# Influence of spin on tunneling times in the super-relativistic regime

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For the relativistic tunneling effect described using Dirac's equation, Winful *et al.* [Phys. Rev. A **70**, 052112 (2004)] presented the deduction of a general result that allows for the determination of the phase time (group delay) as the sum of the particle dwell time inside a potential barrier and of the self-interference delay associated with the incident and reflected wave functions interaction. In this paper, a mathematical model is derived through a construction analogous to the proposal mentioned above, but based on an alternative representation for Dirac's equation. This representation is similar to the one introduced by Ajaib [Found. Phys. **45**, 1586 (2015)]. Thus, from the application of this model in the study of the tunneling effect in the absence of an external magnetic field, the influence of spin on the tunneling times is described. More specifically, the tunneling time is obtained as the sum of the dwell times inside the potential barrier for particles with spin up and spin down and the self-interaction time associated with the incident and reflected wave functions for particles with spin up.

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## I. INTRODUCTION

The unusual behavior of saturation in the tunneling times of nonrelativistic particles in scattering problems, uncovered by Hartman in 1962 [1], is known in the literature as the Hartman effect (HE). The discovery of the HE stimulated an intense debate about what is the correct definition for the traveling time when such incident particles manage to cross the potential barrier, and many related investigations were carried out [2–21]. The definitions obtained from the direct application of the Schrödinger equation in these situations would lead to the apparent paradox of particles tunneling at hyperluminous speeds [22,23].

In fact, the two most acceptable tunneling times definitions in the literature are the dwell time, that is defined as the integral of the probability density within the potential barrier, and the phase time, that is defined as the variation of the transmitted phase with respect to the energy [24,25]. By virtue of such definitions, Winful, Ngom, and Litchinitser generalized the previous tunneling time definitions by deducing an exact relation that defines the phase time, for relativistic particles that satisfy the Dirac equation (DE), as the sum of the dwell time and of the self-interaction time between the incident and reflected wave packets [26]. In the nonrelativistic limit, their expressions lead to the results obtained from the direct application of the Schrödinger equation [25]. In these works, the existence of saturation of the tunneling times with the width of the barrier is indicated, extending Hartman's prediction also to the relativistic regime.

But accepting such definitions as transit timescales would imply accepting that the particle would somehow be aware of the increase in barrier width. The particle would increase its speed by just the right amount to be able to cover a greater distance in the same time, a statement that in the context of quantum mechanics seems untenable. As a result, Winful *et al.* [26,27] made the suggestion to consider the width of the wave packet as a significant quantity to characterize the transit time in quantum tunneling. The wave packet can be much larger than the width of the barrier, which translates into a loss of sense of the notion of the transit time of any individual particle through the potential barrier when the uncertainty of its position is greater than the width of the barrier itself, a situation that can arise in quantum tunneling.

An alternative view for getting a correct definition of such timescales was developed in the works of Büttiker [28] and Rámos *et al.* [29], where definitions are presented that allow for the determination of the tunneling time as a function of the electron-spin evolution when crossing a potential barrier in the presence of a magnetic field and the experimental measurement of such a tunneling time, respectively. However, once one cannot apply this technique for arbitrary systems, it cannot be taken as a general method for quantifying and measuring tunneling times. Furthermore, in Ajaib's works [30,31], it was presented how the influence of spin can be obtained in scattering over potential barriers even without the presence of a magnetic field. In this paper, this result is used to develop a model that allows us to extend the results obtained by Winful *et al.* and with that to determine the influence of the spin in the definitions of the tunneling times deduced from the sensitivity theorem of the wave function with respect to energy variations, but in the super-relativistic regime.

The remainder of this paper is organized as follows. In Sec. II, the deduction of an alternative representation of the DE is presented. Subsequently, in Sec. III, the deduction of the sensitivity theorem of the wave function to energy variations in the alternative representation is discussed. In Sec. IV, we introduce the solution of the DE and apply the expressions deduced in the previous section to determine the tunneling

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times in the lowest-energy solution for relativistic and nonrelativistic regimes in this alternative representation. In Sec. V, by means of a procedure similar to the one applied in the previous section, we report the solution of the DE and the expressions for the tunneling times involving the influence of the spin, introduced in the alternative representation, in the super-relativistic regime and in the absence of an external magnetic field. Finally, in Sec. VI, we give our concluding remarks.

## II. ALTERNATIVE REPRESENTATION FOR DIRAC'S EQUATION

In the works presented by Ajaib in 2015 and 2016 [30,31], an alternative representation for the Dirac equation was deduced in which the Schrödinger-Pauli equation is contained when considering local invariance. Such equation allows for the description of how the spin of the particle affects the reflection and transmission coefficients when applied to quantum scattering problems. This stimulated us to apply this approach to study how the spin of the particles influences the tunneling time when they cross a constant potential barrier.

In this sense, the objective of this section is to determine a suitable alternative representation for the DE [32]:

$$\{i\hbar\gamma^{\mu}\partial_{\mu} - i\gamma_5 mc\}\psi(x^{\kappa}) = 0, \quad \text{for} \quad \mu, \kappa = 0, 1, 2, 3. \tag{1}$$

Above $i = \sqrt{-1}$, $\hbar$ is Planck's constant, $m$ is the rest mass of the particle, and $\gamma^{\mu}$ are the Dirac matrices

$$\gamma^0 = \alpha^4 = \begin{pmatrix} \mathcal{I}_2 & \mathcal{O}_2 \\ \mathcal{O}_2 & -\mathcal{I}_2 \end{pmatrix}, \tag{2}$$

$$\gamma^j = \alpha^4\alpha^j = \begin{pmatrix} \mathcal{O}_2 & \sigma^j \\ -\sigma^j & \mathcal{O}_2 \end{pmatrix}, \quad \text{for} \quad j = 1, 2, 3, \tag{3}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon_{\mu\nu\kappa\lambda}\gamma^{\mu}\gamma^{\nu}\gamma^{\kappa}\gamma^{\lambda}, \tag{4}$$

that satisfy relationships

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}\mathcal{I}_4, \quad \text{for} \quad \mu, \nu = 0, 1, 2, 3, \tag{5}$$

and

$$(\gamma^5)^2 = \mathcal{I}_4, \quad \gamma^5\gamma^{\mu} + \gamma^{\mu}\gamma^5 = \mathcal{O}_4, \quad \text{for} \quad \mu = 0, 1, 2, 3. \tag{6}$$

Above,

$$\varepsilon_{\mu\nu\kappa\lambda} = \begin{cases} +1 & \text{if } (\mu, \nu, \kappa, \lambda) \text{ is an even permutation of (0,1,2,3)}, \\ -1 & \text{if } (\mu, \nu, \kappa, \lambda) \text{ is an odd permutation of (0,1,2,3)}, \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

is the Levi-Civita symbol in four dimensions, $g^{\mu\nu} = diag(1, -1, -1, -1)$ is the Minkowski metric, $\mathcal{I}_d$ and $\mathcal{O}_d$ are the $d \times d$ identity and null matrix, respectively, and $\sigma^j$ are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8}$$

The DE of Eq. (1) can be rewritten as

$$\{E\gamma_0 - i\hbar c\gamma_j\partial_j - i\gamma_5 mc^2\}\varphi(x^l) = 0, \tag{9}$$

with $j, l = 1, 2, 3$. Following a procedure similar to that applied by Ajaib [30,31], we can obtain an alternative representation for the one-dimensional DE,

$$\gamma_0 = \beta = \frac{\eta + \eta^{\dagger}}{\sqrt{2}}, \quad \gamma_3 = \pm\alpha_3 = \eta^{\dagger}\eta - \mathcal{I}_4 \tag{10}$$

and

$$i\gamma_5 = \frac{\eta - \eta^{\dagger}}{\sqrt{2}}, \tag{11}$$

using the matrix

$$\eta = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathcal{I}_2 & \sigma_3 \\ -\sigma_3 & -\mathcal{I}_2 \end{pmatrix}, \tag{12}$$

that is an antisymmetric ($\eta^T \neq \eta$) non-Hermitian matrix ($\eta^{\dagger} \neq \eta$) of null trace and null determinant, with real eigenvalues, that have the properties

$$\eta^2 = (\eta^{\dagger})^2 = 0 \quad \text{and} \quad \{\eta, \eta^{\dagger}\} = 2\mathcal{I}_4. \tag{13}$$

The particular one-dimensional case of Eq. (9) can then be rewritten as

$$\pm\{i\hbar c\alpha_3\partial_3\}\varphi(x^3) + \{E_1\eta + E_2\eta^{\dagger}\}\varphi(x^3) = 0, \tag{14}$$

where $E_1 = (E - mc^2)/\sqrt{2}$ and $E_2 = (E + mc^2)/\sqrt{2}$.

On the other hand, Eq. (9), with $j = l = 3$, written in terms of the alternative representation, allows obtaining the continuity equation. To do this, we multiply Eq. (9) by $\psi^{\dagger}(x^l)\alpha_3\beta$ by the left, obtaining

$$i\hbar\psi^{\dagger}(x^l)\alpha_3\eta\eta^{\dagger}\partial_0\psi(x^l) + i\hbar\psi^{\dagger}(x^l)\mathcal{I}_4\partial_0\psi(x^l) - i\hbar c\psi^{\dagger}(x^l)\alpha_3\beta\alpha_3\partial_3\psi(x^l) - mc^2\psi^{\dagger}(x^l)\mathcal{I}_4\psi(x^l) = 0. \tag{15}$$

Then, we calculate the complex conjugate of Eq. (9), and we multiply it by $\beta\alpha_3\psi(x^l)$ on the right-hand side to get

$$-i\hbar\psi^{\dagger}(x^l)\eta\eta^{\dagger}\alpha_3\psi(x^l) - i\hbar[\partial_0\psi^{\dagger}(x^l)]\mathcal{I}_4\partial_0\psi(x^l) + i\hbar c[\partial_3\psi^{\dagger}(x^l)]\alpha_3\beta\alpha_3\psi(x^l) - mc^2\psi^{\dagger}(x^l)\mathcal{I}_4\psi(x^l) = 0. \tag{16}$$

By subtracting this last equation from Eq. (15), the continuity equation is finally obtained:

$$\partial_t\rho(x^l) + \partial_3 J(x^l) = 0. \tag{17}$$

Above, $\rho(x^l) = \psi^{\dagger}(x^l)\alpha_3\psi(x^l)$ is the probability density while $J(x^l) = \psi^{\dagger}(x^l)c\alpha_3\beta\alpha_3\psi(x^l)$ is the probability current density.

## III. GENERAL THEOREM OF WAVE-FUNCTION SENSITIVITY TO ENERGY VARIATIONS IN THE ALTERNATIVE REPRESENTATION

In this section, various algebraic procedures are applied to Eq. (14) in order to derive the expression for the general sensitivity theorem of the wave function to energy variations, in a representation alternative to the one presented by Winful *et al.* [26]. First, we take the derivative of Eq. (14), with negative sign, with respect to energy, and we multiply it on the left by $\varphi^\dagger(x^3)$, obtaining the following expression:

$$-\varphi^\dagger(x^3)\{i\hbar c\alpha_3\partial_3\}\partial_E\varphi(x^3)$$
$$=-\varphi^\dagger(x^3)\left\{\frac{\eta+\eta^\dagger}{\sqrt{2}}\right\}\varphi(x^3)-\varphi^\dagger(x^3)\{E_1\eta+E_2\eta^\dagger\}\partial_E\varphi(x^3). \tag{18}$$

Secondly, we take the conjugate transpose of Eq. (14), with a positive sign, and multiply this result by $\partial_E\varphi(x^3)$. Thus, we obtain

$$-\{i\hbar c\partial_3\varphi^\dagger(x^3)\alpha_3\}\partial_E\varphi(x^3)$$
$$=-\varphi^\dagger(x^3)\{E_1\eta+E_2\eta^\dagger\}^\dagger\partial_E\varphi(x^3). \tag{19}$$

By subtracting Eqs. (18) and (19), we obtain

$$-i\hbar c\partial_3\{\varphi^\dagger(x^3)\alpha_3\partial_E\varphi(x^3)\}$$
$$=-\varphi^\dagger(x^3)\left\{\frac{\eta+\eta^\dagger}{\sqrt{2}}\right\}\varphi(x^3)-\varphi^\dagger(x^3)[\{E_1\eta+E_2\eta^\dagger\}$$
$$+\{E_1\eta+E_2\eta^\dagger\}^\dagger]\partial_E\varphi(x^3), \tag{20}$$

which allows to analyze two cases of interest in the present paper for the expressions of energy. In the first case, the rest energy value is considered predominant, ($E_j\approx(-1)^j mc^2$ with $j=1,2$), which leads to

$$E_1\eta+E_2\eta^\dagger=\sqrt{2}mc^2\begin{pmatrix}\mathcal{O}_2 & -\sigma_3\\ \sigma_3 & \mathcal{O}_2\end{pmatrix}. \tag{21}$$

Therefore, Eq. (21) implies the equality

$$\{E_1\eta+E_2\eta^\dagger\}^\dagger=-\{E_1\eta+E_2\eta^\dagger\}, \tag{22}$$

and allows us to rewrite Eq. (20) as

$$-i\hbar c\partial_3\{\varphi^\dagger(x^3)\alpha_3\partial_E\varphi(x^3)\}=-\varphi^\dagger(x^3)\left\{\frac{\eta+\eta^\dagger}{\sqrt{2}}\right\}\varphi(x^3). \tag{23}$$

Consequently, if the expression (23) is integrated in the interval from zero to $a$, we obtain

$$-i\hbar c\{\varphi^\dagger(x^3)\alpha_3\partial_E\varphi(x^3)\}|_0^a=-\int_0^a dx^3\varphi^\dagger(x^3)\beta\varphi(x^3). \tag{24}$$

The second case analyzed in this paper is obtained by considering the particle energy being large with respect to its rest energy ($E\approx E_j$ with $j=1,2$), which allows us to write

$$E_1\eta+E_2\eta^\dagger\approx E\{\eta+\eta^\dagger\}, \tag{25}$$

and consequently

$$\{E[\eta+\eta^\dagger]\}^\dagger=E\{\eta+\eta^\dagger\}. \tag{26}$$

We then repeat the procedure applied to obtain Eq. (20), but in this case we consider only the expression (14) with a positive sign and Eq. (26). It can be obtained again the equation corresponding to Eq. (23), and consequently (24), which is the general sensitivity theorem of the wave function to energy variations, but in the alternative representation. Clearly, the expression (24) obtained by considering large-energy regimes ($E\gg mc^2\to E\approx E_j$) is valid only for massive particles (as in the first case) and would also imply the appearance of the Klein tunneling phenomenon [26].

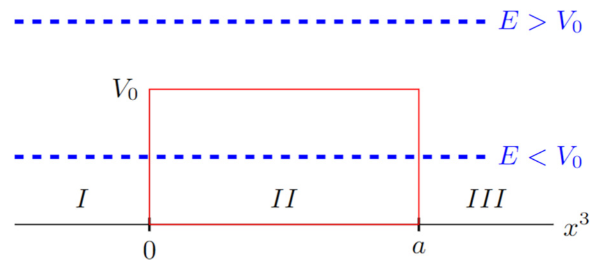


FIG. 1. Schematic representation of a potential barrier with width $a$ and height $V_0$ for a particle with energy $E$. In the figure, we identify the regions I, II, and III.

## IV. SOLVING DIRAC'S EQUATION IN THE ALTERNATIVE REPRESENTATION

In this section, we obtain exact relationships among the tunneling times, similar to the ones obtained by the authors in Ref. [26]. We then study the the manifestation of Hartman's effect in this case. Additionally, we give the correct expressions that should be obtained for the tunneling times when considering the nonrelativistic limit, correcting thus the typos that appear in Ref. [26].

We start by writing Eq. (14) as

$$\begin{pmatrix}\mathbf{0}_2 & i\hbar c\sigma^3\partial_3\\ i\hbar c\sigma^3\partial_3 & \mathbf{0}_2\end{pmatrix}\begin{pmatrix}\varphi_l\\ \varphi_s\end{pmatrix}=\begin{pmatrix}E\mathbf{1}_2 & mc^2\sigma^3\\ -mc^2\sigma^3 & -E\mathbf{1}_2\end{pmatrix}\begin{pmatrix}\varphi_l\\ \varphi_s\end{pmatrix}, \tag{27}$$

with $\varphi_l\equiv\varphi_l(x^3)$ and $\varphi_s\equiv\varphi_s(x^3)$. By its turn, this matrix equation can be recast as the following scalar equations:

$$\varphi_s^j(x^3)=(-1)^{1-j}\frac{1}{E}\{i\hbar c\partial_3\varphi_l^j(x^3)+mc^2\varphi_l^j(x^3)\}, \tag{28}$$

$$\frac{1}{\hbar^2c^2}\{m^2c^4-E^2\}\varphi_l^j(x^3)+\partial_{33}\varphi_l^j(x^3)=0, \tag{29}$$

with $j=1,2$ indicating the $j$th component of $\varphi_l$ or $\varphi_s$. We then consider the problem of particles incident from the left on a constant potential barrier, as illustrated in Fig. 1, with height $V_0$ and width $a$ defined by

$$V(x^3)=V_0\Theta(x^3)\Theta(a-x^3), \tag{30}$$

where $\Theta$ is the Heaviside function defined as

$$\Theta(x^3)=\begin{cases}0 & \text{for } x^3<0\\ 1 & \text{for } x^3\leqslant 0\end{cases}. \tag{31}$$

When solving Eq. (29), considering $j=1$ (stationary solution of the particle scattering problem), the component $l$ can be written as

$$\varphi_l^1(x^3)=Ae^{ikx^3}+Be^{-ikx^3}, \tag{32}$$

where $A$ and $B$ are coefficients to be determined. Then, the $s$ component is obtained by substituting Eq. (32) into Eq. (28). Thus we get

$$\varphi_s^1(x^3)=\Gamma_1Ae^{ikx^3}+\Gamma_2Be^{-ikx^3}, \tag{33}$$

where $\Gamma_1=mc^2/E-\hbar ck/E$ and $\Gamma_2=mc^2/E+\hbar ck/E$.

The expressions (32) and (33) allow us to write the general solution for each of the regions of interest as

$$\varphi_{\mathrm{I}}(x^3) = A\begin{pmatrix}1\\0\\-\Gamma\\0\end{pmatrix}e^{ikx^3} + B\begin{pmatrix}1\\0\\\Gamma\\0\end{pmatrix}e^{-ikx^3}, \tag{34}$$

$$\varphi_{\mathrm{II}}(x^3) = C\begin{pmatrix}1\\0\\i\Gamma'\\0\end{pmatrix}e^{qx^3} + D\begin{pmatrix}1\\0\\-i\Gamma'\\0\end{pmatrix}e^{-qx^3}, \tag{35}$$

$$\varphi_{\mathrm{III}}(x^3) = F\begin{pmatrix}1\\0\\-\Gamma\\0\end{pmatrix}e^{ik(x^3-a)}, \tag{36}$$

where the coefficients $\Gamma$ and $\Gamma'$ are obtained from Eq. (33) when considering the lowest-energy regimes ($\Gamma_1 \approx -\hbar ck/E = -\Gamma$, $\Gamma_2 \approx +\hbar ck/E = \Gamma$ with $\hbar ck = \sqrt{m^2c^4 - E^2}$ and $\Gamma' = \hbar cq/E'$ with $E'^2 = 2m^2c^4 - (E - V_0)^2$ and $q^2 = k^2(E') < 0$), in which the Klein tunneling phenomenon does not occur ($mc^2 \gg E$ and $V_0$).

Applying the continuity conditions

$$\varphi_r(x^3)|_{x^3} = \varphi_{rI}(x^3)|_{x^3}, \tag{37}$$

$$d_{x^3}\varphi_r(x^3)|_{x^3} = d_{x^3}\varphi_{rI}(x^3)|_{x^3}, \tag{38}$$

with $r = \{\mathrm{I}, \mathrm{II}\}$ and $x^3 = \{0, a\}$, we can obtain the expressions for each of the coefficients:

$$B = -\frac{i}{2}\frac{(1+\Xi^2)}{\Xi}\sinh\{qa\}F, \tag{39}$$

$$C = \frac{1}{2}(1 + i\Xi)e^{-qa}F, \tag{40}$$

$$D = \frac{1}{2}(1 - i\Xi)e^{qa}F, \tag{41}$$

$$F = \left\{\cosh\{qa\} - \frac{i}{2}\left[\Xi - \frac{1}{\Xi}\right]\sinh\{qa\}\right\}^{-1}A, \tag{42}$$

with $\Xi = kE'/qE$. For our purposes here, we can assume, without loss of generality, that $A = 1$. Using $|T|^2 + |R|^2 = 1$ we can rewrite the expressions $B = R = |R|\exp i\phi_R$ and $F = T = |T|\exp i\phi_T$. The above considerations allow us to write Eqs. (34)–(36) in terms of the reflection, $R$, and transmission, $T$, coefficients, which when substituted into Eq. (24) give us

$$\begin{aligned}&-2\hbar c\Gamma\left\{|T|^2\partial_E\phi_T + |R|^2\partial_E\phi_R + \frac{1}{\Gamma}Im(R)\partial_E\Gamma\right\}\\ &= -\frac{a|T|^2}{2}\left\{(1+\Xi^2)(1-\Gamma'^2)\frac{\sinh 2qa}{2qa}\right.\\ &\quad\left.+ (1-\Xi^2)(1+\Gamma'^2)\right\}.\end{aligned} \tag{43}$$

Definitions of the group time for particles transmitted and reflected, the self-interaction time, and the time group are given by the expressions

$$\hat{\tau}_{gT} = \hbar\partial_E\phi_T, \tag{44}$$

$$\hat{\tau}_{gR} = \hbar\partial_E\phi_R, \tag{45}$$

$$\hat{\tau}_i = -\hbar Im(R)(\partial_E\Gamma)/\Gamma, \tag{46}$$

$$\hat{\tau}_g = |T|^2\hat{\tau}_{gT} + |R|^2\hat{\tau}_{gR}. \tag{47}$$

From Eq. (43) we can write

$$|T|^2\hat{\tau}_{gT} + |R|^2\hat{\tau}_{gR} - \hat{\tau}_i = \hat{\tau}_g - \hat{\tau}_i = \hat{\tau}_d. \tag{48}$$

Substituting $\Gamma$, $\Gamma'$, and $\mathrm{Im}(R)$ into Eq. (48), we obtain the mathematical expressions for the dwell, interaction, and group times in the alternative representation:

$$\begin{aligned}\hat{\tau}_d = \frac{a|T|^2}{4\hbar c^2 q\Xi E'}&\left\{m^2c^4(1+\Xi^2)\frac{\sinh\{2qa\}}{2qa}\right.\\ &\left.+ [3m^2c^4 - 2(V_0 - E)^2](1-\Xi^2)\right\},\end{aligned} \tag{49}$$

$$\hat{\tau}_i = \frac{m^2c^4|T|^2}{4\hbar c^2k^2\Xi E}(1+\Xi^2)\sinh\{2qa\}, \tag{50}$$

$$\begin{aligned}\hat{\tau}_g = \frac{a|T|^2}{4\hbar c^2 q\Xi E'}&\left\{m^2c^4(1+\Xi^2)\frac{\sinh\{2qa\}}{2qa}\right.\\ &\left.+ [3m^2c^4 - 2(V_0 - E)^2](1-\Xi^2)\right\}\\ &+ \frac{m^2c^4|T|^2}{4\hbar c^2k^2\Xi E}(1+\Xi^2)\sinh\{2qa\}.\end{aligned} \tag{51}$$

When considering very wide potential barriers ($a \to \infty$), these equations are reduced to the expressions

$$\hat{\tau}_d = \frac{m^2c^4}{\hbar c^2q^2E'}\frac{\Xi}{(1+\Xi^2)}, \tag{52}$$

$$\hat{\tau}_i = \frac{2m^2c^4}{\hbar c^2k^2E}\frac{\Xi}{(1+\Xi^2)}, \tag{53}$$

$$\hat{\tau}_g = \frac{m^2c^4}{\hbar c^2}\frac{\Xi}{(1+\Xi^2)}\left(\frac{1}{q^2E'} + \frac{2}{k^2E}\right). \tag{54}$$

The graphical representation of these expressions, shown in Fig. 2, allows us to see how the group time curve, $\hat{\tau}_g$, is composed through the sum of the curve of the dwell time of the particles inside the barrier, $\hat{\tau}_d$, with the curve of the interaction time between the incident and reflected wave pulses, $\hat{\tau}_i$. In addition, the group time saturation is seen when $E/V_0$ goes toward 1. However, it is important to point out that in the alternative representation, Eqs. (10) and (11), the timescales are altered when compared with the the ones obtained in Ref. [26], since here the interaction time, $\hat{\tau}_i$, and dwell time, $\hat{\tau}_d$, are modulated by $E^{-1}$ and $(2E')^{-1}$, respectively. This causes a distortion in the group time curve, that is, a more pronounced growth of $\hat{\tau}_g$ when $E/V_0 \to 1.0$, which can be attributed to the loss of symmetry associated with the matrix $\eta$. On the other hand, it is evident that, as a result of the chosen representation, there is an exchange in the behaviors of the dwell and self-interaction times in relation to what was reported in Ref. [26].

The nonrelativistic limit of the expressions (49)–(51) is obtained by substituting $E = \sqrt{2}E_1 = E_k - mc^2$ in $\hbar ck = \sqrt{m^2c^4 - E^2}$, $E = \sqrt{2}E_2 = E_k + mc^2$ in $\hbar cq = \sqrt{m^2c^4 - (V_0 - E)^2}$, and $E' = \sqrt{2m^2c^4 - (V_0 - E)^2}$. This

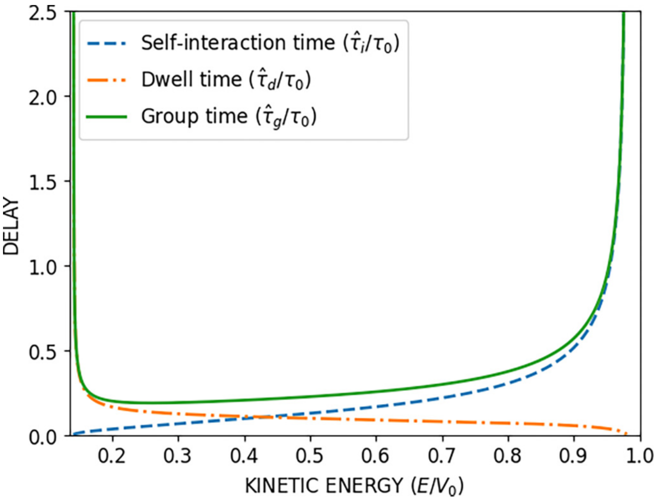


FIG. 2. Group time, $\hat{\tau}_g/\tau_0$, dwell time, $\hat{\tau}_d/\tau_0$, and self-interaction time, $\hat{\tau}_i/\tau_0$, as a function of the ratio of energies $E/V_0$. The expressions for the times are normalized with respect to the transmission time at the vacuum light speed, $\tau_0 = a/c$. Additionally, for these plots, we set $V_0 a/\hbar c = 2\pi$ and $mc^2/V_0 = 0.98$.

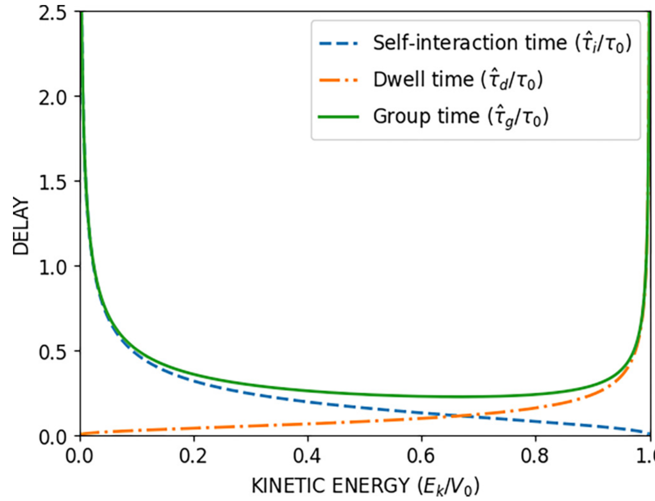


FIG. 3. Group time, $\hat{\tau}_g/\tau_0$, dwell time, $\hat{\tau}_d/\tau_0$, and self-interaction time, $\hat{\tau}_i/\tau_0$, as a function of the energies ratio $E_k/V_0$ in the nonrelativistic limit. The expressions for the times are normalized with respect to the transmission time for the vacuum light speed, $\tau_0 = a/c$. Additionally, for these plots, we set $V_0 a/\hbar c = 2\pi$.

allows us to write $k' = \sqrt{2mE_k}/\hbar$, $q' = \sqrt{2m(V_0 - E_k)}/\hbar$, $E \approx mc^2$, and $E' = mc^2 \Rightarrow \Xi = k'/q'$. Consequently,

$$\hat{\tau}_i = \frac{m|T'|^2 a}{2\hbar k'}\left(1 + \frac{q'^2}{k'^2}\right)\frac{\sinh\{2q'a\}}{2q'a}, \tag{55}$$

$$\hat{\tau}_d = \frac{m|T'|^2 a}{4\hbar k'}\left\{\left(1 + \frac{k'^2}{q'^2}\right)\frac{\sinh\{2q'a\}}{2q'a} + \left(1 - \frac{k'^2}{q'^2}\right)\right\}, \tag{56}$$

$$\hat{\tau}_g = \frac{m|T'|^2 a}{2\hbar k'}\left\{\left[\frac{1}{2}\left(1 + \frac{k'^2}{q'^2}\right) + \left(1 + \frac{q'^2}{k'^2}\right)\right]\frac{\sinh\{2q'a\}}{2q'a} + \left(1 - \frac{k'^2}{q'^2}\right)\right\}, \tag{57}$$

where $T' = (\cosh\{q'a\} - (i/2)[k'/q' - q'/k']\sinh\{q'a\})^{-1}$. Thus, when considering very wide barriers ($a \to \infty$), these equations are reduced to

$$\hat{\tau}_d = \frac{m\Xi}{\hbar q'^2(1 + \Xi^2)}, \tag{58}$$

$$\hat{\tau}_i = \frac{2m\Xi}{\hbar k'^2(1 + \Xi^2)}, \tag{59}$$

$$\hat{\tau}_g = \frac{m\Xi}{\hbar(1 + \Xi^2)}\left(\frac{1}{q'^2} + \frac{2}{k'^2}\right). \tag{60}$$

The graphical representation of the expressions (58)–(60) is given in Fig. 3. We see how the group time curve, in the nonrelativistic limit, is again composed by the sum of the time curve of the dwell time and of the interaction time curve between, as in the model deduced in Refs. [13,26]. Additionally, Fig. 3 shows again the saturation effect of the group time, when $E_k/V_0$ goes toward 1. Furthermore, it is important to point out the shift to the right we observe in the minimum of the curve of the group time, when $E_k/V_0 \to 1.0$. This effect is caused by the halving (in orders of magnitude) of the dwell time $\hat{\tau}_d$ with respect to the dwell time $\tau_d$ obtained in Ref. [26]. This effect can be attributed, in the same way as the relativistic case, to the loss of symmetry associated with the matrix $\eta$, that was used in the alternative representation for Dirac's equation we use here. On the other hand, in the nonrelativistic limit, the dwell time $\hat{\tau}_d$ and interaction time $\hat{\tau}_i$ keep the same behavior they had in the representation chosen in Ref. [26].

In contrast to the expressions (55)–(57), in Ref. [26], the authors present a set of mathematical expressions for the nonrelativistic case, from which it is clearly not possible to recover the characteristic saturation curve of tunneling times (the Hartman effect) when considering a very wide potential barrier. However, such nonrelativistic expressions would be correct if the quantity $m|T'|^2 a/2\hbar k'$ [33] multiplied all the terms that define the phase time $\tau_g$ and the dwell time $\tau_d$, and if, in addition, the first term in brackets in $\tau_g$ were squared (see Refs. [13,27]).

## V. INFLUENCE OF SPIN ON TUNNELING TIMES

By considering the solution of Dirac's equation [Eqs. (34)–(36)] associated with the lower-energy regime where Klein tunneling is not present, in Sec. IV we showed that the alternative representation for Dirac's equation [Eqs. (10) and (11)] leads to exact relationships between tunneling times similar to those reported in Ref. [26]. However, the most relevant aspect of this alternative representation is obtained by writing Eq. (14), with the negative sign, in the momentum basis. That is, by considering $\hat{P} = -i\hbar\partial_3$ and $\varphi = \varphi'(p^l)\exp -i\mathbf{p}\Lambda\mathbf{x} = \varphi'(p^l)\exp -i[Et - p^l x^l] = \varphi(p^l)\exp -iEt$ [34], we derive the equations

$$E_k\varphi_l^j(p^3) + (-1)^{1-j}cp\varphi_s^j(p^3) = 0, \tag{61}$$

$$(-1)^{1-j}cp\varphi_l^j(p^3) - E_k\varphi_s^j(p^3) = 0, \tag{62}$$

with $j = 1, 2$. Considering the super-relativistic energy regime ($E_k \gg mc^2 \to E \approx E_k = \hbar^2 p^2/2m$ and $cp \pm mc^2 \approx cp$), in which the Klein tunneling phenomenon is again not appreciable ($E - mc^2 < V_0 < E + mc^2$), we solve Eqs. (61) and (62) to obtain the general solution for each of the regions:

$$\varphi_{\rm I}(x^3) = \frac{A}{\Upsilon^2}\begin{pmatrix}1\\0\\\Gamma\\0\end{pmatrix}e^{ipx^3} + \frac{A'}{\Upsilon^2}\begin{pmatrix}1\\0\\-\Gamma\\0\end{pmatrix}e^{-ipx^3} + \frac{B'}{\Upsilon^2}\begin{pmatrix}0\\1\\0\\-\Gamma\end{pmatrix}e^{-ipx^3}, \tag{63}$$

$$\varphi_{\rm II}(x^3) = \frac{C}{\Upsilon'^2}\begin{pmatrix}1\\0\\i\Gamma'\\0\end{pmatrix}e^{-p'x^3} + \frac{C'}{\Upsilon'^2}\begin{pmatrix}1\\0\\-i\Gamma'\\0\end{pmatrix}e^{p'x^3}$$

$$+ \frac{D}{\Upsilon'^2}\begin{pmatrix}0\\1\\0\\i\Gamma'\end{pmatrix}e^{-px^3} + \frac{D'}{\Upsilon'^2}\begin{pmatrix}0\\1\\0\\-i\Gamma'\end{pmatrix}e^{p'x^3}, \tag{64}$$

$$\varphi_{\rm III}(x^3) = \frac{F}{\Upsilon^2}\begin{pmatrix}1\\0\\\Gamma\\0\end{pmatrix}e^{ip(x^3-a)} + \frac{G}{\Upsilon^2}\begin{pmatrix}0\\1\\0\\\Gamma\end{pmatrix}e^{ip(x^3-a)}, \tag{65}$$

with $\Gamma = cp/E_k$, $\Gamma' = cp'/E_k'$, $p' = \sqrt{2m(V_0 - E_k)}/\hbar$, $E_k' = V_0 - E_k$, $[\Upsilon^2]^{-1} = 1/\sqrt{1 + (\pm cp)^2/E_k^2}$, and $[\Upsilon'^2]^{-1} = 1/\sqrt{1 + (\pm cp')^2/E_k'^2}$. Furthermore, it becomes evident that the expressions (63)–(65) are written as a linear combination of the vectors

$$\varphi_\pm^\uparrow = \frac{1}{\Upsilon^2}\begin{pmatrix}1\\0\\\pm\Gamma\\0\end{pmatrix} \text{ and } \varphi_\pm^\downarrow = \frac{1}{\Upsilon^2}\begin{pmatrix}0\\1\\0\\\pm\Gamma\end{pmatrix}. \tag{66}$$

These vectors satisfy the orthogonality relations

$$[\varphi_\pm^\uparrow]^\dagger\varphi_\pm^\uparrow = 1, \tag{67}$$

$$[\varphi_\pm^\uparrow]^\dagger\varphi_\pm^\downarrow = [\varphi_\pm^\downarrow]^\dagger\varphi_\pm^\uparrow = [\varphi_\mp^\downarrow]^\dagger\varphi_\mp^\uparrow = 0. \tag{68}$$

On the other hand, the coefficients $A$, $A'$, $B'$, $C$, $C'$, $D$, $D'$, $F$, and $G$ in Eqs. (63)–(65) are determined by applying the continuity condition

$$\varphi_r(x^3)|_{x^3} = \varphi_{rI}(x^3)|_{x^3}, \tag{69}$$

with $r = \{\text{I, II}\}$ and $x^3 = \{0, a\}$. Thus, we obtain

$$A' = -\frac{i}{2}\frac{(1+\Xi^2)}{\Xi}\sinh\{p'a\}F, \tag{70}$$

$$C = \frac{\Upsilon'^2}{2\Upsilon^2}(1 - i\Xi)e^{p'a}F, \tag{71}$$

$$C' = \frac{\Upsilon'^2}{2\Upsilon^2}(1 + i\Xi)e^{-p'a}F, \tag{72}$$

$$D = \frac{\Upsilon'^2}{2\Upsilon^2}(1 - i\Xi)e^{p'a}G, \tag{73}$$

$$D' = \frac{\Upsilon'^2}{2\Upsilon^2}(1 + i\Xi)e^{-p'a}G, \tag{74}$$

$$F = \left\{\cosh\{p'a\} - \frac{i}{2}\left[\Xi - \frac{1}{\Xi}\right]\sinh\{p'a\}\right\}^{-1}, \tag{75}$$

$$G = \{\cosh\{p'a\} - i\Xi\sinh\{p'a\}\}^{-1}B'. \tag{76}$$

Having these equations, for our purposes here, we can choose, without loss of generality, $A = 1$. So $A' \propto |R|e^{i\phi_R}$, $B' \propto |R'|e^{i\phi_{R'}}$, $F \propto |T|e^{i\phi_T}$, and $G \propto |T'|e^{i\phi_{T'}}$. Additionally, $|T|^2 + |T'|^2 + |R|^2 + |R'|^2 = 1$. With this, we can rewrite Eqs. (63)–(65) in terms of these parameters, and substitute them into Eq. (24). With this, we arrive at

$$\begin{aligned}\frac{2\hbar c\Gamma}{[\Upsilon^2]^2}\Big\{&|T|^2\partial_{E_k}\phi_T + |T'|^2\partial_{E_k}\phi_{T'} \\ &+ |R|^2\partial_{E_k}\phi_R + |R'|^2\partial_{E_k}\phi_{R'} - Im(R)\frac{\partial_{E_k}\Gamma}{\Gamma}\Big\} \\ &= -\frac{|T|^2a}{2[\Upsilon^2]^2}\Big\{(1-\Gamma'^2)(1+\Xi^2)\frac{\sinh(2p'a)}{2p'a} \\ &+ (1+\Gamma'^2)(1-\Xi^2)\Big\} \\ &- \frac{|T'|^2a}{2[\Upsilon^2]^2}\Big\{(1-\Gamma'^2)(1+\Xi^2)\frac{\sinh(2p'a)}{2p'a} \\ &+ (1+\Gamma'^2)(1-\Xi^2)\Big\}.\end{aligned} \tag{77}$$

From this last equation, we identify the tunneling times:

$$\hat{\tau}_g^\uparrow = |T|^2(\partial_{E_k}\phi_T) + |R|^2(\partial_{E_k}\phi_R), \tag{78}$$

$$\hat{\tau}_g^\downarrow = |T'|^2(\partial_{E_k}\phi_{T'}) + |R'|^2(\partial_{E_k}\phi_{R'}), \tag{79}$$

$$\hat{\tau}_i = \frac{\text{Im}(R)(\partial_{E_k}\Gamma)}{\Gamma}, \tag{80}$$

$$\hat{\tau}_g = \hat{\tau}_g^\uparrow + \hat{\tau}_g^\downarrow. \tag{81}$$

Then, by substituting $\Gamma$, $\Gamma'$, and Im$(R)$ into Eq. (77), we obtain the following expressions for tunneling times:

$$\begin{aligned}-\frac{2\hbar p'\Xi}{m|T|^2a}\hat{\tau}_d^\uparrow &= [(E_k - V_0)/2mc^2 - 1](1+\Xi^2)\frac{\sinh(2p'a)}{2p'a} \\ &+ [(E_k - V_0)/2mc^2 + 1](1-\Xi^2),\end{aligned} \tag{82}$$

$$\begin{aligned}-\frac{2\hbar p'\Xi}{m|T'|^2a}\hat{\tau}_d^\downarrow &= [(E_k - V_0)/2mc^2 - 1](1+\Xi^2)\frac{\sinh(2p'a)}{2p'a} \\ &+ [(E_k - V_0)/2mc^2 + 1](1-\Xi^2),\end{aligned} \tag{83}$$

$$\hat{\tau}_i = \frac{m|T|^2}{4\hbar p^2\Xi}(1+\Xi^2)\sinh(2p'a), \tag{84}$$

$$\hat{\tau}_g = \hat{\tau}_d^\uparrow + \hat{\tau}_d^\downarrow + \hat{\tau}_i. \tag{85}$$

Now, if we regard very wide barriers ($a \to \infty$), the expressions above are reduced to

$$\hat{\tau}_d^\uparrow = -\frac{2m[(E_k - V_0)/2mc^2 - 1]\Xi}{\hbar p'^2(1+\Xi^2)}, \tag{86}$$

$$\hat{\tau}_d^\downarrow = -\frac{2m[(E_k - V_0)/2mc^2 - 1]\Xi}{\hbar p'^2(1+\Xi^2)}, \tag{87}$$

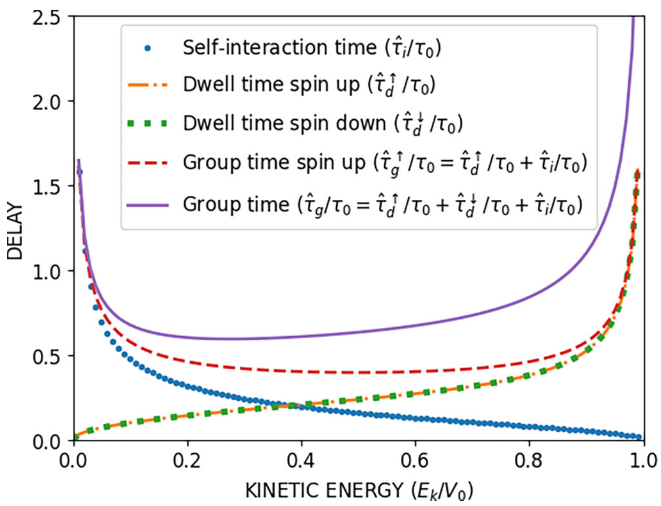


FIG. 4. Group time, $\hat{\tau}_g/\tau_0$, group time considering only the contribution of the spin-up particles, $\hat{\tau}_g^{\uparrow}/\tau_0$, dwell times, $\hat{\tau}_d^{\uparrow}/\tau_0$ and $\hat{\tau}_d^{\downarrow}/\tau_0$, and self-interaction time, $\hat{\tau}_i/\tau_0$, as a function of the ratio of energies $E_k/V_0$ in the super-relativistic limit. The expressions for the times are normalized with respect to the transmission time at the vacuum light speed, $\tau_0 = a/c$. Additionally, for these plots, we set $V_0 a/\hbar c = 2\pi$ and $V_0/mc^2 = 0.98$.

$$\hat{\tau}_i = \frac{2m\Xi}{\hbar p^2(1+\Xi^2)}, \tag{88}$$

$$\hat{\tau}_g = -\frac{2m\Xi}{\hbar(1+\Xi^2)}\left\{\frac{[(E_k - V_0)/mc^2 - 2]}{p'^2} - \frac{1}{p^2}\right\}. \tag{89}$$

The graphical representation of these expressions is presented in Fig. 4. The curves for tunneling times obtained in the super-relativistic regime are seen to be similar to those obtained in Sec. IV for the relativistic and nonrelativistic regimes. In other words, by summing the curves for the dwell times of particles with spin up, $\hat{\tau}_d^{\uparrow}$, and with spin down, $\hat{\tau}_d^{\downarrow}$, and the curve of the interaction time between incident and reflected particles with spin up, $\hat{\tau}_i$, a curve for the group time is obtained, $\hat{\tau}_g$, which is similar to that obtained in Sec. IV for the group time in the relativistic regime. On the other hand, it is evident that if the term of the dwell time for particles with spin down, $\hat{\tau}_d^{\downarrow}$, is eliminated in that sum, a curve is obtained with the shifted minimum for the group time, $\hat{\tau}_g^{\uparrow}$, as obtained in Sec. IV for the nonrelativistic regime. It is interesting that by changing the representation in which we express Eq. (14), the results of the saturation curves can be obtained as a consequence of the contribution of the tunneling times of particles with a specific spin, since in the super-relativistic energy regime the reflection of particles with spin down can occur even in the absence of external magnetic fields.

## VI. CONCLUSION

In this paper, we built a mathematical model to obtain tunneling times through a procedure analogous to the one proposed in Ref. [26], but based on an alternative representation of Dirac's equation. This allowed us to find exact relationships between tunneling times, similar to those obtained in Ref. [26] for relativistic and nonrelativistic energy regimes. Moreover, the use of this alternative representation allowed us to analyze the influence of the particle spin on the exact relationships between tunneling times obtained by studying this phenomenon considering a constant potential barrier, super-relativistic energy regimes, and the absence of external magnetic fields.

From the mathematical expressions obtained in Sec. V, we showed that, in this regime, the group time, $\hat{\tau}_g$, is obtained as the sum of the dwell times inside the potential barrier for particles with spin up, $\hat{\tau}_d^{\uparrow}$, and with spin down, $\hat{\tau}_d^{\downarrow}$, and the self-interaction time, $\hat{\tau}_i$, associated with the incident and reflected wave functions for particles with spin up. Indeed, the above summation resulted in a group time curve similar to that obtained in Sec. IV for relativistic energy regimes. In addition, it was shown that when considering only the contribution of the particles with spin up, the result obtained is similar to the group time curve presented in Sec. IV, with a minimum point shifted to the right and obtained when considering nonrelativistic energy regimes.

However, it is interesting that from a purely mathematical procedure, that is, the choice of an alternative representation for Dirac's equation, and the consideration of the super-relativistic energy regimes, a model is obtained that leads to results involving the influence of particle spin on the exact relationships between tunneling times. Furthermore, it is important to highlight that the results presented in our paper depend on the chosen representation, which is based on the one introduced by Ajaib [30,31].

Consequently, it is this representation that leads to our expressions for the tunneling times, which cannot be obtained by direct application of the Dirac equation written in terms of the standard representation of $\gamma$ matrices. This is so because in such an approach there is no apparent physical or mathematical reason to consider an ansatz as presented in Sec. V. By taking the nonrelativistic limit of the Dirac equation in its standard form, the Schrödinger equation is obtained in which the influence of spin does not appear in the absence of a magnetic field. Therefore, considering an ansatz like the one proposed in Sec. V is interesting only for this type of alternative representation. In addition, for the Dirac equation considered in Sec. IV, from its first order is obtained the equation introduced by Ajaib in Refs. [30,31], where the influence of spin is mathematically appreciable even in the nonrelativistic regime and in the absence of an external magnetic field.

On the other hand, the fact that the results obtained depend on the chosen representation seems to lead to a theoretical model that offers the description of different tunneling times. This may seem like a physically inconsistent result, since, in general, any physical result should be independent of the mathematical tool used. However, it should be noted that the formalism of the Dirac equation dictates only that the representations of Dirac's have to be such that they warrant its invariance under a Lorentz transformation and lead to a Hermitian Hamiltonian.

For that reason, there is no impediment for a description of different tunneling times to emerge as a consequence of the representation change. This apparent controversy between what seems to be the description of different types of tunneling times becomes an open problem that can only be resolved by carrying out the experiment itself. In other words, if it is possible to carry out an experimental measurement of the tunneling time (in the absence of an external magnetic field and for systems that can reach super-relativistic energy regimes), our theoretical predictions could then be tested. This is left as an open problem for future research.

Finally, the results reported in this paper could, in principle, be applied to nuclear physics phenomena in high-energy regimes. However, it is important to highlight that the Hartman effect is present in all our results. Therefore, more work must be done to properly define the tunneling time of quantum systems.

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[33] To clarify the difference in nomenclature used in our paper with respect to that used in Ref. [26], we can define $k' = k$, $f^{-1} = |T'|$, $q' = \kappa$, and $L = a$.

[34] In this paper, we apply the usual designation $\vec{p} = (p_x, p_y, p_z)$ for the linear momentum and $E$ for the energy in the four-momentum $\mathbf{p}$. Also, we consider the coordinate transformation $\mathbf{x}' = \Lambda\mathbf{x} = \Lambda(ct, \vec{x})^T$, where $\Lambda$ has matrix elements $\Lambda_0^0 = \Lambda_1^1 = \gamma$, $\Lambda_1^0 = \Lambda_0^1 = -\gamma v$, and $\Lambda_2^0 = \Lambda_0^2 = \Lambda_3^0 = \Lambda_0^3 = \Lambda_1^2 = \Lambda_2^1 = \Lambda_1^3 = \Lambda_3^1 = \Lambda_3^2 = \Lambda_2^3 = 0$ and belongs to the Lorentz group, $\mathcal{L}$, and leaves the quadratic form $t^2 - |\vec{x}|^2$ invariant. The matrix associated with this quadratic form is $\zeta = \mathrm{diag}(1, -1, -1, -1)$, which satisfies the relation $\Lambda^T \zeta \Lambda = \zeta$.