Exceptional point in self-consistent Markovian master equations

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The cxceptional point (EP) denotes the non-Hermitian degeneracy, in which both eigenvalues and eigenstates become identical. By the conventional local Markovian master equation, the EP can be constructed by parity-time ($\mathscr{PT}$) or anti-$\mathscr{PT}$ symmetry in a system composed of coupled subsystems. However, the coupling between two systems makes the conventional local Markovian master equation become inconsistent. By using the self-consistent Markovian master equation, we show that there is no EP in the system composed of two bosonic subsystems suffering from the incoherent gain and loss. We further prove that the conventional local Markovian master equation can be valid when the coupling strength is much smaller than the difference in resonance frequency between the two subsystems, rather than the resonance frequencies. In a system composed of three bosonic subsystems, the EP can be obtained by adiabatically eliminating one of the three subsystems.

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I. INTRODUCTION

The Hamiltonian governing the evolution of the closed system is Hermitian, and thus only degeneracy of the energy levels is possible. The inevitable coupling to the surrounding environment makes the effective Hamiltonian of the open system become non-Hermitian. The non-Hermitian degeneracy, known as the exceptional point (EP) [1,2], denotes that both eigenvalues and eigenstates coalesce. EPs have recently attracted more and more research, mainly by finding a large number of meaningful applications and exotic phenomena such as loss-induced lasing [3], stopped light [4], quantum state control [5,6], asymmetric backscattering [7], asymmetric mode switching [8,9], energy transfer [10], phase accumulation [11,12], and enhancement of quantum heat engine [13]. More importantly, EPs have been found to play an important role in improving the measurement sensitivity [14–21].

EPs can appear in $\mathscr{PT}$- and anti-$\mathscr{PT}$-symmetrical systems. In the presence of $\mathscr{PT}$ symmetry [22,23], non-Hermitian Hamiltonians can have entirely real eigenvalues. EPs are the separate points between purely real eigenvalues and the normally complex eigenvalues. Similarly, EPs are the separate points between the purely imaginary eigenvalues and the normally complex eigenvalues of anti-$\mathscr{PT}$-symmetrical non-Hermitian Hamiltonians [24,25].

$\mathscr{PT}$- and anti-$\mathscr{PT}$-symmetrical non-Hermitian effective Hamiltonians can be constructed by the coupled bosonic systems suffering from the local Markovian dissipation and driving [26–31]. By transforming the conventional Lindblad master equation into the quantum Heisenberg-Langevin equation, the effective $\mathscr{PT}$- and anti-$\mathscr{PT}$-symmetrical non-Hermitian effective Hamiltonians for the evolution of bosonic modes can be obtained. However, the local master equation may fail when there is coupling between the systems. It has been shown that the local master equation may violate the second law of thermodynamics [32] and give rise to nonphysical results [33–36], even in the limit of small bath couplings. Recently, it is shown that the local master equation may fail to describe dissipative critical behavior [37]. It is, therefore, necessary to be careful about the coupling between systems when constructing EPs.

Taking into account light-matter interaction, a self-consistent nonlocal Markovian master equation in the dressed picture has been proposed [38,39]. Recently, a self-consistent nonlocal Markovian dissipation master equation for an open quadratic quantum system has been derived [40]. In this article, we further derive the self-consistent nonlocal Markovian driving master equation for constructing the $\mathscr{PT}$-symmetrical system. We show that a fermionic bath with a strong enough chemical potential is required to obtain the incoherent driving. By transforming the self-consistent equation into the corresponding quantum Heisenberg-Langevin equation, we prove that EPs cannot appear in a system composed of two subsystems suffering from the incoherent gain and loss. The conventional local Markovian master equation is reasonable, which not only requires the coupling strength to be much smaller than the bare resonance frequency difference but also requires the baths tobe symmetric. Finally, we show that adiabatically eliminating one of the three coupled subsystems can construct EPs.

This article is organized as follows. In Sect. II, we introduce the EPs in a $\mathscr{PT}$-symmetrical non-Hermitian Hamiltonian by the conventional local Markovian master equation. In Sec. III, the self-consistent Markovian master equation for the general system is reviewed. In Sec. IV, we obtain the dressed Markovian master equation for the general quadratic system and the condition of incoherent driving is proposed. In Sec. V, we show that there is no EP in the system composed of two bosonic subsystems by the incoherent gain and loss. In Sec. IV, we show that the EP can be obtained by adiabatically eliminating one of the three coupled subsystems.

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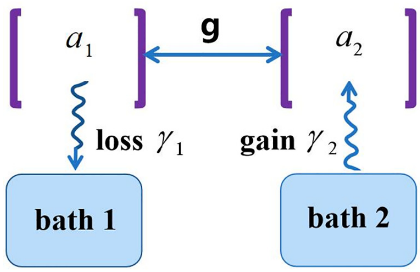

FIG. 1. Schematic diagram of the non-Hermitian Hamiltonian. Two cavity modes are coupled with the strength g. Cavity mode 1 suffers from dissipation bath 1 with the loss rate γ_1. Cavity mode 2 is incoherently driven by bath 2 with the gain rate γ_2.

A simple summary and the possibility of experimentation are proposed in Sec. VII.

II. EPs OBTAINED BY THE CONVENTIONAL LOCAL MARKOVIAN MASTER EQUATION

A typical non-Hermitian system is composed of coupled cavities with two resonant modes a_1 and a_2, as shown in Fig. 1, with the non-Hermitian Hamiltonian (setting $\hbar = 1$)

$$H = (\omega_1 - i\gamma_1)a_1^\dagger a_1 + (\omega_2 + i\gamma_2)a_2^\dagger a_2 + g(a_1^\dagger a_2 + a_2^\dagger a_1), \tag{1}$$

where ω_1 and ω_2 are the resonance frequencies of modes 1 and 2, respectively; γ_1 and γ_2 are the total loss and gain rates of modes 1 and 2, respectively.

When resonance frequencies are tuned to be equal (i.e., $\omega_1 = \omega_2 = \omega$) and the gain rate is equal to the loss rate (i.e., $\gamma_1 = \gamma_2 = \gamma$), the non-Hermitian Hamiltonian is $\mathscr{PT}$ symmetrical. The eigenvalues of the non-Hermitian Hamiltonian H are given by

$$E_\pm = \omega \pm \sqrt{g^2 - \gamma^2}. \tag{2}$$

When $g = \gamma$, the eigenvalues and the eigenstates are degenerate. Hence, $g = \gamma$ represents the EP.

The non-Hermitian Hamiltonian comes from the conventional local Markovian master equation [26]

$$\frac{d\rho}{dt} = -i[H_S, \rho] + \gamma\mathcal{L}(a_1)\rho + \gamma\mathcal{L}(a_2^\dagger)\rho, \tag{3}$$

where the superoperator $\mathcal{L}(a)\rho = 2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a$, with $a = \{a_1, \ a_2\}$, and the Hamiltonian $H_S = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g(a_1^\dagger a_2 + a_2^\dagger a_1)$. The quantum Langevin equation can be achieved by the formula [41–43]

$$\begin{aligned}\frac{da}{dt} &= i[H_S, a] - [a, a_1^\dagger](\gamma a_1 - \sqrt{2\gamma}a_{1\text{in}}) + (\gamma a_1^\dagger - \sqrt{2\gamma}a_{1\text{in}}^\dagger)\\ &\quad\times [a, a_1] - [a, a_2](\gamma a_2^\dagger - \sqrt{2\gamma}a_{2\text{in}}^\dagger)\\ &\quad+ (\gamma a_2 - \sqrt{2\gamma}a_{2\text{in}})[a, a_2^\dagger],\end{aligned} \tag{4}$$

where the noise operators satisfy that

$$\langle a_{j\text{in}}\rangle = 0, \quad \langle a_{j\text{in}}a_{k\text{in}}\rangle = 0, \tag{5}$$

$$\langle a_{j\text{in}}^\dagger a_{k\text{in}}\rangle = 0, \quad \langle a_{j\text{in}}a_{k\text{in}}^\dagger\rangle = \delta_{jk}. \tag{6}$$

III. SELF-CONSISTENT MARKOVIAN MASTER EQUATION

However, the master equation in Eq. (3), is not self-consistent due to the coupling between the two subsystems. A self-consistent master equation can be derived by the dressed master equation [38–40,44]

$$\frac{d\rho}{dt} = -i[H_S + H_{\text{LS}}, \rho] + \mathcal{D}[\rho], \tag{7}$$

where $H_{\text{LS}} = \sum_{\alpha,\beta,\omega} L_{\alpha\beta}(\omega)O_\alpha^\dagger(\omega)O_\beta(\omega)$ is a Lamb-shift correction, and the superoperator $\mathcal{D}[\rho]$ is described by

$$\begin{aligned}\mathcal{D}[\rho] = \sum_{\alpha,\beta,\omega}\lambda_{\alpha\beta}(\omega)[2O_\beta(\omega)\rho O_\alpha^\dagger(\omega) - O_\beta(\omega)O_\alpha^\dagger(\omega)\rho\\ - \rho O_\beta(\omega)O_\alpha^\dagger(\omega)].\end{aligned} \tag{8}$$

Here, the dressed operator is given by

$$O_\alpha(\omega) = \sum_{k,q}\delta_{\omega_q-\omega_k}|k\rangle\langle k|Q_\alpha|q\rangle\langle q|, \tag{9}$$

and the factors are

$$\lambda_{\alpha\beta}(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} d\tau e^{i\omega\tau}\langle\tilde{R}_\alpha^\dagger(\tau)R_\beta\rangle, \tag{10}$$

$$L_{\alpha\beta}(\omega) = \frac{1}{2i}\int_0^\infty d\tau[e^{i\omega\tau}\langle\tilde{R}_\alpha^\dagger(\tau)R_\beta\rangle - e^{-i\omega\tau}\langle\tilde{R}_\alpha^\dagger(\tau)R_\beta\rangle], \tag{11}$$

where ω_k and $|k\rangle$ are the kth eigenvalue and eigenstate of the system Hamiltonian H_{S}, the undressed operator O_α acts on the system in the interaction Hamiltonian $H_{\text{int}} = \sum_\alpha O_\alpha \bigotimes R_\alpha$ (R_α acts on the environment), $\langle.\rangle$ denotes the expectation value calculated with the environment density operator ρ_{E}, and $\tilde{R}_\alpha(\tau) = e^{iH_{\text{E}}\tau}R_\alpha e^{-iH_E\tau}$, with H_{E} being the environment Hamiltonian.

IV. DRESSED MASTER EQUATION FOR A GENERAL QUADRATIC SYSTEM

For a general quadratic bosonic system, the Hamiltonian is described by [45]

$$H' = \sum_{n=1}^{N} H_n + \sum_{i=1,i<j}^{N} H_{ij}, \tag{12}$$

in which

$$H_n = \omega_n a_n^\dagger a_n + \left(\frac{\chi_n}{2}a_n^2 + \text{H.c.}\right), \tag{13}$$

$$H_{ij} = (g_{ij}a_i a_j + \lambda_{ij}a_i a_j^\dagger) + \text{H.c.}, \tag{14}$$

where H_n denotes the Hamiltonian for the nth subsystem with $n = \{1, \ldots, N\}$, ω_n is the resonance frequency of the bosonic subsystem with the annihilation operator a and the creation operator $a^\dagger$, $|\chi_n|$ denotes the strength of two-photon driving, and λ_{ij} (g_{ij}) denotes the coupling strength of the rotating (counter-rotating)-wave interaction between the two subsystems.

By using a Hopfield-Bogoliubov (HB) transformation [46,47], for the stable normal phase, the total Hamiltonian can be rewritten as a diagonal form:

$$H' = \sum_{n=1}^{N} \Omega_n b_n^\dagger b_n + \frac{1}{2}(\Omega_n - \omega_n), \tag{15}$$

where the collective bosonic mode operators b_n satisfy the commutation relation $[b_i, b_j^\dagger] = \delta_{ij}$,

$$b_n = \sum_{i=1}^{N} (\mu_{ni} a_i + \nu_{ni} a_i^\dagger)/\xi_n, \tag{16}$$

in which the normalization factor is described by $\xi_n = \sqrt{\sum_{i=1}^{N}(|\mu_{ni}|^2 - |\nu_{ni}|^2)}$. In the Nambu space, it can be rewritten as

$$\vec{\mathbf{b}} = \mathbb{T}\vec{\mathbf{a}}, \tag{17}$$

where the Nambu field vector is defined as

$$\vec{\mathbf{x}} = (x_1, \dots, x_N, x_1^\dagger, \dots, x_N^\dagger), \tag{18}$$

and the canonical transformation matrix

$$\mathbb{T} = \begin{pmatrix} \tilde{\mu} & \tilde{\nu} \\ \tilde{\nu}^* & \tilde{\mu}^* \end{pmatrix}, \tag{19}$$

where the elements of the matrix are $\tilde{\mu}_{ni} = \mu_{ni}/\xi_n$ and $\tilde{\nu}_{ni} = \nu_{ni}/\xi_n$. The coefficient vectors $(\mu_{n1}, \dots, \mu_{nN}, \nu_{n1}, \dots, \nu_{nN})^T$ are eigenvectors of the HB matrix $\mathbf{M}$, which is derived by the commutation relation $[b_n, H'] = \Omega_n b_n$ [45]:

$$\mathbf{M} = \begin{pmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B}^* & -\mathbf{A}^* \end{pmatrix}, \tag{20}$$

with submatrices

$$\mathbf{A} = \begin{pmatrix} \omega_1 & \lambda_{12} & \dots & \lambda_{1N} \\ \lambda_{12}^* & \omega_2 & \dots & \lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1N}^* & \lambda_{2N}^* & \dots & \omega_N \end{pmatrix}, \tag{21}$$

$$\mathbf{B} = \begin{pmatrix} \chi_1 & g_{12} & \dots & g_{1N} \\ g_{12} & \chi_2 & \dots & g_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{1N} & g_{2N} & \dots & \chi_N \end{pmatrix}. \tag{22}$$

Incoherent dissipation and driving

For the dissipation environment, there are N independent thermal baths with the Hamiltonian described by

$$H_{\mathrm{E},n} = \int dk(\epsilon_n(k) - \eta_n) c_n^\dagger(k) c_n(k), \tag{23}$$

where the spectrum $\epsilon_n(k) \geqslant 0$ is non-negative and η_n denotes the chemical potential of the nth thermal bath at temperature T_n.

The bath operators $c_n(k)$ satisfy the rules

$$\{c_n(k), c_n^\dagger(q)\}_{\zeta_n} = \delta(k - q), \tag{24}$$

$$\{c_n(k), c_n(q)\}_{\zeta_n} = 0, \quad [c_n(k), c_m(q)] = 0, \tag{25}$$

where $\{X, Y\}_{\zeta_n} = XY + \zeta_n YX$, and $\zeta_n = +1$ (-1) belongs to fermionic (bosonic) systems. The density matrix of the thermal bath can be described by

$$\rho_{\mathrm{E}} = \bigotimes_{n=1}^{N} \frac{e^{-H_{\mathrm{E},n}/T_n}}{\mathrm{Tr}(e^{-H_{\mathrm{E},n}/T_n})}. \tag{26}$$

Based on the above equations, the two-point expectation values can be obtained as

$$\langle c_n(k) c_m(q)\rangle = 0, \tag{27}$$

$$\langle c_n^\dagger(k) c_m(q)\rangle = \delta_{mn}\delta(k - q) f_n(\epsilon_n(k)), \tag{28}$$

in which

$$f_n(\epsilon) = [\zeta_n + e^{(\epsilon - \eta_n)/T_n}]^{-1}. \tag{29}$$

We consider that N subsystems interact linearly with the corresponding bath separately, which are described by

$$\begin{aligned} H_{\mathrm{int}} &= \sum_{n=1}^{N} (a_n + a_n^\dagger) \otimes \int dk g_n(k)[c_n(k) + c_n^\dagger(k)] \\ &= \sum_{n=1}^{N} O_n \otimes R_n. \end{aligned} \tag{30}$$

In the basis of the Hamiltonian H_{S}, the eigenoperator associated with O_n can be obtained as

$$O_n(\omega) = \sum_{k=1}^{N} [\phi_{n,k}\delta_{\omega,\Omega_k} b_k + \phi_{n,k}^* \delta_{\omega,-\Omega_k} b_k^\dagger], \tag{31}$$

where the element $\phi_{n,k} = (\mathbb{T}^{-1})_{n,k} + (\mathbb{T}^{-1})_{n,k+N}^*$.

The factor λ_{nm} in Eq. (10) can be derived by the correlation functions [40]

$$\lambda_{nm}(\omega) = \delta_{nm}\lambda_{nn}(\omega), \tag{32}$$

$$\lambda_{nn}(\omega) = \begin{Bmatrix} \mathcal{J}_n(\omega)[1 - \zeta_n f_n(\omega)] & \text{if } \omega > 0 \\ \mathcal{J}_n(-\omega) f_n(-\omega) & \text{if } \omega < 0 \\ \mathcal{J}_n(0)[1 + (1 - \zeta_n) f_n(0)] & \text{if } \omega = 0 \end{Bmatrix}, \tag{33}$$

where the spectral density for the nth bath is given by

$$\mathcal{J}_n(\omega) = \pi \int dk |g_n(k)|^2 \delta(\omega - \epsilon_n(k)). \tag{34}$$

In a general case, where the system is not degenerate (i.e., $\Omega_n \neq \Omega_m$ when $n \neq m$) and the eigenspectrum is not 0 (i.e., $\Omega_n \neq 0$), the superoperator $\mathcal{D}(\rho)$ can be described by

$$\mathcal{D}(\rho) = \sum_{n,k} \gamma_{n,k}\{[1 - \zeta_n f_n(\Omega_k)]\mathcal{L}(b_k)\rho + f_n(\Omega_k)\mathcal{L}(b_k^\dagger)\rho\}, \tag{35}$$

where the coupling constants $\gamma_{n,k} = \mathcal{J}_n(\Omega_k)|\phi_{n,k}|^2$.

Combining Eqs. (4) and (35), we can see that the total change rate of the nth mode b_k is

$$\Gamma_k = \sum_n \Gamma_{k,n} = \sum_n \gamma_{n,k}[1 - \zeta_n f_n(\Omega_k) - f_n(\Omega_k)]. \tag{36}$$

When all the baths are composed of bosons, the total change rate $\Gamma_k = \sum_n \gamma_{n,k} > 0$ denotes that the mode is suffering from the dissipation process.

In order to obtain the incoherent gain, several baths must be composed of fermions, i.e., $\zeta_n = 1$. In this case, the change rate $\Gamma_{k,n} = \gamma_{n,k}[1 - 2f_n(\Omega_k)]$ should be negative. As a result, it leads to $\Omega_k < \eta_n$, which means that a strong enough chemical potential is needed to drive the system incoherently. It can also be considered that the bath is composed of spins, which are in the excited states. In other words, a bosonic bath is not suitable for implementing incoherent driving.

V. THE EP DOES NOT EXIST IN TWO-BOSON SYSTEMS

In this section, we investigate whether an EP can exist in the system composed of two bosonic subsystems with the self-consistent Markovian master equation.

The bosonic system composed of two subsystems is dominated by the Hamiltonian

$$H_{S1} = \omega a_1^\dagger a_1 + \omega a_2^\dagger a_2 + g(a_1^\dagger a_2 + a_2^\dagger a_1). \tag{37}$$

Subsystem 1 interacts with bath 1 composed of bosons at zero temperature, and subsystem 2 interacts with bath 2 composed of fermions at zero temperature. The corresponding interaction Hamiltonian is described by

$$H_{\text{int}1} = \sum_{n=1}^{2}(a_n + a_n^\dagger) \otimes \int dk g_n(k)[c_n(k) + c_n^\dagger(k)]. \tag{38}$$

The Hamiltonian can be diagonalized as

$$H_{S1} = (\omega + g)b_1^\dagger b_1 + (\omega - g)b_2^\dagger b_2, \tag{39}$$

where the dressed operators are $b_1 = (a_1 + a_2)/\sqrt{2}$ and $b_2 = (a_1 - a_2)/\sqrt{2}$. Then, using the consistent master equation in Eq. (35) for $g \neq 0$, we can achieve

$$\begin{aligned}\frac{d\rho}{dt} = &- i[H_{S1}, \rho] + [J_1(\omega + g)\mathcal{L}(b_1)\rho + J_2(\omega + g)\mathcal{L}(b_1^\dagger)\rho \\ &+ J_1(\omega - g)\mathcal{L}(b_2)\rho + J_2(\omega - g)\mathcal{L}(b_2^\dagger)\rho].\end{aligned} \tag{40}$$

We assume that baths have a very large bandwidth, leading to $\gamma = J_j(\omega + g) = J_j(\omega - g)$, with $j = 1$ and 2, and the Lamb-shift correction $H_{LS1} = 0$. The quantum-Heisenberg Langevin equation is given as follows by using Eq. (4):

$$i\begin{pmatrix}\dot{b}_1\\ \dot{b}_2\end{pmatrix} = \begin{pmatrix}\omega + g & 0\\ 0 & \omega - g\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix} + \sqrt{2\gamma}\begin{pmatrix}b_{1\text{in}} + b_{1\text{in}}^\dagger\\ b_{2\text{in}} + b_{2\text{in}}^\dagger\end{pmatrix}, \tag{41}$$

where b_{in} denotes the noise operator. The effective Hamiltonian is described by

$$H_{\text{eff}1} = \begin{pmatrix}\omega + g & 0\\ 0 & \omega - g\end{pmatrix}. \tag{42}$$

Because $H_{\text{eff}1}$ is Hermitian, the EP does not exist. This result shows that the EP cannot be constructed in a resonance-coupled incoherent driven-dissipative system. Because of the resonant coupling, both the driving and the dissipation act synchronously on each subsystem. The asymmetric effects of driving and dissipation cannot be obtained.

The self-consistent master equations with degenerate eigenvalues

In the nondegenerate case, the dressed modes b_k are independent, leading to the EP not existing. Then, we consider the degenerate eigenenergies of the system Hamiltonian.

The system possesses M different energy eigenspaces, labeled by an index $\iota = 1, \dots, M$. There are N_ι eigenvectors for the eigenvalue ω_ι. The consistent Markovian master equation for the baths with large bandwidth can be given by [40]

$$\begin{aligned}\frac{d\rho}{dt} = &- i[H_S, \rho] + \sum_{n,\iota}\sum_{\alpha=1,\beta=1}^{N_\iota}\big[\Phi_{\mu\nu}^{(n,\iota)}\lambda_{nn}(\omega_\iota)(2b_\mu\rho b_\nu^\dagger \\ &- \{b_\nu^\dagger b_\mu, \rho\}_+) + \Phi_{\nu\mu}^{(n,\iota)}\lambda_{nn}(-\omega_\iota) \\ &\times (2b_\mu^\dagger\rho b_\nu - \{b_\nu b_\mu^\dagger, \rho\}_+)\big],\end{aligned} \tag{43}$$

where the factors $\Phi_{\mu\nu}^{(n,\iota)} = \phi_{n,\mu}\phi_{n,\nu}^*$.

A simple degenerate case is one in which there is only pairing coupling between two bosonic subsystems with the Hamiltonian

$$H_{S2} = \omega a_1^\dagger a_1 + \omega a_2^\dagger a_2 + g(a_1 a_2 + a_1^\dagger a_2^\dagger). \tag{44}$$

Subsystem 1 interacts with bosonic bath 1 at zero temperature, and subsystem 2 interacts with fermionic bath 2 at zero temperature.

The transformation matrix between Nambu field vectors $\vec{\mathbf{a}} = (a_1, a_2, a_1^\dagger, a_2^\dagger)$ and $\vec{\mathbf{b}} = (b_1, b_2, b_1^\dagger, b_2^\dagger)$ can be expressed as

$$\mathbb{T}^{-1} = \begin{pmatrix}W_+ & 0 & 0 & W_-\\ 0 & W_+ & W_- & 0\\ 0 & W_- & W_+ & 0\\ W_- & 0 & 0 & W_+\end{pmatrix}, \tag{45}$$

where the values are defined as $W_\pm = \pm\sqrt{\frac{\omega}{2\sqrt{\omega^2-g^2}} \pm \frac{1}{2}}$.

In the rotating reference frame, we can obtain the dynamic of the expectation values of the dressed operators b_1 and b_2 according to Eq. (43):

$$i\begin{pmatrix}\langle\dot{b}_1\rangle\\ \langle\dot{b}_2\rangle\end{pmatrix} = \begin{pmatrix}\frac{-i(W\lambda_- + \lambda_+)}{2} & \frac{-i\sqrt{W^2-1}\lambda_-}{2}\\ \frac{-i\sqrt{W^2-1}\lambda_-}{2} & \frac{-i(W\lambda_- - \lambda_+)}{2}\end{pmatrix}\begin{pmatrix}\langle b_1\rangle\\ \langle b_2\rangle\end{pmatrix}, \tag{46}$$

where the factors are defined as $W = \frac{\omega}{2\sqrt{\omega^2-g^2}}$ and $\lambda_\pm = \mathcal{J}_1(\sqrt{\omega^2 - g^2}) \pm \mathcal{J}_2(\sqrt{\omega^2 - g^2})$. Therefore, the effective Hamiltonian $H_{\text{eff}2}$ can be expressed as

$$H_{\text{eff}2} = \begin{pmatrix}\frac{-i(W\lambda_- + \lambda_+)}{2} & \frac{-i\sqrt{W^2-1}\lambda_-}{2}\\ \frac{-i\sqrt{W^2-1}\lambda_-}{2} & \frac{-i(W\lambda_- - \lambda_+)}{2}\end{pmatrix}. \tag{47}$$

The eigenvalues of $H_{\text{eff}2}$ can be derived and are given by

$$E_\pm = \frac{i}{2}[\sqrt{\lambda_+^2 + (W^2 - 1)\lambda_-^2} \pm W\lambda_-]. \tag{48}$$

Because the eigenvalues are still imaginary, there are no EPs that separate the real and the imaginary values. Therefore, in the degenerate eigenspace, EPs are not allowed in the self-consistent Markovian master equation.

As a summary, EPs cannot appear in the driven-dissipative bosonic system irrespective of whether the eigenvalues of the eigensystem are degenerate or not.

VI. EPs BY ADIABATIC ELIMINATION

In this section, we try to construct the EP by adiabatic elimination in multiple boson systems.

First, we find the condition that the conventional local Markovian master equation can be close to the nonlocal self-consistent master equation. For two nonresonant coupled subsystems, the Hamiltonian is described by

$$H_{S3} = \omega_1 a_1^\dagger a_1 + \omega_2 a_2^\dagger a_2 + g(a_1^\dagger a_2 + a_2^\dagger a_1). \tag{49}$$

In the diagonalized form, the Hamiltonian is rewritten as $H_{S3} = \Omega_+ b_1^\dagger b_1 + \Omega_- b_2^\dagger b_2$ with the eigenvalues $\Omega_\pm = \frac{\omega_1+\omega_2 \pm\sqrt{4g^2+\Delta^2}}{2}$. The canonical transformation matrix is given by

$$\mathbb{T}^{-1} = \begin{pmatrix} \sqrt{\frac{1+Y}{2}} & -\sqrt{\frac{1-Y}{2}} & 0 & 0 \\ \sqrt{\frac{1-Y}{2}} & \sqrt{\frac{1+Y}{2}} & 0 & 0 \\ 0 & 0 & \sqrt{\frac{1+Y}{2}} & -\sqrt{\frac{1-Y}{2}} \\ 0 & 0 & \sqrt{\frac{1-Y}{2}} & \sqrt{\frac{1+Y}{2}} \end{pmatrix}, \tag{50}$$

where the value Y is defined as $Y = \frac{\Delta}{\sqrt{\Delta^2+4g^2}}$, and the resonance frequency difference Δ is given by $\Delta = \omega_1 - \omega_2$. Without loss of generality, next we consider that both the resonance frequency difference and the coupling strength are larger than 0, i.e., $\Delta > 0$ and $g > 0$.

When the coupling strength is much less than the resonance frequency difference ($g \ll \Delta$), we can obtain

$$a_1 \simeq b_1 - \frac{g}{\Delta} b_2 + O\left(\frac{g^2}{\Delta^2}\right) b_1 + O\left(\frac{g^3}{\Delta^3}\right) b_2, \tag{51}$$

$$a_2 \simeq b_2 + \frac{g}{\Delta} b_1 + O\left(\frac{g^2}{\Delta^2}\right) b_2 + O\left(\frac{g^3}{\Delta^3}\right) b_1, \tag{52}$$

$$\Omega_1 \simeq \omega_1 + \frac{g^2}{\Delta} + O\left(\frac{g^3}{\Delta^3}\right), \tag{53}$$

$$\Omega_2 \simeq \omega_2 - \frac{g^2}{\Delta} - O\left(\frac{g^3}{\Delta^3}\right), \tag{54}$$

$$\gamma_{21} = \gamma_{12} \simeq O\left(\frac{g^2}{\Delta^2}\right), \tag{55}$$

where $O(\frac{g^2}{\Delta^2})$ denotes the second-order small quantity, and $O(\frac{g^3}{\Delta^3})$ denotes the third-order small quantity. Ignoring all the small quantities, the local Markovian master equation is recovered as

$$\frac{d\rho}{dt} \approx \sum_{k=1}^{2} \{-i[\omega_k a_k^\dagger a_k, \rho] + \mathcal{L}_{\text{loc}}(\rho)\}, \tag{56}$$

in which,

$$\mathcal{L}_{\text{loc}}(\rho) = \gamma_{k,k}\{[1-\zeta_k f_k(\omega_k)]\mathcal{L}(a_k)\rho + f_k(\omega_k)\mathcal{L}(a_k^\dagger)\rho\}. \tag{57}$$

In the above local Markovian master equation, the two bosonic modes a_k are independent. Therefore, EPs cannot occur in the local Markovian master equation.

Up to the first-order small quantity (i.e., on the order of g/Δ), the self-consistent Markovian master equation can be given by

$$\begin{aligned} \frac{d\rho}{dt} \approx & -i[H_{S3}, \rho] + \mathcal{L}_{\text{loc}}(\rho) + \frac{g}{\Delta}\sum_{k=1}^{2}(-1)^{k-1}\gamma_{k,k} \\ & \times \{[1-\zeta_k f_k(\omega_k)](2a_1\rho a_2^\dagger + 2a_2\rho a_1^\dagger) \\ & + f_k(\omega_k)(2a_1^\dagger \rho a_2 + 2a_2^\dagger \rho a_1) \\ & - [1+(1-\zeta_k)f_k(\omega_k)](a_1^\dagger a_2 + a_2^\dagger a_1)\rho \\ & + \rho(a_1^\dagger a_2 + a_2^\dagger a_1)\}. \end{aligned} \tag{58}$$

When the two baths are identical and their temperatures are close to zero degrees $[\zeta_1 f_1(\omega_1) = \zeta_2 f_2(\omega_2)]$, and the couplings between the subsystems and the corresponding baths are the same ($\gamma_{11} = \gamma_{22}$), we can achieve the following conventional Markovian master equation from the above equation:

$$\frac{d\rho}{dt} \approx -i[H_{S3}, \rho] + \mathcal{L}_{\text{loc}}(\rho). \tag{59}$$

It shows that the conventional local Markovian master equation is reasonable, which not only requires the coupling strength g to be much smaller than the resonance frequency difference Δ (rather than the resonance frequencies) but also requires the baths to be symmetric.

Then, we consider a system composed of three bosonic subsystems, with the Hamiltonian described by

$$H_{S4} = \sum_{i=1}^{3} \omega_i a_i^\dagger a_i + g(a_1^\dagger a_3 + a_3^\dagger a_1) + g'(a_2^\dagger a_3 + a_3^\dagger a_2), \tag{60}$$

where ω_i denotes the frequencies of the ith subsystem, and g (g') denotes the coupling strength between modes 1 (2) and 3.

In the rotating reference frame, the Hamiltonian H_{S4} can be rewritten as

$$\begin{aligned} H_{S4} = & \Delta' a_1^\dagger a_1 + (\Delta' - \varepsilon) a_2^\dagger a_2 + g(a_1^\dagger a_3 + a_3^\dagger a_1) \\ & + g'(a_2^\dagger a_3 + a_3^\dagger a_2), \end{aligned} \tag{61}$$

where $\Delta' = \omega_1 - \omega_3$ and $\varepsilon = \omega_1 - \omega_2$.

We consider that the condition satisfies $\Delta' \gg g$, so that the self-consistent Markovian master equation is close to the local Markovian master equation according to Eq. (56). Then, we obtain the unitary transformation matrix U defined in $(b_1, b_2, b_3)^T = U(a_1, a_2, a_3)^T$ for $\varepsilon \simeq O(g)$, which is close to

$$U \approx \begin{pmatrix} 1 & \frac{g}{\Delta'} & \frac{g}{\Delta'} \\ \frac{-g}{\Delta'} & 1 & \frac{g}{\Delta'} \\ \frac{-g}{\Delta'} & \frac{-g}{\Delta'} & 1 \end{pmatrix}. \tag{62}$$

When the temperature in all three baths is 0, the self-consistent Markovian master equation can give the Langevin equation with the diagonalized modes

$$i\begin{pmatrix} \langle \dot{b}_1 \rangle \\ \langle \dot{b}_2 \rangle \\ \langle \dot{b}_3 \rangle \end{pmatrix} \approx -\begin{pmatrix} \Upsilon_1 & 0 & 0 \\ 0 & \Upsilon_2 & 0 \\ 0 & 0 & \Gamma_2 \end{pmatrix} \begin{pmatrix} \langle b_1 \rangle \\ \langle b_2 \rangle \\ \langle b_3 \rangle \end{pmatrix}, \tag{63}$$

where the values are defined as $\Upsilon_1 = \Gamma_1 + i\Delta'$, $\Upsilon_2 = \Gamma_2 + i(\Delta' - \varepsilon)$, $\Gamma_1 = \mathcal{J}_1(\omega_1) = \Gamma_2 = \mathcal{J}_2(\omega_2)$, and $\Gamma_3 = \mathcal{J}_3(\omega_3)$.

Based on the above equation, the evolution of modes a_j can be derived by

$$i\begin{pmatrix}\langle\dot{a_1}\rangle\\ \langle\dot{a_2}\rangle\\ \langle\dot{a_3}\rangle\end{pmatrix}\approx -U^{-1}\begin{pmatrix}\Upsilon_1 & 0 & 0\\ 0 & \Upsilon_2 & 0\\ 0 & 0 & \Gamma_2\end{pmatrix}U\begin{pmatrix}\langle a_1\rangle\\ \langle a_2\rangle\\ \langle a_3\rangle\end{pmatrix}. \tag{64}$$

After calculation from the above equation, we can achieve the detailed evolution dynamic

$$i\begin{pmatrix}\langle\dot{a_1}\rangle\\ \langle\dot{a_2}\rangle\\ \langle\dot{a_3}\rangle\end{pmatrix}\approx -\Lambda\begin{pmatrix}\langle a_1\rangle\\ \langle a_2\rangle\\ \langle a_3\rangle\end{pmatrix}, \tag{65}$$

where the evolution matrix is given by

$$\Lambda=\begin{pmatrix}\Upsilon' & \frac{g^2(\Gamma_3-\Gamma_1)}{\Delta'^2} & \frac{g(\Gamma_1-\Gamma_3)}{\Delta'}\\ \frac{g^2(\Gamma_3-\Gamma_1)}{\Delta'^2} & \Upsilon' & \frac{g(\Gamma_1-\Gamma_3)}{\Delta'}\\ \frac{g(\Gamma_1-\Gamma_2)}{\Delta'} & \frac{g(\Gamma_1-\Gamma_3)}{\Delta'} & \Gamma_3\end{pmatrix}, \tag{66}$$

in which $\Upsilon'=\Gamma_1+\frac{g^2(2\Gamma_1+\Gamma_3)}{\Delta'^2}$. When $\Gamma_2\gg\Gamma_1\gg\Delta'$, mode $\langle a_3\rangle$ can be eliminated because mode $\langle a_3\rangle$ reaches the steady state much faster than the other two modes. By assuming $\langle\dot{a}_3\rangle=0$, we can obtain

$$\langle a_3\rangle\approx\frac{g}{\Delta'\Gamma_2}(\Gamma_2-\Gamma_1)(\langle a_1\rangle+\langle a_2\rangle). \tag{67}$$

Using the above equation, we can obtain the evolution equation for modes 1 and 2:

$$i\begin{pmatrix}\langle\dot{a_1}\rangle\\ \langle\dot{a_2}\rangle\end{pmatrix}=H_{\mathrm{eff3}}\begin{pmatrix}\langle a_1\rangle\\ \langle a_2\rangle\end{pmatrix}, \tag{68}$$

where the effective Hamiltonian of the reduced two subsystems is depicted as

$$H_{\mathrm{eff3}}=i\begin{pmatrix}-\Gamma_1-i\Delta' & -\frac{g^2\Gamma_1}{\Delta'^2}\\ -\frac{g^2\Gamma_1}{\Delta'^2} & -\Gamma_1-i(\Delta'-\varepsilon)\end{pmatrix}. \tag{69}$$

Moving to the reference frame rotating with frequency $\Delta'-\varepsilon/2$, the effective Hamiltonian can be reduced to

$$H_{\mathrm{eff3}}=i\begin{pmatrix}-\Gamma_1-i\varepsilon/2 & -\frac{g^2\Gamma_1}{\Delta'^2}\\ -\frac{g^2\Gamma_1}{\Delta'^2} & -\Gamma_1+i\varepsilon/2\end{pmatrix}. \tag{70}$$

The eigenvalues of H_{eff3} are given by $\Omega_\pm=i(-\Gamma_1\pm\sqrt{g^4\Gamma_1^2/\Delta'^4-\varepsilon^2/4})$. As a consequence, the EP appears at $\varepsilon=2g^2\Gamma_1/\Delta'^2$, which separates the purely imaginary eigenvalue and the normally complex eigenvalue. Therefore, the EP appears in the effective anti-$\mathscr{P}\mathscr{T}$-symmetrical Hamiltonian.

Then, we try to construct the EP without the condition $g\ll\Delta'$; i.e., there is no need to approach the local Markovian master equation. We redefine $\varepsilon=2\Delta'$. In this case, the unitary transformation matrix U can be exactly derived, which is described by

$$U=\begin{pmatrix}\frac{\Delta'+\Delta_g}{2\Delta_g} & \frac{g^2}{\Delta_g(\Delta_g+\Delta')} & \frac{g}{\Delta_g}\\ \frac{\Delta'-\Delta_g}{2\Delta_g} & \frac{g^2}{\Delta_g(\Delta'-\Delta_g)} & \frac{g}{\Delta_g}\\ \frac{-g\Delta_g}{\Delta'^2} & \frac{g\Delta_g}{\Delta'^2} & \frac{\Delta_g}{\Delta'}\end{pmatrix}, \tag{71}$$

where $\Delta_g=\sqrt{\Delta'^2+2g^2}$. In this case, the evolution matrix Λ defined in Eq. (65) can be reformulated as

$$\Lambda=U^{-1}\begin{pmatrix}\Gamma_1+i\Delta' & 0 & 0\\ 0 & \Gamma_1-i\Delta' & 0\\ 0 & 0 & \Gamma_3\end{pmatrix}U$$
$$=\begin{pmatrix}\frac{Z_-}{\Delta_g^2} & \frac{ig^2(\Gamma_1-\Gamma_3)}{\Delta_g^2} & \frac{ig\Delta'(\Gamma_3-\Gamma_1)-g\Delta_g^2}{\Delta_g^2}\\ \frac{ig^2(\Gamma_1-\Gamma_3)}{\Delta_g^2} & \frac{Z_+}{\Delta_g^2} & \frac{ig\Delta'(\Gamma_1-\Gamma_3)-g\Delta_g^2}{\Delta_g^2}\\ \frac{ig\Delta'(\Gamma_3-\Gamma_1)-g\Delta_g^2}{\Delta_g^2} & \frac{ig\Delta'(\Gamma_1-\Gamma_3)-g\Delta_g^2}{\Delta_g^2} & \frac{-i(2g^2\Gamma_1+\Gamma_3\Delta'^2)}{\Delta_g^2}\end{pmatrix}, \tag{72}$$

where $Z_\pm=i[g^2(\Gamma_1+\Gamma_2)+\Gamma_1\Delta'^2]\pm\Delta'\Delta_g^2$. For $|\Gamma_3|\gg|\Gamma_1|$ and $|\Delta'|$, we can adiabatically eliminate mode a_3. As a result, the effective Hamiltonian for modes a_1 and a_2 can be described by

$$H_{\mathrm{eff4}}=\begin{pmatrix}\Gamma_2\Delta'\Delta_g^2-i\kappa & -i\chi\\ -i\chi & -\Gamma_2\Delta'\Delta_g^2-i\kappa\end{pmatrix}, \tag{73}$$

where $\kappa=\Delta'^2(g^2+\Gamma_1\Gamma_3)+g^2(2g^2+\Gamma_1^2+\Gamma_1\Gamma_3)$ and $\chi=g^2(\Delta_g^2+\Gamma_1^2-\Gamma_1\Gamma_3)$. The eigenvalues of H_{eff4} are given by $E_\pm=-i\kappa\pm\sqrt{(\Gamma_3\Delta'\Delta_g^2)^2-\chi^2}$. When modes 1, 2, and 3 are suffering from incoherent dissipation, i.e., $\Gamma_1>0$ and $\Gamma_3>0$, $|\chi|=\Gamma_3\Delta'\Delta_g^2$ denotes the anti-$\mathscr{P}\mathscr{T}$-symmetrical EP. Therefore, the relation $\Gamma_1\approx\frac{\Delta'(\Delta'^2+2g^2)}{g^2}$ is required to find the anti-$\mathscr{P}\mathscr{T}$-symmetrical EP.

In order to find the $\mathscr{P}\mathscr{T}$-symmetrical EP, the condition $\kappa=0$ is necessary. Modes 1 and 2 are incoherent driven, i.e., $\Gamma_1<0$. As a result, we can achieve $\Gamma_2=\frac{g^2(2g^2+\Delta'^2+\Gamma_1^2)}{(g^2+\Delta'^2)|\Gamma_1|}$. The $\mathscr{P}\mathscr{T}$-symmetrical EP can also appear at $\Gamma_1\approx\frac{\Delta'(\Delta'^2+2g^2)}{g^2}$.

VII. CONCLUSION

We investigate the construction of EPs in the effective non-Hermitian Hamiltonian from the self-consistent Markovian master equation. Unlike the result from the conventional local Markovian master equation, we prove that the EPs cannot exist in the system composed of two bosons suffering from the incoherent gain and loss. For constructing the $\mathscr{P}\mathscr{T}$-symmetrical system, we further derive the self-consistent nonlocal Markovian driving master equation. We show that a fermionic bath with a strong enough chemical potential is required to obtain the incoherent driving. It can also be considered that the bath is composed of spins, which are in the excited states by extra controls. A bosonic bath is not suitable for implementing the incoherent driving, and we show that the conventional local Markovian master equation is reasonable, which not only requires the coupling strength to be much smaller than the resonance frequency difference (rather than the resonance frequencies) but also requires the baths to be symmetrical. By adiabatically eliminating one of the three coupled subsystems, we can reconstruct the EPs with two different parameter choices: one is that the coupling strength is much less than the resonance frequency difference, and the other is that the coupling strength is not much less than the resonance frequency difference. The former can only construct anti-$\mathcal{PT}$-symmetrical EPs. The latter can construct anti-$\mathcal{PT}$- and $\mathcal{PT}$-symmetrical EPs.

The system composed of three coupled bosonic subsystems in this article can be realized in a magnon-cavity-magnon coupled system [25,48] or in a variety of different photonic and phononic systems [49]. Our work lays the foundation for constructing real EPs in non-Hermitian systems.

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