Multispin Clifford codes for angular momentum errors in spin systems

Sivaprasad Omanakuttan [1,*] and Jonathan A. Gross [2,†]
[1] Center for Quantum Information and Control, Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico 87131, USA
[2] Google Quantum AI, Venice, California 90291, USA

(Received 1 May 2023; accepted 28 July 2023; published 25 August 2023)

The physical symmetries of a system play a central role in quantum error correction. In this work we encode a qubit in a collection of systems with angular momentum symmetry (spins), extending the tools developed by Gross [J. A. Gross, Phys. Rev. Lett. **127**, 010504 (2021)] for single large spins. By considering large spins present in atomic systems and focusing on their collective symmetric subspace, we develop codes with octahedral symmetry capable of correcting errors up to second order in angular momentum operators. These errors include the most physically relevant noise sources such as microwave control errors and optical pumping. We additionally explore qubit codes that exhibit distance scaling commensurate with the surface code while permitting transversal single-qubit Clifford operations.

DOI: 10.1103/PhysRevA.108.022424

I. INTRODUCTION

Quantum error correction (QEC) is an essential ingredient for implementing quantum computation reliably. Put simply, QEC uses a large Hilbert space to encode a smaller-dimensional system to overcome the detrimental effects of decoherence and recover the ideal state of an encoded system. One standard strategy for QEC, analogous to classical error correction, where the major error is the bit flip, is to encode a qubit of information in multiple qubits. However, due to the fact that for QEC one needs to account for both bit-flip and phase-flip errors, the number of physical qubits required to encode a logical qubit is very large. In spite of this difficulty, these techniques are widely considered for QEC and have found a great deal of success including recent experimental implementation using the surface codes and color codes [1–3].

Another approach for QEC is to encode a qubit in a single system with a large Hilbert space, for example, the standard Gottesman-Kitaev-Preskill (GKP) code where a qubit is encoded in a simple harmonic oscillator, whose large Hilbert space provides natural protection from many errors native to this system [4,5]. This approach in general reduces the overhead and thus makes the scaling easier. There have been many recent ideas about quantum computation using GKP states [6–9] and a recent experiment where real-time quantum error correction beyond the break-even point was demonstrated [10].

In [11], quantum error-correcting codes native to spin systems with spin larger than $\frac{1}{2}$ were developed using the special symmetries associated with these systems. In particular, the binary octahedral symmetry was used; however, one needs a very large spin ($j \geqslant \frac{13}{2}$) to build a fully error-correcting code for this symmetry. In this work we find a way to obviate this need for big spins by using the tensor product of multiple spins for spin larger than $j = \frac{1}{2}$ and using the irreducible SU(2) representations in the symmetric subspace of these tensor products. Since these codes exist in multiple spins and have transversal Clifford gates, we call them multispin Clifford codes. These systems could generally have great potential as they are easier to scale and systems with an order of 100 spins have been used for quantum simulation experiments with neutral atoms [12,13]. In spin systems, the main source of decoherence are random rotations, which contribute to the first-order errors in angular momentum, and optical pumping, which is a second-order effect in angular momentum involving vector and tensor light shifts [14,15]. Accordingly, designing codes in these composite spin systems that correct for first- and second-order angular momentum errors could reduce the overhead required to achieve fault-tolerant regimes of quantum computation and thus accelerate the path to useful quantum computation.

Similarly, we also consider the case of the tensor product of qubit systems. We encode a qubit in the symmetric subspace of multiple qubits to find codes that have transversal Clifford gates and correct arbitrarily large errors. Using the binary octahedral symmetry, we demonstrate explicit code words with distance 3 and distance 5 and generally find that the minimum number of qubits required for a given distance scales similarly to the surface code while allowing full single-qubit transversal Clifford operations.

The remainder of this article is organized as follows. In Sec. II we give a brief introduction to the binary octahedral code and the natural symmetry associated with these quantum error-correcting codes. In Sec. III we study the Knill-Laflamme condition for a general spin system by using the spherical tensor operators. In Sec. IV we find the relevant SU(2) irreducible representations in the symmetric subspace for the tensor product of spin systems by mapping the problem to bosons. We use these approaches to find useful codes that correct for first-order angular momentum [small random

*somanakuttan@unm.edu
†jarthurgross@google.com

SU(2)] errors in Sec. V and the second-order (light shift) errors in Sec. VI. In Sec. VII we study how one can apply these approaches to the tensor product of multiple spin $j = \frac{1}{2}$ (qubit) systems and create error-correcting codes in the symmetric subspace of this multipartite system, finding explicit codes with distances 3 and 5. We provide a summary and possible directions for future work in Sec. VIII.

II. INTRODUCTION TO BINARY OCTAHEDRAL CODE

We build upon work [11] done to encode information against random SU(2) rotations in large single spins [irreducible representations of SU(2)]. This task is simplified by restricting ourselves to codespaces that are preserved under the action of a finite subgroup of SU(2), such as the single-qubit Clifford group (binary octahedral group). If the finite subgroup is rich enough, the full set of Knill-Laflamme conditions for first-order rotation errors reduces to a single expectation value, which is simple to check. The single-qubit Clifford group is one such rich subgroup, in that it can map any of $\{J_x, J_y, J_z\}$ to any other, with either sign. These symmetries allow us to consolidate the conditions to

$$\langle i|J_z|j\rangle = C_{0z}\delta_{ij}, \tag{1}$$

$$\langle i|J_xJ_y|j\rangle = C_{xy}\delta_{ij}, \tag{2}$$

$$\langle i|J_z^2|j\rangle = C_{zz}\delta_{ij}. \tag{3}$$

The fact that a π rotation about J_z must put a relative phase between logical 0 and 1 means that we must have odd support on the J_z basis states and the other must have even support, which further reduces the conditions to

$$\langle 0|J_z|0\rangle = 0. \tag{4}$$

It turns out the binary tetrahedral group (a subgroup of the binary octahedral group) has enough symmetries for the above argument to go through as well, so we will also consider codes with that symmetry in this work. The binary octahedral group, having additionally the S gate, a $\pi/2$ rotation about J_z, further constrains the support of the code words in the J_z basis such that the J_z eigenvalues included in logical 0 are either $4\mathbf{Z} + \frac{1}{2}$ or $4\mathbf{Z} - \frac{3}{2}$, where $\mathbf{Z}$ indicates the set of all integers, depending on the code, and the eigenvalues for logical 1 are the negatives.

III. DERIVATION OF KNILL-LAFLAMME CONDITIONS

In this section we extend the Knill-Laflamme condition derived for small random SU(2) rotations in large single spins in [11] to general errors which are powers of angular momentum operators. Since products of angular momentum operators up to a given order are not linearly independent (due to equivalence relations such as the commutation relations), it can be convenient to use spherical tensors [16–18] as an error basis

$$T_q^k(j) = \sqrt{\frac{2k+1}{2j+1}}\sum_m \langle j, m+q|k, q; j, m\rangle |j, m+q\rangle\langle j, m|, \tag{5}$$

which are basically the sums of powers of the angular momentum operators and are related to spherical harmonics. Using this as our basis of errors, the Knill-Laflamme conditions [19] require that

$$\langle i|E_a^\dagger E_b|j\rangle = \delta_{ij}C_{ab}, \tag{6}$$

$$E_a, E_b \in \left\{T_q^k\right\}_{0\leqslant k\leqslant N} \tag{7}$$

if we want to be able to correct angular momentum errors of orders up to N. Because products of spherical tensors are sums of spherical tensors

$$T_q^k T_{q'}^{k'} = \sqrt{(2k+1)(2k'+1)}\sum_{\tilde{k}} c_{\tilde{q}}^{\tilde{k}} T_{\tilde{q}}^{\tilde{k}}, \tag{8}$$

where $\tilde{q} = q + q'$, the sum over $\tilde{k}$ is restricted over $|k - k'| \leqslant \tilde{k} \leqslant k + k'$, and $c_{\tilde{q}}^{\tilde{k}}$ is defined in terms of 6-j symbols and Clebsch-Gordon coefficients [17]

$$c_{\tilde{q}}^{\tilde{k}} = (-1)^{2j+\tilde{k}}\begin{Bmatrix} k & k' & \tilde{k} \\ j & j & j \end{Bmatrix} C_{k,q,k'q'}^{\tilde{k}\tilde{q}}, \tag{9}$$

we can equivalently consider the conditions

$$\langle j|T_{\tilde{q}}^{\tilde{k}}|k\rangle = \delta_{jk}C_{\tilde{q}}^{\tilde{k}}, \tag{10}$$

$$0 \leqslant \tilde{k} \leqslant 2N. \tag{11}$$

Consider the unitary $U_X = \exp(-i\pi J_x)$ the octahedral symmetry of the states gives us, an overall global phase that is irrelevant,

$$\begin{aligned} U_X|0\rangle &= |1\rangle, \\ U_X|1\rangle &= |0\rangle, \end{aligned} \tag{12}$$

and we can find that

$$U_X T_q^k U_X^\dagger = (-1)^k T_{-q}^k. \tag{13}$$

The details of this calculation are given in Appendix A. Using this, we see that for the code words

$$\langle 0|T_q^k|0\rangle = (-1)^k\langle 1|T_{-q}^k|1\rangle. \tag{14}$$

For the case of the code words with octahedral symmetry, the code words are real in the angular momentum basis (see Appendix B) and so is T_q^k; thus when we have two states $|\psi\rangle$ and $|\phi\rangle$ which are real linear combinations of the code words that respect the binary octahedral symmetry,

$$\langle\psi|T_{-q}^k|\phi\rangle = (-1)^q\langle\phi|T_q^k|\psi\rangle, \tag{15}$$

which we prove in Appendix B. Thus we get

$$\langle 0|T_q^k|0\rangle = (-1)^k\langle 1|T_{-q}^k|1\rangle = (-1)^{k-q}\langle 1|T_q^k|1\rangle, \tag{16}$$

and from this equation the error condition is trivially satisfied unless

$$(k - q)\text{mod } 2 = 1. \tag{17}$$

However, the code words have support on the J_z eigenstates that are separated by $q \bmod 4 = 0$, as described in Sec. II and given in Fig. 1, and hence the expression is identically zero unless q is even and thus the only diagonal conditions we need to check are those when k is odd: $\{T_0^1, T_0^3, T_0^5, T_4^5, \ldots\}$.

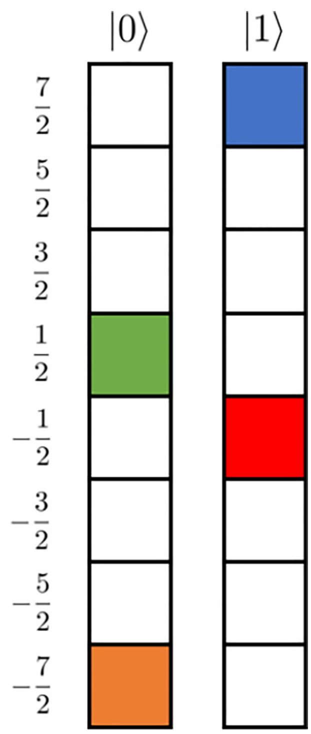

FIG. 1. Code words $|0\rangle$ and $|1\rangle$ for the ϱ_4 irreducible representation of the binary octahedral symmetry for $j = \frac{7}{2}$ in the angular momentum basis. The colored boxes indicate the states occupied whereas the blank ones indicate those states are not occupied for the code word. The states in the code word are spaced by four units of angular momentum $m_z = \pm 4$, a standard property of the octahedral symmetry, and contribute to the error-correction condition. The code words $|0\rangle$ and $|1\rangle$ are separated a single unit of angular momentum and hence overlap at $\langle 0|T_1^k|1\rangle = (-1)^k\langle 1|T_{-1}^k|0\rangle \neq 0$ for odd values of k and at $\langle 0|T_{-1}^k|1\rangle = (-1)^k\langle 1|T_1^k|0\rangle = 0$ for even values of k. This contributes to the off-diagonal terms to consider for error correction in Eq. (21).

Now thinking about the next error-correction condition, we get

$$\begin{aligned}\langle 0|T_q^k|1\rangle &= (-1)^k\langle 1|T_{-q}^k|0\rangle \\ &= (-1)^{k-q}\langle 0|T_q^k|1\rangle.\end{aligned} \tag{18}$$

Equation (17) states that when $(k-q)\text{mod }2 = 1$ we automatically get

$$\langle 0|T_q^k|1\rangle = 0. \tag{19}$$

Now again the support of the different code words is separated by odd shifts in angular momentum and hence we also automatically get that

$$\langle 0|T_q^k|1\rangle = 0 \tag{20}$$

when $q\,\text{mod}4 = 1$, which can be seen from Fig. 1. Thus the only off-diagonal conditions we need to check are when both k and q are odd: $\{T_1^1, T_1^3, T_{-3}^3, T_5^5, T_1^5, T_{-3}^5, \ldots\}$. Hence the error-correction conditions can be written as

$$\begin{aligned}&\langle 0|T_q^{(k)}|0\rangle = (-1)^{(k-q)}\langle 1|T_q^{(k)}|1\rangle \\ &\quad\Rightarrow \text{consider only } (k \in \text{odd and } q \equiv 0\,\text{mod}4), \\ &\langle 0|T_q^{(k)}|1\rangle = (-1)^{(k-q)}\langle 0|T_q^{(k)}|1\rangle \\ &\quad\Rightarrow \text{consider only } (k \in \text{odd and } q \equiv 1\,\text{mod}\,4).\end{aligned} \tag{21}$$

This gives the general error-correction conditions we need to check for the binary octahedral codes. We can easily see that a large number of conditions are trivially satisfied accounting for the symmetry of the code words. In the following sections we will see how these correction conditions will help us obtain useful quantum error-correction codes.

IV. THE SU(2) IRREDUCIBLE REPRESENTATIONS IN THE SYMMETRIC SUBSPACE OF THE TENSOR PRODUCT OF *n* SPIN-*j* SYSTEMS

Now consider the tensor product of n spin-j systems. This forms a Hilbert space $\mathcal{H}$ of dimension d^n where $d = 2j+1$. We focus on the symmetric subspace [20] where expectation values are unchanged by permuting the subsystems, so for any arbitrary operators $A_1, A_2, \ldots, A_n$ we have

$$\langle A_1 \otimes A_2 \otimes \cdots \otimes A_n\rangle = \langle A_{\pi(1)} \otimes A_{\pi(2)} \otimes \cdots \otimes A_{\pi(n)}\rangle \tag{22}$$

for any permutation π. Restricting our attention to the symmetric subspace simplifies the Knill-Laflamme conditions, as many of the error terms $E_a^\dagger E_b$ that arise are permutations of each other and need only be verified once within the symmetric subspace. The dimension of the symmetric subspace for the tensor product of n spin-j systems is

$$\dim[S_n(d)] = \frac{d(d+1)\cdots(d+n-1)}{n!}. \tag{23}$$

Since we are interested in encoding qubits in the symmetric subspace, we need to identify how the symmetric subspace decomposes into SU(2) irreducible representations. For $j = \frac{1}{2}$ the decomposition is simple, as the symmetric subspace is itself a spin-$(n+1)/2$ irreducible representation. For larger spins, we must work harder, as the symmetric subspace decomposes into multiple SU(2) irreducible representations.

One way to see that we must get multiple SU(2) irreducible representations in the symmetric subspace is to notice that the operator J_z gains some degeneracies for $j > \frac{1}{2}$. For example, $|+1,-1\rangle + |-1,+1\rangle$ and $|0,0\rangle$ are both symmetric states that are also eigenstates of J_z with eigenvalue $m_z = 0$. Since J_z is nondegenerate within any SU(2) irreducible representation, this means the symmetric subspace of two spin-1 systems must decompose into multiple SU(2) irreducible representations.

A useful perspective on the decomposition is to consider the symmetric subspace as n bosonic modes with at most $2j$ bosons in each mode [21]. Each mode is associated with one of the spins, and the number of bosons in a mode corresponds to the J_z eigenvalue of the associated spin (adding j to the eigenvalue so the number of bosons ranges from 0 to $2j$). The total J_z eigenvalue is then given by the total number of bosons, and the degeneracy of that eigenvalue in the symmetric subspace is given by the number of partitions of those bosons into n distinct modes, restricted to putting no more than $2j$ bosons in a single mode. These can be counted using restricted Young diagrams, where the number of columns must not exceed $2j$

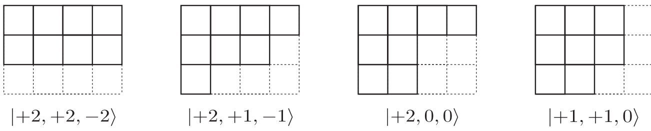

FIG. 2. Restricted Young diagram showing a basis for the three-dimensional subspace of the totally symmetric subspace of three spin-2 systems for which $J_z = 2$. The associated states are obtained by converting the number of boxes in each row to a J_z eigenvalue by subtracting $j = -2$. Once symmetrized over the three subsystems, these states form a basis for the $J_z = 2$ symmetric subspace.

and the number of rows must not exceed n. An example of such restricted Young diagrams and their associated states is given in Fig. 2.

For example, consider the symmetric subspace of two spin-$\frac{1}{2}$ particles, where the symmetric subspace is spanned by the triplet states and has a total spin $J = 1$ (the largest possible angular momentum under the tensor product). Mapping this to the two bosonic modes with at most $2j = 1$ boson each, we enumerate all partitions of N bosons among these modes for $N \in \{0, 1, 2\}$. The possible partitions are given in Table I. Each total photon number N corresponds only to a single restricted partition, consistent with our previous statement that the symmetric subspace is a single SU(2) irreducible representation.

As a first nontrivial example consider the case of spin $j = 1$ and $n = 2$. The restricted partitions of bosons into two modes are given in Table II. As we can see from the table, there are two partitions of $N = 2$ bosons into two modes, revealing a degeneracy of the J_z operator for eigenvalue $m_z = 0$. Since a one-dimensional subspace of this degenerate subspace must belong to the spin-2 irreducible representation and there are no degeneracies for larger m_z, we see that the symmetric subspace decomposes into one copy of spin 2 and one copy of spin 0.

Using this same approach, we can numerically find that for the case of the tensor product of any two spins j,

$$j \otimes j \overset{\text{SS}}{=} 2j \oplus (2j-2) \oplus (2j-4) \oplus \cdots , \tag{24}$$

where SS denotes symmetric subspace. Simple counting of the total dimensions verifies this and is given in detail in Appendix C.

TABLE I. Symmetric subspace of two spin $j = \frac{1}{2}$ systems for $n = 2$ as two bosonic modes. Here n_1 and n_2 are the numbers of bosons in each of the modes (symmetrized combinations as they are bosons) and $N = n_1 + n_2$. There is only one possible partition for each of the values of N and accordingly there exists only a single SU(2) irreducible representation in the symmetric subspace. (Note that our restriction on the number of bosons allowed per mode disallows the partition of 2 into 2,0.)

N	n_1	n_2
0	0	0
1	1	0
2	1	1

TABLE II. Symmetric subspace of $n = 2$ spin $j = 1$ systems. We find we need two columns to account for the distinct partitions of $N = 2$ bosons. Filling in the columns from left to right for each N, we can identify the SU(2) irreducible representations present by the number of occupied rows in each column. Here the first column has five occupied rows, corresponding to the five-dimensional spin-2 irreducible representation, and the second column has one occupied entry, corresponding to the spin-0 irreducible representation. The particular partition of N appearing in each column here has no special meaning, as the actual basis states of the irreducible representations are generally superpositions of these partitions.

N	n_1	n_2	n_1	n_2
0	0	0		
1	1	0		
2	1	1	2	0
3	2	1		
4	2	2		

Similarly, we can use the same approach for more complex cases. For example, consider the case of $n = 3$ and $j = 1$; the possible restricted partitions are given in Table III. As we can see from the table, we have two occupied columns with $d = 7$ and $d = 3$, which yields the two SU(2) irreducible representations for spin 3 and spin 1.

Since the specific symmetries we are interested in are present only for half-integer spins [11], the tensor product of two spins will not give us valid codespaces as it produces only integer spins. Hence the first nontrivial cases of interest are three copies of a half-integer spin. The decompositions into SU(2) irreducible representations for the cases of $j = \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$, and $\frac{9}{2}$ are given by

$$\begin{aligned}
&\frac{3}{2} \otimes \frac{3}{2} \otimes \frac{3}{2} \overset{\text{SS}}{=} \frac{9}{2} \oplus \frac{5}{2} \oplus \frac{3}{2},\\
&\frac{5}{2} \otimes \frac{5}{2} \otimes \frac{5}{2} \overset{\text{SS}}{=} \frac{15}{2} \oplus \frac{11}{2} \oplus \frac{9}{2} \oplus \frac{7}{2} \oplus \frac{5}{2} \oplus \frac{3}{2},\\
&\frac{7}{2} \otimes \frac{7}{2} \otimes \frac{7}{2} \overset{\text{SS}}{=} \frac{21}{2} \oplus \frac{17}{2} \oplus \frac{15}{2} \oplus \frac{13}{2} \oplus \frac{11}{2} \oplus \frac{9}{2}^{(2)}\\
&\quad \oplus \frac{7}{2} \oplus \frac{5}{2} \oplus \frac{3}{2},
\end{aligned} \tag{25}$$

TABLE III. Symmetric subspace of $n = 3$ spin $j = 1$ systems. Three values of N have multiple partitions, resulting in the second column having three occupied rows and giving us a decomposition of the symmetric subspace into one copy of spin 3 and one copy of spin 1.

N	n_1	n_2	n_3	n_1	n_2	n_3
0	0	0	0			
1	1	0	0			
2	1	1	0	2	0	0
3	1	1	1	2	1	0
4	2	1	1	2	2	0
5	2	2	1			
6	2	2	2			

$$\frac{9}{2}\otimes\frac{9}{2}\otimes\frac{9}{2}\overset{\mathrm{SS}}{=}\frac{27}{2}\oplus\frac{23}{2}\oplus\frac{21}{2}\oplus\frac{19}{2}\oplus\frac{17}{2}\oplus\frac{15}{2}^{(2)}\oplus\frac{13}{2}\oplus\frac{11}{2}^{(2)}\oplus\frac{9}{2}^{(2)}\oplus\frac{7}{2}\oplus\frac{3}{2},$$

where the superscript to the spins represents the multiplicity.

V. CORRECTING SMALL RANDOM SU(2) ERRORS

In the case of errors that are small random SU(2) rotations, the error operators to first order in the rotation angle will be linear in the angular momentum operators $\{J_x, J_y, J_z\}$ or equivalently first-rank tensor operators $T_q^{(1)}$ with $-1 \leqslant q \leqslant 1$. Tensor products of these errors to first order are permutations of

$$\mathcal{E} = A \otimes 1 \otimes 1, \tag{26}$$

where $A \in \{J_x, J_y, J_z\}$ or $T_q^{(1)}$. Thus the Knill-Laflamme conditions we need to check are

$$\begin{aligned}&\langle i|T_q^1 \otimes T_{q'}^1 \otimes \mathbb{1}|j\rangle,\\ &\langle i|T_q^1 T_{q'}^1 \otimes \mathbb{1} \otimes \mathbb{1}|j\rangle,\\ &\langle i|T_q^1 \otimes \mathbb{1} \otimes \mathbb{1}|j\rangle,\end{aligned} \tag{27}$$

where $i, j = \{0, 1\}$ and $-1 \leqslant q, q' \leqslant 1$. However, using the unitary operator

$$U_X^{\mathrm{tot}} = \otimes_i U_X, \tag{28}$$

where $U_X = \exp(-i\pi J_x)$, for two states $|\psi\rangle$ and $|\phi\rangle$ which are real linear combinations of the states that respect the binary octahedral symmetry, we get

$$\langle\psi| \otimes_i T_{-q_i}^{k_i}|\phi\rangle = (-1)^{\sum_i q_i}\langle\phi| \otimes_i T_{q_i}^{k_i}|\psi\rangle, \tag{29}$$

which leaves us with the error-correction conditions

$$\begin{aligned}&\langle 0| \otimes_i T_{q_i}^{k_i}|0\rangle = (-1)^{\sum_i k_i - \sum_i q_i}\langle 1| \otimes_i T_{q_i}^{k_i}|1\rangle \Rightarrow \text{consider only}\\ &\qquad\times\left(\sum_i k_i \in \text{odd and } \sum_i q_i \equiv 0 \,\mathrm{mod} 4\right),\\ &\langle 0| \otimes_i T_{q_i}^{k_i}|1\rangle = (-1)^{\sum_i k_i - \sum_i q_i}\langle 0| \otimes_i T_{q_i}^{k_i}|1\rangle \Rightarrow \text{consider only}\\ &\qquad\times\left(\sum_i k_i \in \text{odd and } \sum_i q_i \equiv 1 \,\mathrm{mod} 4\right).\end{aligned} \tag{30}$$

Here we used the fact that the tensor product of spherical tensors shifts the total angular momentum by the sum of the individual shifts

$$\begin{aligned}&\otimes_i T_{q_i}^{k_i}|j_z = m_1, j_z = m_2, \ldots, j_z = m_N\rangle\\ &\propto |j_z = m_1 + q_1, j_z = m_2 + q_2, \ldots, j_z = m_N + q_N\rangle,\end{aligned} \tag{31}$$

and hence the spacing arguments we used to get the mod4 are still valid for a code respecting the binary octahedral group.

Returning our attention to the case of the Knill-Laflamme conditions for the first-order errors in the angular momentum operators in Eq. (27), the condition $\langle i|T_q^1 \otimes T_{q'}^1 \otimes \mathbb{1}|j\rangle$ is trivially satisfied when $\sum_i k_i$ is even. Now using the fact that when we multiply two spherical tensors of ranks k_1 and k_2, the decomposition consists of all the spherical tensors with rank k, where $|k_1 - k_2| \leqslant k \leqslant k_1 + k_2$; the condition $\langle i|T_q^1 T_{q'}^1 \otimes \mathbb{1} \otimes \mathbb{1}|j\rangle$ leaves us with spherical tensors with ranks 0, 1, and 2. However, from Eq. (27) the rank 0 and 2 cases are trivially satisfied, and hence the only term to check is $\langle i|T_q^1 \otimes 1 \otimes 1|j\rangle$. We recall that when correcting for total angular momentum errors on binary octahedral codes, it is sufficient to check

$$\langle 0|J_{z,\mathrm{total}}|0\rangle = 0. \tag{32}$$

Since we are considering codes in the symmetric subspace, we have

$$\begin{aligned}\tfrac{1}{3}\langle 0|J_{z,\mathrm{total}}|0\rangle &= \langle 0|J_z \otimes \mathbb{1} \otimes \mathbb{1}|0\rangle &(33)\\ &= \langle 0|\mathbb{1} \otimes J_z \otimes \mathbb{1}|0\rangle &(34)\\ &= \langle 0|\mathbb{1} \otimes \mathbb{1} \otimes J_z|0\rangle, &(35)\end{aligned}$$

so correcting first-order single-system angular momentum errors in a binary octahedral code is equivalent to correcting first-order global angular momentum errors.

A. Case of three $j = \frac{3}{2}$ systems

According to Eq. (25), the symmetric subspace of three spin-$\frac{3}{2}$ systems decomposes into three SU(2) irreducible representations. Faithful two-dimensional binary octahedral irreducible representations are present in both the $j = \frac{9}{2}$ and the $j = \frac{5}{2}$ SU(2) irreducible representations. However, these irreducible representations are incompatible with each other. Using the ϱ_i notation of [11] to designate irreducible representations of the binary octahedral group, $j = \frac{9}{2}$ has a single copy of ϱ_4 while $j = \frac{5}{2}$ has a single copy of ϱ_5. While this prevents us from engineering a code with binary octahedral symmetry, we obtain more freedom by relaxing to binary tetrahedral symmetry [11].

For the binary tetrahedral symmetry, the error condition becomes

$$\begin{aligned}&\langle 0| \otimes_i T_{q_i}^{k_i}|0\rangle = (-1)^{\sum_i k_i - \sum_i q_i}\langle 1| \otimes_i T_{q_i}^{k_i}|1\rangle \Rightarrow \text{consider only}\\ &\qquad\times\left(\sum_i k_i \in \text{odd and } \sum_i q_i \equiv 0 \bmod 2\right),\\ &\langle 0| \otimes_i T_{q_i}^{K_i}|1\rangle = (-1)^{\sum_i k_i - \sum_i q_i}\langle 0| \otimes_i T_{q_i}^{k_i}|1\rangle \Rightarrow \text{consider only}\\ &\qquad\times\left(\sum_i k_i \in \text{odd and } \sum_i q_i \equiv 1 \bmod 2\right).\end{aligned} \tag{36}$$

The factor of mod 2 appears as the spacing of the binary tetrahedral code words is 2 instead of the 4 for the binary octahedral code words. However, for the case of first-order errors in the angular momentum, the only nontrivial condition we need to satisfy is $\langle i|T_q^1 \otimes 1 \otimes 1|j\rangle$.

Making this relaxation, we find that $j = \frac{9}{2}$ and $j = \frac{5}{2}$ each have a copy of the faithful two-dimensional binary tetrahedral irreducible representation ϱ_4 (again in the notation of the Appendix of [11]). The expectation values of J_z for the logical 0′s of these two irreducible representations have opposite signs, so we engineer a combined code word with vanishing

J_z expectation value to satisfy the error-correction conditions

$$|0\rangle = \frac{1}{\sqrt{16}}(\sqrt{5}|0\rangle_{9/2} + \sqrt{11}|0\rangle_{5/2}), \tag{37}$$

where

$$|0\rangle_{9/2} = \frac{\sqrt{6}}{4}\left|\frac{9}{2},\frac{9}{2}\right\rangle + \frac{\sqrt{21}}{6}\left|\frac{9}{2},\frac{1}{2}\right\rangle + \frac{\sqrt{6}}{12}\left|\frac{9}{2},\frac{-7}{2}\right\rangle,$$
$$|0\rangle_{5/2} = -\frac{\sqrt{6}}{6}\left|\frac{5}{2},\frac{5}{2}\right\rangle + \frac{\sqrt{30}}{6}\left|\frac{5}{2},\frac{-3}{2}\right\rangle. \tag{38}$$

The projectors onto the irreducible representations in $j = \frac{9}{2}$ and $j = \frac{5}{2}$ can be constructed from the character for ϱ_4 along with the representatives for the binary tetrahedral group elements provided by the SU(2) irreducible representations as discussed in [11].

B. Case of three $j = \frac{5}{2}$ systems

Next consider the case of three spin-$\frac{5}{2}$ systems whose symmetric-subspace decomposition is also given in Eq. (25). Again we are looking for multiple copies of one of the faithful two-dimensional irreducible representations of the binary octahedral group. For this case, we have multiple options and for simplicity we choose the irreducible representation ϱ_4 appearing in $j = \frac{9}{2}$ and $j = \frac{11}{2}$. The corresponding logical zero states are

$$|0\rangle_{11/2} = \frac{\sqrt{21}}{12}\left|\frac{11}{2};\frac{9}{2}\right\rangle - \frac{\sqrt{2}}{4}\left|\frac{11}{2};\frac{1}{2}\right\rangle + \frac{\sqrt{105}}{12}\left|\frac{11}{2};\frac{-7}{2}\right\rangle,$$
$$|0\rangle_{9/2} = \frac{\sqrt{6}}{4}\left|\frac{9}{2};\frac{9}{2}\right\rangle + \frac{\sqrt{21}}{6}\left|\frac{9}{2};\frac{1}{2}\right\rangle + \frac{\sqrt{6}}{12}\left|\frac{7}{2};\frac{1}{2}\right\rangle. \tag{39}$$

These code words have equal and opposite expectation values

$$\langle 0|J_z \otimes \mathbb{1} \otimes \mathbb{1}|0\rangle_{11/2} = -\tfrac{11}{18},$$
$$\langle 0|J_z \otimes \mathbb{1} \otimes \mathbb{1}|0\rangle_{9/2} = \tfrac{11}{18}, \tag{40}$$

meaning we get a code word that corrects for first-order errors by simply taking a uniform superposition:

$$|0\rangle_L = \frac{1}{\sqrt{2}}(|0\rangle_{11/2} + |0\rangle_{9/2}). \tag{41}$$

VI. CORRECTING OPTICAL PUMPING

In the case of the error that is similar to optical pumping [14], the error operators are of the form $J_i^l J_j^m$, where $\{i, j = x, y, z\}$ and $l + m \leqslant 2$. However, we find it convenient again to express these errors in terms of the spherical tensors $\{T_q^k; -k \leqslant q \leqslant k\}$ as they form an orthogonal basis for errors and can be written in terms of angular momentum operators as given in Appendix A. Errors of this type acting on a single spin are permutations of

$$\mathcal{E} = A \otimes \mathbb{1} \otimes \mathbb{1}, \tag{42}$$

where $A \in \{T_q^k; 1 \leqslant k \leqslant 2, -k \leqslant q \leqslant k\}$. We see the Knill-Laflamme conditions in Eq. (30) are trivially satisfied except the ones given in Table IV. The errors with total $\sum k \bmod 2 = 0$ are trivially satisfied by Eq. (30).

TABLE IV. Relevant errors we need to satisfy for the error correction up to the second order for the tensor product of three spins. The table is constructed using Eq. (21) and the tensor product structure.

Diagonal errors	Off-diagonal errors
$\langle 0\vert T_0^1 \otimes \mathbb{1} \otimes \mathbb{1}\vert 0\rangle_L$	$\langle 0\vert T_1^1 \otimes \mathbb{1} \otimes \mathbb{1}\vert 1\rangle_L$
$\langle 0\vert T_0^2 T_0^1 \otimes \mathbb{1} \otimes \mathbb{1}\vert 0\rangle_L$	$\langle 0\vert T_{-1}^1 \otimes T_2^2 \otimes \mathbb{1}\vert 1\rangle_L$
$\langle 0\vert T_{-1}^1 \otimes T_1^2 \otimes \mathbb{1}\vert 0\rangle_L$	$\langle 0\vert T_1^1 \otimes T_0^2 \otimes \mathbb{1}\vert 1\rangle_L$
$\langle 0\vert T_1^1 \otimes T_{-1}^2 \otimes \mathbb{1}\vert 0\rangle_L$	$\langle 0\vert T_0^1 \otimes T_1^2 \otimes \mathbb{1}\vert 1\rangle_L$
$\langle 0\vert T_0^1 \otimes T_0^2 \otimes \mathbb{1}\vert 0\rangle_L$	$\langle 0\vert T_{-1}^1 \otimes T_{-2}^2 \otimes \mathbb{1}\vert 1\rangle_L$
	$\langle 0\vert T_{-1}^1 T_{-2}^2 \otimes \mathbb{1} \otimes \mathbb{1}\vert 1\rangle_L$

In our numerical simulations, we observed that we need to satisfy either the diagonal or the off-diagonal condition for the codes respecting the binary octahedral symmetry. Thus if we find a code satisfying the diagonal conditions, the off-diagonal conditions will be trivially satisfied and vice versa, which is also true for the error operators that are linear in the angular momentum operators. Unlike the case of linear angular momentum errors, finding the code word analytically is hard and we need to rely on numerical methods to find the code words; the method is described in detail in Appendix D. Also, as we are interested in the local rather than global errors we need to transform the basis from $|j_{\text{tot}}, j_z^{\text{tot}}\rangle \to |j_1, m_1; j_2, m_2; j_3, m_3\rangle$ using the Clebsch-Gordan coefficients, where $\{j_i, m_i\}$ refers to the angular momentum basis of the individual spins.

From Eq. (25) there are multiple SU(2) irreducible representations within the symmetric subspace of the threefold tensor product of spin-j systems. Decomposing these further into binary octahedral irreducible representations gives us high multiplicities for the two faithful two-dimensional irreducible representations and therefore many degrees of freedom with which to satisfy the error-correction conditions. For example, consider the case of spin $j = \frac{7}{2}$. A possible code word obtained numerically for the ϱ_4 irreducible representation [11] is

$$|0\rangle \propto \sqrt{\frac{70}{849}}|0\rangle_{21/2} + \sqrt{\frac{1}{4468}}|0\rangle_{17/2}^1 + \sqrt{\frac{338}{1251}}|0\rangle_{17/2}^2 + \sqrt{\frac{112}{479}}|0\rangle_{15/2} + \sqrt{\frac{515}{1246}}|0\rangle_{13/2}, \tag{43}$$

where $|0\rangle_{17/2}^1$ and $|0\rangle_{17/2}^2$ are orthogonal choices for $|0\rangle$ within the multiplicity-2 ϱ_4 irreducible representation of the binary octahedral representation derived from $j = \frac{17}{2}$, where the degeneracy is broken by diagonalizing J_z in the subspace spanned by the logical $|0\rangle'$s.

Similarly for the case of $j = \frac{9}{2}$, we can use the SU(2) irreducible representations given in Eq. (25) and find a code numerically as

$$|0\rangle \propto -\sqrt{\frac{2}{439}}|0\rangle_{27/2}^1 + \sqrt{\frac{55}{739}}|0\rangle_{27/2}^2 - \sqrt{\frac{216}{349}}|0\rangle_{23/2}^1 + \sqrt{\frac{133}{1090}}|0\rangle_{23/2}^2 - \sqrt{\frac{237}{1316}}|0\rangle_{21/2}, \tag{44}$$

where again we have used the ϱ_4 irreducible representation, the superscripts in the code word represent the multiplicities for $j = \frac{27}{2}$ and $j = \frac{23}{2}$, and degeneracy is broken by diagonalizing J_z in the subspace spanned by the logical $|0\rangle$'s.

Thus using the tensor-product structure of a minimum of three spins with individual spins $j > \frac{1}{2}$, we can encode a qubit correcting the most significant error in these physical platforms, which are rotation errors and optical pumping. This in turn provides an alternate approach for error correction with very low overhead, the number of physical systems to encode a logical qubit, by caring about the most significant error mechanisms.

VII. CORRECTING MULTIBODY ERRORS WITH SPIN $j = \frac{1}{2}$

Now we turn our attention to the case of the N-fold tensor product of $j = \frac{1}{2}$. Here the only irreducible representation in the symmetric subspace is spin $N/2$. Hence we shift away from the paradigm of local (one-body) first- and second-order angular momentum errors and consider nonlocal (multibody) errors in this section. For this case, we can work with collective spin operators

$$J_k = \frac{1}{2}\sum_{i=1}^{N}\sigma_{k,i}, \tag{45}$$

where $\sigma_{k,i}$ is the Pauli matrix acting on the ith location and $k \in \{x, y, z\}$.

Using the property of the symmetric subspace in Eq. (22), we get

$$\langle J_k\rangle = \frac{N}{2}\langle\sigma_{k,1}\rangle = \frac{N}{2}\langle\sigma_{k,2}\rangle = \cdots = \frac{N}{2}\langle\sigma_{k,N}\rangle. \tag{46}$$

Thus making the expectation value of the collective spin operator vanish makes all the local expectation values vanish, which is the condition we studied for small random SU(2) errors in Sec. V.

Now looking for codes for the qubit with the capacity to correct individual qubit errors, we can think of the same in terms of the collective spin operators. For example, consider the case of the code corrects for all single-body Pauli errors, i.e., a code with distance 3. The Knill-Laflamme conditions we need to consider are $\langle i|\sigma_{k,p}|j\rangle$ and $\langle i|\sigma_{k,p}\sigma_{l,p'}|j\rangle$, where we use the fact that $(\sigma_{k,i})^2 = \mathbb{1}$; $p, p' = \{1, 2, \dots, N\}$; and $k, l = \{x, y, z\}$. However, if we restrict ourselves to the case of the codes respecting the binary octahedral symmetry and using the error-correction condition derived in Eq. (30) where we have all the operators with rank $k_i = 1$, the only conditions remaining to check are

$$\langle i|\sigma_{k,p}|j\rangle = \frac{2}{N}\langle i|J_k|j\rangle. \tag{47}$$

However, for the binary octahedral symmetry for the collective spin operators the only condition we need to satisfy is [11] $\langle 0|J_z|0\rangle$. For example, we can think of a code with parameter $[[n, k, d]] = [13, 1, 3]]$ in the ϱ_5 irreducible representation for the octahedral symmetry and the code word is

$$|0\rangle = \frac{\sqrt{105}}{14}|0\rangle_0 + \frac{\sqrt{91}}{14}|0\rangle_1, \tag{48}$$

where the states in the basis $|J, J_z\rangle$ are

$$\begin{aligned}|0\rangle_0 = {}& \frac{\sqrt{910}}{56}\left|\frac{13}{2},\frac{13}{2}\right\rangle - \frac{3\sqrt{154}}{56}\left|\frac{13}{2},\frac{5}{2}\right\rangle \\ & - \frac{\sqrt{770}}{56}\left|\frac{13}{2},-\frac{3}{2}\right\rangle + \frac{\sqrt{70}}{56}\left|\frac{13}{2},-\frac{11}{2}\right\rangle, \\ |0\rangle_1 = {}& \frac{\sqrt{231}}{84}\left|\frac{13}{2},\frac{13}{2}\right\rangle - \frac{3\sqrt{1365}}{84}\left|\frac{13}{2},\frac{5}{2}\right\rangle \\ & - \frac{\sqrt{273}}{84}\left|\frac{13}{2},-\frac{3}{2}\right\rangle + \frac{\sqrt{3003}}{84}\left|\frac{13}{2},-\frac{11}{2}\right\rangle.\end{aligned} \tag{49}$$

Next we can consider the case of the error-correcting code that corrects two Pauli errors, otherwise known as a distance-5 code. We start by considering correcting global angular momentum errors up to the second order. The octahedral symmetry of the codes reduces the Knill-Laflamme conditions (21) we need to satisfy to

$$\langle i|J_z|j\rangle = C_z\delta_{ij}, \tag{50}$$

$$\langle i|J_z^3|j\rangle = C_{zz}\delta_{ij}, \tag{51}$$

$$\langle i|J_zJ_x^2|j\rangle = C_{xz}\delta_{ij}, \tag{52}$$

$$\langle i|J_xJ_yJ_z|j\rangle = C_{xyz}\delta_{ij}, \tag{53}$$

where $i, j = \{0, 1\}$. Now, as we have seen in Sec. II, the condition $\langle i|J_z|j\rangle$ is equivalent to just satisfying $\langle 0|J_z|0\rangle = 0$. Again invoking the support structure of octahedral codes in Sec. II and the operator U_X defined in Eq. (12) yields

$$\begin{aligned}\langle 0|J_z^3|1\rangle &= \langle 1|J_z^3|0\rangle = 0, \\ \langle 0|J_z^3|0\rangle &= -\langle 1|J_z^3|1\rangle.\end{aligned} \tag{54}$$

Thus the condition needed to satisfy Eq. (51) reduces to $\langle 0|J_z^3|0\rangle = 0$.

Now using the fact that $J_\pm = J_x \pm iJ_y$, we get

$$J_x^2 = \tfrac{1}{4}\big[J_+^2 + J_-^2 + 2j(j+1)\mathbb{1} + 2J_z^2\big], \tag{55}$$

and therefore $J_zJ_x^2 = \frac{1}{4}[J_zJ_+^2 + J_zJ_-^2 + 2j(j+1)\mathbb{1} + J_z^3]$. Again invoking the support property of the binary octahedral symmetry yields

$$\begin{aligned}\langle 0|J_zJ_\pm^2|1\rangle &= \langle 1|J_zJ_\pm^2|0\rangle = 0, \\ \langle 0|J_zJ_\pm^2|0\rangle &= \langle 1|J_zJ_\pm^2|1\rangle = 0.\end{aligned} \tag{56}$$

Thus, to satisfy Eq. (52) it is sufficient to satisfy Eq. (51).

Now for Eq. (53) we can use

$$J_xJ_y = \frac{-i}{4}(J_+^2 - J_-^2 - J_z) \tag{57}$$

to show $J_xJ_yJ_z = \frac{-i}{4}(J_+^2J_z - J_-^2J_z - J_z^2)$. However, from Eq. (56) and using

$$\begin{aligned}\langle 0|J_z^2|1\rangle &= \langle 1|J_z^2|0\rangle = 0, \\ \langle 0|J_z^2|0\rangle &= \langle 1|J_z^2|1\rangle,\end{aligned} \tag{58}$$

from [11], we see that Eq. (53) is trivially satisfied, and thus to correct all the errors up to the second power in angular momentum we only need to satisfy

$$\langle 0|J_z|0\rangle = 0,$$
$$\langle 0|J_z^3|0\rangle = 0. \tag{59}$$

Armed with this result, we turn our attention to the local errors that actually concern us. For a collection of spin-$\frac{1}{2}$ systems

$$J_z^2 = \frac{1}{4}\sum_{i,j}\sigma_{z,i}\sigma_{z,j}$$
$$= \frac{1}{4}\sum_{i=j}\mathbb{1} + \frac{1}{4}\sum_{i\neq j}\sigma_{z,i}\sigma_{z,j}. \tag{60}$$

Again using the fact that $(\sigma_{z,i})^2 = \mathbb{1}$, we get

$$J_z^3 = \frac{1}{8}\sum_{i,j,k}\sigma_{z,i}\sigma_{z,j}\sigma_{z,k}$$
$$= \frac{1}{8}\left(4\sum_{k}\sigma_{z,k} + \sum_{i\neq j\neq k}\sigma_{z,i}\sigma_{z,j}\sigma_{z,k}\right). \tag{61}$$

For a state in the symmetric subspace for N spins,

$$\langle J_z^3\rangle = \frac{1}{8}\left(4\sum_{k}\langle\sigma_{z,k}\rangle + \sum_{i\neq j\neq k}\langle\sigma_{z,i}\sigma_{z,j}\sigma_{z,k}\rangle\right)$$
$$= \langle J_z\rangle + N(N-1)(N-2)\langle\sigma_{z,1}\sigma_{z,2}\sigma_{z,3}\rangle. \tag{62}$$

Thus a code that follows Eq. (59) satisfies the Knill-Laflamme conditions for the errors of the form $\sigma_{z,i}\sigma_{z,j}\sigma_{z,k}$.

Now consider a general Knill-Laflamme condition $\langle i|\sigma_{p,k}\sigma_{q,l}\sigma_{r,m}|j\rangle$, where $p, q, r = \{x, y, z\}$ and $k, l, m = \{1, 2, \dots, N\}$ for N spin-$\frac{1}{2}$ systems. We can again look at the collective spin operators and the expansion of $J_xJ_yJ_z$ and $J_zJ_x^2$ in terms of Pauli operators. We have

$$J_xJ_yJ_z = \frac{1}{8}\sum_{i,j,k}\sigma_{x,i}\sigma_{y,j}\sigma_{z,k}. \tag{63}$$

Using the fact that $\sigma_x = \sigma_+ + \sigma_-$ and $\sigma_y = -i(\sigma_+ - \sigma_-)$, we have

$$J_xJ_yJ_z = -\frac{i}{8}\left(\sum_{i,j,k}\sigma_{+,i}\sigma_{+,j}\sigma_{z,k} - \sigma_{-,i}\sigma_{-,j}\sigma_{z,k}\right)$$
$$+ \frac{i}{8}\left(\sum_{i,j,k}\sigma_{+,i}\sigma_{-,j}\sigma_{z,k} - \sigma_{-,i}\sigma_{+,j}\sigma_{z,k}\right). \tag{64}$$

However, the Knill-Laflamme condition for the first two terms is trivially satisfied using Eq. (30) and we need not consider the case when i, j, or k is repeated as the total rank $\sum_i k_i$ is even for that case and those cases are trivially satisfied again by Eq. (30). Thus the only nontrivial terms to consider are

$$\frac{i}{8}\left(\sum_{i,j,k}\sigma_{+,i}\sigma_{-,j}\sigma_{z,k} - \sigma_{-,i}\sigma_{+,j}\sigma_{z,k}\right) = i[J_+, J_-]J_z$$
$$= -J_z^2. \tag{65}$$

Thus the condition for $\sigma_{x,i}\sigma_{y,j}\sigma_{z,k}$ is satisfied if the global condition for J_z^2 is satisfied and for the binary octahedral symmetry the condition for J_z^2 is trivially satisfied. Now we can look at the expansion of $J_zJ_x^2$ and we get

$$J_zJ_x^2 = \frac{1}{8}\sum_{i,j,k}\sigma_{z,i}\sigma_{x,j}\sigma_{x,k}. \tag{66}$$

Again expanding the σ_x and ignoring the trivially satisfied cases, we are left with the terms

$$\frac{1}{8}\left(\sum_{i,j,k}\sigma_{+,i}\sigma_{-,j}\sigma_{z,k} + \sigma_{-,i}\sigma_{+,j}\sigma_{z,k}\right) = 2J_z\left(J_x^2 + J_y^2\right)$$
$$= 2J_z^3 + 2J_z[j(j+1)], \tag{67}$$

where $j = N/2$ is the spin of the totally symmetric subspace. Thus if we satisfy the global condition of J_z^3 and J_z, the condition for $\sigma_{z,i}\sigma_{x,j}\sigma_{x,k}$ is satisfied, and hence the only condition we need to check to satisfy all the errors up to distance 5 is to check the global conditions given in Eq. (59).

The minimum spin we need to find conditions to correct for J_z and J_z^3 is $j = \frac{25}{2}$ in the ϱ_4 irreducible representation, i.e., we need 25 qubits and to form a [[25,1,5]] code. The code word is approximately

$$|0\rangle \propto -\sqrt{\frac{267}{1213}}|0\rangle_1 + \sqrt{\frac{701}{1457}}|0\rangle_2 + \sqrt{\frac{337}{1128}}|0\rangle_3, \tag{68}$$

where

$$|0\rangle_1 = -\sqrt{\frac{1377}{4132}}\left|\frac{25}{2}\frac{25}{2}\right\rangle - \sqrt{\frac{1}{674}}\left|\frac{25}{2}\frac{17}{2}\right\rangle - \sqrt{\frac{109}{1169}}\left|\frac{25}{2},\frac{9}{2}\right\rangle - \sqrt{\frac{803}{1918}}\left|\frac{25}{2},\frac{1}{2}\right\rangle$$
$$-\sqrt{\frac{103}{690}}\left|\frac{25}{2},-\frac{7}{2}\right\rangle - \sqrt{\frac{1}{263}}\left|\frac{25}{2},-\frac{13}{2}\right\rangle - \sqrt{\frac{1}{3608}}\left|\frac{25}{2},-\frac{21}{2}\right\rangle,$$
$$|0\rangle_2 = \sqrt{\frac{1}{4402}}\left|\frac{25}{2}\frac{25}{2}\right\rangle - \sqrt{\frac{2}{839}}\left|\frac{25}{2}\frac{17}{2}\right\rangle \quad - \sqrt{\frac{293}{983}}\left|\frac{25}{2},\frac{9}{2}\right\rangle - \sqrt{\frac{11}{1264}}\left|\frac{25}{2},\frac{1}{2}\right\rangle$$
$$\times\sqrt{\frac{913}{2925}}\left|\frac{25}{2},-\frac{7}{2}\right\rangle + \sqrt{\frac{21}{412}}\left|\frac{25}{2},-\frac{13}{2}\right\rangle - \sqrt{\frac{1069}{3264}}\left|\frac{25}{2},-\frac{21}{2}\right\rangle,$$

$$|0\rangle_3 = -\sqrt{\frac{1}{61\,408}}\left|\frac{25}{2}\frac{25}{2}\right\rangle + \sqrt{\frac{1750}{2781}}\left|\frac{25}{2}\frac{17}{2}\right\rangle - \sqrt{\frac{325}{3548}}\left|\frac{25}{2},\frac{9}{2}\right\rangle + \sqrt{\frac{43}{763}}\left|\frac{25}{2},\frac{1}{2}\right\rangle$$
$$- \sqrt{\frac{47}{551}}\left|\frac{25}{2},-\frac{7}{2}\right\rangle + \sqrt{\frac{183}{1349}}\left|\frac{25}{2},-\frac{13}{2}\right\rangle + \sqrt{\frac{2}{1011}}\left|\frac{25}{2},-\frac{21}{2}\right\rangle. \quad (69)$$

The distance-5 code for the binary octahedral code has the same code parameters as the distance-5 surface code [1,22]. These codes have another interesting correspondence in that they both belong to efficiently representable subsets of the full Hilbert space. The codes we study in this article all belong to the symmetric subspace, which is spanned by the Dicke basis and has dimension $N + 1$, which is linear instead of exponential in the number of qubits N. The code words for the surface code are stabilizer states, which we can efficiently represent by specifying a generating set of stabilizers of size $N - 1$ [23]. One notable difference is that, unlike the surface code, the binary octahedral codes have full transversal single-qubit Clifford gates.

Using the same approach as we did for the distance-3 and -5 codes, we can build codes that have larger distances. In Fig. 3 the number of physical qubits as a function of distance is given for both the binary octahedral (Clifford) codes and the surface codes. Both scale quadratically in the distance, though the Clifford codes have an improved constant factor.

We can use the binary tetrahedral symmetry to find code words with even fewer qubits. For example, we can construct a [[7,1,3]] code with code word

$$|0\rangle = \sqrt{\frac{7}{16}}|0\rangle_0 + \sqrt{\frac{16}{16}}|0\rangle_1, \quad (70)$$

where

$$|0\rangle_0 = -\frac{\sqrt{3}}{2}\left|\frac{7}{2},\frac{5}{2}\right\rangle + \frac{1}{2}\left|\frac{7}{2},-\frac{3}{2}\right\rangle,$$
$$|0\rangle_1 = \sqrt{\frac{7}{12}}\left|\frac{7}{2},\frac{1}{2}\right\rangle + \sqrt{\frac{5}{12}}\left|\frac{7}{2},-\frac{7}{2}\right\rangle. \quad (71)$$

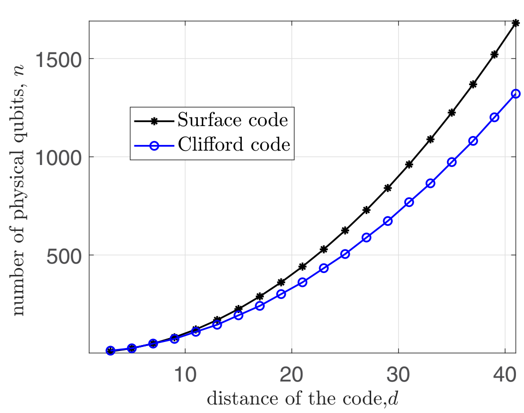

FIG. 3. Scaling of distance for binary octahedral codes. The plot shows the number of physical qubits required for correcting errors up to a distance d for the surface code (rotated) and the binary octahedral code.

The smallest distance-3 stabilizer code that has transversal Clifford gates is the Steane code [24] with code parameters [7,1,3] and also with binary octahedral symmetry, as it has transversal Clifford operators. The Steane code lies outside our classification as it does not exist entirely within the symmetric subspace (being a superposition of spin $\frac{1}{2}$ and spin $\frac{7}{2}$), suggesting that more interesting codes might be found by looking beyond the symmetric subspace.

VIII. CONCLUSION AND OUTLOOK

In this work we focused on using binary octahedral symmetry to construct useful quantum error-correcting codes extending the ideas in [11]. In [11] the codes were designed to protect against SU(2) errors in a single large spin. In this article we developed a technique for designing codes for multiple copies of spins. We leveraged the multiple SU(2) irreducible representations within the symmetric subspace of the tensor product of several large spins to correct for the additional physically relevant error channel of tensor light shifts. This resulted in numerically derived codes correcting tensor light shifts in three copies of spin $j = \frac{7}{2}$ and in three copies of spin $j = \frac{9}{2}$. We derived general simplified error-correction conditions for correcting errors at arbitrary order using the structure of spherical tensors (21) and (30), which are polynomials of the angular momentum operators and well studied in the spin systems.

We additionally studied the case of qubits ($j = \frac{1}{2}$) and extended the framework to multibody errors. Again we used the symmetric subspace for a large number of spin-$\frac{1}{2}$ systems and used the symmetries to find codes with distance 3 for $n = 7$ and distance 5 for $n = 25$. The distance-5 code contrasts interestingly with the distance-5 surface code, which has the same code parameters but gives up the transversal Clifford gates of the binary octahedral code in favor of its stabilizer structure.

The techniques outlined in this work can easily be extended to further develop codes with larger distances with octahedral symmetry. An important open question is whether one can develop fault-tolerant schemes for these kinds of codes, as their highly non-Abelian nature makes applying existing fault-tolerant strategies difficult. Finally, it would be interesting to explore whether binary octahedral codes might have use as nonstabilizer versions of the metrological codes discussed in [25].

ACKNOWLEDGMENTS

The authors acknowledge fruitful discussions with I. Deutsch and M. Marvian about quantum error correction for spin systems. S.O. acknowledges useful discussions with T. Thurtel during various stages of the work. S.O. also thanks

P. Poggi and K. Chinni for useful discussions about spherical tensors and their interesting properties, in particular K. Chinni for helping numerically create the spherical tensors as a polynomial of angular momentum operators. The derivation of the counting of parameters for finding the symmetric subspace of two spins in the Appendixes was proven as a follow-up to a discussion with A. Daniel. This material is based upon work supported by the U.S. Department of Energy, Office of Science, National Quantum Information Science Research Centers, Quantum Systems Accelerator.

APPENDIX A: SPHERICAL TENSORS

The spherical tensor operators for a spin j are defined in terms of the commutator relations [16,17]

$$\left[J_z, T_q^k\right] = qT_q^k,$$
$$\left[J_\pm, T_q^k\right] = \sqrt{k(k+1) - q(q\pm 1)}T_{q\pm1}^k. \tag{A1}$$

Using these relations, the irreducible spherical tensors can be explicitly written in terms of the angular momentum basis as [16,18]

$$T_q^k(j) = \sqrt{\frac{2k+1}{2j+1}}\sum_m \langle j, m+q|k, q; j, m\rangle |j, m+q\rangle\langle j, m|, \tag{A2}$$

where $0 \leqslant k \leqslant 2j$ and $-k \leqslant q \leqslant k$. The spherical tensor operators of rank k can be expressed as order-k polynomials in the angular momentum operators [18,26]. The spherical tensor operators also form an orthonormal basis for the operators on an SU(2) irreducible representation with respect to the Hilbert-Schmidt inner product

$$\mathrm{Tr}\left(T_{q_1}^{k_1}T_{q_2}^{k_2}\right) = \delta_{k_1,k_2}\delta_{q_1,q_2}. \tag{A3}$$

Now consider the unitary transformation given as $U_X = \exp(-i\pi J_x)$, which can also be written in terms of the angular momentum basis as

$$U_X = -i\sum_{m=-j}^{j} |j, m\rangle\langle j, -m|. \tag{A4}$$

Thus the action of the unitary operator on the irreducible spherical tensor gives

$$U_X T_q^k U_X^\dagger = \sum_{m=-j}^{j} \langle j, m+q|k, q; j, m\rangle |j, -m-q\rangle\langle -m|. \tag{A5}$$

Using the transformation $m \to -m$ and the fact that

$$\langle j, m+q|k, q; j, m\rangle = (-1)^k\langle j, -m-q|k, -q; j, -m\rangle, \tag{A6}$$

we get

$$U_X T_q^k U_X^\dagger = (-1)^k\sum_m \langle j, m-q|k, -q; j, m\rangle |j, m-q\rangle\langle j, m|$$
$$= (-1)^k T_{-q}^k. \tag{A7}$$

Thus the action of U_X on the spherical tensor operators is to flip the sign of q and to add a rank-dependent phase of ±1 to the operator.

APPENDIX B: ERROR CORRECTION CONDITION

The logical Pauli Z operator on an irreducible representation ϱ of the binary octahedral group is given by [11]

$$\sigma_z = P_\varrho[i\exp(-i\pi J_z)]P_\varrho. \tag{B1}$$

Logical $|0\rangle$ is taken to be a $+1$ eigenstate of the logical Pauli Z operator. The projector for the binary octahedral group is given as

$$P_\varrho = \frac{\dim\varrho}{|2O|}\sum_{g\in 2O}\chi_\varrho(g)^* D(g), \tag{B2}$$

where $2O$ is the single-qubit Clifford group [27], also called the binary octahedral group. Now from [11], the $\chi_\varrho(g)$ for the SU(2) irreducible representations of interest are real. For the binary octahedral group, we also have that every representative $D(g)$ is in the same conjugacy class as $D(g)^\dagger$, $D(g)^T$, and $D(g)^*$. Restricting the sum to a fixed conjugacy class $[g]$ gives

$$\frac{1}{4}\chi_\varrho(g)^*\sum_{h\in[g]}[D(h) + D(h)^\dagger + D(h)^T + D(h)^*]. \tag{B3}$$

The term for each conjugacy class is real symmetric since χ_ρ is real and $D(g) + D(g)^\dagger + D(g)^T + D(g)^*$ is manifestly real and symmetric. Thus we get P_ϱ to be a real symmetric matrix. The term sandwiched by the projectors in Eq. (B1) is also real and symmetric for half-integer spins

$$i\exp(-i\pi J_z) = [i\exp(-i\pi J_z)]^\dagger = [i\exp(-i\pi J_z)]^T; \tag{B4}$$

hence σ_z is a real-symmetric operator.

Now the eigenvector of a real symmetric matrix A can be found by solving the eigenvalue equation

$$(A - \lambda\mathbb{1})|\psi\rangle = 0. \tag{B5}$$

Since the eigenvalue λ is real from A being Hermitian, when solved by Gaussian elimination we get a real vector and hence the eigenvectors of a real symmetric matrix are also real (up to an overall constant which is not important).

Consider the expectation value for two states $|\psi\rangle = \sum_i \alpha_i|i\rangle$ and $|\phi\rangle = \sum_i \beta_i|i\rangle$, where $|i\rangle$ is in the angular momentum basis,

$$\langle\psi|T_{-q}^k(j)|\phi\rangle = d_j^k\sum_{i,i',m}\alpha_i^*\beta_{i'}C_{j,m,j,m-q}^{k,-q}\langle i'|j, m-q\rangle\langle j, m|i\rangle, \tag{B6}$$

where $d_j^k = \sqrt{2k+1/2j+1}$ and

$$C_{j_1,m_1,j_2,m_2}^{j_3,m_3} = \langle j_3, m_3|j_1, m_1; j_2, m_2\rangle \tag{B7}$$

is the Clebsch-Gordan coefficient. Now using the property that $\langle i|j, m+q\rangle = \langle j, m+q|i\rangle$, as they are both in the angular momentum basis, we get

$$\langle\psi|T_{-q}^k|\phi\rangle = d_j^k\sum_{i,i',m}\alpha_i^*\beta_{i'}C_{j,m,j,m-q}^{k,-q}\langle j, m-q|i'\rangle\langle i|j, m\rangle. \tag{B8}$$

Also, by transforming this equation by $m \to m+q$ we get

$$\langle\psi|T_{-q}^k|\phi\rangle = d_j^k\sum_{i,i',m}\alpha_i^*\beta_{i'}C_{j,m+q,j,m}^{k,-q}\langle j, m|i'\rangle\langle i|j, m+q\rangle. \tag{B9}$$

Using the property of the Clebsch-Gordan coefficients

$$C^{j_3,m_3}_{j_1,m_1,j_2,m_2} = (-1)^{j_1+j_2+j_3} C^{j_3,m_3}_{j_2,m_2,j_1,m_1}, \tag{B10}$$

we get

$$\langle\psi|T^k_{-q}|\phi\rangle = (-1)^k d^k_j \sum_{i,i',m} \alpha^*_i \beta_{i'} C^{k,-q}_{j,m,j,m+q} \langle J, m|i'\rangle \times \langle i|J, m+q\rangle. \tag{B11}$$

Again using another property of Clebsch-Gordan coefficients

$$C^{j_3,m_3}_{j_1,m_1,j_2,m_2} = \sqrt{\frac{2j_1+1}{2j_2+1}}(-1)^{j_2+m_2} C^{j_3,-m_3}_{j_1,m_1,j_2,m_2}, \tag{B12}$$

we get

$$\langle\psi|T^k_{-q}|\phi\rangle = (-1)^q d^k_j \sum_{i,i',m} \alpha^*_i \beta_{i'} C^{k,q}_{j,m,j,m+q} \langle j, m|i'\rangle \times \langle i|j, m+q\rangle. \tag{B13}$$

Since the computational-basis code words for the binary octahedral case are real the amplitudes α_i and β_i are real when $|\psi\rangle$ and $|\phi\rangle$ are computational-basis code words, as when we are checking error-correction conditions, and thus

$$\langle\psi|T^k_{-q}|\phi\rangle = (-1)^q \langle\phi|T^k_q|\psi\rangle. \tag{B14}$$

APPENDIX C: SYMMETRIC SUBSPACE UNDER THE TENSOR PRODUCT OF TWO SPINS

It is known that the SU(2) irreducible representations under the addition of two spin-j systems is given as

$$j \otimes j = 2j \oplus (2j-1) \oplus (2j-1) \oplus \cdots. \tag{C1}$$

Focusing our attention on the symmetric subspace, in Eq. (24), we numerically find that the symmetric subspace of two spin-j systems is composed of all SU(2) subspaces interleaving one in between starting from the highest possible angular momentum. To verify this we could do dimension counting of these subspaces. First, consider the case of even multiple of spin $\frac{1}{2}$ and thus the dimension of the alternate SU(2) subspaces is given as

$$\begin{aligned}
\dim &= \sum_{k=0}^{j} 4j+1-4k \\
&= 4j(j+1)+j+1-2j(j+1) \\
&= 2j^2+3j+1 = \frac{(2j+1)(2j+2)}{2} \\
&= \dim[S_2(2j+1)].
\end{aligned} \tag{C2}$$

For the case of odd multiples of $\frac{1}{2}$ we have

$$\begin{aligned}
\dim &= \sum_{k=0}^{j-1/2} 4j+1-4k \\
&= 4j\left(j+\frac{1}{2}\right)+j+\frac{1}{2}-2\left[\left(j-\frac{1}{2}\right)\left(j+\frac{1}{2}\right)\right] \\
&= 4j^2+2j+j+\frac{1}{2}-2j^2+\frac{1}{2} \\
&= 2j^2+3j+1 \\
&= \frac{(2j+1)(2j+2)}{2} \\
&= \dim[S_2(2j+1)].
\end{aligned} \tag{C3}$$

Thus we get that for both even and odd multiple of spin $\frac{1}{2}$ the dimension of the symmetric subspace is SU(2) subspaces interleaving one in between starting from the highest possible angular momentum.

APPENDIX D: ALGORITHM FOR FINDING THE CODE WORD FOR THE CASE OF SECOND-ORDER ERRORS

The simple algorithm for finding the code word follows three steps.

Step 1. Write the code words as

$$|0\rangle_L = \sum_{i=1}^{n} c_i|0\rangle_i, \quad |1\rangle_L = \sum_{i=1}^{n} c_i|1\rangle_i, \tag{D1}$$

where i corresponds to the two-dimensional qubit spaces one has access to and $c_i \in \mathbf{R}$.

Step 2. Define the cost function

$$\mathcal{F}[(\boldsymbol{c})] = \sum_{\text{constraints}} |f(\boldsymbol{c})|, \tag{D2}$$

where $f(\boldsymbol{c})$ is the value we get for each constraint we need to satisfy according to the Knill-Laflamme conditions in Eq. (21).

Step 3. Minimize the cost function to obtain the right code word where $\boldsymbol{c} \in \mathbf{R}^n$ such that

$$\boldsymbol{c}_{\text{opt}} = \arg\min_{\boldsymbol{c}\in\mathbf{R}} \mathcal{F}[(\boldsymbol{c})], \tag{D3}$$

which in turn gives the code words as

$$|0\rangle_L = \sum_i c^{\text{opt}}_i|0\rangle_i, \quad |1\rangle_L = \sum_i c^{\text{opt}}_i|1\rangle_i. \tag{D4}$$

[1] R. Acharya, I. Aleiner, R. Allen, T. I. Andersen, M. Ansmann, F. Arute, K. Arya, A. Asfaw, J. Atalaya, R. Babbush *et al.*, Suppressing quantum errors by scaling a surface code logical qubit, arXiv:2207.06431.

[2] C. Ryan-Anderson, N. C. Brown, M. S. Allman, B. Arkin, G. Asa-Attuah, C. Baldwin, J. Berg, J. G. Bohnet, S. Braxton, N. Burdick *et al.*, Implementing fault-tolerant entangling gates on the five-qubit code and the color code, arXiv:2208.01863.

[3] S. Krinner, N. Lacroix, A. Remm, A. Di Paolo, E. Genois, C. Leroux, C. Hellings, S. Lazar, F. Swiadek, J. Herrmann *et al.*, Realizing repeated quantum error correction in a distance-three surface code, Nature (London) **605**, 669 (2022).

[4] D. Gottesman, A. Kitaev, and J. Preskill, Encoding a qubit in an oscillator, Phys. Rev. A **64**, 012310 (2001).

[5] V. V. Albert, K. Noh, K. Duivenvoorden, D. J. Young, R. T. Brierley, P. Reinhold, C. Vuillot, L. Li, C. Shen, S. M. Girvin, B. M. Terhal, and L. Jiang, Performance and structure of single-mode bosonic codes, Phys. Rev. A **97**, 032346 (2018).

[6] J. E. Bourassa, R. N. Alexander, M. Vasmer, A. Patil, I. Tzitrin, T. Matsuura, D. Su, B. Q. Baragiola, S. Guha, G. Dauphinais, K. K. Sabapathy, N. C. Menicucci, and I. Dhand, Blueprint for a scalable photonic fault-tolerant quantum computer, Quantum **5**, 392 (2021).

[7] Y. Xu, Y. Wang, E.-J. Kuo, and V. V. Albert, Qubit-oscillator concatenated codes: Decoding formalism & code comparison, PRX Quantum **4**, 020342 (2023).

[8] K. Noh, S. M. Girvin, and L. Jiang, Encoding an Oscillator into Many Oscillators, Phys. Rev. Lett. **125**, 080503 (2020).

[9] S. Omanakuttan and T. J. Volkoff, Spin squeezed GKP codes for quantum error correction in atomic ensembles, arXiv:2211.05181.

[10] V. V. Sivak, A. Eickbusch, B. Royer, S. Singh, I. Tsioutsios, S. Ganjam, A. Miano, B. L. Brock, A. Z. Ding, L. Frunzio *et al.*, Real-time quantum error correction beyond break-even, Nature **616**, 50 (2023).

[11] J. A. Gross, Designing Codes around Interactions: The Case of a Spin, Phys. Rev. Lett. **127**, 010504 (2021).

[12] S. Ebadi, T. T. Wang, H. Levine, A. Keesling, G. Semeghini, A. Omran, D. Bluvstein, R. Samajdar, H. Pichler, W. W. Ho *et al.*, Quantum phases of matter on a 256-atom programmable quantum simulator, Nature (London) **595**, 227 (2021).

[13] P. Scholl, M. Schuler, H. J. Williams, A. A. Eberharter, D. Barredo, K.-N. Schymik, V. Lienhard, L.-P. Henry, T. C. Lang, T. Lahaye *et al.*, Quantum simulation of 2D antiferromagnets with hundreds of Rydberg atoms, Nature (London) **595**, 233 (2021).

[14] I. H. Deutsch and P. S. Jessen, Quantum control and measurement of atomic spins in polarization spectroscopy, Opt. Commun. **283**, 681 (2010).

[15] S. Omanakuttan, A. Mitra, M. J. Martin, and I. H. Deutsch, Quantum optimal control of ten-level nuclear spin qudits in ^{87}Sr, Phys. Rev. A **104**, L060401 (2021).

[16] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics, 2nd ed.* (Pearson, London, 2014).

[17] A. B. Klimov and P. Espinoza, Moyal-like form of the star product for generalized SU(2) Stratonovich-Weyl symbols, J. Phys. A: Math. Gen. **35**, 8435 (2002).

[18] K. Chinni, Reliability of quantum simulation on NISQ-era devices, Ph.D. thesis, University of New Mexico, 2022.

[19] E. Knill and R. Laflamme, Theory of quantum error-correcting codes, Phys. Rev. A **55**, 900 (1997).

[20] A. W. Harrow, The church of the symmetric subspace, arXiv:1308.6595.

[21] R. E. F. Silva and J. Feist, Permutational symmetry for identical multilevel systems: A second-quantized approach, Phys. Rev. A **105**, 043704 (2022).

[22] A. G. Fowler, M. Mariantoni, J. M. Martinis, and A. N. Cleland, Surface codes: Towards practical large-scale quantum computation, Phys. Rev. A **86**, 032324 (2012).

[23] D. Gottesman, Stabilizer codes and quantum error correction, Ph.D. thesis, California Institute of Technology, 1997.

[24] A. Steane, Multiple-particle interference and quantum error correction, Proc. R. Soc. London A **452**, 2551 (1996).

[25] P. Faist, M. P. Woods, V. V. Albert, J. M. Renes, J. Eisert, and J. Preskill, Time-energy uncertainty relation for noisy quantum metrology, in *Quantum Information and Measurement VI 2021*, edited by F. Sciarrino, N. Treps, M. Giustina, and C. Silberhorn (Optica Publishing Group, Washington, DC, 2021), paper W2A.3.

[26] S. Omanakuttan, K. Chinni, P. D. Blocher, and P. M. Poggi, Scrambling and quantum chaos indicators from long-time properties of operator distributions, Phys. Rev. A **107**, 032418 (2023).

[27] D. Gottesman, The Heisenberg representation of quantum computers, arXiv:quant-ph/9807006.