dot{\Omega} = \tau_0 - \Omega\tau_1', \tag{5.7}$$

where I is the moment of inertia of the body, we immediately obtain

$$\Omega(t) = \frac{\tau_0}{\tau_1'}(1 - e^{-\tau_1' t/I}) \tag{5.8}$$

if the body is not rotating at time $t = 0$. The terminal velocity is $\Omega_T = \Omega(t \to \infty) = \tau_0/\tau_1'$, which might be expected to be small. (This of course assumes that the particle and environmental temperatures do not change. We will address the tendency toward thermal equilibrium in Sec. IX.)

To proceed, let us again use the model (3.2), with $\omega_0 = 0$ and $\omega_c \ll \eta$. Then we have

$$\text{Im}(\alpha_{xx} + \alpha_{yy})(\omega) = \frac{2\omega_p^2\eta}{\omega(\omega^2 + \eta^2)}V. \tag{5.9}$$

The result is [see Eq. (A2)] ($x = \omega/\eta$)

$$\begin{aligned}\tau_1' &= \frac{2\omega_p^2\eta V}{3\pi^2}\left(3 + \beta\frac{\partial}{\partial\beta}\right)\int_0^\infty dx \frac{x}{x^2+1}\left(\frac{1}{e^{\beta'\eta x} - 1} - \frac{1}{e^{\beta\eta x} - 1}\right) \\ &= \frac{\omega_p^2\eta V}{3\pi^2}\left[\frac{\pi}{\eta}(2T - 3T') + 3\ln\frac{T}{T'} - 1 + 3\psi\left(\frac{\eta}{2\pi T}\right) - 3\psi\left(\frac{\eta}{2\pi T'}\right) + \frac{\eta}{2\pi T}\psi'\left(\frac{\eta}{2\pi T}\right)\right].\end{aligned} \tag{5.10}$$

Let us write, from Eqs. (4.15) and (5.10),

$$\tau_0 = \frac{\eta\omega_c\omega_p^2 V}{\pi^2}f(T, T'), \quad \tau_1' = \frac{\eta\omega_p^2 V}{\pi^2}g(T, T'). \tag{5.11}$$

Then the terminal angular velocity is

$$\Omega_T = \frac{\tau_0}{\tau_1'} = \omega_c\frac{f(T, T')}{g(T, T')} \sim \omega_c \sim 10^{-4}\ \text{eV} \sim 10^{11}\ \text{s}^{-1}, \tag{5.12}$$

perhaps surprisingly high, but very small compared to atomic frequencies. (The terminal circumferential speed in this case is $\Omega_T R \sim 10^4$ m/s for a gold nanosphere of radius $R = 100$ nm.) The relaxation time required to reach this velocity is

$$t_0 = \frac{I}{\tau_1'} \sim \frac{MR^2}{\omega_p^2\eta V} \sim 10^6\ \text{s} \tag{5.13}$$

for the same parameters. The temperature dependence of the terminal angular velocity Ω_T is shown in Fig. 2. Note that if the temperature of the body is lower than that of the environment, the terminal angular velocity is negative. Whether the body is hotter or not too much colder than the environment, it will reach a terminal velocity if the temperature difference is maintained. If the temperature of the body is much colder than that of the environment (for the example shown, about half room temperature), the frictional torques reverse sign and no bound to the angular velocity can be reached. A negative τ_1' means exponential growth. Of course, before the angular velocity gets too large, the nonrelativistic approximation used here breaks down, even if the temperature difference can be maintained by some external or internal agent. However, it is not necessary to wait a long time to reach the terminal velocity, because the initial angular acceleration

$$\dot{\Omega}(0) = \frac{\Omega_T}{t_0} \sim 10^5\ \text{s}^{-2} \tag{5.14}$$

should be easily discernible.

[7]Specifically, $\tau_0 = \tau_z(\Omega = 0)$ and $\tau_1' = -\frac{d\tau_z}{d\Omega}(\Omega = 0)$.

VI. TORQUE IN THE PRESENCE OF A DIELECTRIC PLATE

What happens if the background is less trivial, say, consisting of an isotropic dielectric plate filling the half space $z < 0$, while the body lies a distance a above it? Then, of course, there will be a torque on a nonspherical body as well as a force, due to ordinary Casimir forces, even when the body is made of ordinary reciprocal material. What would be unusual is if there were a torque around the z axis, since the environment possesses rotational symmetry about that direction. For simplicity, we will assume that the entire background, vacuum plus dielectric plate, is in equilibrium at temperature T, while the nanoparticle has temperature T'. In that case, the z component of the torque coming from field fluctuations is given by Eq. (4.6). Now both terms can contribute, but only

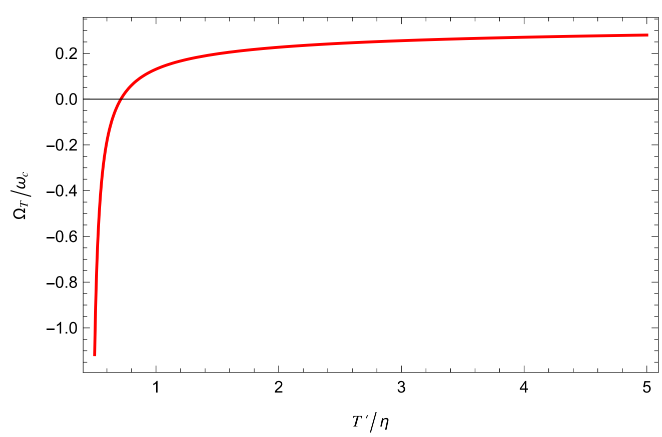

FIG. 2. Terminal angular velocity of the nonreciprocal body when it is hotter or colder than its environment, which is taken to be at room temperature. The limit of Ω_T for high particle temperature is universally $\omega_c/3$, independent of the background temperature. When the temperature is lower than that of the background, the frictional torque initially acts as a drag, but for sufficiently low temperature, the frictional terms change sign and the angular velocity increases exponentially without bound.

for a nonreciprocal body, which might be irregularly shaped. For such a body, where

$$\hat{\chi}_{ij}(\mathbf{r};\omega) = \mathrm{Re}(\chi_{ij} - \chi_{ji})(\mathbf{r};\omega) \tag{6.1}$$

is nonzero, the normal torque component can be computed from the explicit construction of the Green's dyadic (as given, for example, in Ref. [12]). Using the Fourier representation (2.8), the integration over k_x and k_y will vanish except for the g_{xx} and g_{yy} terms, for the first term in Eq. (4.6), while $g_{xz,zx}$ and $g_{yz,zy}$ contribute for the second term, yielding the scattering part of the torque

$$\tau_z^s = \int (d\mathbf{r})\frac{d\omega}{2\pi}\left(\coth\frac{\beta\omega}{2} - \coth\frac{\beta'\omega}{2}\right)\frac{1}{4}\int\frac{(d\mathbf{k}_\perp)}{(2\pi)^2} \times \left\{\hat{\chi}_{xy}(\mathbf{r};\omega)\mathrm{Im}\left[\left(\kappa r^H + \frac{\omega^2}{\kappa}r^E\right)e^{-2\kappa z}\right] + [\hat{\chi}_{yz}(\mathbf{r};\omega)x - \hat{\chi}_{xz}(\mathbf{r};\omega)y]\,k^2\mathrm{Im}\left(r^H e^{-2\kappa z}\right)\right\}, \tag{6.2}$$

where $\kappa = \sqrt{k^2-\omega^2}$ and the transverse magnetic and electric reflection coefficients are

$$r^H = \frac{\kappa - \kappa'/\varepsilon(\omega)}{\kappa + \kappa'/\varepsilon(\omega)}, \quad r^E = \frac{\kappa-\kappa'}{\kappa+\kappa'}, \quad \kappa' = \sqrt{k^2 - \omega^2\varepsilon(\omega)}. \tag{6.3}$$

Now the torque depends on the distribution of the anisotropic material across the body and so is not describable by simply an effective nonreciprocal polarizability.

Note further that the last two terms in Eq. (6.2), proportional to x and y, respectively, depend on the position of the body as well as the distribution of material within the body. If we write $\mathbf{r} = \mathbf{R} + \mathbf{r}'$, where $\mathbf{R}$ locates the center of mass of the body, we can read off the force on the center of mass of the body from $\tau_z = XF_y - YF_x$ so that

$$F_x = \int (d\mathbf{r})\int_0^\infty \frac{d\omega}{2\pi}\left(\frac{1}{e^{\beta\omega}-1} - \frac{1}{e^{\beta'\omega}-1}\right)\hat{\chi}_{xz}(\mathbf{r};\omega) \times \int\frac{(d\mathbf{k}_\perp)}{(2\pi)^2}k^2\mathrm{Im}\left(r^H e^{-2\kappa z}\right). \tag{6.4}$$

We will derive this result directly in Sec. VIII.

Torque for a nanoparticle above a perfectly conducting plate

A very simple example is provided by a perfectly conducting surface lying in the $z=0$ plane. This means $r^{H,E} = \pm 1$. Consider only the case with $\hat{\chi}_{xy} \neq 0$, that is, for our model, the magnetic field lying in the z direction. The imaginary part comes only from the region where $\omega^2 > k^2$, where

$$\mathrm{Im}\kappa = -\mathrm{sgn}(\omega)\sqrt{\omega^2-k^2}, \tag{6.5}$$

and then the integral in Eq. (6.2) over transverse wave numbers is (provided the body is of negligible extent, a nanoparticle, so $z=a$)

$$\frac{1}{(2\pi)^2}\int_0^{|\omega|}dk\,k\int_0^{2\pi}d\theta\left(-\sqrt{\omega^2-k^2} - \frac{\omega^2}{\sqrt{\omega^2-k^2}}\right) \times \cos\left(2a\sqrt{\omega^2-k^2}\right)$$
$$= -\frac{\omega^3}{2\pi}\int_0^1 dy(y^2+1)\cos 2\omega a y$$
$$= -\frac{1}{\pi(2a)^2}[u\cos u + (u^2-1)\sin u]. \tag{6.6}$$

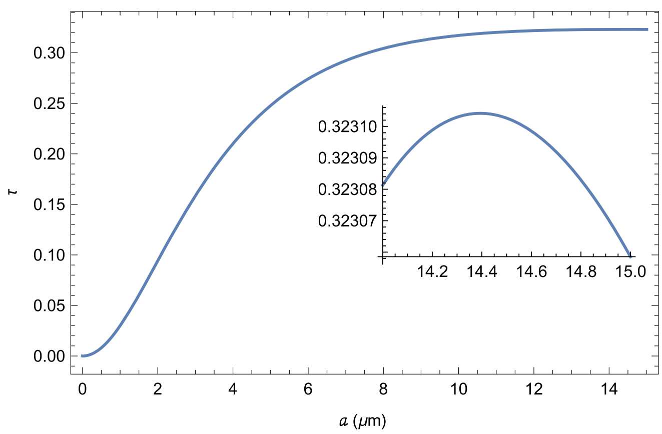

FIG. 3. Torque [apart from the prefactor in Eq. (6.9)] as a function of separation of the nanoparticle a from the perfectly conducting plate in microns, for a damping parameter of $\eta = 0.035$ eV. The temperatures are taken to be $T = 300$ K and $T' = 600$ K. The torque vanishes, as expected, close to the plate and approaches the vacuum value far from the plate. Most interesting is the appearance of a very weak maximum at about 14.4 μm, just before the decrease to the vacuum torque value, as displayed in the inset.

Here we have defined $u = 2\omega a$. Then we write the scattering part of the torque in the direction perpendicular to the plate as

$$\tau_z^s = \frac{2}{\pi^2}V\omega_c\omega_p^2\eta\int_0^\infty du\frac{u\cos u + (u^2-1)\sin u}{[u^2+(2\eta a)^2]^2} \times \left(\frac{1}{e^{\beta u/2a}-1} - \frac{1}{e^{\beta' u/2a}-1}\right). \tag{6.7}$$

Close to the plate, $2\eta a \ll 1$, $\beta/2a \gg 1$, and $\beta'/2a \gg 1$, where the integral is dominated by small values of u, for which the numerator in the integrand approaches

$$u\cos u + (u^2-1)\sin u \sim \tfrac{2}{3}u^3, \tag{6.8}$$

we obtain precisely the negative of the torque coming from the vacuum contribution (4.14). That the total normal torque vanishes as the perfectly conducting plate is approached is expected, because the tangential electric field must vanish there.

The total torque is then

$$\tau_z^{\mathrm{vac}} + \tau_z^s = \frac{4}{3\pi^2}V\omega_c\omega_p^2\eta\int_0^\infty du\frac{u^3}{[u^2+(2a\eta)^2]^2} \times \left(1 - \frac{3}{2}\frac{u\cos u + (u^2-1)\sin u}{u^3}\right) \times \left(\frac{1}{e^{\beta' u/2a}-1} - \frac{1}{e^{\beta u/2a}-1}\right). \tag{6.9}$$

This is plotted in Fig. 3.

VII. QUANTUM VACUUM FORCE

Let us start by writing the force on a dielectric body on which an electric field is impressed, writing the fields in terms of their frequency transforms

$$\mathbf{F} = \int (d\mathbf{r}) \int \frac{d\omega}{2\pi}\frac{d\nu}{2\pi} e^{-i(\omega+\nu)t}\left[-[\nabla\cdot\mathbf{P}(\mathbf{r};\omega)]\mathbf{E}(\mathbf{r};\nu) - i\omega\mathbf{P}(\mathbf{r};\omega)\times\left(\frac{1}{i\nu}\nabla\times\mathbf{E}(\mathbf{r};\nu)\right)\right]$$
$$= \int (d\mathbf{r}) \int \frac{d\omega}{2\pi}\frac{d\nu}{2\pi} e^{-i(\omega+\nu)t}\left[-[\nabla\cdot\mathbf{P}(\mathbf{r};\omega)]\mathbf{E}(\mathbf{r};\nu)\left(1+\frac{\omega}{\nu}\right) - \frac{\omega}{\nu}\mathbf{P}(\mathbf{r};\omega)\cdot(\nabla)\cdot\mathbf{E}(\mathbf{r};\nu)\right], \tag{7.1}$$

where in the second line we integrated spatially by parts. Unlike for the torque, the total divergence does not contribute. Now we expand either **P** in terms of **E**, using Eq. (4.3b), or **E** in terms of **P**, using Eq. (4.3a), and then use the fluctuation-dissipation theorem on the two parts. This yields rather immediately for the force on the body[8]

$$F_i^{\rm EE} + F_i^{\rm PP} = \int \frac{d\omega}{2\pi}(d\mathbf{r})\left(\chi_{jl}(\mathbf{r};\omega)\nabla_i'(\Im\boldsymbol{\Gamma})_{lj}(\mathbf{r},\mathbf{r}';\omega)\Big|_{\mathbf{r}'=\mathbf{r}} \coth\frac{\beta\omega}{2} + (\Im\boldsymbol{\chi})_{jl}(\mathbf{r};\omega)\nabla_i\Gamma_{lj}(\mathbf{r},\mathbf{r}';-\omega)\Big|_{\mathbf{r}'=\mathbf{r}} \coth\frac{\beta'\omega}{2}\right). \tag{7.2}$$

However, for vacuum, Eq. (4.8) describes the vacuum Green's dyadic. So, again in the coincidence limit, it is then clear that the gradient of the Green's dyadic vanishes, and thus there is no vacuum force. The conclusion appears to be opposite to that of Refs. [30,31], but evidently the self-propulsion found there arises as a second-order effect. There, nonreciprocity is not required. The only necessary conditions are that the system be out of thermal equilibrium and that the body be extended and inhomogeneous.

VIII. TRANSVERSE FORCE ON A NONRECIPROCAL NANOPARTICLE INDUCED BY A DIELECTRIC SURFACE

In contrast to the result found in the preceding section, a nonreciprocal body does experience, in first order, a force transverse to another ordinary body, even when both bodies are at rest, provided they are not in thermal equilibrium with each other. This was observed in Ref. [31] and more recently in Refs. [21,32].

This is still described by the formula (7.2), but requires the scattering part of the Green's function. We will consider the second, ordinary body to be a planar dielectric, with permittivity $\varepsilon(\omega)$, lying in the half space $z<0$, while the nonreciprocal body lies at a distance $z=a$ above the plane. It is convenient then to introduce a two-dimensional Fourier transform in the transverse coordinates x and y. Then the force in the x direction, say, is

$$F_x = -\int (d\mathbf{r})\frac{d\omega}{2\pi}\frac{(d\mathbf{k}_\perp)}{(2\pi)^2} ik_x\left(\chi_{jk}(\mathbf{r};\omega)(\Im\mathbf{g}^s)_{kj}(z,z;\omega,\mathbf{k}_\perp)\right.$$
$$\left.\times\coth\frac{\beta\omega}{2} - (\Im\boldsymbol{\chi})_{jk}(\mathbf{r};\omega)g^s_{kj}(z,z;-\omega,\mathbf{k}_\perp)\coth\frac{\beta'\omega}{2}\right). \tag{8.1}$$

Here the s superscripts on the reduced Green's functions represent the scattering parts, since it is evident that the bulk (vacuum) part does not contribute, as already demonstrated in the preceding section. Now the integration over k_x and k_y will vanish except for g_{xz} and g_{zx}.[9] Hence, unlike for the torque, only the TM Green's function contributes. Using the properties of $\Im\boldsymbol{\chi}$ given in Sec. II, we immediately obtain

$$F_x = 2\int_0^\infty \frac{d\omega}{2\pi}\int (d\mathbf{r})\frac{(d\mathbf{k}_\perp)}{(2\pi)^2}\hat{\chi}_{xz}(\mathbf{r};\omega)\,k_x^2{\rm Im}\left(r^H e^{-2\kappa z}\right)$$
$$\times\left(\frac{1}{e^{\beta\omega}-1}-\frac{1}{e^{\beta'\omega}-1}\right). \tag{8.2}$$

This force is precisely that inferred from the torque in Eq. (6.4).

As with Casimir friction, this force will vanish unless dissipation occurs somewhere. This could be due to dissipation in the dielectric slab or to radiation. We will consider these in the following sections.

A. Dissipation in a metallic slab

We will describe the metallic substrate by a Drude model

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2+i\omega\nu}, \tag{8.3}$$

where ω_p is the plasma frequency and ν the damping parameter. For simplicity, we will consider the regime

$$\nu \ll \omega \ll \omega_p, \quad \omega \ll k, \tag{8.4}$$

so that

$${\rm Im}\,\varepsilon(\omega) \approx \frac{\omega_p^2\nu}{\omega^3}, \tag{8.5}$$

and then, for low frequencies,

$${\rm Im} r^H = {\rm Im}\frac{\kappa-\kappa'/\varepsilon}{\kappa+\kappa'/\varepsilon} \approx {\rm Im}\frac{\varepsilon-1}{\varepsilon+1} \approx \frac{2\omega\nu}{\omega_p^2}. \tag{8.6}$$

Thus, in this approximation, where we crudely replace κ and κ' by k, the force on a nanoparticle of negligible extent is

$$F_x = -2\frac{\nu}{\omega_p^2}\int\frac{d\omega}{2\pi}\frac{(d\mathbf{k}_\perp)}{(2\pi)^2}\hat{\alpha}_{xz}(\omega)\,\omega\,k_x^2e^{-2\kappa a}$$
$$\times\left(\frac{1}{e^{\beta\omega}-1}-\frac{1}{e^{\beta'\omega}-1}\right)$$
$$\approx -\frac{12}{(2\pi)^2}\frac{1}{(2a)^4}\frac{\nu}{\omega_p^2}\int_0^\infty d\omega\,\hat{\alpha}_{xz}(\omega)\,\omega$$
$$\times\left(\frac{1}{e^{\beta\omega}-1}-\frac{1}{e^{\beta'\omega}-1}\right). \tag{8.7}$$

[8] This general formula can be derived immediately from the torque on the center of mass inferred from Eq. (4.6).

[9] For the bulk (vacuum) contribution, the force would involve the symmetric limit $\lim_{z\to z'}{\rm sgn}(z-z')=0$.

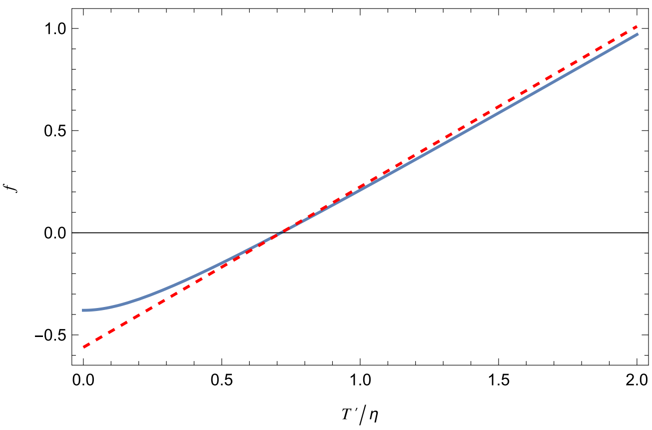

FIG. 4. Force, defined in Eq. (8.8), between a nonreciprocal nanoparticle and a metal plate with finite conductivity out of thermal equilibrium. The temperature of both the plate and the background electromagnetic field is fixed at room temperature, which corresponds to $T/\eta = 0.714$ for gold. The polarizability of the nanoparticle is described by the model (3.5). The temperature of the nanoparticle, T', is given in units of the damping parameter for gold, η. For comparison, the dashed straight line shows the force when both temperatures are large, demonstrating that the high-temperature approximation is quite good across the temperature range displayed.

Now for the nonreciprocal polarizability, let us use the model (3.5), where we now assume that the magnetic field (confined to the particle) lies in the y direction. This leads directly to the formula for the force ($x = \omega/\eta$)

$$F_x = \frac{3V}{4\pi^2 a^4}\frac{\nu}{\eta}\omega_c f(\beta\eta, \beta'\eta),$$
$$f(\beta\eta, \beta'\eta) = -\int_0^\infty dx \frac{x}{(x^2+1)^2}\left(\frac{1}{e^{\beta\eta x}-1} - \frac{1}{e^{\beta'\eta x}-1}\right), \tag{8.8}$$

where the integral follows from Appendix A,

$$\int_0^\infty dx \frac{x}{(x^2+1)^2}\frac{1}{e^{\beta\eta x}-1} = \frac{1}{4}\left[1 + \frac{\pi}{\beta\eta} - \frac{\beta\eta}{2\pi}\psi'\left(\frac{\beta\eta}{2\pi}\right)\right]. \tag{8.9}$$

The dimensionless force f is plotted in Fig. 4 and compared to the high-temperature approximation. The prefactor, for $\nu = \eta$, $a = 1$ μm, and the radius of the nanosphere being 100 nm, is 5×10^{-21} N.

B. Transverse force in the presence of a perfectly conducting plate

If the slab is a perfect conductor, with $r^H = 1$, the formula for the transverse force simplifies considerably. The imaginary part of the Green's function then requires that $\omega^2 > k^2$, and so the integral over the wave number is

$$\int \frac{(d\mathbf{k}_\perp)}{(2\pi)^2} k_x^2 \text{Im}\, e^{-2\kappa a} = -\frac{1}{4\pi}\frac{1}{(2a)^4}[6u\cos u + 2(u^2-3)\sin u], \tag{8.10}$$

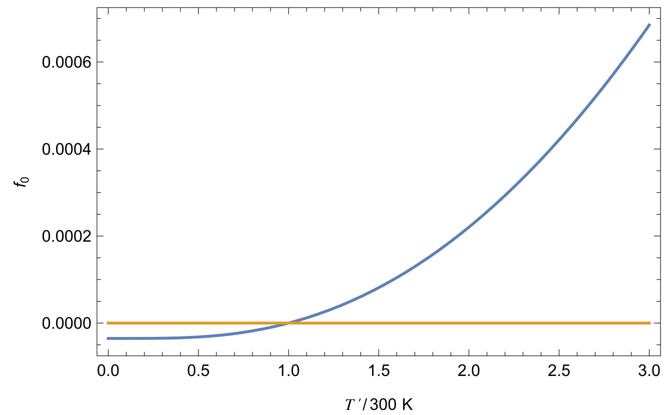

FIG. 5. Transverse force given in Eq. (8.11) as a function of the ratio of the temperature of the nonreciprocal nanoparticle relative to that of the environment and the perfectly conducting plate, 300 K. Here we take the separation a of the nanoparticle and the plate to be 100 nm and the damping to be that appropriate for gold, 0.035 eV. In this case, the prefactor in the force in Eq. (8.11) for a gold nanosphere of 10 nm radius is 1.2×10^{-20} N, so this would be challenging to observe.

where $u = 2\omega a$. Then the transverse force is

$$F_x = \frac{\omega_c \eta \omega_p^2 V}{2\pi^2 a} f_0(\epsilon, b, b'),$$
$$f_0(\epsilon, b, b') = \int_0^\infty du \frac{6u\cos u + 2(u^2-3)\sin u}{(u^2+\epsilon^2)^2} \times \left(\frac{1}{e^{ub}-1} - \frac{1}{e^{ub'}-1}\right), \tag{8.11}$$

where $\epsilon = 2\eta a$, $b = 1/2aT$, and $b' = 1/2aT'$. The integral f_0 is plotted in Fig. 5 as a function of the nanoparticle temperature T' for the environment at room temperature, for a separation of $a = 100$ nm, with a damping parameter appropriate for gold, $\eta = 0.035$ eV. Note that the high-temperature limit for the force is given by $f_0 \sim \frac{\pi}{8b'}$ for $b' \ll 1/\epsilon, b$, but this limit requires very high temperatures which are not accessible in practice. It is noteworthy that Figs. 4 and 5 are qualitatively (but not quantitatively) similar, given that the physical mechanisms invoked are rather different. It is easily seen that the lateral force rapidly vanishes as $a \to \infty$, consistent with the absence of a quantum vacuum force.

We expect, as we saw for the torque, that this force will be resisted by the quantum vacuum friction in the presence of the plate, which for low velocities will lead to a terminal velocity, according to

$$m\frac{dv}{dt} = F_0 - vF_1' \Rightarrow v(t) = \frac{F_0}{F_1'}(1 - e^{-F_1' t/m}). \tag{8.12}$$

We require, then, the nonequilibrium frictional force in the presence of a conducting plate, which we derive in Appendix B. Using the same model for the permittivity of the nanoparticle, the linear term in the friction is

$$F_1' = \frac{\omega_p^2 \eta V}{\pi^2 (2a)^2} f_1(\epsilon, b, b'), \tag{8.13}$$

where

$$f_1(\epsilon, b, b') = \int_0^\infty du \frac{u^3}{u^2+\epsilon^2}\left[\left(1-\frac{2\cos u+(u^2-2)\sin u}{u^3}\right)\left(\frac{1}{e^{b'u}-1}-\frac{1}{e^{bu}-1}\right) + \frac{1}{12}\frac{bu}{\sinh^2(bu/2)}\left(1+3\frac{u(u^2-12)\cos u-(5u^2-12)\sin u}{u^5}\right)\right]. \tag{8.14}$$

Indeed, f_1 is always positive, corresponding to a frictional drag, and the corresponding terminal velocity is

$$v_T = \frac{F_0}{F_1'} = 2\omega_c a \frac{f_0}{f_1}. \tag{8.15}$$

The scale factor here is small compared to the speed of light: For a particle 100 nm above the plate, $2\omega_c a = 10^{-4}$. The ratio of f_0/f_1 is shown in Fig. 6. The apparent saturation of the terminal velocity near 0.2 is illusory; for still larger temperatures, the terminal velocity tends to zero, since the frictional force rapidly increases with temperature. However, for these nominal values, the damping time is long,

$$t_0 = \frac{m}{F_1'} \sim 2\times 10^3 \text{ s}\frac{1}{f_1} \sim 10^6 \text{ s}, \tag{8.16}$$

if the particle is at twice room temperature.

IX. RELAXATION TO THERMAL EQUILIBRIUM

All of the above considerations assume that the temperatures of the body and of the background are constant. Of course, this will not be so unless some mechanism keeps the system out of thermal equilibrium. Here we will calculate the time it would take for such a body at rest to come to thermal equilibrium with its environment. We cannot regard the body to be a blackbody, but we can calculate the rate at which it loses heat from the power[10] (for an isotropic body) [13]

$$P(T, T') = \frac{1}{\pi^2}\int_0^\infty d\omega\, \omega^4 \operatorname{Im}\alpha(\omega)\left(\frac{1}{e^{\beta\omega}-1}-\frac{1}{e^{\beta'\omega}-1}\right) = \frac{dQ}{dt}. \tag{9.1}$$

This is related to the rate of change of temperature of the body by its heat capacity:

$$\frac{dQ}{dt} = C_V(T')\frac{dT'}{dt}. \tag{9.2}$$

Thus, the time it takes for the body to cool from temperature T_0' to temperature T_1', where $T_0' > T_1' > T$, is

$$t = \int_{T_0'}^{T_1'} dT' \frac{C_V(T')}{P(T, T')}. \tag{9.3}$$

To proceed, we need a model for the heat capacity of the body, which is provided by the Debye model,[11] which is satisfactory for simple crystals (see Ref. [33]),

$$C_V(T) = 9N\left(\frac{T}{\Theta}\right)^3\int_0^{\Theta/T} dx \frac{x^4 e^x}{(e^x-1)^2}, \tag{9.4}$$

where N is the number of atoms constituting the body and Θ is the Debye temperature. This interpolates between the low- and high-temperature limits

$$C_V(T) \sim 3N \times \begin{cases} 1, & T \gg \Theta \\ \frac{4\pi^4}{5}\left(\frac{T}{\Theta}\right)^3, & T \ll \Theta. \end{cases} \tag{9.5}$$

Since the Debye temperature for gold is about $\Theta = 170$ K, the high-temperature approximation would seem appropriate for an estimate at room temperature and above.

We finally need a model for the imaginary part of the polarizability of the body. The Lorenz-Lorentz model would give

$$\operatorname{Im}\alpha(\omega) = \frac{V\omega_p^2\omega\eta}{\left(\omega_1^2-\omega^2\right)^2+\omega^2\eta^2} \approx \frac{V\omega_p^2\omega\eta}{\omega_1^4}, \tag{9.6}$$

where, for a metal (Drude model), $\omega_1 = \omega_p/\sqrt{3}$. The approximation here is appropriate if, as expected, $\omega_1 \gg \omega, \eta$.

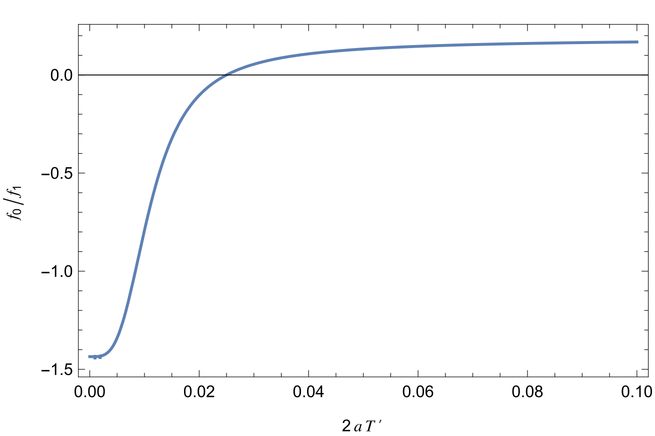

FIG. 6. Terminal velocity of a nonreciprocal nanoparticle near a perfectly conducting plate in units of $2\omega_c a$. Here it is assumed that the plate and the background are at temperature $T = 300$ K and that the particle is made of gold and it is a distance $a = 100$ nm above the surface of the plate. The temperature of the nanoparticle is $T' = 1/2ab'$. Thus, the highest particle temperature displayed on the graph is 1200 K.

[10] For the purpose of the rough calculation presented here, we will ignore the small nonreciprocal effects.

[11] We consider bulk effects only, and ignore surface effects, for the purpose of a rough estimate. Since we are considering vacuum, we also disregard possible near-field enhancements that would occur if the body were near another object.

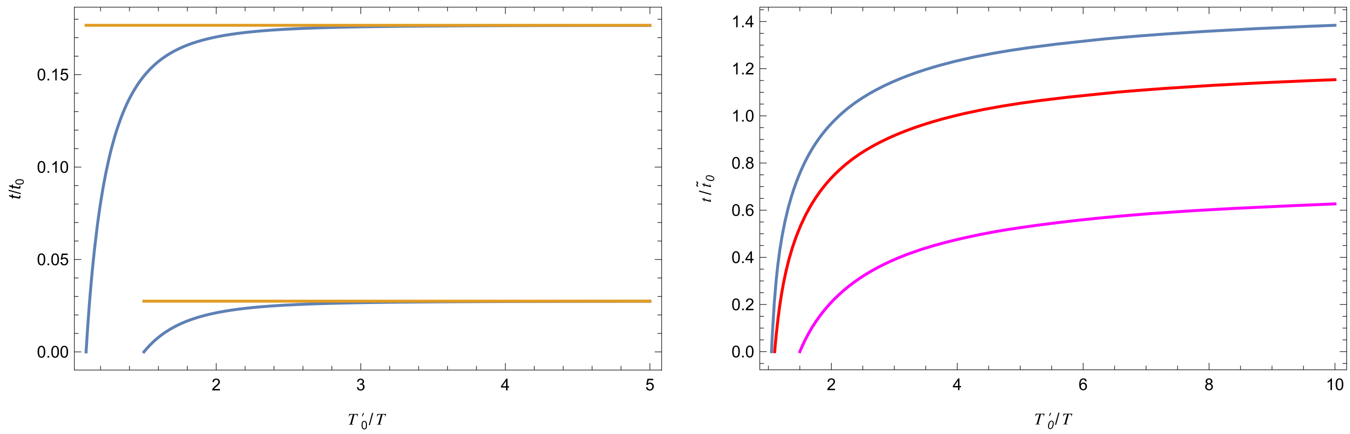

FIG. 7. Time required for a body to cool from temperature T_0' to temperature T_1' for different environmental temperatures T: (a) $T = 300$ K and (b) $T = 1$ K. Here $T_0' > T_1' > T$. (a) The upper set of curves is for $T_1'/T = 1.1$ and the lower curves are for $T_1'/T = 1.5$. (b) The three curves from top to bottom are for $T_1'/T = 1.05$, 1.1, and 1.5. The times are scaled by the prefactor t_0 [Eq. (9.8)], which for a gold body evaluates to about 10^4 s, for the environment at room temperature, and by $\tilde{t}_0$ [Eq. (9.10)], which is about 10^{11} s, for $T = 1$ K.

Inserting this approximation into the formula (9.1), we obtain

$$P(T, T') \approx \frac{8\pi^4}{7}\frac{V\eta}{\omega_p^2}(T^6 - T'^6). \tag{9.7}$$

Now we compute the cooling time from Eq. (9.3):

$$t = t_0 \int_{T_0'/T}^{T_1'/T} du \frac{1}{1-u^6}, \quad T_0' > T_1' > T, \tag{9.8a}$$

where

$$t_0 = \frac{21}{8\pi^4} n \frac{\omega_p^2}{\eta}\frac{1}{T^5}. \tag{9.8b}$$

Here n is the number density of atoms in the body. The relaxation scale t_0 is independent of the volume of the particle and is about 10^4 s for gold, for an environmental temperature of 300 K. The cooling time diverges as $T_1' \to T$, but cooling to a temperature slightly above the environmental temperature takes a finite time. The integral here is elementary, but the resulting expression is not very illuminating. We content ourselves by showing some representative values in Fig. 7(a). It will be seen that if T_0' is appreciably larger than T, the cooling time rapidly saturates to an asymptotic value. If we then take T_0' to be large, Fig. 8 shows how long it will take to reach a multiple of the environmental temperature. Thus, we see that the terminal angular velocity seen in Eq. (5.12) and the terminal linear velocity obtained in Eq. (8.15) will not be achievable unless some mechanism maintains the thermal imbalance, because the timescales for achieving those velocities [Eqs. (5.13) and (8.16)] are much longer than the cooling time found here.

If the environmental temperature is very low $T \ll \Theta$, the cooling time is very much longer. The analysis proceeds as above, using the low-temperature limit in Eq. (9.5), with the result for the cooling time being

$$t = \tilde{t}_0 \int_{(T_0'/T)^2}^{(T_1'/T)^2} dy \frac{y}{1-y^3}, \quad \tilde{t}_0 = \frac{21}{20} n \frac{\omega_p^2}{\eta}\left(\frac{T}{\Theta}\right)^3 \frac{1}{T^5}. \tag{9.9}$$

The integral, which has a relatively simple analytic form, is shown in Fig. 7(b). The ratio of the timescales in the two cases is

$$\frac{\tilde{t}_0}{t_0} = \frac{2\pi^4}{5}\left(\frac{T_{\rm low}}{\Theta}\right)^3\left(\frac{T_{\rm high}}{T_{\rm low}}\right)^5 \sim 10^7 \tag{9.10}$$

for $T_{\rm low} = 1$ K, $T_{\rm high} = 300$ K, and $\Theta = 170$ K for gold, so terminal velocities might be achievable.

X. LORENZ-LORENTZ CORRECTION

Hitherto, we have ignored the Lorenz-Lorentz correction familiar in passing from the permittivity of a body to its polarizability. This was because the forces and torques were derived directly from the macroscopic susceptibilities appropriate for a dissipative metal body. In the case of small bodies, we could always pass from the susceptibility to the mean polarizability by integrating over the volume of the body. However, as is evident from the discussion in the preceding section [see Eq. (9.6)], the effect of the medium on the local electric field can result in a large correction in the case of metal bodies.

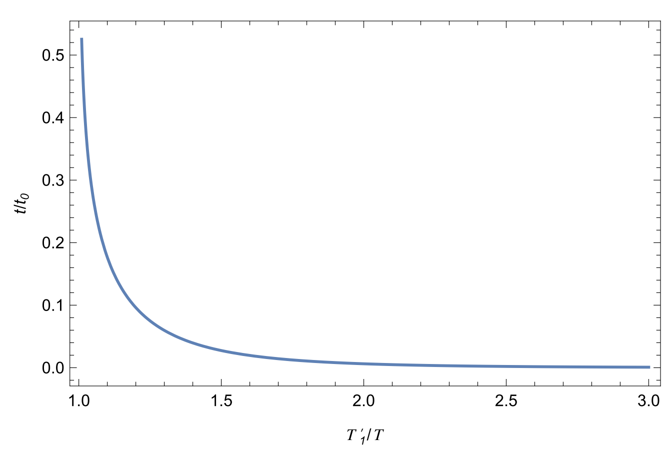

FIG. 8. Cooling time, in terms of the scale t_0, for the nanoparticle to cool from a high temperature to T_1', in the regime where Eq. (9.8) is valid. It takes an increasingly long time to get very close to the environmental temperature.

The difficulty is that the simple Lorenz-Lorentz model is ordinarily derived using spherical symmetry. The relation between the electric polarizability and the permittivity is, in the isotropic case, in HL units,

$$\alpha = \frac{\varepsilon - 1}{\varepsilon + 2} 4\pi a^3. \tag{10.1}$$

This is not valid for a nonsymmetric permittivity (see, for example, Ref. [34]). However, for ω_c small, the nonsymmetric nature is small, so, for the purpose of an estimate, we use, as in Refs. [25,26], the matrix generalization of the above:

$$\boldsymbol{\alpha} = (\boldsymbol{\varepsilon} - 1)(\boldsymbol{\varepsilon} + 2)^{-1} 4\pi a^3. \tag{10.2}$$

It is quite straightforward to compute the components of this matrix: The term we need for the torque in Eq. (4.13) is, for $\omega_p \gg \omega \sim T$,

$$\mathrm{Re}\,\alpha_{xy} \approx 54V \frac{\omega^2 \omega_c \eta}{\omega_p^4}. \tag{10.3}$$

When this is inserted into Eq. (4.13), we obtain

$$\tau_z = \frac{32}{7}\pi^4 V \frac{\omega_c \eta}{\omega_p^4} T^6 \left[1 - \left(\frac{T'}{T}\right)^6\right]. \tag{10.4}$$

Putting in the numbers for a 100-nm gold nanosphere, the coefficient of $1 - T'^6/T^6$ is about 5×10^{-36} N m, some 11 orders of magnitude smaller than that found at the end of Sec. IV.

We can also repeat the calculation of the terminal angular velocity in Sec. V in this Lorenz-Lorentz model. Then

$$\tau_1' = -\frac{1}{6\pi^2}\left(3 + \beta\frac{\partial}{\partial\beta}\right)\int_0^\infty d\omega\, \omega^2 \mathrm{Im}(\alpha_{xx} + \alpha_{yy})(\omega) \times \left(\frac{1}{e^{\beta'\omega} - 1} - \frac{1}{e^{\beta\omega} - 1}\right), \tag{10.5}$$

where in our model

$$\mathrm{Im}(\alpha_{xx} + \alpha_{yy}) \approx 18V \frac{\omega\eta}{\omega_p^2}, \tag{10.6}$$

which implies

$$\tau_1' = \frac{2\pi^2}{5}\frac{V\eta}{\omega_p^2} T^4 \left[1 + 3\left(\frac{T'}{T}\right)^4\right]. \tag{10.7}$$

Note that τ_1' is always positive, indicating that it always opposes the rotation. The corresponding terminal angular velocity is

$$\Omega_T = \frac{\tau_0}{\tau_1'} = \frac{80}{7}\pi^2 \frac{\omega_c}{\omega_p^2} T^2 \frac{1 - (T'/T)^6}{1 + 3(T'/T)^4}, \tag{10.8}$$

where the prefactor, independent of T', implies a substantial angular velocity for gold at room temperature: approximately 10^8 s^{-1}. Although the time required to reach such a velocity is very long, $t_0 = I/\tau_1' \sim 10^{13}$ s, the initial angular acceleration is not so small,

$$\dot{\Omega}(0) = \frac{\Omega_T}{t_0} \sim 10^{-5}\ \mathrm{s}^{-2}. \tag{10.9}$$

While this angular acceleration is 10 orders of magnitude smaller than that found without the Lorenz-Lorentz correction in Eq. (5.14), the body will acquire a measurable angular velocity after a relatively small period of observation.

However, is this correction valid or even necessary for a metal nanoparticle? The discussion in Secs. IV–VIII is based on describing the susceptibility of a metal by the phenomenological Drude model, which should include, approximately, all internal effects. There is a large amount of literature on the subject of ordinary polarizabilities of metal nanoparticles (see, for example, Refs. [35,36]), where it is seen that both classical and approximate quantum-mechanical treatments are inadequate. We are unaware of comparable work in the nonreciprocal case. So, to some extent, the issue of applying the Lorenz-Lorentz correction remains open. In this paper we are interested in the interaction between the electromagnetic field and the body, the electromagnetic properties of which are specified by a given susceptibility, so the crude models for the latter should only be taken as illustrative.

XI. CONCLUSION

In this paper we have concentrated on analysis to first order in the susceptibility, to better understand the effects of nonreciprocal materials on torque and on forces for bodies out of thermal equilibrium with their environment. Time-reversal symmetry is broken by these materials, so spontaneous forces and torques are possible. Of course, time-reversal symmetry is not broken by electrodynamics, whether classical or quantum; rather, the nonreciprocity is a consequence of an external agent, such as a magnetic field, that is encoded in the dielectric response of the materials.

Interestingly, potentially observable phenomena are nevertheless predicted. A nonreciprocal body out of thermal equilibrium will spontaneously start to rotate and reach a substantial terminal angular velocity. Such a body will not feel a net force to first order in the susceptibility. However, if the body is placed near a translationally invariant surface, even a perfect conductor, then a force parallel to the surface would arise. The presence of such a surface would tend to suppress the vacuum torque. A potentially observable terminal linear velocity arises here as well, although the timescales are such that it would be difficult to keep the system out of thermal equilibrium. A possible drastic reduction in the strength of these nonreciprocal effects, due to the Lorenz-Lorentz correction for dielectric susceptibilities, was discussed in the penultimate section, although it seems the angular and linear accelerations might still be amenable to observation.

We leave for future work the examination of higher-order effects, to see how phenomena such as vacuum self-propulsion can arise, even for a reciprocal body [30,31].

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APPENDIX A: EVALUATION AND EXPANSION OF INTEGRALS

Here we express the integrals encountered in the text in terms of the digamma and trigamma functions $\psi(z)$ and $\psi'(z)$, respectively, and provide corresponding low- and high-temperature expansions. Differentiation of Binet's second integral representation of the logarithmic γ function immediately yields the integral representation

$$\psi(z) = \ln z - \frac{1}{2z} - 2\int_0^\infty dx \frac{x}{x^2+1}\frac{1}{e^{2\pi zx}-1}. \tag{A1}$$

Thus,

$$\begin{aligned} I_1(\beta\eta) &\equiv \int_0^\infty dx \frac{x}{x^2+1}\frac{1}{e^{\beta\eta x}-1} \\ &= \frac{1}{2}\left[-\frac{\pi}{\beta\eta} + \ln\left(\frac{\beta\eta}{2\pi}\right) - \psi\left(\frac{\beta\eta}{2\pi}\right)\right]. \end{aligned} \tag{A2}$$

Since

$$\beta\frac{\partial}{\partial\beta}\frac{1}{e^{\beta\eta x}-1} = \eta\frac{\partial}{\partial\eta}\frac{1}{e^{\beta\eta x}-1} = x\frac{\partial}{\partial x}\frac{1}{e^{\beta\eta x}-1}, \tag{A3}$$

it follows that

$$\begin{aligned} \beta\frac{\partial}{\partial\beta}I_1(\beta\eta) &= \eta\frac{\partial}{\partial\eta}I_1(\beta\eta) = \int_0^\infty dx \frac{x^2}{x^2+1}\frac{\partial}{\partial x}\frac{1}{e^{\beta\eta x}-1} \\ &= 2\int_0^\infty dx \left(\frac{x^3}{(x^2+1)^2} - \frac{x}{x^2+1}\right)\frac{1}{e^{\beta\eta x}-1}. \end{aligned} \tag{A4}$$

Thus,

$$\begin{aligned} I_2(\beta\eta) &\equiv \int_0^\infty dx \frac{x^3}{(x^2+1)^2}\frac{1}{e^{\beta\eta x}-1} = \left(1 + \frac{\beta}{2}\frac{\partial}{\partial\beta}\right)I_1(\beta\eta) \\ &= \left(1 + \frac{\eta}{2}\frac{\partial}{\partial\eta}\right)I_1(\beta\eta). \end{aligned} \tag{A5}$$

Hence, from Eq. (A2),

$$\begin{aligned} I_2(\beta\eta) = \frac{1}{4}\bigg[&-\frac{\pi}{\beta\eta} + 2\ln\left(\frac{\beta\eta}{2\pi}\right) \\ &+ 1 - 2\psi\left(\frac{\beta\eta}{2\pi}\right) - \frac{\beta\eta}{2\pi}\psi'\left(\frac{\beta\eta}{2\pi}\right)\bigg]. \end{aligned} \tag{A6}$$

Using the series representation

$$\psi(z) = -\frac{1}{z} - \gamma_E + z\sum_{k=1}^\infty \frac{1}{k(z+k)}, \tag{A7}$$

where γ_E is the Euler-Mascheroni constant, we readily obtain from Eqs. (A2) and (A6) the small-$\beta\eta$ (or high-temperature) expansions

$$I_1(\beta\eta) \sim \frac{1}{2}\left[\frac{\pi}{\beta\eta} + \ln\left(\frac{\beta\eta}{2\pi}\right) + \gamma_E\right] \quad (\beta\eta \to 0) \tag{A8a}$$

and

$$I_2(\beta\eta) \sim \frac{1}{4}\left[\frac{\pi}{\beta\eta} + 2\ln\left(\frac{\beta\eta}{2\pi}\right) + 1 + 2\gamma_E\right] \quad (\beta\eta \to 0). \tag{A8b}$$

Likewise, using the asymptotic representation

$$\psi(z) \sim \ln z - \frac{1}{2z} - \sum_{k=1}^\infty \frac{B_{2k}}{2kz^{2k}} \quad (z \to \infty), \tag{A9}$$

there follow from Eqs. (A2) and (A6) the large $\beta\eta$ (or low-temperature) expansions

$$I_1(\beta\eta) \sim \frac{\pi^2}{6\beta^2\eta^2} - \frac{\pi^4}{15\beta^4\eta^4} \quad (\beta\eta \to \infty) \tag{A10a}$$

and

$$I_2(\beta\eta) \sim \frac{\pi^4}{15\beta^4\eta^4} \quad (\beta\eta \to \infty). \tag{A10b}$$

APPENDIX B: OUT-OF-EQUILIBRIUM FRICTIONAL FORCE NEAR PERFECTLY CONDUCTING PLATE

Following the discussion in Ref. [13], it is easy to derive the expression for the frictional force used in Sec. VIII B. The general expression for the force is

$$\begin{aligned} F = &\int \frac{d\omega}{2\pi}\frac{(d\mathbf{k}_\perp)}{(2\pi)^2}(k_x + \omega v)\,\mathrm{tr}\,\Im\boldsymbol{\alpha}(\omega)\Im\mathbf{g}'(\omega, \mathbf{k}_\perp) \\ &\times \left(\coth\frac{\beta\gamma(\omega + k_x v)}{2} - \coth\frac{\beta'\omega}{2}\right). \end{aligned} \tag{B1}$$

Here $\mathbf{g}'$ is the reduced Green's function in the rest frame of the particle. For a perfectly conducting plate, as with the vacuum, $\mathbf{g}' = \mathbf{g}$, the Green's function in the frame of the plate and vacuum. From this we can calculate both the frictional force and the propulsive force. The latter is present at $v = 0$, arises from the antisymmetric parts of the polarizability and the Green's dyadic, and is given in Sec. VIII. The diagonal parts of both these tensors correspond to friction. According to the model (3.2) with ω_c neglected, the diagonal terms of the imaginary part of the polarizability are all equal to

$$\mathrm{Im}\,\alpha_d = \frac{V\omega_p^2\omega\eta}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\eta^2}. \tag{B2}$$

That leaves us with the imaginary part of the trace of $\mathbf{g}$:

$$\begin{aligned} \mathrm{Im}\,\mathrm{tr}\,\mathbf{g} = \mathrm{sgn}(\omega)&\left(\frac{\omega^2}{\sqrt{\omega^2-k^2}} - \sqrt{\omega^2-k^2}\cos\left(2a\sqrt{\omega^2-k^2}\right)\right) \\ &\times \theta\left(\omega^2 - k^2\right). \end{aligned} \tag{B3}$$

Now when we expand Eq. (B1) to first order in v we obtain two terms

$$F = F^{(1)} + F^{(2)}. \tag{B4}$$

Here the first term comes from expanding the hyperbolic cotangent

$$F^{(1)} = -\frac{\beta v}{8\pi^2}\int_0^\infty d\omega\,\mathrm{Im}\,\alpha_d(\omega)\frac{\omega^5}{\sinh^2(\beta\omega/2)} \times\left(\frac{2}{3}-\frac{2}{u^5}\left[-u(u^2-12)\cos u+(5u^2-12)\sin u\right]\right), \tag{B5}$$

where $u = 2\omega a$ and we have carried out the elementary integration over $\mathbf{k}_\perp$. Note that the first term here corresponds to the usual Einstein-Hopf effect. The $F^{(2)}$ contribution to the force corresponds to the ωv prefactor in Eq. (B1) and is a nonequilibrium friction contribution

$$F^{(2)} = \frac{v}{2\pi^2}\int_0^\infty d\omega\,\mathrm{Im}\,\alpha_d(\omega)\,\omega^4\left(\coth\frac{\beta\omega}{2}-\coth\frac{\beta'\omega}{2}\right) \times\left(1-\frac{1}{u^3}\left[2u\cos u+(u^2-2)\sin u\right]\right), \tag{B6}$$

again after carrying out the wave-number integration. The sum of these two terms is Eq. (8.14) if we set $\omega_0 = 0$, appropriate for a metal nanoparticle.

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