Distinguishability-based genuine nonlocality with genuine multipartite entanglement

Zong-Xing Xiong,[1,*] Mao-Sheng Li,[2] Zhu-Jun Zheng,[2] and Lvzhou Li[1,†]

[1]*Institute of Quantum Computing and Software, School of Computer Science and Engineering, Sun Yat-Sen University, Guangzhou 510006, China*

[2]*School of Mathematics, South China University of Technology, Guangzhou 510641, China*

(Received 8 May 2023; accepted 7 July 2023; published 8 August 2023)

A set of orthogonal multipartite quantum states is said to be distinguishability-based genuinely nonlocal (also genuinely nonlocal, for abbreviation) if the states are locally indistinguishable across any bipartition of the subsystems. This form of multipartite nonlocality, although more naturally arising than the recently popular "strong nonlocality" in the context of local distinguishability, receives much less attention. In this work, we study the distinguishability-based genuine nonlocality of a typical type of genuine multipartite entangled states—the d-dimensional Greenberger-Horne-Zeilinger (GHZ) states, featuring systems with local dimension not limited to two. In the three-partite case, we find the existence of small genuinely nonlocal sets consisting of these states: we show that the cardinality can at least scale down to linear in the local dimension d, with the linear factor $l = 1$. Specifically, the method we use is semidefinite programming and the GHZ states to construct these sets are special ones which we call "GHZ lattices". This result might arguably suggest a significant gap between the strength of strong nonlocality and the distinguishability-based genuine nonlocality. Moreover, we put forward the notion of (s, n)-threshold distinguishability and, utilizing a similar method, we successfully construct (2,3)-threshold sets consisting of GHZ states in three-partite systems.

DOI: 10.1103/PhysRevA.108.022405

I. INTRODUCTION

Quantum nonlocality is one of the most surprising properties in quantum mechanics. The most well-known manifestation of nonlocality—Bell nonlocality [1–3], which is revealed by violation of Bell-type inequalities [3–8]—can only arise from entangled states. Apart from Bell nonlocality, there are also other forms of nonlocality. Among them, the distinguishability-based nonlocality, which concerns the problem of locally distinguishing a certain set of orthogonal quantum states, has attracted much attention. It serves to explore the fundamental question about locally accessing global information and also the relation between entanglement and locality, which are of central interest in quantum information theory. Unlike Bell nonlocality, however, entanglement is not necessary for this kind of nonlocality. In [9], Bennett *et al.* presented the first example of orthogonal product states that are indistinguishable when only local operations and classical communications are allowed, which was known as "quantum nonlocality without entanglement". From then on, such nonlocality based on distinguishability has been studied extensively (see [9–39] for an incomplete list). Moreover, the local discrimination of quantum states has been practically applied in a number of distributed quantum protocols such as quantum data hiding [40–43] and quantum secret sharing [44–50].

Although not necessary for distinguishability-based nonlocality, evidences have been found showing that entanglement can somehow raise difficulty in quantum state discrimination in many occasions. To what extent entanglement is responsible for indistinguishability has been one of the main focuses [14–17]. In [15], Hayashi *et al.* observed that the number of pure states that can be perfectly distinguished locally is bounded above by the total dimension over the average entanglement of the states. This result quantitatively shows that states with more entanglement will generally (while not always) be more difficult to be distinguished. Meanwhile, lots of studies had focused on local distinguishability of the maximally entangled states [19–25], as opposed to studying the distinguishability of product states. Notably, it can be deduced directly from the results of Hayashi *et al.* [15] that any $k > d$ orthogonal maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$ are not locally distinguishable. This fact was actually first revealed by Nathanson, who also showed that any three orthogonal maximally entangled states in $\mathbb{C}^3 \otimes \mathbb{C}^3$ are locally distinguishable [26]. In spite of these quantitative results indicating the maximal number of perfectly distinguishable states, we have, on the other hand, little idea on the minimal number of locally indistinguishable states. What is known to us is that any two orthogonal pure states can always be locally distinguished, no matter whether the states are entangled or not [11]. Frustratingly, whether there exist three locally indistinguishable maximally entangled states in system $\mathbb{C}^d \otimes \mathbb{C}^d$ for $d > 3$ remains unknown today. Therefore, efforts had been paid in seeking sets of $k \leqslant d$ orthogonal maximally entangled states that are not perfectly distinguishable by local operations and classical communication (LOCC) [26–33]. The best result so far is due to Yu and Oh [33], who showed that the cardinality of indistinguishable sets of maximally entangled states

*hongbe123@outlook.com

†lilvzh@mail.sysu.edu.cn

in $\mathbb{C}^d \otimes \mathbb{C}^d$ can asymptotically scale down to $3d/4$ when d is large.

Not long ago, Halder *et al.* [51] introduced the concept of local irreducibility, which is a stronger form of local indistinguishability. A set of orthogonal multipartite quantum states is said to be locally irreducible if one cannot eliminate one or more states from the whole set, with the restriction that only orthogonality-preserving local measurements are allowed. They revealed the phenomenon of "strong nonlocality without entanglement", which is a nontrivial generation of Bennett's nonlocality without entanglement [9]. Strong nonlocality of a set of orthogonal multipartite quantum states refers to local irreducibility of these states through every bipartition of the subsystems and the authors of [51] presented the first examples of strong nonlocal sets of product states in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$. After that, much attention has been paid to study the strong nonlocality of multipartite quantum states [52–62]. In [56], the authors constructed $6(d-1)^2$ orthogonal product states in three-partite systems $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ that is strongly nonlocal. Soon after, Wang *et al.* successfully constructed strongly nonlocal sets with genuine multipartite entanglement [57]. The strongly nonlocal sets they constructed consist of GHZ states in the three-partite systems $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ ($d \geqslant 3$) and have cardinality $d^3-(d-2)^3$ $[d^3-(d-2)^3+2]$ when d is odd (even). Shi *et al.* gave more general results in $(\mathbb{C}^d)^{\otimes N}$ ($N \geqslant 3$), showing the existence of strongly nonlocal orthogonal entangled sets with size $d^N-(d-1)^N+1$ [58]. They also constructed 18 strongly nonlocal genuine multipartite entangled states in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, outperforming the former result of 19 strongly nonlocal unextendible product basis in the same tripartite system [59].

In [63], Rout *et al.* first studied the concept of distinguishability-based genuine nonlocality (there the authors named it genuine nonlocality simply; we also adopt this abbreviation here sometimes when no ambiguity occurs): a set of orthogonal multipartite quantum states is said to be (distinguishability-based) genuinely nonlocal if the states are locally indistinguishable across any bipartition of the subsystems. Later, in [64,65], the authors constructed sets of multipartite product states which are genuinely nonlocal. Namely, they found genuine nonlocality without entanglement. This concept of multipartite nonlocality is weaker but more natural than strong nonlocality in the context of distinguishability. It is obvious that strong nonlocality always implies genuine nonlocality [51], yet to what extent the former is stronger than the latter, little is known.

In this paper, we are to study the distinguishability-based genuine nonlocality of a typical type of genuine multipartite entangled states—the (d-dimensional) Greenberger-Horne-Zeilinger (GHZ) states on qudit systems. More explicitly, we focus on the three-partite case, where three-qudit GHZ states are maximally entangled in the sense that they are maximally entangled in all bipartitions. Similar to the bipartite case when maximally entangled states are studied, one can derive from the result of Hayashi *et al.* [15] that any $k > d^2$ three-qudit GHZ states are genuinely nonlocal. However, we have yet no idea about the minimal number of such states having the property of genuine nonlocality. Here, we make a step forward by showing the existence of certain genuinely nonlocal sets of GHZ states that have rather small cardinality. We find that the cardinality can at least scale down to $d+3$, for the cases when local dimension d are powers of 2. The method we use here is inspired by Cosentino [30,31], and the special kind of GHZ states to construct these sets is referred to as "GHZ lattices" by us. Until now, since no existing strongly nonlocal sets that have been constructed have such small magnitude, it is reasonable for us to argue that there might exist a substantial difference between the strength of strong nonlocality and the distinguishability-based genuine nonlocality. Furthermore, we put forward the notion of (s,n)-threshold distinguishability in n-partite systems as an extension of genuine nonlocality. Adopting the similar method of semidefinite programming in our former discussion, we successfully construct (2,3)-threshold sets consisting of GHZ states in three-partite systems.

The rest of this paper is organized as follows: In Sec. II, we present some necessary notations and definitions. In Sec. III, we study the genuine nonlocality of the GHZ states in three-partite system $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$. Taking advantage of the structure of "GHZ lattices" and also the method of semidefinite programming, we construct genuinely nonlocal sets of these states with conspicuously small cardinality. In Sec. IV, we extend the notion of genuine nonlocality to the more general (s,n)-threshold distinguishability in multipartite systems and we construct (2,3)-threshold sets consisting of GHZ lattices in three-partite systems. Finally, we draw our conclusion and present some related problems in Sec. V.

II. PRELIMINARIES

Genuine multipartite entanglement. In bipartite systems $\mathcal{H}_A \otimes \mathcal{H}_B$, a pure state $|\Psi\rangle_{AB}$ is called entangled if it cannot be written as the tensor product of two pure states of the two subsystems. Namely, it is not of the form $|\Psi\rangle_{AB} = |\alpha\rangle_A \otimes |\beta\rangle_B$. For pure state $|\Psi\rangle_{A_1\cdots A_n}$ in multipartite system $\mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_n}$, it is called genuinely multipartite entangled if it is entangled for any bipartition of the subsystems $\{A_1,\ldots,A_n\}$. The most well-known kind of genuinely (three-partite) entangled states are the GHZ states and the W states in three-qubit systems $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. They are in fact the only two types of genuine tripartite entangled states in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ up to stochastic LOCC (SLOCC) equivalence [66,67].

d-dimensional GHZ states [68]. In the n-partite system $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$, the states of the form

$$\frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \left|\xi_k^{(1)} \xi_k^{(2)} \cdots \xi_k^{(n)}\right\rangle \tag{1}$$

are called d-dimensional GHZ states, where each $\{|\xi_k^{(i)}\rangle\}_{k=0}^{d-1}$ is any set of orthogonal basis for the ith subsystem. These states can be proved to be genuinely entangled and they have been considered generic resources in many quantum information processing tasks [44–50,69–71].

Local distinguishability. A set of orthogonal (pure) multipartite quantum states, which is priorly known to several spatially separated parties, is said to be locally distinguishable if the parties are able to tell exactly which state they share through some protocol, provided only local measurements and classical communications are allowed. In the literature, since the mathematical structure of LOCC measurement is rather complicated, it is usually the separable measurement or the positive-partial-transpose (PPT) measurement that is considered [72–76].

Local irreducibility. A set of multipartite orthogonal quantum states is said to be locally irreducible if it is not possible to locally eliminate one or more states from the set while preserving orthogonality of the postmeasurement states. Typical examples of locally irreducible sets include the two-qubit Bell basis and the three-qubit GHZ basis [51]. While a locally irreducible set must be locally indistinguishable, the opposite is generally not true.

Distinguishability-based genuine nonlocality and strong nonlocality. A set of orthogonal multipartite quantum states is called (distinguishability-based) genuinely nonlocal if the states are locally indistinguishable across every bipartition of the subsystems. Furthermore, if the states are locally irreducible across every bipartition, then they are called strongly nonlocal. Straightforwardly, a strongly nonlocal set must be also genuinely nonlocal, while the converse is not true. As it is shown immediately in the next section, while the three-qubit GHZ basis (2) failed to be strongly nonlocal [51], it is genuinely nonlocal obviously.

III. GENUINE NONLOCALITY FOR THE THREE-QUDIT GHZ STATES

In three-qudit system $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, the d-dimensional GHZ states are maximally entangled in the sense that all their reductions to $\lfloor \frac{3}{2} \rfloor$-qudit are maximally mixed. Here in this section, we discuss the genuine nonlocality of this typical kind of genuine three-partite entangled state. Roughly speaking, a set of more states might often appear harder for distinguishing while a set with less states is usually more likely to be distinguishable. For example, all supersets of an indistinguishable set are always indistinguishable, while, on the other hand, all subsets of a distinguishable one are certainly also distinguishable. Therefore, one is interested in both the maximal number of states that might be distinguishable and also the minimal number of states that are indistinguishable (in our case, genuinely nonlocal). The following lemma that can be derived from Hayashi *et al.* [15] shows that the number of d-dimensional GHZ states in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ that are not genuinely nonlocal is at most d^2.

Lemma 1 ([15]). In three-partite systems $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, any set of $s > d^2$ orthogonal d-dimensional GHZ states is genuinely nonlocal.

Here, however, we will mainly focus on seeking small number of these states that are genuinely nonlocal. We first discuss the simplest situation when $d = 2$, namely, the three-qubit case. In $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, the following eight orthogonal states,

$$|\psi_{0,7}\rangle = \frac{|000\rangle \pm |111\rangle}{\sqrt{2}}, \quad |\psi_{1,6}\rangle = \frac{|001\rangle \pm |110\rangle}{\sqrt{2}},$$
$$|\psi_{2,5}\rangle = \frac{|010\rangle \pm |101\rangle}{\sqrt{2}}, \quad |\psi_{3,4}\rangle = \frac{|011\rangle \pm |100\rangle}{\sqrt{2}}, \qquad (2)$$

consist a set of orthogonal basis. Here, we place the conjugate pairs together and assign them subscript indices which sum up to $2^3 - 1$. They are referred to as the three-qubit GHZ basis and are important in a number of quantum information processing scenarios. In [51], the authors showed that these states are not strongly nonlocal. Namely, when certain two of the three parties are allowed to join together, it is possible to locally eliminate one or more states from the whole set while preserving the orthogonality of the postmeasurement states. Here, instead of discussing the reducibility, we focus on the bipartite distinguishability problem of the three-qubit GHZ basis. It is immediate from Lemma 1 that any $s \geqslant 5$ states of the three-qubit GHZ basis (2) are distinguishability-based genuinely nonlocal. For the other side, results of [17] showed that this bound is tight, in the sense that there exist four states among the three-qubit GHZ basis (2) that are not genuinely nonlocal. Here, we further show that any four states among the three-qubit GHZ basis are not genuinely nonlocal. We give an elegant and visible way to demonstrate this fact.

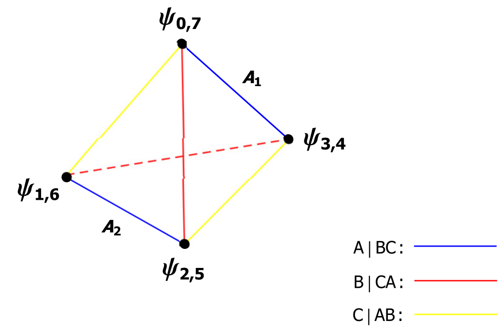

FIG. 1. A three-dimensional schematic picture for local distinguishability of the three-qubit GHZ basis (2) in all bipartitions. Each vertex of the tetrahedron represents one conjugate pair of the basis. Four states on one edge are locally equivalent to the two-qubit Bell basis $\{|00\rangle \pm |11\rangle, |01\rangle \pm |10\rangle\}$, within the bipartition shown by the color of that edge. For instance, in bipartition $A|BC$, the four states of $\mathcal{A}_1$ are locally equivalent to the Bell basis and the same holds for $\mathcal{A}_2$. These two sets lie on the opposite edge of the tetrahedron colored blue, and states from different sets can be told apart if BC perform a two-outcome joint measurement $\{P_1^{(BC)} = |00\rangle\langle 00| + |11\rangle\langle 11|, P_2^{(BC)} = |01\rangle\langle 01| + |10\rangle\langle 10|\}$.

Proposition 1. Any four states of the three-qubit GHZ basis (2) can be locally distinguished by at least one of the bipartition $A|BC$, $B|CA$, or $C|AB$. That is, they are not genuinely nonlocal.

Proof. We demonstrate this fact with the aid of a schematic picture shown by Fig. 1. The four conjugate pairs are represented by four vertices of the tetrahedron and the colored edges represent different bipartitions. Obviously, the four states to be distinguished must be located on no less than two vertices of the tetrahedron in Fig. 1. If the four states are located on four vertices of the tetrahedron, they can obviously be distinguished across all the bipartitions. For example, in bipartition $A|BC$, the qubit holders BC can first perform local measurement $\{P_1^{(BC)} = |00\rangle\langle 00| + |11\rangle\langle 11|, P_2^{(BC)} = |01\rangle\langle 01| + |10\rangle\langle 10|\}$ to reduce the states into two disjoint sets, with each being a 2-ary subset of $\{|\psi_0\rangle, |\psi_7\rangle, |\psi_3\rangle, |\psi_4\rangle\}$ and $\{|\psi_1\rangle, |\psi_6\rangle, |\psi_2\rangle, |\psi_5\rangle\}$ respectively. Since any two states are always locally distinguishable [11], the four states can be locally distinguished across bipartition $A|BC$. The same holds for $B|CA$ and $C|AB$. If the four states to be distinguished are distributed on three vertices, they must be located on a certain face of the tetrahedron. Without loss of generality, suppose that these states are $\{|\psi_0\rangle, |\psi_7\rangle, |\psi_1\rangle, |\psi_2\rangle\}$. In this case, only in bipartition $A|BC$ can they be distinguished, for in $B|CA$ ($C|AB$), the subset $\{|\psi_0\rangle, |\psi_7\rangle, |\psi_2\rangle\}$ ($\{|\psi_0\rangle, |\psi_7\rangle, |\psi_1\rangle\}$) is locally equivalent to three states among the Bell basis, which are known to be locally indistinguishable. If the four states to be distinguished are on two vertices, they must be located on a certain edge. As an example, for $\{|\psi_0\rangle, |\psi_7\rangle, |\psi_3\rangle, |\psi_4\rangle\}$, they are not locally distinguishable through $A|BC$ while they can be distinguished through $B|CA$ and $C|AB$. To sum up, any four states of the three-qubit GHZ basis (2) can be distinguished through at least one of the three bipartitions, i.e., they are not genuinely nonlocal. ■

Despite the nonexistence of genuinely nonlocal subsets of the three-qubit GHZ basis with cardinality $s < 2^2 + 1$, in systems of higher local dimension, we are using them to construct genuinely nonlocal sets of three-partite GHZ states with small cardinality—much smaller than the trivial cardinality $d^2 + 1$ (indicated by Lemma 1). What we are to discuss here are the special cases where the local dimension $d = 2^t$, and the GHZ states are "lattice-type" ones, which are tensor products of the three-qubit GHZ basis (2). The idea here is primarily inspired by [30], where small indistinguishable sets in bipartite systems were constructed using lattice-type maximally entangled states. Let $\vec{v} = (v_1, \dots, v_t) \in \{0, 1, \dots, 7\}^t$ be a t-element vector ($t \geqslant 1$). Then the t-level *lattice-type GHZ state* with index $\vec{v}$ in the three-partite system $\mathbb{C}^{2^t} \otimes \mathbb{C}^{2^t} \otimes \mathbb{C}^{2^t}$ is defined by $|\Psi_{\vec{v}}\rangle = |\psi_{v_1}\rangle \otimes \cdots \otimes |\psi_{v_t}\rangle$. One can easily check that these 8^t states are 2^t-dimensional GHZ states and they make up a set of orthogonal basis in $\mathbb{C}^{2^t} \otimes \mathbb{C}^{2^t} \otimes \mathbb{C}^{2^t}$. For abbreviation, here and after we also call this form of GHZ states "GHZ lattices". It turns out when the multipartite states to be distinguished are all GHZ lattices, the special structure of these states enables us to discuss their genuine nonlocality with much less effort.

Since LOCC distinguishability is difficult to tackle mathematically, here instead we study the PPT distinguishability of the multipartite states, for different bipartition, respectively. For bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$, if a set of orthogonal states is locally distinguishable, the states must be also PPT distinguishable. To be more precise, the orthogonal bipartite states $\{|\phi_1\rangle, \dots, |\phi_s\rangle\}$ (here and after, it is also denoted as $\{\phi_1, \dots, \phi_s\}$ where $\phi_i = |\phi_i\rangle\langle\phi_i|$'s are the corresponding density operators) are PPT distinguishable if and only if the semidefinite program

$$\begin{aligned}\alpha = \max_{P_1,\dots,P_s} \frac{1}{s} \sum_{k=1}^{s} \mathrm{Tr}(P_k \phi_k)\\ \text{such that } \quad & P_1 + \cdots + P_s = I_{AB},\\ & P_1, \dots, P_s \geqslant 0,\\ & \mathrm{T}_A(P_1), \dots, \mathrm{T}_A(P_s) \geqslant 0, \qquad (3)\end{aligned}$$

has optimal value $\alpha = 1$ [30]. Here we also apply this semidefinite program to construct genuinely nonlocal sets on multipartite systems. For a set of orthogonal quantum states $\mathcal{S} = \{|\Phi_1\rangle, \dots, |\Phi_s\rangle\}$ on three-partite systems $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, a sufficient condition for their genuine nonlocality is that across each bipartition these states are PPT indistinguishable. More explicitly, consider the family of semidefinite programs

$$\begin{aligned}\alpha_X = \max_{P_1^{(X)},\dots,P_s^{(X)}} \frac{1}{s} \sum_{k=1}^{s} \mathrm{Tr}\big(P_k^{(X)} \Phi_k\big)\\ \text{such that } \quad & P_1^{(X)} + \cdots + P_s^{(X)} = I_{ABC},\\ & P_1^{(X)}, \dots, P_s^{(X)} \geqslant 0,\\ & \mathrm{T}_X\big(P_1^{(X)}\big), \dots, \mathrm{T}_X\big(P_s^{(X)}\big) \geqslant 0, \qquad (4)\end{aligned}$$

where $X = A, B$ or C represents that we are dealing with the bipartition $A|BC$, $B|CA$ or $C|AB$ respectively. If for a specific set $\mathcal{S}$ we have $\alpha_X < 1 (\forall X \in \{A, B, C\})$, namely, PPT indistinguishable through all the bipartitions, then the set is genuinely nonlocal. For convenience of calculation, we further consider the dual problems

$$\begin{aligned}\beta_X = \min_{Y^{(X)},Q_1^{(X)},\dots,Q_s^{(X)}} \frac{1}{s}\, Tr(Y^{(X)})\\ \text{such that } \quad & Y^{(X)} - \Phi_k \geqslant \mathrm{T}_X\big(Q_k^{(X)}\big),\\ & Q_k^{(X)} \geqslant 0 \qquad (1 \leqslant k \leqslant s), \qquad (5)\end{aligned}$$

where $X \in \{A, B, C\}$. By the Slater condition, we know that strong duality holds: $\alpha_X = \beta_X$ $(X \in \{A, B, C\})$. Therefore, given a set $\mathcal{S} = \{|\Phi_1\rangle, \dots, |\Phi_s\rangle\}$ of orthogonal three-partite states, once we found that $\beta_X < 1 (\forall X \in \{A, B, C\})$, then the genuine nonlocality of $\mathcal{S}$ will be concluded.

To use these semidefinite programs to construct genuinely nonlocal sets of GHZ lattices, notice that for subsystems $X \in \{A, B, C\}$, $\mathrm{T}_{X_1 \cdots X_t}(\Psi_{\vec{v}}) = \mathrm{T}_{X_1}(\psi_{v_1}) \otimes \cdots \otimes \mathrm{T}_{X_t}(\psi_{v_t})$, where $\psi_{v_i} = |\psi_{v_i}\rangle\langle\psi_{v_i}|$ ($\Psi_{\vec{v}} = |\Psi_{\vec{v}}\rangle\langle\Psi_{\vec{v}}|$) are the corresponding density operators of the three-qubit GHZ basis (t-level GHZ lattices). The subscripts "$X_1 \cdots X_t$" indicate that t qubits are being held by $X = A$, B, or C and we sometimes abbreviate it as "X" when no ambiguity occurs. We should also note that the partial-transpose operations T_X are linear maps on $\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$, the space of all linear operators on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Moreover, when GHZ lattices are considered, we have the following property about T_A, T_B, and T_C that is crucial for all subsequent discussions.

Lemma 2. In the linear space $\mathcal{L}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$ of all linear operators on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, the linear maps T_A, T_B and T_C act invariantly on the subspace spanned by $\{\psi_0, \dots, \psi_7\}$.

Moreover, on this subspace they have matrix representations:

$$\mathrm{T}_A\begin{bmatrix}\psi_0\\\psi_7\\\psi_3\\\psi_4\\\psi_1\\\psi_6\\\psi_2\\\psi_5\end{bmatrix}=\begin{bmatrix}\frac12&\frac12&\frac12&-\frac12&&&&\\\frac12&\frac12&-\frac12&\frac12&&0&&\\\frac12&-\frac12&\frac12&\frac12&&&&\\-\frac12&\frac12&\frac12&\frac12&&&&\\&&&&\frac12&\frac12&\frac12&-\frac12\\&0&&&\frac12&\frac12&-\frac12&\frac12\\&&&&\frac12&-\frac12&\frac12&\frac12\\&&&&-\frac12&\frac12&\frac12&\frac12\end{bmatrix}\begin{bmatrix}\psi_0\\\psi_7\\\psi_3\\\psi_4\\\psi_1\\\psi_6\\\psi_2\\\psi_5\end{bmatrix},$$

$$\mathrm{T}_B\begin{bmatrix}\psi_0\\\psi_7\\\psi_2\\\psi_5\\\psi_3\\\psi_4\\\psi_1\\\psi_6\end{bmatrix}=\begin{bmatrix}\frac12&\frac12&\frac12&-\frac12&&&&\\\frac12&\frac12&-\frac12&\frac12&&0&&\\\frac12&-\frac12&\frac12&\frac12&&&&\\-\frac12&\frac12&\frac12&\frac12&&&&\\&&&&\frac12&\frac12&\frac12&-\frac12\\&0&&&\frac12&\frac12&-\frac12&\frac12\\&&&&\frac12&-\frac12&\frac12&\frac12\\&&&&-\frac12&\frac12&\frac12&\frac12\end{bmatrix}\begin{bmatrix}\psi_0\\\psi_7\\\psi_2\\\psi_5\\\psi_3\\\psi_4\\\psi_1\\\psi_6\end{bmatrix},$$

$$\mathrm{T}_C\begin{bmatrix}\psi_0\\\psi_7\\\psi_1\\\psi_6\\\psi_2\\\psi_5\\\psi_3\\\psi_4\end{bmatrix}=\begin{bmatrix}\frac12&\frac12&\frac12&-\frac12&&&&\\\frac12&\frac12&-\frac12&\frac12&&0&&\\\frac12&-\frac12&\frac12&\frac12&&&&\\-\frac12&\frac12&\frac12&\frac12&&&&\\&&&&\frac12&\frac12&\frac12&-\frac12\\&0&&&\frac12&\frac12&-\frac12&\frac12\\&&&&\frac12&-\frac12&\frac12&\frac12\\&&&&-\frac12&\frac12&\frac12&\frac12\end{bmatrix}\begin{bmatrix}\psi_0\\\psi_7\\\psi_1\\\psi_6\\\psi_2\\\psi_5\\\psi_3\\\psi_4\end{bmatrix}. \tag{6}$$

The proof is straightforward by routine calculation, which is shown in the Appendix. Now starting from the case $t=1$, consider the subset $\mathcal{S}_5=\{\psi_0,\psi_1,\psi_2,\psi_3,\psi_4\}$ of the three-qubit GHZ basis, namely, $\Phi_k=\psi_{k-1}$ $(k=1,\dots,5)$. By the matrix representations (6) in Lemma 2, one can check that what we list in the following is a set of feasible solutions to the family of dual problems (5) [namely, the constraints $Y^{(X)}-\Phi_k\geqslant \mathrm{T}_X(Q_k^{(X)})$ are satisfied for $1\leqslant k\leqslant 5$ and $X\in\{A,B,C\}$]:

$$\begin{aligned}&Y^{(A)}=\tfrac12\psi_0+\psi_1+\psi_2+\tfrac12\psi_3+\tfrac12\psi_4+\tfrac12\psi_7,\\&Q_1^{(A)}=\psi_4,\ \ Q_2^{(A)}=Q_3^{(A)}=0,\ \ Q_4^{(A)}=\psi_7,\ \ Q_5^{(A)}=\psi_0;\\&Y^{(B)}=\psi_0+\tfrac12\psi_1+\psi_2+\tfrac12\psi_3+\tfrac12\psi_4+\tfrac12\psi_6,\\&Q_1^{(B)}=0,\ \ Q_2^{(B)}=\psi_4,\ \ Q_3^{(B)}=0,\ \ Q_4^{(B)}=\psi_6,\ \ Q_5^{(B)}=\psi_1;\\&Y^{(C)}=\psi_0+\psi_1+\tfrac12\psi_2+\tfrac12\psi_3+\tfrac12\psi_4+\tfrac12\psi_5,\\&Q_1^{(C)}=Q_2^{(C)}=0,\ \ Q_3^{(C)}=\psi_4,\ \ Q_4^{(C)}=\psi_5,\ \ Q_5^{(C)}=\psi_2.\end{aligned}\tag{7}$$

Here as an example, we check only the inequality constraints for bipartition $A|BC$, and the others are similar by routine calculation:

$$\begin{aligned}\mathrm{T}_A\big(Q_1^{(A)}\big)&=-\tfrac12\psi_0+\tfrac12\psi_7+\tfrac12\psi_3+\tfrac12\psi_4\\&<-\tfrac12\psi_0+\tfrac12\psi_7+\tfrac12\psi_3+\tfrac12\psi_4+(\psi_1+\psi_2)\\&=Y^{(A)}-\psi_0,\\\mathrm{T}_A\big(Q_2^{(A)}\big)&=0<\tfrac12\psi_0+\psi_2+\tfrac12\psi_3+\tfrac12\psi_4+\tfrac12\psi_7\\&=Y^{(A)}-\psi_1,\\\mathrm{T}_A\big(Q_3^{(A)}\big)&=0<\tfrac12\psi_0+\psi_1+\tfrac12\psi_3+\tfrac12\psi_4+\tfrac12\psi_7\\&=Y^{(A)}-\psi_2,\\\mathrm{T}_A\big(Q_4^{(A)}\big)&=\tfrac12\psi_0+\tfrac12\psi_7-\tfrac12\psi_3+\tfrac12\psi_4\\&<\tfrac12\psi_0+\tfrac12\psi_7-\tfrac12\psi_3+\tfrac12\psi_4+(\psi_1+\psi_2)\\&=Y^{(A)}-\psi_3,\\\mathrm{T}_A\big(Q_5^{(A)}\big)&=\tfrac12\psi_0+\tfrac12\psi_7+\tfrac12\psi_3-\tfrac12\psi_4\\&<\tfrac12\psi_0+\tfrac12\psi_7+\tfrac12\psi_3-\tfrac12\psi_4+(\psi_1+\psi_2)\\&=Y^{(A)}-\psi_4.\end{aligned}\tag{8}$$

The target values for the three bipartitions are $4/5$, $4/5$ and $4/5$, respectively. Therefore, the set $\mathcal{S}_5$ is genuinely nonlocal, which is consistent with with our former discussion. Now with these, we give a simple construction of a genuinely nonlocal set of GHZ lattices in $\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4$, of which the cardinality is considerably smaller than 4^2+1 (which is a trivial cardinality by Lemma 1).

Proposition 2. In $\mathbb{C}^4\otimes\mathbb{C}^4\otimes\mathbb{C}^4$, the following set of orthogonal lattice-type GHZ states:

$$\begin{aligned}\mathcal{S}_{10}=\{&\psi_0\otimes\psi_0,\psi_1\otimes\psi_0,\psi_2\otimes\psi_0,\psi_3\otimes\psi_0,\psi_4\otimes\psi_0,\\&\psi_0\otimes\psi_7,\psi_1\otimes\psi_7,\psi_2\otimes\psi_7,\psi_3\otimes\psi_7,\psi_4\otimes\psi_7\},\end{aligned}$$

is genuinely nonlocal.

Proof. Notice that by matrix representations (6), we have $\mathrm{T}_X(\psi_0+\psi_7)=\psi_0+\psi_7$ $\forall X\in\{A,B,C\}$. From the above discussion, the constraints $Y^{(X_1)}-\psi_{k-1}\geqslant \mathrm{T}_{X_1}(Q_k^{(X_1)})$ are satisfied for $1\leqslant k\leqslant 5$ and $X_1\in\{A_1,B_1,C_1\}$, so we have

$$\begin{aligned}&Y^{(X_1)}\otimes(\psi_0+\psi_7)-\psi_{k-1}\otimes(\psi_0+\psi_7)\\&\geqslant \mathrm{T}_{X_1}\big(Q_k^{(X_1)}\big)\otimes(\psi_0+\psi_7)\\&=\mathrm{T}_{X_1X_2}\big[Q_k^{(X_1)}\otimes(\psi_0+\psi_7)\big].\end{aligned}$$

If we set $Y^{(X_1X_2)}=Y^{(X_1)}\otimes(\psi_0+\psi_7)$ and $Q_k^{(X_1X_2)}=Q_{k+5}^{(X_1X_2)}=Q_k^{(X_1)}\otimes(\psi_0+\psi_7)$ $(1\leqslant k\leqslant 5)$, then

$$Y^{(X_1X_2)}-\Psi_j>\mathrm{T}_{X_1X_2}\big(Q_j^{(X_1X_2)}\big)\ \ (j=1,\dots,10)$$

also satisfy the inequality constraints of dual problems (5), where

$$\Psi_j=\begin{cases}\psi_{j-1}\otimes\psi_0, & 1\leqslant j\leqslant 5\\\psi_{j-6}\otimes\psi_7, & 6\leqslant j\leqslant 10\end{cases}$$

are just elements of $\mathcal{S}_{10}$. Since $\mathrm{Tr}(Y^{(A)})=\mathrm{Tr}(Y^{(B)})=\mathrm{Tr}(Y^{(C)})=8$, all the minimums β_X $(X\in\{A,B,C\})$ are smaller than 1. This means that $\mathcal{S}_{10} = \{|\Psi_1\rangle, \ldots, |\Psi_{10}\rangle\}$ is genuinely nonlocal. ■

The reader may be curious of the above construction procedure about $\mathcal{S}_5$ and $\mathcal{S}_{10}$, for the feasible solutions to semidefinite programs (5) we have presented (the $Y^{(X)}$'s and the $Q_k^{(X)}$'s) are all lattice-type operators (diagonal in the basis $\{|\psi_0\rangle, \ldots, |\psi_7\rangle\}$ or their tensor products). To explain why, we present the following lemma that is an analog to Theorem 2 of [30]. It turns out that when GHZ lattices are considered, any set of feasible solutions to the semidefinite programs (5) corresponds to a set of lattice-type ones that have the same target values.

Lemma 3. If the states $|\Phi_1\rangle, \ldots, |\Phi_s\rangle$ to be distinguished are all GHZ lattices, then the semidefinite programs (5) [and (4) likewise] can be reduced to linear programs.

Proof. We check here the dual problem case and the primal case is similar. First, note that the set of three-qubit GHZ basis (2) is the common eigenbasis of the following set of commutative product operators on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$:

$$C = \{\mathrm{III}, \mathrm{XXX}, \mathrm{IZZ}, \mathrm{ZIZ}, \mathrm{ZZI}, \mathrm{XYY}, \mathrm{YXY}, \mathrm{YYX}\},$$

where $\{\mathrm{X}, \mathrm{Y}, \mathrm{Z}\}$ are the one-qubit Pauli operators. More explicitly, they have spectral decomposition:

$$\begin{aligned}
\mathrm{III} &= \psi_0 + \psi_1 + \psi_2 + \psi_3 + \psi_4 + \psi_5 + \psi_6 + \psi_7,\\
\mathrm{XXX} &= \psi_0 + \psi_1 + \psi_2 + \psi_3 - \psi_4 - \psi_5 - \psi_6 - \psi_7,\\
\mathrm{IZZ} &= \psi_0 - \psi_1 - \psi_2 + \psi_3 + \psi_4 - \psi_5 - \psi_6 + \psi_7,\\
\mathrm{ZIZ} &= \psi_0 - \psi_1 + \psi_2 - \psi_3 - \psi_4 + \psi_5 - \psi_6 + \psi_7,\\
\mathrm{ZZI} &= \psi_0 + \psi_1 - \psi_2 - \psi_3 - \psi_4 - \psi_5 + \psi_6 + \psi_7,\\
\mathrm{XYY} &= -\psi_0 + \psi_1 + \psi_2 - \psi_3 + \psi_4 - \psi_5 - \psi_6 + \psi_7,\\
\mathrm{YXY} &= -\psi_0 + \psi_1 - \psi_2 + \psi_3 - \psi_4 + \psi_5 - \psi_6 + \psi_7,\\
\mathrm{YYX} &= -\psi_0 - \psi_1 + \psi_2 + \psi_3 - \psi_4 - \psi_5 + \psi_6 + \psi_7.
\end{aligned} \tag{9}$$

Consider the quantum operation

$$\Delta(\rho) = \frac{1}{|C|} \sum_{U_i \in C} U_i \rho U_i^\dagger \tag{10}$$

that acts on system $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Substituting (9) into (10), one can check that all the cross terms in expression are eliminated and hence Δ acts as the dephasing operation under the three-qubit GHZ basis:

$$\Delta(\rho) = \sum_{i=0}^{7} |\psi_i\rangle\langle\psi_i|\rho|\psi_i\rangle\langle\psi_i|. \tag{11}$$

It is obvious that Δ is positive and trace preserving. What is more, it commutes with the partial transpose operations T_X's ($X \in \{A, B, C\}$). To see this, just write Δ in the form (10), and check if each T_X commutes with each term $U_i \rho U_i^\dagger$. This is routine; for example,

$$\begin{aligned}
\mathrm{T}_B(\mathrm{YYX} \cdot \rho \cdot \mathrm{YYX}) &= \mathrm{YY^TX} \cdot \mathrm{T}_B(\rho) \cdot \mathrm{YY^TX}\\
&= (-\mathrm{YYX}) \cdot \mathrm{T}_B(\rho) \cdot (-\mathrm{YYX})\\
&= \mathrm{YYX} \cdot \mathrm{T}_B(\rho) \cdot \mathrm{YYX},
\end{aligned} \tag{12}$$

and so on. Hence, $\mathrm{T}_X[\Delta(\rho)] = \Delta[\mathrm{T}_X(\rho)]$ for $X \in \{A, B, C\}$. Now let $\Lambda = \Delta^{\otimes t}$, that is,

$$\Lambda(\gamma) = \sum_{i_1=0}^{7} \cdots \sum_{i_t=0}^{7} \gamma_{i_1\cdots i_t} \cdot |\psi_{i_1} \cdots \psi_{i_t}\rangle\langle\psi_{i_1} \cdots \psi_{i_t}|,$$

where γ is any state in the tripartite system $\mathbb{C}^{2^t} \otimes \mathbb{C}^{2^t} \otimes \mathbb{C}^{2^t}$ and $\gamma_{i_1\cdots i_t} = \langle\psi_{i_1} \cdots \psi_{i_t}|\gamma|\psi_{i_1} \cdots \psi_{i_t}\rangle$. It also acts as the dephasing operation under the basis $\{|\psi_{i_1}\rangle \otimes \cdots \otimes |\psi_{i_t}\rangle\}_{i_1,\ldots,i_t}$ and has the same property as $\Delta(\cdot)$. Since this operation is invariant on the GHZ lattices $\Phi_k = \psi_{k_1} \otimes \cdots \otimes \psi_{k_t}$, by acting it on any specific feasible solution of (5), one can achieve another feasible solution consisting only of diagonal operators while the target values stay still. That is to say, the semidefinite programs (5) reduce to linear programs,

$$\begin{aligned}
\beta_X &= \min \frac{1}{s} \sum_{l_1=0}^{7} \cdots \sum_{l_t=0}^{7} (\overrightarrow{y^{(X)}})_{l_1\cdots l_t}\\
\text{such that } & (\overrightarrow{y^{(X)}})_{i_1\cdots i_t} - \sum_{l_1\cdots l_t} (T^{(X)})^{\otimes t}_{i_1\cdots i_t, l_1\cdots l_t} \cdot (\overrightarrow{q_k^{(X)}})_{l_1\cdots l_t}\\
&\geqslant \delta_{i_1\cdots i_t, k_1\cdots k_t},\\
&(\overrightarrow{q_k^{(X)}})_{i_1\cdots i_t} \geqslant 0\\
&(i_1, \ldots, i_t \in \{0, \ldots, 7\}; 1 \leqslant k \leqslant s),
\end{aligned} \tag{13}$$

where $X \in \{A, B, C\}$. Therein, "δ" is the Kronecker delta and we suppose that the GHZ lattices $|\Phi_k\rangle$ have index $k_1, \ldots, k_t$ $(1 \leqslant k \leqslant s)$. The $(\overrightarrow{y^{(X)}})$'s and $(\overrightarrow{q_k^{(X)}})$'s $(1 \leqslant k \leqslant s,\ X \in \{A, B, C\})$ are $8^t \times 1$ vectors representing the diagonal elements of $Y^{(X)}$'s and $Q_k^{(X)}$'s in problem (5), respectively (which are the optimization variables); $\mathrm{T}^{(X)}$'s $(X \in \{A, B, C\})$ are the transformation matrices of linear maps T_X's under basis $\{\psi_0, \ldots, \psi_7\}$, as shown by (6) in Lemma 2 (but with the basis rearranged in ascending order). ■

This result greatly reduces the complexity of the problem, for all operators we need to consider now are just diagonal (lattice-type) ones. We should also point out here that the feasible solutions we presented in the case $\mathcal{S}_5$ are actually optimal. Moreover, by running the linear programs (13) numerically, we found the existence of genuinely nonlocal subsets of $\mathcal{S}_{10}$ with cardinality down to 7. In other words, the genuine nonlocality of $\mathcal{S}_{10}$ can actually be derived from its subset.

Theorem 1. In $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, the following 7-ary subset of $\mathcal{S}_{10}$:

$$\begin{aligned}
\mathcal{S}_7 = \{&\psi_0 \otimes \psi_0, \psi_1 \otimes \psi_0, \psi_2 \otimes \psi_0,\\
&\psi_3 \otimes \psi_0, \psi_4 \otimes \psi_0, \psi_3 \otimes \psi_7, \psi_4 \otimes \psi_7\},
\end{aligned}$$

already has the property of genuine nonlocality ($\psi_0 \otimes \psi_7, \psi_1 \otimes \psi_7, \psi_2 \otimes \psi_7$ from $\mathcal{S}_{10}$ have been excluded).

The proof is placed in the Appendix, where we present a set of feasible solutions to the dual problems (5) with target values smaller than 1. Notably, although these solutions come out of numerical evaluation, it turns out that they can also be constructed inductively from the feasible solutions (7) for $\mathcal{S}_5$. This inspires us to further construct such genuinely nonlocal sets inductively when $t > 2$, simply duplicating the procedure from $\mathcal{S}_5$ to $\mathcal{S}_7$. That is to say, in $\mathbb{C}^8 \otimes \mathbb{C}^8 \otimes \mathbb{C}^8$ we will achieve

$$\begin{aligned}\mathcal{S}_{11} = \{&\psi_0 \otimes \psi_0 \otimes \psi_0, \psi_1 \otimes \psi_0 \otimes \psi_0, \psi_2 \otimes \psi_0 \otimes \psi_0,\\ &\psi_3 \otimes \psi_0 \otimes \psi_0, \psi_4 \otimes \psi_0 \otimes \psi_0, \psi_3 \otimes \psi_7 \otimes \psi_0,\\ &\psi_4 \otimes \psi_7 \otimes \psi_0, \psi_3 \otimes \psi_0 \otimes \psi_7, \psi_4 \otimes \psi_0 \otimes \psi_7,\\ &\psi_3 \otimes \psi_7 \otimes \psi_7, \psi_4 \otimes \psi_7 \otimes \psi_7\}\end{aligned}$$

[see (16) in the proof of Theorem 2] and more generally, the following theorem.

Theorem 2. In three-partite systems $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$, where $d = 2^t (t \geqslant 1)$, there exist genuinely nonlocal sets of lattice-type GHZ states with cardinality $d + 3$.

Proof. Inspired by the proof of Theorem 1, we explicitly construct these sets $\mathcal{S}^{(t)}$ by mathematical induction on t. For $t = 1, 2$, they are just $\mathcal{S}_5$, $\mathcal{S}_7$ in the main text. Now assume that $\mathcal{S}^{(t)} = \{\eta_1, \dots, \eta_{2^t+3}\}$, together with the $Y^{(X_1\cdots X_t)}$, $Q_j^{(X_1\cdots X_t)}$ $(1 \leqslant j \leqslant d + 3$ where $d = 2^t)$, have been constructed, such that

$$\begin{aligned}&Y^{(A_1\cdots A_t)} - \eta_2 - \eta_3\\ &\quad\geqslant 0 = \mathrm{T}_A\big(Q_2^{(A_1\cdots A_t)}\big) = \mathrm{T}_A\big(Q_3^{(A_1\cdots A_t)}\big),\\ &Y^{(B_1\cdots B_t)} - \eta_1 - \eta_3\\ &\quad\geqslant 0 = \mathrm{T}_B\big(Q_1^{(B_1\cdots B_t)}\big) = \mathrm{T}_B\big(Q_3^{(B_1\cdots B_t)}\big),\\ &Y^{(C_1\cdots C_t)} - \eta_1 - \eta_2\\ &\quad\geqslant 0 = \mathrm{T}_C\big(Q_1^{(C_1\cdots C_t)}\big) = \mathrm{T}_C\big(Q_2^{(C_1\cdots C_t)}\big),\end{aligned} \tag{14}$$

and

$$\begin{aligned}&Y^{(A_1\cdots A_t)} - \eta_2 - \eta_3 \geqslant \mathrm{T}_A\big(Q_j^{(A_1\cdots A_t)}\big) + \eta_j\\ &\qquad\qquad (1 \leqslant j \leqslant d + 3,\ j \neq 2, 3),\\ &Y^{(B_1\cdots B_t)} - \eta_1 - \eta_3 \geqslant \mathrm{T}_B\big(Q_j^{(B_1\cdots B_t)}\big) + \eta_j\\ &\qquad\qquad (1 \leqslant j \leqslant d + 3,\ j \neq 1, 3),\\ &Y^{(C_1\cdots C_t)} - \eta_1 - \eta_2 \geqslant \mathrm{T}_C\big(Q_j^{(C_1\cdots C_t)}\big) + \eta_j\\ &\qquad\qquad (1 \leqslant j \leqslant d + 3,\ j \neq 1, 2).\end{aligned} \tag{15}$$

Moreover, assume that $\mathrm{Tr}[Y^{(A_1\cdots A_t)}] = \mathrm{Tr}[Y^{(B_1\cdots B_t)}] = \mathrm{Tr}[Y^{(C_1\cdots C_t)}] = 2^t + 2 = d + 2$. It can be seen from (7), (8), and also the proof of Theorem 1 that these assumptions hold for $t = 1$.

Now, we let $\mathcal{S}^{(t+1)} = \{\chi_1, \dots, \chi_{2^{t+1}+3}\}$, where

$$\chi_j = \begin{cases}\eta_j \otimes \psi_0, & 1 \leqslant j \leqslant d + 3\\ \eta_{j-d} \otimes \psi_7, & d + 4 \leqslant j \leqslant 2d + 3.\end{cases} \tag{16}$$

Meanwhile, we set

$$\begin{aligned}Y^{(A_1\cdots A_{t+1})} &= Y^{(A_1\cdots A_t)} \otimes (\psi_0 + \psi_7) - \eta_2 \otimes \psi_7 - \eta_3 \otimes \psi_7,\\ Y^{(B_1\cdots B_{t+1})} &= Y^{(B_1\cdots B_t)} \otimes (\psi_0 + \psi_7) - \eta_1 \otimes \psi_7 - \eta_3 \otimes \psi_7,\\ Y^{(C_1\cdots C_{t+1})} &= Y^{(C_1\cdots C_t)} \otimes (\psi_0 + \psi_7) - \eta_1 \otimes \psi_7 - \eta_2 \otimes \psi_7,\end{aligned}$$

and

$$Q_j^{(X_1\cdots X_{t+1})} = \begin{cases}Q_j^{(X_1\cdots X_t)} \otimes (\psi_0 + \psi_7), & 1 \leqslant j \leqslant d + 3\\ Q_{j-d}^{(X_1\cdots X_t)} \otimes (\psi_0 + \psi_7), & d + 4 \leqslant j \leqslant 2d + 3,\end{cases}$$

where $X \in \{A, B, C\}$. Acting $\otimes(\psi_0 + \psi_7)$ on the right-hand sides of (14) and (15), we get

$$\begin{aligned}&Y^{(A_1\cdots A_{t+1})} - \chi_2 - \chi_3\\ &\quad\geqslant 0 = \mathrm{T}_A\big(Q_2^{(A_1\cdots A_{t+1})}\big) = \mathrm{T}_A\big(Q_3^{(A_1\cdots A_{t+1})}\big),\\ &Y^{(B_1\cdots B_{t+1})} - \chi_1 - \chi_3\\ &\quad\geqslant 0 = \mathrm{T}_B\big(Q_1^{(B_1\cdots B_{t+1})}\big) = \mathrm{T}_B\big(Q_3^{(B_1\cdots B_{t+1})}\big),\\ &Y^{(C_1\cdots C_{t+1})} - \chi_1 - \chi_2\\ &\quad\geqslant 0 = \mathrm{T}_C\big(Q_1^{(C_1\cdots C_{t+1})}\big) = \mathrm{T}_C\big(Q_2^{(C_1\cdots C_{t+1})}\big),\end{aligned}$$

and

$$\begin{aligned}&Y^{(A_1\cdots A_{t+1})} - \chi_2 - \chi_3\\ &\quad\geqslant \mathrm{T}_A\big(Q_j^{(A_1\cdots A_{t+1})}\big) + \eta_j \otimes \psi_0\\ &\big(\text{also }\ \mathrm{T}_A\big(Q_j^{(A_1\cdots A_{t+1})}\big) + \eta_j \otimes \psi_7\big)\\ &\qquad\qquad (1 \leqslant j \leqslant d + 3,\ j \neq 2, 3),\end{aligned}$$

$$\begin{aligned}&Y^{(B_1\cdots B_{t+1})} - \chi_1 - \chi_3\\ &\quad\geqslant \mathrm{T}_B\big(Q_j^{(B_1\cdots B_{t+1})}\big) + \eta_j \otimes \psi_0\\ &\big(\text{also }\ \mathrm{T}_B\big(Q_j^{(B_1\cdots B_{t+1})}\big) + \eta_j \otimes \psi_7\big)\\ &\qquad\qquad (1 \leqslant j \leqslant d + 3,\ j \neq 1, 3),\end{aligned}$$

$$\begin{aligned}&Y^{(C_1\cdots C_{t+1})} - \chi_1 - \chi_2\\ &\quad\geqslant \mathrm{T}_C\big(Q_j^{(C_1\cdots C_{t+1})}\big) + \eta_j \otimes \psi_0\\ &\big(\text{also }\ \mathrm{T}_C\big(Q_j^{(C_1\cdots C_{t+1})}\big) + \eta_j \otimes \psi_7\big)\\ &\qquad\qquad (1 \leqslant j \leqslant d + 3,\ j \neq 1, 2).\end{aligned}$$

That is, assumptions (14) and (15) also hold for $\mathcal{S}^{(t+1)}$. Besides, $\mathrm{Tr}[Y^{(A_1\cdots A_{t+1})}] = \mathrm{Tr}[Y^{(B_1\cdots B_{t+1})}] = \mathrm{Tr}[Y^{(C_1\cdots C_{t+1})}] = 2(d + 2) - 2 = 2^{t+1} + 2 < 2^{t+1} + 3$. Since the constraints of the dual problems (5) are all satisfied for any $\mathcal{S}^{(t)}$ $(t \geqslant 1)$, these sets are consequently genuinely nonlocal. ■

This result is somewhat unexpected, for the size of such genuinely nonlocal sets can scale down to linear in the local dimension d, conspicuously with a small linear factor $l = 1$. In sharp contrast, when strong nonlocality is considered, all existing nonlocal sets that have been constructed in three-partite systems $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ have cardinality $\Theta(d^2)$ (with the leading coefficient $c > 1$), to our best knowledge [56–60]. This comparison might arguably imply a substantial difference between strong nonlocality and distinguishability-based genuine nonlocality.

IV. CONSTRUCTING (2,3)-THRESHOLD SETS IN THREE-PARTITE SYSTEMS

In the previous section, we studied the genuine nonlocality of the GHZ states on three-qudit systems. Namely, the distinguishability when only two of the three parties A, B and C are allowed to combine. In the n-partite scenario, if the system is in one of a priorly known set of quantum states that are genuinely nonlocal, then the actual state of the system cannot be revealed perfectly unless all n parties are joined together to make global measurements. It is then natural and interesting to extend this notion of multipartite distinguishability to the more general case: a set of orthogonal states in the n-partite system is called (s, n)-*threshold distinguishable* if the states are distinguishable when any s or more parties are combined ($s \leqslant n$), but indistinguishable when at most $s-1$ parties can combine. Such sets are called (s, n)-*threshold sets* here and genuinely nonlocal sets are just (n, n)-threshold sets in this sense. This notion of multipartite distinguishability is an analog to the resembling notion in quantum secret sharing (QSS), where an (s, n)-*threshold QSS scheme* means that the secret has been distributed into n shares and any group of s or more shares can collaboratively reconstruct the secret while no group of fewer than s shares can [47–50].

In this section, we consider the problem of constructing (2,3)-threshold sets consisting of GHZ states in three-partite systems, as a complement to our studies of (3,3)-threshold sets in Sec. III. It is routine to first consider the simplest case of three-qubit systems. Unfortunately, it turns out that (2,3)-threshold sets consisting of states from the three-qubit GHZ basis do not exist.

Proposition 3. In systems $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, there exists no (2,3)-threshold set consisting of states from the three-qubit GHZ basis (2).

Proof. Suppose that $\mathcal{T}$ is a (2,3)-threshold set consisting of states from the three-qubit GHZ basis; namely, it is distinguishable across all bipartitions $A|BC$, $B|CA$ and $C|AB$ but indistinguishable when the three parties are fully separated. Now if $\mathcal{T}$ contains a certain conjugate pair, one can easily see from the schematic picture in Fig. 1 that the set cannot contain a third state, for otherwise $\mathcal{T}$ will be indistinguishable across one of the bipartitions. However, all conjugate pairs are distinguishable by the three fully separated parties. As an example, for $|\psi_{0,7}\rangle = \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)$, since

$$\frac{|000\rangle + |111\rangle}{\sqrt{2}} = \frac{|+++\rangle + |+--\rangle + |-+-\rangle + |--+\rangle}{2}$$

and

$$\frac{|000\rangle - |111\rangle}{\sqrt{2}} = \frac{|++-\rangle + |+-+\rangle + |-++\rangle + |---\rangle}{2}$$

$$\left(|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}},\ \ |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}\right),$$

the three parties can do local positive operator valued measure (POVM) $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ on their own subsystems and tell them apart by classical communications. Therefore, $\mathcal{T}$ must not contain any conjugate pair. Unfortunately, when no conjugate pair exists in $\mathcal{T}$, the three fully separated parties can distinguish the set by simply doing local POVM $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on their subsystems and communicating classically. As a result, no (2,3)-threshold set exists in this case.■

Although such (2,3)-threshold sets consisting of GHZ states do not exist in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, it is natural to further consider their existence in systems of higher local dimension. Fortunately, by applying the method of semidefinite program similarly as before, we find the existence of (2,3)-threshold sets consisting of lattice-type GHZ states. If a set of orthogonal quantum states $S = \{|\Phi_1\rangle, \dots, |\Phi_s\rangle\}$ in three-partite systems $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ is locally distinguishable across the $A|B|C$ (fully separated) partition, then the semidefinite program

$$\begin{aligned}
\alpha_{A|B|C} = \max_{P_1,\dots,P_s} \ & \frac{1}{s}\sum_{k=1}^{s} \mathrm{Tr}(P_k\Phi_k) \\
\text{such that} \quad & P_1 + \dots + P_s = I_{ABC}, \\
& P_1, \dots, P_s \geqslant 0, \\
& \mathrm{T}_A(P_k), \mathrm{T}_B(P_k), \mathrm{T}_C(P_k) \geqslant 0 \quad (1 \leqslant k \leqslant s)
\end{aligned} \tag{17}$$

must have optimal value $\alpha_{A|B|C} = 1$. To obtain an upper bound of $\alpha_{A|B|C}$, it is natural to consider its dual problem which we present in the following lemma.

Lemma 4. The dual problem of semidefinite program (17) has the form

$$\begin{aligned}
\beta_{A|B|C} = \min_{Y, Q_k^{(A)}, Q_k^{(B)}, Q_k^{(C)}} \ & \frac{1}{s}\,\mathrm{Tr}(Y) \\
\text{such that} \quad & Y - \Phi_k \geqslant \mathrm{T}_A(Q_k^{(A)}) + \mathrm{T}_B(Q_k^{(B)}) + \mathrm{T}_C(Q_k^{(C)}), \\
& Q_k^{(A)}, Q_k^{(B)}, Q_k^{(C)} \geqslant 0 \quad (1 \leqslant k \leqslant s).
\end{aligned} \tag{18}$$

For consideration of readability, we place the proof in the Appendix. Since $\alpha_{A|B|C} \leqslant \beta_{A|B|C}$ (actually $\alpha_{A|B|C} = \beta_{A|B|C}$ by Slater's condition), once we find a feasible solution to the dual problem (18) with target value smaller than 1, the indistinguishability of $S = \{|\Phi_1\rangle, \dots, |\Phi_s\rangle\}$ across the $A|B|C$ partition will be deduced immediately. For the distinguishability across the $2-1$ bipartitions, a necessary condition is that the family of semidefinite programs (5) have optimal values $\beta_X = 1$ for $X \in \{A, B, C\}$ ($\alpha_X = \beta_X$ by strong duality). These facts indicate that we can search for sets $\mathcal{T}$ such that $\beta_{A|B|C} < 1$ and $\beta_X = 1$ ($X \in \{A, B, C\}$) as candidates for (2,3)-threshold sets in three-partite systems (we should further check their distinguishability across the $2-1$ partitions as PPT distinguishability is just a necessary condition). In our case where GHZ lattices are considered, these semidefinite programs reduce to linear programs by Lemma 2 and Lemma 3, and the complexity will be greatly reduced. By running numerical search, we find plenty of such sets in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ and we present here only one of them with the smallest cardinality.

Theorem 3. In system $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$, the 5-ary set

$$\mathcal{T}_5^{(2,3)} = \{\psi_0 \otimes \psi_1, \psi_0 \otimes \psi_6, \psi_0 \otimes \psi_3, \psi_3 \otimes \psi_0, \psi_4 \otimes \psi_0\}$$

consisting of three-partite lattice-type GHZ states is a (2,3)-threshold set.

Proof. First, we prove that $\mathcal{T}_5^{(2,3)}$ is indistinguishable in the $1-1-1$ partition. This can be done by presenting a feasible solution to the dual problem (18), with target value smaller than 1 (19/20 in this case):

$$\begin{aligned}
Y = & \tfrac{1}{2}\psi_0 \otimes (\psi_1 + \psi_6) + \psi_0 \otimes \psi_3 + \tfrac{1}{2}(\psi_3 + \psi_4) \otimes \psi_0 \\
& + \tfrac{1}{4}\psi_1 \otimes (\psi_0 + \psi_7) + \tfrac{1}{4}\psi_6 \otimes (\psi_0 + \psi_7) \\
& + \tfrac{1}{4}\psi_7 \otimes (\psi_3 + \psi_4) + \tfrac{1}{4}\psi_0 \otimes \psi_4,
\end{aligned}$$

$$Q_1^{(A)} = Q_2^{(A)} = Q_3^{(A)} = 0,$$
$$Q_4^{(A)} = \tfrac{1}{2}\psi_0 \otimes \psi_4 + \tfrac{1}{2}\psi_7 \otimes \psi_3,$$
$$Q_5^{(A)} = \tfrac{1}{2}\psi_0 \otimes \psi_3 + \tfrac{1}{2}\psi_7 \otimes \psi_4,$$

$$Q_1^{(B)} = \tfrac{1}{2}(\psi_0 + \psi_7) \otimes \psi_4,$$
$$Q_2^{(B)} = \tfrac{1}{2}(\psi_0 + \psi_7) \otimes \psi_3, \; Q_3^{(B)} = 0,$$
$$Q_4^{(B)} = \tfrac{1}{2}\psi_6 \otimes (\psi_0 + \psi_7),$$
$$Q_5^{(B)} = \tfrac{1}{2}\psi_1 \otimes (\psi_0 + \psi_7),$$

$$Q_1^{(C)} = \tfrac{1}{2}\psi_1 \otimes \psi_7 + \tfrac{1}{2}\psi_6 \otimes \psi_0,$$
$$Q_2^{(C)} = \tfrac{1}{2}\psi_1 \otimes \psi_0 + \tfrac{1}{2}\psi_6 \otimes \psi_7,$$
$$Q_3^{(C)} = Q_4^{(C)} = Q_5^{(C)} = 0.$$

This solution is actually an optimal one but we only need to check its feasibility. As an instance, we check here the constraint $Y - \psi_0 \otimes \psi_1 \geqslant \mathrm{T}_A(Q_1^{(A)}) + \mathrm{T}_B(Q_1^{(B)}) + \mathrm{T}_C(Q_1^{(C)})$ [with the help of expressions (6)]:

$$\begin{aligned}
&\mathrm{T}_A(Q_1^{(A)}) + \mathrm{T}_B(Q_1^{(B)}) + \mathrm{T}_C(Q_1^{(C)}) \\
&\quad = 0 + \tfrac{1}{2}(\psi_0 + \psi_7) \otimes \tfrac{1}{2}(\psi_3 + \psi_4 - \psi_1 + \psi_6) \\
&\qquad + \tfrac{1}{8}(\psi_0 - \psi_7 + \psi_1 + \psi_6) \otimes (\psi_0 + \psi_7 - \psi_1 + \psi_6) \\
&\qquad + \tfrac{1}{8}(-\psi_0 + \psi_7 + \psi_1 + \psi_6) \otimes (\psi_0 + \psi_7 + \psi_1 - \psi_6) \\
&\quad = \tfrac{1}{4}(\psi_1 + \psi_6) \otimes (\psi_0 + \psi_7) + \tfrac{1}{4}(\psi_0 + \psi_7) \otimes (\psi_3 + \psi_4) \\
&\qquad + \tfrac{1}{2}\psi_0 \otimes \psi_6 - \tfrac{1}{2}\psi_0 \otimes \psi_1 \\
&\quad \leqslant Y - \psi_0 \otimes \psi_1,
\end{aligned}$$

and the others can be checked similarly. For the distinguishability of $\mathcal{T}_5^{(2,3)}$ across the $2-1$ bipartitions, we present the explicit distinguishing protocol respectively:

(i) For $A|BC$: BC can first perform a two-outcome joint measurement $\{|00\rangle\langle 00| + |11\rangle\langle 11|, |01\rangle\langle 01| + |10\rangle\langle 10|\}_{B_2C_2}$ on the second share of the qubit GHZ state to reduce $\mathcal{T}_5^{(2,3)}$ into $\{\psi_0 \otimes \psi_1, \psi_0 \otimes \psi_6\}$ and $\{\psi_0 \otimes \psi_3, \psi_3 \otimes \psi_0, \psi_4 \otimes \psi_0\}$ (see Fig. 1), of which the former can apparently be distinguished across $A|BC$ and the latter can be further distinguished into $\{\psi_0 \otimes \cancel{\psi_3}\}$ or $\{\psi_3 \otimes \cancel{\psi_0}, \psi_4 \otimes \cancel{\psi_0}\}$ if $A|BC$ measure on the second share and communicate classically (we use the delete line "–" to denote that the share of state has collapsed after measurements). Last, $\{\psi_3 \otimes \cancel{\psi_0}, \psi_4 \otimes \cancel{\psi_0}\}$ can also be distinguished by measuring on the first share.

(ii) For $B|CA$: CA can perform a joint measurement $\{|00\rangle\langle 00| + |11\rangle\langle 11|, |01\rangle\langle 01| + |10\rangle\langle 10|\}_{C_2A_2}$ on the second share of the state to reduce $\mathcal{T}_5^{(2,3)}$ into $\{\psi_3 \otimes \psi_0, \psi_4 \otimes \psi_0\}$ and $\{\psi_0 \otimes \psi_1, \psi_0 \otimes \psi_6, \psi_0 \otimes \psi_3\}$, where the former set is apparently distinguishable across $B|CA$. For the latter, B can use the first share ψ_0 as resource, to teleport his second share of qubit to CA and then CA can distinguish $\{\cancel{\psi_0} \otimes \psi_1, \cancel{\psi_0} \otimes \psi_6, \cancel{\psi_0} \otimes \psi_3\}$ by their own.

(iii) For $C|AB$: AB first perform a joint measurement $\{|00\rangle\langle 00| + |11\rangle\langle 11|, |01\rangle\langle 01| + |10\rangle\langle 10|\}_{A_1B_1}$ on the first share of the state to reduce the set into $\{\psi_3 \otimes \psi_0, \psi_4 \otimes \psi_0\}$ and $\{\psi_0 \otimes \psi_1, \psi_0 \otimes \psi_6, \psi_0 \otimes \psi_3\}$, where the former is distinguishable across $C|AB$. For the other, AB can further perform a measurement $\{|00\rangle\langle 00| + |11\rangle\langle 11|, |01\rangle\langle 01| + |10\rangle\langle 10|\}_{A_2B_2}$ on the second share to reduce it into $\{\psi_0 \otimes \psi_3\}$ and $\{\psi_0 \otimes \psi_1, \psi_0 \otimes \psi_6\}$, where the latter one can be distinguished simply through the second share. ■

It is also straightforward to construct (2,3)-threshold sets in systems $\mathbb{C}^{2^t} \otimes \mathbb{C}^{2^t} \otimes \mathbb{C}^{2^t}$ $(t > 2)$ inductively from $\mathcal{T}_5^{(2,3)}$, similarly as that of Proposition 2. What is more, we have also found a 16-ary superset of $\mathcal{T}_5^{(2,3)}$ that is PPT distinguishable in all three $2-1$ bipartitions:

$$\begin{aligned}
\mathcal{T}_{16} = \{&\psi_0 \otimes \psi_1, \psi_0 \otimes \psi_6, \psi_0 \otimes \psi_3, \psi_3 \otimes \psi_0, \\
&\psi_0 \otimes \psi_5, \psi_5 \otimes \psi_5, \psi_5 \otimes \psi_0, \psi_4 \otimes \psi_0, \\
&\psi_6 \otimes \psi_5, \psi_6 \otimes \psi_3, \psi_6 \otimes \psi_6, \psi_6 \otimes \psi_0, \\
&\psi_2 \otimes \psi_2, \psi_2 \otimes \psi_3, \psi_3 \otimes \psi_3, \psi_3 \otimes \psi_1\}.
\end{aligned}$$

Note that 16 achieves the upper bound of cardinality of such sets indeed (it is not hard to prove that any $k > d^2$ GHZ states in $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ are PPT indistinguishable across the $2-1$ partitions). Yet, we do not know whether $\mathcal{T}_{16}$ is locally distinguishable in the $2-1$ bipartitions. It is then interesting to find out the maximal intermediate (2,3)-threshold set(s) $\mathcal{T}_5^{(2,3)} \subseteq \mathcal{T}_{\max}^{(2,3)} \subseteq \mathcal{T}_{16}$ with the largest cardinality $|\mathcal{T}_{max}^{(2,3)}|$, such that the intermediate sets between $\mathcal{T}_5^{(2,3)}$ and $\mathcal{T}_{\max}^{(2,3)}$ are all (2,3)-threshold sets immediately.

V. CONCLUSION AND DISCUSSION

In this paper, we have studied the distinguishability-based genuine nonlocality of a typical type of genuine multipartite entangled states—the GHZ states. We first study the genuine nonlocality of the three-qubit GHZ basis. Then, using the result in the three-qubit case, we construct genuinely nonlocal sets of "GHZ lattices" in three-partite systems where the local dimension are powers of 2. It turns out that the size of these genuinely nonlocal sets with genuine multipartite entanglement can at least scale down to linear in the local dimension d, with a conspicuously small linear factor $l = 1$. The concept of genuine nonlocality, which concerns the distinguishability of multipartite quantum states through any possible bipartition of the subsystems, is in our opinion more naturally arising than the recently more popular "strong nonlocality". However, little is known about this form of distinguishability-based nonlocality. A natural and interesting question is to ask: In what extent is strong nonlocality stronger than the more normal genuine nonlocality? As far as we know presently, no strongly nonlocal set with such small scale has been constructed yet, no matter product states or genuinely entangled states that were considered. It is therefore reasonable to argue that there might exist a significant gap between the strength of strong nonlocality and the distinguishability-based genuine nonlocality. Besides that, our result might possibly also illuminate the relation between entanglement and distinguishability in multipartite scenarios, substantiating the perspective that entanglement can somehow raise difficulty in state discrimination.

We also discuss the concept of (s, n)-threshold distinguishability, which extends the notion of genuine nonlocality to characterize the local accessibility of global information more comprehensively. Although there are no (2,3)-threshold sets in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ consisting of states from the three-qubit GHZ basis, we have found their existence in systems of higher local dimension. We stress here that the techniques we use can be easily generalized to more-partite cases. For example, in the four-partite case, when distinguishability in bipartitions is considered, we can consider semidefinite programs (4) and (5) similarly by letting $X \in \{A, B, C, D, AB, BC, AC\}$. When considering distinguishability through the $1-1-1-1$ partition, semidefinite program (17) can be adjusted to

$$\alpha_{A|B|C|D} = \max_{P_1,\cdots,P_s} \frac{1}{s}\sum_{k=1}^{s} \mathrm{Tr}(P_k\Phi_k)$$
$$\text{such that}\quad P_1+\cdots+P_s = I_{ABCD},$$
$$P_1,\ldots,P_s \geqslant 0,$$
$$\mathrm{T}_A(P_k), \mathrm{T}_B(P_k), \mathrm{T}_C(P_k), \mathrm{T}_D(P_k) \geqslant 0,$$
$$\mathrm{T}_{AB}(P_k), \mathrm{T}_{BC}(P_k), \mathrm{T}_{AC}(P_k) \geqslant 0 \quad (1\leqslant k\leqslant s),$$

and so forth. It is interesting to derive some more universal results and this will be the main focus in our future work.

There are also some questions left to be considered. For example, can the technique we use be generalized to more general cases where the local dimensions d are not powers of 2? Can we find genuinely nonlocal sets with even smaller cardinality? What is the situation when a large number of parties is considered? Besides, notice that we have constructed the genuinely nonlocal sets or threshold sets mainly through PPT indistinguishability in this work. It is then natural and interesting to ask whether there exist other ways to construct these sets, such that they have even smaller cardinality or some other novel properties.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grant No. 62272492, the Guangdong Basic and Applied Basic Research Foundation under Grant No. 2020B1515020050, Key Research and Development Project of Guangdong Province under Grant No. 2020B0303300001, and the Guangdong Basic and Applied Basic Research Foundation under Grants No. 2020B1515310016 and No. 2023A1515012074.

APPENDIX

1. Proof of Lemma 2

Proof. It is routine to check this result. For example, notice that $\mathrm{T}_A(\psi_0) = (|000\rangle\langle000| + |111\rangle\langle111| + |100\rangle\langle011| + |011\rangle\langle100|)/2$, and $\psi_4 = (|011\rangle\langle011| + |100\rangle\langle100| - |011\rangle\langle100| - |100\rangle\langle011|)/2$. Then by adding these two operators we eliminate the cross terms:

$$\mathrm{T}_A(\psi_0)+\psi_4 = \frac{|000\rangle\langle000| + |111\rangle\langle111| + |011\rangle\langle011| + |100\rangle\langle100|}{2}. \tag{A1}$$

Moreover, $\psi_0+\psi_7 = |000\rangle\langle000| + |111\rangle\langle111|$ while $\psi_3+\psi_4 = |011\rangle\langle011| + |100\rangle\langle100|$. Therefore, we get $\mathrm{T}_A(\psi_0) = (\psi_0+\psi_7+\psi_3-\psi_4)/2$ as it is shown in (6). By acting T_A on (A1), we get $\psi_0 + \mathrm{T}_A(\psi_4) = (\psi_0+\psi_7+\psi_3+\psi_4)/2$ and so $\mathrm{T}_A(\psi_4) = (-\psi_0+\psi_7+\psi_3+\psi_4)/2$. Similarly,

$$\mathrm{T}_A(\psi_3)+\psi_7 = \mathrm{T}_A(\psi_7)+\psi_3 = \frac{\psi_0+\psi_7+\psi_3+\psi_4}{2},$$

hence we have $\mathrm{T}_A(\psi_7) = (\psi_0+\psi_7-\psi_3+\psi_4)/2$ and $\mathrm{T}_A(\psi_3) = (\psi_0-\psi_7+\psi_3+\psi_4)/2$. Actually, within the bipartition $A|BC$, $\mathcal{A}_1 = \{\psi_0,\psi_7,\psi_3,\psi_4\}$ is locally equivalent to the Bell basis (see Fig. 1). Similarly for $\mathcal{A}_2 = \{\psi_1,\psi_6,\psi_2,\psi_5\}$, one can easily check that

$$\begin{aligned}\mathrm{T}_A(\psi_1)+\psi_5 &= \mathrm{T}_A(\psi_5)+\psi_1\\ &= \mathrm{T}_A(\psi_2)+\psi_6 = \mathrm{T}_A(\psi_6)+\psi_2\\ &= \frac{|001\rangle\langle001| + |110\rangle\langle110| + |010\rangle\langle010| + |101\rangle\langle101|}{2}\\ &= \frac{\psi_1+\psi_6+\psi_2+\psi_5}{2}.\end{aligned}$$

Hence, T_A has the same matrix representation on $\mathcal{A}_2$ and so forth for the other two bipartitions. ■

2. Proof of Theorem 1

Proof. We present here a set of feasible solutions (actually optimal ones) to the dual problems for $\mathcal{S}_7$, inductively from the solutions (7) for $\mathcal{S}_5$.

For $A|BC$,

$$\begin{aligned}Y^{(A_1A_2)} &= Y^{(A_1)}\otimes(\psi_0+\psi_7) - \psi_1\otimes\psi_7 - \psi_2\otimes\psi_7,\\ Q_i^{(A_1A_2)} &= Q_i^{(A_1)}\otimes(\psi_0+\psi_7),\\ Q_6^{(A_1A_2)} &= Q_4^{(A_1)}\otimes(\psi_0+\psi_7),\\ Q_7^{(A_1A_2)} &= Q_5^{(A_1)}\otimes(\psi_0+\psi_7)\ \ (1\leqslant i\leqslant 5).\end{aligned} \tag{A2}$$

For $B|CA$,

$$\begin{aligned}Y^{(B_1B_2)} &= Y^{(B_1)}\otimes(\psi_0+\psi_7) - \psi_0\otimes\psi_7 - \psi_2\otimes\psi_7,\\ Q_i^{(B_1B_2)} &= Q_i^{(B_1)}\otimes(\psi_0+\psi_7),\\ Q_6^{(B_1B_2)} &= Q_4^{(B_1)}\otimes(\psi_0+\psi_7),\\ Q_7^{(B_1B_2)} &= Q_5^{(B_1)}\otimes(\psi_0+\psi_7)\ \ (1\leqslant i\leqslant 5).\end{aligned} \tag{A3}$$

For $C|AB$,

$$\begin{aligned}Y^{(C_1C_2)} &= Y^{(C_1)}\otimes(\psi_0+\psi_7) - \psi_0\otimes\psi_7 - \psi_1\otimes\psi_7,\\ Q_i^{(C_1C_2)} &= Q_i^{(C_1)}\otimes(\psi_0+\psi_7),\\ Q_6^{(C_1C_2)} &= Q_4^{(C_1)}\otimes(\psi_0+\psi_7),\\ Q_7^{(C_1C_2)} &= Q_5^{(C_1)}\otimes(\psi_0+\psi_7)\ \ (1\leqslant i\leqslant 5).\end{aligned} \tag{A4}$$

Now the feasibility of these solutions can be checked inductively from the feasibility of (7). Taking $A|BC$ for example, it is routine to check from (8) that

$$Y^{(A_1)} - (\psi_1+\psi_2) > 0 = \mathrm{T}_A(Q_2^{(A_1)}) = \mathrm{T}_A(Q_3^{(A_1)}),$$

and

$$Y^{(A_1)} - (\psi_1+\psi_2) \geqslant \psi_{k-1} + \mathrm{T}_A(Q_k^{(A_1)})\ \ (k=1,4,5).$$

Therefore, we have

$$\begin{aligned}Y^{(A_1A_2)} &- (\psi_1 \otimes \psi_0 + \psi_2 \otimes \psi_0)\\ &= Y^{(A_1)} \otimes (\psi_0 + \psi_7) - (\psi_1 + \psi_2) \otimes (\psi_0 + \psi_7)\\ &> 0 = \mathrm{T}_A(Q_2^{(A_1A_2)}) = \mathrm{T}_A(Q_3^{(A_1A_2)}),\end{aligned}$$

and

$$\begin{aligned}Y^{(A_1A_2)} &- (\psi_1 \otimes \psi_0 + \psi_2 \otimes \psi_0)\\ &= Y^{(A_1)} \otimes (\psi_0 + \psi_7) - (\psi_1 + \psi_2) \otimes (\psi_0 + \psi_7)\\ &> \left[\psi_{k-1} + \mathrm{T}_A(Q_k^{(A_1)})\right] \otimes (\psi_0 + \psi_7)\\ &> \psi_{k-1} \otimes \psi_0 + \mathrm{T}_A\left(Q_k^{(A_1A_2)}\right)\\ &\left(\text{also } \ \psi_{k-1} \otimes \psi_7 + \mathrm{T}_A\left(Q_k^{(A_1A_2)}\right)\right) \qquad (k = 1, 4, 5).\end{aligned}$$

This means that $Y^{(A_1A_2)} - \Psi_j \geqslant \mathrm{T}_A(Q_j^{(A_1A_2)})$ for all $1 \leqslant j \leqslant 7$, where $\{\Psi_1, \ldots, \Psi_7\} = \mathcal{S}_7$. Hence, the solution (A2) for bipartition $A|BC$ is feasible and so forth for the other two bipartitions. Since the corresponding target values for these solutions are all smaller than 1 (6/7, 6/7, 6/7, respectively), the set $\mathcal{S}_7$ is genuinely nonlocal. ■

3. Proof of Lemma 4

A general standard form of semidefinite program is defined as

$$\begin{aligned}\alpha = \max_X \ &\mathrm{Tr}(MX)\\ \text{such that} \quad &\Omega(X) = N,\\ &X \geqslant 0,\end{aligned}$$

where M, N, and X (called optimization variables) are Hermitian operators and $\Omega(\cdot)$ is a linear map that preserves Hermiticity. The problem is called the primal problem and its dual problem is defined as

$$\begin{aligned}\beta = \min_Y \ &\mathrm{Tr}(NY)\\ \text{such that} \quad &\Omega^*(Y) \leqslant M,\end{aligned}$$

where the optimization variable Y is also Hermitian and Ω^* is the adjoint map of Ω satisfying $\mathrm{Tr}[\Omega(X) \cdot Y] = \mathrm{Tr}[X \cdot \Omega^*(Y)]$ for arbitrary Hermitian X and Y.

Putting the semidefinite program (17) into this standard form, we have that

$$M = \begin{pmatrix} \Phi_1 & & & & & & \\ & \ddots & & & & 0\prime s & \\ & & \Phi_s & & & & \\ & & & 0 & & & \\ & & & & \ddots & & \\ & 0\prime s & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} I_{ABC} & & & 0\prime s \\ & 0 & & \\ & & \ddots & \\ 0\prime s & & & \ddots \\ & & & & 0 \end{pmatrix},$$

and

$$\Omega(X) = \begin{pmatrix} X_1 + \cdots + X_s & & & & & & & & & & \\ & \mathrm{T}_A(X_1) - X_1' & & & & & & 0\prime s & & & \\ & & \ddots & & & & & & & & \\ & & & \mathrm{T}_A(X_s) - X_s' & & & & & & & \\ & & & & \mathrm{T}_B(X_1) - X_{s+1}' & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & \mathrm{T}_B(X_s) - X_{2s}' & & & & \\ & & & & & & & \mathrm{T}_C(X_1) - X_{2s+1}' & & & \\ & & 0's & & & & & & \ddots & & \\ & & & & & & & & & \mathrm{T}_C(X_s) - X_{3s}' \end{pmatrix},$$

where the optimization variable $X \geqslant 0$ has been written as

$$X = \frac{1}{s}\begin{pmatrix} X_1 & & & & & & \\ & \ddots & & & & * & \\ & & X_s & & & & \\ & & & X_1' & & & \\ & & & & \ddots & & \\ & * & & & & \ddots & \\ & & & & & & X_{3s}' \end{pmatrix}.$$

Namely, $X_1,\ldots,X_s$ and $X'_1,\ldots,X'_{3s}$ are the diagonal blocks of sX and they are therefore semidefinite positive. For its dual problem, denote

$$Y=\frac{1}{s}\begin{pmatrix} Y' & & & & \\ & -Y_1 & & * & \\ & & \ddots & & \\ & * & & \ddots & \\ & & & & -Y_{3s}\end{pmatrix}$$

the optimization variable and we have

$$\begin{aligned}\mathrm{Tr}[\Omega(X)\cdot Y] &= \frac{1}{s}\mathrm{Tr}[(X_1+\cdots+X_s)\cdot Y'] - \frac{1}{s}\sum_{i=1}^{s}\mathrm{Tr}\{[\mathrm{T}_A(X_i)-X'_i]\cdot Y_i\} - \frac{1}{s}\sum_{i=1}^{s}\mathrm{Tr}\{[\mathrm{T}_B(X_i)-X'_{s+i}]\cdot Y_{s+i}\}\\ &\quad - \frac{1}{s}\sum_{i=1}^{s}\mathrm{Tr}\{[\mathrm{T}_C(X_i)-X'_{2s+i}]\cdot Y_{2s+i}\}\\ &= \mathrm{Tr}\left[\underbrace{\frac{1}{s}\begin{pmatrix} X_1 & & & & & & \\ & \ddots & & & & * & \\ & & X_s & & & & \\ & & & X'_1 & & & \\ & & & & \ddots & & \\ & * & & & & \ddots & \\ & & & & & & X'_{3s}\end{pmatrix}}_{=X}\right.\\ &\quad\left.\cdot\underbrace{\begin{pmatrix} Y'-\mathrm{T}_A(Y_1)-\mathrm{T}_B(Y_{s+1})-\mathrm{T}_C(Y_{2s+1}) & & & 0's & & \\ & \ddots & & & & \\ & & Y'-\mathrm{T}_A(Y_s)-\mathrm{T}_B(Y_{2s})-\mathrm{T}_C(Y_{3s}) & & & \\ & & & Y_1 & & \\ & & & & \ddots & \\ & 0's & & & & \ddots & \\ & & & & & & Y_{3s}\end{pmatrix}}_{=\Omega^*(Y)}\right],\end{aligned}$$

where we apply $\mathrm{Tr}[\mathrm{T}_A(V)\cdot W]=\mathrm{Tr}[V\cdot\mathrm{T}_A(W)]$ (also for T_B, T_C) in the second equality. Consequently, by the constraint $\Omega^*(Y)\geqslant M$ we get the dual problem as

$$\begin{aligned}\beta = &\min_{Y',Y_1,\ldots,Y_{3s}} \frac{1}{s}\mathrm{Tr}(Y')\\ \text{such that } &Y'-\Phi_i \geqslant \mathrm{T}_A(Y_i)+\mathrm{T}_B(Y_{s+i})+\mathrm{T}_C(Y_{2s+i})\ \ (1\leqslant i\leqslant s),\\ &Y_1,\ldots,Y_{3s}\geqslant 0\end{aligned}$$

which is equivalent to problem (18).

[1] J. S. Bell, On the Einstein-Podolsky-Rosen paradox, Physics **1**, 195 (1964)

[2] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Rev. Mod. Phys. **86**, 419 (2014)

[3] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969)

[4] S. J. Freedman and J. F. Clauser, Experimental Test of Local Hidden-Variable Theories, Phys. Rev. Lett. **28**, 938 (1972).

[5] A. Aspect, F. Grangier, and G. Roger, Experimental Tests of Realistic Local Theories via Bell's Theorem, Phys. Rev. Lett. **47**, 460 (1981).

[6] A. Aspect, J. Dalibard, and G. Roger, Experimental Test of Bell's Inequalities Using Time-Varying Analyzers, Phys. Rev. Lett. **49**, 1804 (1982).

[7] B. Hensen *et al.*, Loophole-free Bell inequality violation using electron spins separated by 1.3 kilometres, Nature (London) **526**, 682 (2015).

[8] BIG Bell Test Collaboration, Challenging local realism with human choices, Nature (London) **557**, 212 (2018).

[9] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Quantum nonlocality without entanglement, Phys. Rev. A **59**, 1070 (1999).

[10] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Unextendible Product Bases and Bound Entanglement, Phys. Rev. Lett. **82**, 5385 (1999).
[11] J. Walgate, A. J. Short, L. Hardy, and V. Vedral, Local Distinguishability of Multipartite Orthogonal Quantum States, Phys. Rev. Lett. **85**, 4972 (2000).
[12] J. Walgate and L. Hardy, Nonlocality, Asymmetry, and Distinguishing Bipartite States, Phys. Rev. Lett. **89**, 147901 (2002).
[13] D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Unextendible product bases, uncompletable product bases and bound entanglement, Commun. Math. Phys. **238**, 379 (2003).
[14] M. Horodecki, A. Sen(De), U. Sen, and K. Hoeodecki, Local Indistinguishability: More Nonlocality with Less Entanglement, Phys. Rev. Lett. **90**, 047902 (2003).
[15] M. Hayashi, D. Markham, M. Murao, M. Owari, and S. Virmani, Bounds on Multipartite Entangled Orthogonal State Discrimination Using Local Operations and Classical Communication, Phys. Rev. Lett. **96**, 040501 (2006).
[16] S. Bandyopadhyay and J. Walgate, Local distinguishability of any three quantum states, J. Phys. A: Math. Theor. **42**, 072002 (2009).
[17] S. Bandyopadhyay, Entanglement and perfect discrimination of a class of multiqubit states by local operations and classical communication, Phys. Rev. A **81**, 022327 (2010).
[18] J. Niset and N. J. Cerf, Multipartite nonlocality without entanglement in many dimension, Phys. Rev. A **74**, 052103 (2006).
[19] S. Ghosh, G. Kar, A. Roy, A. Sen(De), and U. Sen, Distinguishability of Bell States, Phys. Rev. Lett. **87**, 277902 (2001).
[20] S. Ghosh, G. Kar, A. Roy, and D. Sarkar, Distinguishability of maximally entangled states, Phys. Rev. A **70**, 022304 (2004).
[21] H. Fan, Distinguishability and Indistinguishability by Local Operations and Classical Communication, Phys. Rev. Lett. **92**, 177905 (2004).
[22] G. Tian, S. Yu, F. Gao, Q. Wen, and C. H. Oh, Local discrimination of qudit lattice states via commutativity, Phys. Rev. A **92**, 042320 (2015).
[23] G. Tian, S. Yu, F. Gao, Q. Wen, and C. H. Oh, Local discrimination of four or more maximally entangled states, Phys. Rev. A **91**, 052314 (2015).
[24] Z.-X. Xiong, M.-S. Li, Z.-J. Zheng, C.-J. Zhu, and S.-M. Fei, Positive-partial-transpose distinguishability for lattice-type maximally entangled states, Phys. Rev. A **99**, 032346 (2019).
[25] Y.-L. Wang, M.-S. Li, and Z.-X. Xiong, One-way local distinguishability of generalized Bell states in arbitrary dimension, Phys. Rev. A **99**, 022307 (2019).
[26] M. Nathanson, Distinguishing bipartite orthogonal states using LOCC: Best and worst cases, J. Math. Phys. **46**, 062103 (2005).
[27] S. Bandyopadhyay, S. Ghosh, and G. Kar, LOCC distinguishability of unilaterally transformable quantum states, New J. Phys. **13**, 123013 (2011).
[28] N. Yu, R. Duan, and M. Ying, Four Locally Indistinguishable Ququad-Ququad Orthogonal Maximally Entangled States, Phys. Rev. Lett. **109**, 020506 (2012).
[29] M. Nathanson, Three maximally entangled states can require two-way local operations and classical communication for local discrimination, Phys. Rev. A **88**, 062316 (2013).
[30] A. Cosentino, Positive-partial-transpose indistinguishable states via semidefinite programming, Phys. Rev. A **87**, 012321 (2013).
[31] A. Cosentino and V. Russo, Small sets of locally indistinguishable orthogonal maximally entangled states, Quantum Inf. Comput. **14**, 1098 (2014).
[32] M.-S. Li, Y.-L. Wang, S.-M. Fei, and Z.-J. Zheng, d locally indistinguishable maximally entangled states in $\mathbb{C}^d \otimes \mathbb{C}^d$, Phys. Rev. A **91**, 042318 (2015).
[33] S. Yu and C. H. Oh, Detecting the local indistinguishability of maximally entangled states, arXiv:1502.01274.
[34] G.-B. Xu, Q.-Y. Wen, S.-J. Qin, Y.-H. Yang, and F. Gao, Quantum nonlocality of multipartite orthogonal product states, Phys. Rev. A **93**, 032341 (2016).
[35] Z.-C. Zhang, F. Gao, Y. Cao, S.-J. Qin, and Q.-Y. Wen, Local indistinguishability of orthogonal product states, Phys. Rev. A **93**, 012314 (2016).
[36] Z.-C. Zhang, K.-J. Zhang, F. Gao, Q.-Y. Wen, and C. H. Oh, Construction of nonlocal multipartite quantum state, Phys. Rev. A **95**, 052344 (2017).
[37] S. Halder, Several nonlocal sets of multipartite pure orthogonal product states, Phys. Rev. A **98**, 022303 (2018).
[38] M.-S. Li and Y.-L. Wang, Alternative method for deriving nonlocal multipartite product states, Phys. Rev. A **98**, 052352 (2018).
[39] D.-H. Jiang and G.-B. Xu, Nonlocal sets of orthogonal product states in an arbitrary multipartite quantum system, Phys. Rev. A **102**, 032211 (2020).
[40] B. M. Terhal, D. P. Divicenzo, and D. W. Leung, Hiding Bits in Bell States, Phys. Rev. Lett. **86**, 5807 (2001).
[41] D. P. DiVicenzo, D. W. Leung, and B. M. Terhal, Quantum data hiding, IEEE Trans. Inf. Theory **48**, 580 (2002).
[42] T. Eggeling and R. F. Werner, Hiding Classical Data in Multipartite Quantum States, Phys. Rev. Lett. **89**, 097905 (2002).
[43] W. Matthews, S. Wehner, and A. Winter, Distinguishability of quantum states under restricted families of measurements with an application to quantum data hiding, Commun. Math. Phys. **291**, 813 (2009).
[44] M. Hillery, V. Bužek, and A. Berthiaume, Quantum secret sharing, Phys. Rev. A **59**, 1829 (1999).
[45] D. Markham and B. C. Sanders, Graph states for quantum secret sharing, Phys. Rev. A **78**, 042309 (2008).
[46] S. Schauer, M. Huber, and B. C. Hiesmayr, Experimentally feasible security check for n-qubit quantum secret sharing, Phys. Rev. A **82**, 062311 (2010).
[47] R. Cleve, D. Gottesman, and H.-K. Lo, How to Share a Quantum Secret, Phys. Rev. Lett. **83**, 648 (1999).
[48] D. Gottesman, Theory of quantum secret sharing, Phys. Rev. A **61**, 042311 (2000).
[49] R. Rahaman and M. G. Parker, Quantum scheme for secret sharing based on local distinguishability, Phys. Rev. A **91**, 022330 (2015).
[50] J. Wang, L. Li, H. Peng, and Y. Yang, Quantum-secret-sharing scheme based on local distinguishability of orthogonal multiqudit entangled states, Phys. Rev. A **95**, 022320 (2017).
[51] S. Halder, M. Banik, S. Agrawal, and S. Bandyopadhyay, Strong Quantum Nonlocality without Entanglement, Phys. Rev. Lett. **122**, 040403 (2019).
[52] Z.-C. Zhang and X. Zhang, Strong quantum nonlocality in multipartite quantum systems, Phys. Rev. A **99**, 062108 (2019).
[53] F. Shi, M. Hu, L. Chen, and X. Zhang, Strong quantum nonlocality with entanglement, Phys. Rev. A **102**, 042202 (2020).

[54] H. Zhou, T. Gao, and F. Yan, Orthogonal product sets with strong quantum nonlocality on a plane structure, Phys. Rev. A **106**, 052209 (2022).
[55] H. Zhou, T. Gao, and F. Yan, Strong quantum nonlocality without entanglement in an n-partite system with even n, Phys. Rev. A **107**, 042214 (2023).
[56] P. Yuan, G. Tian, and X. Sun, Strong quantum nonlocality without entanglement in multipartite quantum systems, Phys. Rev. A **102**, 042228 (2020).
[57] Y.-L. Wang, M.-S. Li, and M.-H. Yung, Graph-connectivity-based strong quantum nonlocality with genuine entanglement, Phys. Rev. A **104**, 012424 (2021).
[58] F. Shi, Z. Ye, L. Chen, and X. Zhang, Strong quantum nonlocality in N-partite systems, Phys. Rev. A **105**, 022209 (2022).
[59] F. Shi, M.-S. Li, M. Hu, L. Chen, M.-H. Yung, Y.-L. Wang, and X. Zhang, Strongly nonlocal unextendible product bases do exist, Quantum **6**, 619 (2022).
[60] M.-S. Li and Y.-L. Wang, Bounds on the smallest sets of quantum states with special quantum nonlocality, arXiv:2202.09034.
[61] S. Bandyopadhyay and S. Halder, Genuine activation of nonlocality: From locally available to locally hidden information, Phys. Rev. A **104**, L050201 (2021).
[62] M.-S. Li and Z.-J. Zheng, Genuine hidden nonlocality without entanglement: From the perspective of local discrimination, New J. Phys. **24**, 043036 (2022).
[63] S. Rout, A. G. Maity, A. Mukherjee, S. Halder, and M. Banik, Genuinely nonlocal product bases: Classification and entanglement-assisted discrimination, Phys. Rev. A **100**, 032321 (2019).
[64] M.-S. Li, Y.-L. Wang, F. Shi, and M.-H. Yung, Local distinguishability based genuinely quantum nonlocality without entanglement, J. Phys. A: Math. Theor. **54**, 445301 (2021).
[65] S. Rout, A. G. Maity, A. Mukherjee, S. Halder, and M. Banik, Multiparty orthogonal product states with minimal genuine nonlocality, Phys. Rev. A **104**, 052433 (2021).
[66] W. Dür, V. Vidal, and J. I. Cirac, Three qubits can be entangled in two inequivalent ways, Phys. Rev. A **62**, 062314 (2000).
[67] A. í, D. Bruß, M. Lewenstein, and A. Sanpera, Classification of Mixed Three-Qubit States, Phys. Rev. Lett. **87**, 040401 (2001).
[68] D. M. Greenberger, M. A. Horne, and A. Zeilinger, Bell's theorem without inequalities, Am. J. Phys. **58**, 1131 (1990).
[69] A. Karlsson and M. Bourennane, Quantum teleportation using three-particle entanglement, Phys. Rev. A **58**, 4394 (1998).
[70] B. Shi, Y. Jiang, and G. Guo, Probabilistic teleportation of two-particle entangled state, Phys. Lett. A **268**, 161 (2000).
[71] J. Joo, Y. Park, S. Oh, and J. Kim, Quantum teleportation via a W state, New J. Phys. **5**, 136 (2003).
[72] S. Bandyopadhyay and M. Nathanson, Tight bounds on the distinguishability of quantum states under separable measurements, Phys. Rev. A **88**, 052313 (2013).
[73] S. Bandyopadhyay, A. Cosentino, N. Johnston, V. Russo, J. Watrous, and N. Yu, Limitations on separable measurements by convex optimization, IEEE Trans. Inf. Theory **61**, 3593 (2015).
[74] R. Duan, Y. Feng, Y. Xin, and M. Ying, Distinguishability of quantum states by separable operations, IEEE Trans. Inf. Theory **55**, 1320 (2009).
[75] N. Yu, R. Duan, and M. Ying, Distinguishability of quantum states by positive operator-valued measures with positive partial transpose, IEEE Trans. Inf. Theory **60**, 2069 (2014).
[76] Y. Li, X. Wang, and R. Duan, Indistinguishability of bipartite states by positive-partial-transpose operations in the many-copy scenario, Phys. Rev. A **95**, 052346 (2017).