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Exclusion principle for nonlocal advantage of quantum coherence

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Coherences in mutually unbiased bases of states of an isolated quantum system follow a complementarity relation. The nonlocal advantage of quantum coherence (NAQC), defined in a bipartite scenario, is a situation in which the average quantum coherences of the ensembles of one subsystem, effected by a measurement performed on the other subsystem, violate the complementarity relation. We analyze two criteria to detect NAQC for bipartite quantum states. We construct a more generalized version of the criterion to detect NAQC that is better than the standard criterion as it can capture more states exhibiting NAQC. We prove the local unitary invariance of these NAQC criteria. Further on, we focus on investigating the monogamy properties of NAQC in the tripartite scenario. We check for monogamy of NAQC from two perspectives, differentiated by whether or not the nodal observer in the monogamy relation performs the measurement for the nonlocal advantage. We find in particular that in the case where the nodal observer does not perform the measurement, a strong monogamy relation—an exclusion principle—is exhibited by NAQC.

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I. INTRODUCTION

In recent years, quantum coherence (QC) [1] has emerged as a key notion of nonclassicality. The underlying concept of QC is the wave-like nature of systems which allows two distinct pure states of a system to interfere coherently with each other, forming a quantum superposition. QC is crucial in areas such as quantum information theory [1], metrology [2,3], quantum biology [4–8], and quantum thermodynamics [9–12]. In the literature, there exist different measures of QC, and the resource theory of QC has also been developed [13–20]. However, understanding about the ability to manipulate and utilize QC as a resource is far from complete, especially in the multipartite setting.

There exists a complementarity relation for coherences of an isolated qubit system in that the sum of quantum coherences of its states in mutually unbiased bases (MUBs) [21–24] is nontrivially bounded from above [25,26]. The notion of nonlocal advantage of quantum coherence (NAQC) was introduced in the bipartite scenario to detect steerability, as captured by coherence [26]. That is, the average coherence of the ensemble of states of one of the parties, affected by the measurement on the other party, was shown to violate the complementarity bound for quantum coherence of an isolated system. While a state that shows NAQC is steerable, the converse is not always true, making the set of states exhibiting NAQC strictly smaller than the set of steerable states [27].

Unlike its classical counterpart, nonclassical correlations of multipartite systems have restricted sharability and are referred to as the monogamous property [28], for the quantum state and the nonclassical shared physical quantity considered. For research on monogamy of entanglement, see Refs. [28–33], and for that of quantum discord, see [34,35]. See Refs. [36] and [37,38] for reviews of entanglement and quantum discord, and Refs. [39,40] for their monogamy properties.

The current work focuses on investigating the monogamy properties of NAQC, as an integral part of characterizing it as a nonclassical correlation. We consider two criteria of NAQC and refer to them as the standard and generalized NAQC criteria. The standard criterion includes measurements only in bases belonging to an arbitrary MUB on one of the subsystems of the bipartite system, while the generalized version includes measurements belonging to an arbitrary set of bases on a subsystem. Both criteria involve optimizations over the relevant sets of projective measurements (MUBs for standard criterion and an arbitrary set of bases for generalized criterion) and the set of MUBs for quantum coherence measurement, which makes both the NAQC functionals local unitary invariant. We show that the generalized NAQC criterion allows to capture more states that exhibit NAQC. We also provide a lower bound of both the NAQC functionals for any bipartite system.

Next, we check the monogamy properties of NAQC which is motivated by the following consideration. The phenomenon of NAQC and its quantifications utilize a single-site quantum characteristic, viz. quantum coherence, to create a two-body physical quantity, which exhibits certain "nonlocal" aspects. Typical quantum correlations are often connected with nonlocality, and are almost universally observed to exhibit certain monogamy properties [39,40]. With this motivation, we consider the monogamy properties of NAQC, for the set of pure three-qubit states in two separate scenarios. The first case corresponds to the situation where the ensemble-generating measurements are performed on different subsystems of the tripartite system, while we switch parties to define the monogamy relation. The measurements are therefore performed on the "non-nodal" parties in this case. We show that the generalized NAQC is strongly monogamous in this scenario. Indeed, there appears an exclusion principle in this case. In the second scenario, the nodal party performs the ensemble-generating measurements. The strong monogamy is no more valid here.

II. PRELIMINARIES

We begin by introducing some of the key elements essential for analyzing the monogamy of NAQC. The central notion of this work is *quantum coherence* (QC). QC, assuredly a basis-dependent notion, is the underlying concept of quantum entanglement and other quantum phenomena. It arises from coherent superposition of states of a quantum system. Several measures can be opted for measuring QC, such as relative entropy of quantum coherence [14], l_1-norm of quantum coherence [14], geometric quantum coherence [41], etc.

For the current work, we will opt for the l_1-norm of quantum coherence [14], which, for any quantum state ρ, is defined as the sum of absolute values of all nondiagonal elements of the state corresponding to a chosen basis M, i.e., $C_M(\rho) := \sum_{i\neq j} |\rho_{ij}|$, where ρ_{ij} represents the component of the state ρ for the ith row and jth column, in the computational basis. The l_1-norm of coherence of ρ in a given basis is zero if all the nondiagonal elements of the state in the corresponding basis are zero, and such states are termed *incoherent states* for that basis, in the resource theory of coherence.

The subsequent key ingredient to define NAQC is the concept of *mutually unbiased bases* [21–24]. Let $\{\{|e_i^a\rangle\}^a\}_i$ represent a set of bases, where the subscript i indicates different bases and a indicates different elements of a basis. If for a set of orthonormal bases on the Hilbert space $\mathcal{H}$ with dimension d, the elements of any two bases satisfy the relation $|\langle e_i^a|e_j^b\rangle|^2 = \frac{1}{d}$, for $i \neq j$ and all $a, b \in \{1, 2, ..., d\}$, then this set of bases is said to form a set of MUBs. There can be at most three bases forming a set of MUBs in a two-dimensional complex Hilbert space. The set of eigenbases of σ_x, σ_y, and σ_z (Pauli) matrices is an example. We will exclusively be considering MUBs in the qubit Hilbert space.

At this point, we are ready to set forth the *quantum coherence complementarity relation for a single-qubit system*, presented in Refs. [25,26], which states that the sum of quantum coherences for any qubit, over a set of three bases forming a set of mutually unbiased bases, has the following nontrivial upper bound:

$$\sum_{i=1}^{3} C_{M_i} \leqslant \sqrt{6}. \tag{1}$$

Here, C_{M_i} denotes the value of quantum coherence of the corresponding state measured in the basis M_i, where $\{M_i, i = 1, 2, 3\}$ forms a set of MUBs. This bound is nontrivial as the maximum value of the l_1-norm of coherence of a qubit over all bases is unity.

This sets the stage for us to briefly recapitulate the notion of nonlocal advantage of quantum coherence. Consider two spatially separated parties, Alice and Bob, sharing a two-qubit state ρ_{AB}. Alice performs measurements on her part of the shared state, resulting in ensembles of states of Bob's subsystem. As Alice communicates her measurement outcomes to Bob, the average quantum coherence of the ensembles of states at Bob's end may violate the quantum coherence complementarity relation given in (1), exhibiting a "nonlocal advantage" of QC. Formally, NAQC can be defined as follows.

Definition 1 (Nonlocal advantage of quantum coherence [26]). Violation of the quantum coherence complementarity relation for average quantum coherences of one of the subsystems of a bipartite system by measurements on the other subsystem is defined as the nonlocal advantage of quantum coherence.

It has been established that only entangled bipartite states can exhibit NAQC [26], and indeed, nonsteerable states cannot show NAQC, so the set of states providing NAQC is a subset of the set of steerable states [27]. For states on $\mathbb{C}^2 \otimes \mathbb{C}^2$ with diagonal correlation matrix, NAQC also captures stronger quantum correlation than "Bell nonlocality" [42].

Lastly, we mention the standard form of the three-qubit pure state that will be used in this work. In the computational basis, any three-qubit pure state $|\Psi\rangle_{ABC}$, up to local unitaries, can be written as

$$|\Psi\rangle_{ABC} = \lambda_0 |000\rangle + \lambda_1 e^{i\beta} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle , \tag{2}$$

where $\lambda_i \geqslant 0$, $\sum_{i=0}^4 \lambda_i^2 = 1$, $\beta \in [0, \pi]$ [43–45]. Equation (2) represents W-class states when $\lambda_4 = 0$ and $\beta = 0$.

III. DETECTION OF NONLOCAL ADVANTAGE OF QUANTUM COHERENCE

In this section, we present and analyze two criteria to detect NAQC. In the standard criterion to detect NAQC, Alice performs measurements only in the bases belonging to a set of MUBs, $\{\{\Lambda_i^a\}^a\}_i$. She then communicates her measurement settings and outcomes to Bob, creating three ensembles of states, $\mathcal{E}_i \equiv \{p(\rho_{B|\Lambda_i^a}),\ \rho_{B|\Lambda_i^a}\}^a$, at Bob's end, where $p(\rho)$ is the probability of getting the state ρ, in the relevant measurement (at Alice). Now consider an independent set of MUBs, viz. $\{M_i\}_i$, and Bob measures the quantum coherence of the ensemble $\mathcal{E}_i$ in the basis M_i, so that the average coherence of ensemble $\mathcal{E}_i$ in the basis M_i is given by $\sum_a p(\rho_{B|\Lambda_i^a})C_{M_i}(\rho_{B|\Lambda_i^a})$. Notice that the average of coherences is taken over the outcomes (at Alice) for a given measurement setting i (at Bob). Finally, we sum these average quantum coherences over all values of i to obtain $\sum_{i,a} p(\rho_{B|\Lambda_i^a})C_{M_i}(\rho_{B|\Lambda_i^a})$. This sum is to be compared with the left-hand side of the complementarity relation (1). This sum can be maximized over the choice of the sets of MUBs, $\{\Lambda_i\}_i \equiv \{\{\Lambda_i^a\}^a\}_i$ and $\{M_i\}_i$, to obtain

$$\mathcal{N}^{\rightarrow}(\rho_{AB}) := \max_{\{M_i\}_i, \{\Lambda_i\}_i} \sum_{i,a} p\big(\rho_{B|\Lambda_i^a}\big)C_{M_i}\big(\rho_{B|\Lambda_i^a}\big), \tag{3}$$

where the $\rightarrow$ indicates that the (ensemble-generating) measurement is done at party A and quantum coherence is measured by party B. We now formally state the standard criterion for detecting NAQC as follows.

Criterion 1. [Standard criterion for NAQC detection] Any two-qubit state ρ_{AB} exhibits advantage in quantum coherence "nonlocally" if

$$\mathcal{N}^{\rightarrow}(\rho_{AB}) > \sqrt{6}. \tag{4}$$

We refer to $\mathcal{N}^{\rightarrow}(\rho_{AB})$ as an NAQC functional. The above NAQC detection criterion is only a sufficient criterion, and not a necessary one, since Alice performs a measurement only in the bases contained in a set of MUBs, $\{\Lambda_i^a\}^a$, to detect NAQC. For the criterion to become necessary and sufficient, one needs to perform optimization over all possible measurements on Alice's side.

Next, in order to capture a larger set of states exhibiting NAQC, we consider a generalized version of the standard NAQC criterion. In the generalized criterion, Alice is not restricted to performing measurements only in bases forming a set of MUBs. That is, she may choose a set of arbitrary three bases which, in general, may not form a set of MUBs, for measurement on her subsystem.

Criterion 2. [Generalized criterion for NAQC detection] Any two-qubit state ρ_{AB} shows advantage in quantum coherence "nonlocally" if

$$\mathsf{N}^{\rightarrow}(\rho_{AB}) := \max_{\{M_i\}_i,\{\Pi_i\}_i} \sum_{i,a} p\big(\rho_{B|\Pi_i^a}\big) C_{M_i}\big(\rho_{B|\Pi_i^a}\big) > \sqrt{6}, \tag{5}$$

where $\{\{\Pi_i^a\}^a\}_i$ are a set of three arbitrary projective measurements, while $\{M_i\}_i$ are an arbitrary set of MUBs.

This criterion is potentially also sufficient and not necessary since not all measurements are spanned on Alice's side. But nevertheless, by construction, it is no less strong than the standard criterion, i.e.,

$$\mathsf{N}^{\rightarrow}(\rho_{AB}) \geqslant \mathcal{N}^{\rightarrow}(\rho_{AB}). \tag{6}$$

We will explicitly show below that the generalized criterion can detect the nonlocal advantage of quantum coherence of states which are not detected by the standard criterion.

Let us now demonstrate a useful property of invariance of the functionals, $\mathcal{N}^{\rightarrow}(\rho_{AB})$ and $\mathsf{N}^{\rightarrow}(\rho_{AB})$, under the action of local unitaries on the state in their arguments.

Theorem 1. The NAQC functionals, $\mathcal{N}^{\rightarrow}(\rho_{AB})$ and $\mathsf{N}^{\rightarrow}(\rho_{AB})$, are invariant under the action of local unitaries for any quantum state, ρ_{AB}.

Proof. Consider the transformation $\rho'_{AB} = (U \otimes V)\rho_{AB}(U^\dagger \otimes V^\dagger)$, where U and V are arbitrary unitary operators. We first want to show that $\mathcal{N}^{\rightarrow}(\rho_{AB}) = \mathcal{N}^{\rightarrow}(\rho'_{AB})$. Let $\{\{\Lambda_i^a\}^a\}_i$ and $\{M_i\}_i$ be sets of MUBs responsible for the maximization in (3) for the state ρ_{AB}. Let $\rho_{B|\Lambda_i^a}$ be the conditional states at Bob's end, after Alice's measurements.

Now, consider the set of measurements, $\{\{\Lambda_i'^a\}^a\}_i = \{\{U\Lambda_i^a U^\dagger\}^a\}_i$, performed by Alice, and $\{M'_i\}_i = \{VM_iV^\dagger\}_i$, used by Bob on the state ρ'_{AB} to obtain $\sum_{i,a} p(\rho'_{B|\Lambda_i'^a}) C_{M'_i}(\rho'_{B|\Lambda_i'^a})$. Note that the reduced states at Bob's end, created due to measurements, $\{\{\Lambda_i'^a\}^a\}_i$, by Alice on ρ'_{AB} are $V\rho_{B|\Lambda_i^a}V^\dagger$ and its occurrence probability, $p(V\rho_{B|\Lambda_i^a}V^\dagger) = p(\rho_{B|\Lambda_i^a})$. Also note that

$$C_{M_i}(\chi) = C_{VM_iV^\dagger}(V\chi V^\dagger), \tag{7}$$

where χ is a single-qubit density matrix. Thus, for the measurements, $\{\{\Lambda_i'^a\}^a\}_i$ and $\{M'_i\}_i$, which are sets of MUBs as their parents were so,

$$\sum_{i,a} p\big(\rho'_{B|\Lambda_i'^a}\big) C_{M'_i}\big(\rho'_{B|\Lambda_i'^a}\big) = \mathcal{N}^{\rightarrow}(\rho_{AB}).$$
$$\Rightarrow \mathcal{N}^{\rightarrow}(\rho'_{AB}) \geqslant \mathcal{N}^{\rightarrow}(\rho_{AB}). \tag{8}$$

Now, $\rho_{AB} = (U^\dagger \otimes V^\dagger)\rho'_{AB}(U \otimes V)$, where $U^\dagger$ and $V^\dagger$ are also unitary, and thus a similar treatment as above leads to

$$\mathcal{N}^{\rightarrow}(\rho_{AB}) \geqslant \mathcal{N}^{\rightarrow}(\rho'_{AB}). \tag{9}$$

Therefore, (8) and (9) prove that

$$\mathcal{N}^{\rightarrow}(\rho'_{AB}) = \mathcal{N}^{\rightarrow}(\rho_{AB}), \tag{10}$$

and thus we have proven invariance under local unitaries, of $\mathcal{N}^{\rightarrow}$.

Notice that the above proof can be repeated without requiring the measurements to belong to sets of MUBs, proving that the generalized NAQC functional, $\mathsf{N}^{\rightarrow}(\rho'_{AB})$, is also unaffected by local unitary action on the states. ■

Now we present a lemma providing a lower bound of the quantity $\mathcal{N}^{\rightarrow}(\rho_{AB})$, and thus a lower bound of the quantity $\mathsf{N}^{\rightarrow}(\rho_{AB})$.

Lemma 1. The NAQC functional of an arbitrary state ρ_{AB} of two qubits is lower bounded by the sum of the quantum coherences of the reduced state, $\text{Tr}_A[\rho_{AB}]$ of ρ_{AB}, measured in an arbitrary set of mutually unbiased bases.

Proof. Let $\{M_i\}_i$ be an arbitrary set of mutually unbiased bases and $\{\Lambda_i^a\}^a$ be another. Then, the NAQC functional for any bipartite state ρ_{AB} can be expressed as

$$\begin{aligned}
\mathcal{N}^{\rightarrow}(\rho_{AB}) &:= \max_{M_i,\Lambda_i^a} \sum_{i,a} p\big(\rho_{B|\Lambda_i^a}\big) C_{M_i}\big(\rho_{B|\Lambda_i^a}\big)\\
&= \max_{M_i,\Lambda_i^a} \sum_{i,a} p\big(\rho_{B|\Lambda_i^a}\big) C_{M_i}\left(\text{Tr}_A\left[\frac{\langle \Lambda_i^a \otimes \mathbb{1}_B|\, \rho_{AB}\, |\Lambda_i^a \otimes \mathbb{1}_B\rangle}{p\big(\rho_{B|\Lambda_i^a}\big)}\right]\right)\\
&\geqslant \max_{M_i,\Lambda_i^a} \sum_{i} C_{M_i}\left(\sum_a p\big(\rho_{B|\Lambda_i^a}\big)\, \text{Tr}_A\left[\frac{\langle \Lambda_i^a \otimes \mathbb{1}_B|\, \rho_{AB}\, |\Lambda_i^a \otimes \mathbb{1}_B\rangle}{p\big(\rho_{B|\Lambda_i^a}\big)}\right]\right)\\
&= \max_{M_i} \sum_i C_{M_i}(\text{Tr}_A[\rho_{AB}]) = \max_{M_i} \sum_i C_{M_i}(\rho_B),\\
\Rightarrow \mathcal{N}^{\rightarrow}(\rho_{AB}) &\geqslant \max_{M_i} \sum_i C_{M_i}(\rho_B),
\end{aligned}$$

where the first inequality is due to the convexity of quantum coherence [14] and the following equality is due to the fact that any measurement on any subsystem of a system cannot disturb the average state of the other subsystems.

We have used the notation $\rho_B := \mathrm{Tr}_A[\rho_{AB}]$ for the reduced density matrix of the state ρ_{AB} in the B part. This completes the proof. ■

Consider now an arbitrary set of mutually unbiased bases, $\{M_i\}_i$, for $i \in \{1, 2, 3\}$, of a single qubit, for which the elements of the bases can be expressed as follows:

$$\{|M_1^{\pm}\rangle\} = \Big\{\cos\frac{\theta'}{2}|0\rangle + e^{i\phi'}\sin\frac{\theta'}{2}|1\rangle, \times \sin\frac{\theta'}{2}|0\rangle - e^{i\phi'}\cos\frac{\theta'}{2}|1\rangle\Big\}, \tag{11}$$

$$|M_2^{\pm}\rangle = \frac{|M_1^+\rangle \pm |M_1^-\rangle}{\sqrt{2}}, \tag{12}$$

$$|M_3^{\pm}\rangle = \frac{|M_1^+\rangle \pm i\,|M_1^-\rangle}{\sqrt{2}}, \tag{13}$$

where $|0\rangle, |1\rangle$ are eigenvectors of the σ_z matrix, and θ', ϕ' are azimuthal and polar angles in spherical polar coordinates. Quantum coherences of a general single-qubit state in these bases are obtained in Appendix A. These are important for evaluation of the NAQC functionals later in the paper.

It has been realized that nonsteerable bipartite states can never exhibit nonlocal advantage of quantum coherence [27], and thus the same follows for the standard NAQC as well as the generalized NAQC functionals (see Appendix B). Therefore, separable states will never exhibit NAQC, as indicated by using the standard NAQC and generalized NAQC functionals [26].

A classical (probabilistic) mixture of two Bell states, say, $|\phi^+\rangle$ $(:= \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle])$ and $|\psi^+\rangle$ $(:= \frac{1}{\sqrt{2}}[|10\rangle + |01\rangle])$, can be represented as

$$\rho_{AB} = p\,|\phi^+\rangle\langle\phi^+| + (1-p)\,|\psi^+\rangle\langle\psi^+|, \tag{14}$$

with $p \in [0, 1]$. Using this set of states, we find that the generalized NAQC criterion detects strictly more two-qubit states exhibiting NAQC than the standard NAQC criterion. This is shown in Fig. 1. As another example, consider the two-qubit Werner states,

$$\rho_W = p\,|\phi^+\rangle\langle\phi^+| + \frac{(1-p)}{4}\mathbb{1}, \tag{15}$$

with $p \in [0, 1]$ and $\mathbb{1}$ representing the identity operator on the $\mathbb{C}^2 \otimes \mathbb{C}^2$ Hilbert space. It is known that two-qubit Werner states are entangled if $p > \frac{1}{3}$ [46,47] and steerable if $p > \frac{1}{2}$ [48]. Interestingly, we observe in Fig. 2 that there exist entangled and steerable states which do not exhibit nonlocal advantage of quantum coherence when the standard NAQC or the generalized NAQC functionals are taken into account. So, the set of states showing NAQC, using both the NAQC functionals, is still a strict subset of the set of entangled states, as well as that of the set of steerable states.

IV. MONOGAMY OF NAQC

Since a NAQC functional of a bipartite system involves quantum measurements on one of the subsystems and quantum coherence of reduced states on the other subsystem, monogamy of NAQC for tripartite state ρ_{ABC} can be seen from two different perspectives. In one of them, quantum coherence is measured on a fixed subsystem, A, while measurements for

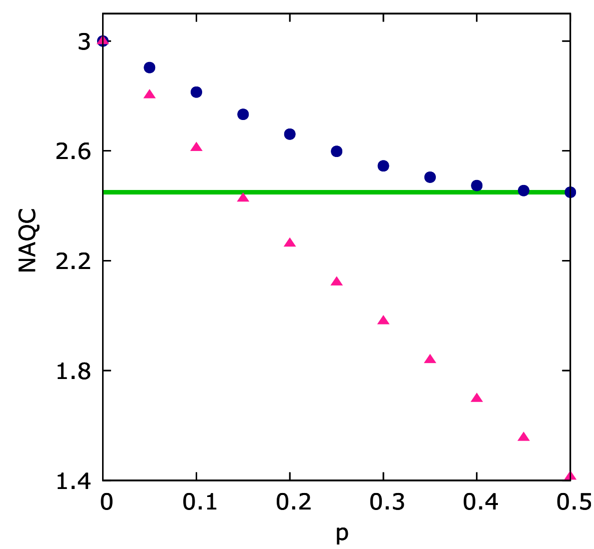

FIG. 1. NAQC for probabilistic mixtures of two Bell states. We plot the NAQC functionals (vertical axis) for mixtures of two Bell states, parametrized by the mixing parameter p (horizontal axis). Pink triangles and blue circles denote the standard and generalized NAQC functionals, respectively, whereas the green line represents the upper bound beyond which NAQC occurs in a two-qubit system. We find that the states with $p \lesssim 0.144$ exhibit NAQC using the standard criterion, whereas all states with $p \lesssim 0.5$ exhibit NAQC using the generalized criterion. This shows that the generalized criterion detects more two-qubit states exhibiting NAQC than the standard criterion. All quantities used are dimensionless.

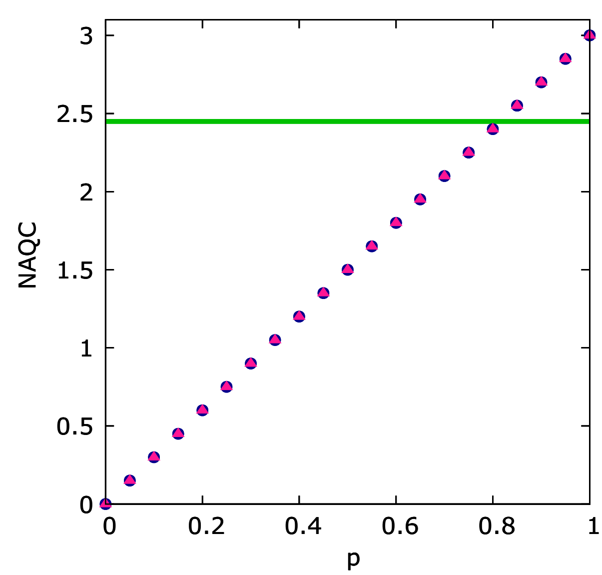

FIG. 2. Nature of NAQC for two-qubit Werner states. All considerations are the same as in Fig. 1 except that here the states under discussion are two-qubit Werner states. They exhibit NAQC for the mixing parameter, $p \gtrsim 0.815$, using both the NAQC functionals considered in this paper.

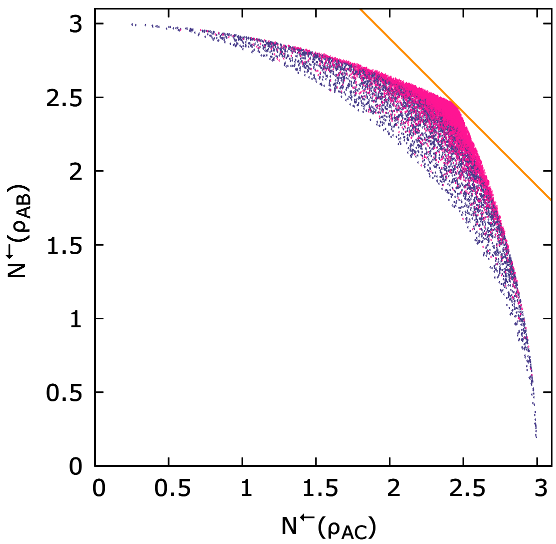

FIG. 3. Exclusion principle for NAQC in three-qubit system. We analyze here the nature of monogamy of the generalized NAQC functional when the reduced states of the fixed subsystem A are subjected to coherence measurement for any pure three-qubit state of ABC. We plot $\mathsf{N}^{\leftarrow}(\rho_{AC})$ along the horizontal axis and $\mathsf{N}^{\leftarrow}(\rho_{AB})$ along the vertical axis. GHZ- and W-class genuine three-qubit entangled pure states are represented by pink and dark blue circles, respectively, while the condition $\mathsf{N}^{\leftarrow}(\rho_{AC}) + \mathsf{N}^{\leftarrow}(\rho_{AB}) = 2\sqrt{6}$ is denoted by the orange straight line. All quantities used are dimensionless.

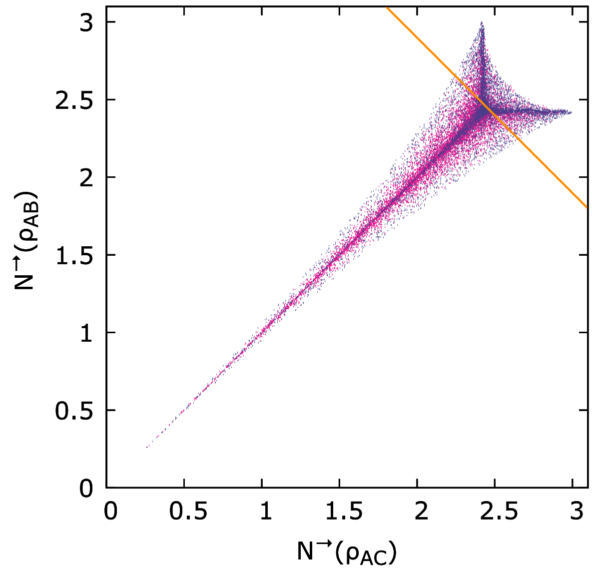

FIG. 4. Sharing of nonlocal advantage of quantum coherence, when ensemble-generating measurements are performed on a fixed party. We plot $\mathsf{N}^{\rightarrow}(\rho_{AB})$ on the vertical axis against $\mathsf{N}^{\rightarrow}(\rho_{AC})$ represented on the horizontal axis, for Haar-uniformly generated pure genuine three-qubit entangled states, viz. the states of the GHZ and W classes. GHZ- and W-class states are denoted by pink and dark blue points, respectively, whereas the orange line represents the equation of the straight line, $\mathsf{N}^{\rightarrow}(\rho_{AC}) + \mathsf{N}^{\rightarrow}(\rho_{AB}) = 2\sqrt{6}$. All quantities used are dimensionless.

creating ensembles at A are performed on one of the remaining subsystems, viz. B and C, so that the quantity under study is

$$\mathsf{N}^{\leftarrow}(\rho_{AB}) + \mathsf{N}^{\leftarrow}(\rho_{AC}). \tag{16}$$

In the other one, monogamy of NAQC is studied in the situation where quantum measurements for generating ensembles are performed on a fixed subsystem A of the multipartite system, and quantum coherence of states is measured by switching the remaining subsystems, i.e., the quantity studied is

$$\mathsf{N}^{\rightarrow}(\rho_{AB}) + \mathsf{N}^{\rightarrow}(\rho_{AC}). \tag{17}$$

Here, ρ_{AB} and ρ_{AC} denote reduced states of ρ_{ABC} by tracing out parties C and B, respectively. We now note that $2\sqrt{6}$ would be a nontrivial upper bound for the above sums, for monogamy of NAQC in the tripartite system ABC. The inequalities would then suggest that if the pair AB shows NAQC, then the pair AC will not exhibit NAQC, and vice versa. Such monogamy has previously been observed for Bell correlations [49–52] and for quantum dense coding [53].

A. Monogamy when coherence is measured on a fixed subsystem

We examine here monogamy of the generalized NAQC functional for three-qubit pure states ρ_{ABC}, where quantum coherence is measured on the fixed subsystem, A. We therefore focus on the expression in (16).

Note that the upper bound of (16) that can be achieved, by fully separable pure three-qubit states, is $2\sqrt{6}$. For all biseparable pure three-qubit states, the maximum of $\mathsf{N}^{\leftarrow}(\rho_{AB}) + \mathsf{N}^{\leftarrow}(\rho_{AC})$ is also $(2\sqrt{6})$. By employing a nonlinear optimization routine, we numerically analyze monogamy of the generalized NAQC functional for all GHZ- and W-class three-qubit pure states. We find that the expression in (16) can reach a maximum of $\mathsf{N}^{\leftarrow}(\rho_{AB}) + \mathsf{N}^{\leftarrow}(\rho_{AC}) \approx 4.899 \approx 2\sqrt{6}$. Since the upper bound of $\mathsf{N}^{\leftarrow}(\rho_{AB}) + \mathsf{N}^{\leftarrow}(\rho_{AC})$ for all three-qubit pure states is $2\sqrt{6}$, we therefore have a strong monogamy—an exclusion principle—for such states, as if ρ_{AB} exhibits NAQC, ρ_{AC} does not, and vice versa, where the quantum coherence measurements in both cases are at the party A. In Fig. 3, we also plot a scatter diagram of $\mathsf{N}^{\leftarrow}(\rho_{AB})$ versus $\mathsf{N}^{\leftarrow}(\rho_{AC})$ for Haar-uniformly generated Greenberger-Horne-Zeilinger–(GHZ-) [54,55] and W-class [44,56] three-qubit pure states.

B. When ensemble-preparing measurement is performed on a fixed subsystem

Finally, we analyze monogamy properties of the generalized NAQC functional for any three-qubit pure state under the assumption that the ensemble-generating projective measurements are performed on a fixed subsystem of the tripartite system.

We notice that the maximum of (17), for all fully separable and biseparable three-qubit pure states, would be $2\sqrt{6}$ and $(3 + \sqrt{6})$, respectively. We also observe that, unlike the previous case, a generic pure three-qubit state may not always satisfy the strong monogamy—the exclusion principle—for the generalized NAQC, when the ensemble-generating projective measurements are performed on a fixed subsystem. In other words, there exist genuine three-party entangled pure states that violate the inequality, $\mathsf{N}^{\rightarrow}(\rho_{AC}) + \mathsf{N}^{\rightarrow}(\rho_{AB}) \leqslant 2\sqrt{6}$. This is demonstrated in Fig. 4 by plotting $\mathsf{N}^{\rightarrow}(\rho_{AB})$

with respect to $\mathsf{N}^{\rightarrow}(\rho_{AC})$ for Haar uniformly generated pure three-qubit states.

V. CONCLUSION

Nonlocal advantage of quantum coherence captures a form of "steerability" of bipartite quantum states in terms of quantum coherence. There exists a quantum coherence complementarity relation for an isolated single-qubit system, stating that the sum of quantum coherences in any set of mutually unbiased bases is nontrivially bounded from above. A two-qubit state can exhibit NAQC if for some measurements on one of the subsystems, the sum of the average quantum coherences of the other subsystem violates the isolated single-qubit complementarity relation.

In this paper, we have considered detection of NAQC by using two criteria. The first one is referred to as the standard criterion, which allows measurements in a arbitrary set of MUBs on one of the subsystems of the bipartite system, for creating ensembles. The second criterion is termed the generalized criterion, as the restriction of the ensemble-generating measurement bases to be a set of MUBs is relaxed. In both the criteria, an optimization of the NAQC functionals over all the relevant measurement bases is considered, which makes both the NAQC functionals local unitarily invariant over the states. We also explicitly demonstrated that the generalized NAQC criterion performs better than the standard NAQC criterion, in that the former can capture a greater number of bipartite states exhibiting nonlocal advantage of quantum coherence. In addition, we provided a lower bound on both the NAQC functionals for bipartite systems in terms of the quantum coherence of a reduced state.

Lastly, we examine the monogamy of NAQC in tripartite systems in two different cases: first, when coherence of states of a fixed subsystem of the tripartite system is measured, and second, when the ensemble-generating projective measurements are carried out on a fixed subsystem of the tripartite system. In the first case, it is shown that all three-qubit pure states follow a strong monogamy—an exclusion principle—of NAQC. However, in the second case, the strong monogamy of NAQC may or may not be followed for general pure three-qubit states.

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APPENDIX A: QUANTUM COHERENCE OF A QUBIT

We provide here the quantum coherence of a qubit state with respect to a basis chosen from an arbitrary set of MUBs on the qubit Hilbert space. A single-qubit state, ρ, expressed in the σ_z basis, will have the form

$$\rho := \begin{pmatrix} \rho_{00} & \rho_{10} \\ \rho_{01} & \rho_{11} \end{pmatrix}.$$

The l_1-norm of quantum coherence with respect to the bases in Eqs. (11), (12), and (13) will be

$$C_{M_1^\pm} = 2\left|\frac{1}{2}\sin\theta'\rho_{00} - \frac{1}{2}\sin\theta'\rho_{11} - e^{i\phi'}\cos^2\frac{\theta'}{2}\rho_{01} + e^{-i\phi'}\sin^2\frac{\theta'}{2}\rho_{10}\right|, \tag{A1}$$

$$C_{M_2^\pm} = |\cos\theta'\rho_{00} - \cos\theta'\rho_{11} + e^{i\phi'}(1+\sin\theta')\rho_{01} - e^{-i\phi'}(1-\sin\theta')\rho_{10}|, \tag{A2}$$

$$C_{M_3^\pm} = |\rho_{00} - \rho_{11} + ie^{i\phi'}\rho_{01} + ie^{-i\phi'}\rho_{10}|. \tag{A3}$$

APPENDIX B: PROOF OF NONEXHIBITION OF GENERALIZED NAQC BY ANY NONSTEERABLE TWO-QUBIT STATE

In quantum information tasks, steerability [57] is a significant quantum resource. Let us assume that Alice and Bob are situated in two distant labs, and share a state, ρ_{AB}, between them. Suppose that Alice makes a measurement in the setting "x" and obtains the outcome "a." Let the conditional state obtained thereby at Bob be $\rho(a|x)$. Assume also that there exists a hidden variable λ, distributed as p_λ, such that there exists a "hidden" state, $\sigma_B(\lambda)$ at Bob and a conditional probability $p(a|x,\lambda)$ with

$$\rho(a|x) = \sum_\lambda p_\lambda p(a|x,\lambda)\sigma_B(\lambda).$$

Then we call the state ρ_{AB} nonsteerable if the above relation is available for every setting x and every outcome a. Otherwise, it is steerable.

The NAQC for any nonsteerable bipartite state can be determined as

$$\begin{aligned}
&\mathcal{N}^{\rightarrow}(\rho_{AB}) \\
&:= \max_{M_i,\Lambda_i^a}\sum_{i,a} p\big(\rho_{B|\Lambda_i^a}\big)C_{M_i}\big(\rho_{B|\Lambda_i^a}\big) \\
&= \max_{M_i,\Lambda_i^a}\sum_{i,a} p\big(\rho_{B|\Lambda_i^a}\big)C_{M_i}\left(\frac{\sum_\lambda p_\lambda p\big(a|\Lambda_i^a,\lambda\big)\sigma_B(\lambda)}{p\big(\rho_{B|\Lambda_i^a}\big)}\right) \\
&\leqslant \max_{M_i,\Lambda_i^a}\sum_{i,a,\lambda} p_\lambda p\big(a|\Lambda_i^a,\lambda\big)C_{M_i}[\sigma_B(\lambda)], \\
&= \max_{M_i}\sum_{i,\lambda} p_\lambda C_{M_i}[\sigma_B(\lambda)] \\
&\leqslant \sum_{i,\lambda} p_\lambda\sqrt{6} = \sqrt{6},
\end{aligned}$$

where the first and second inequalities are obtained, respectively, by utilizing the convexity of quantum coherence and the coherence complementarity relation of single-qubit systems.

Hence, any nonsteerable two-qubit state will never exhibit NAQC. Analogously, it can be proved for the generalized NAQC functional.

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