Gauge dependence of the Aharonov-Bohm phase in a quantum electrodynamics framework

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The Aharonov-Bohm (AB) phase is usually associated with a line integral of the electromagnetic vector potential generated by an external current source, such as a solenoid. According to this interpretation, the AB phase of a nonclosed path cannot be observed, as the integral depends on the gauge choice of the vector potential. Recent attempts to explain the AB effect through the interaction between a charged particle and an external current, mediated by the exchange of quantum photons, have assumed that the AB phase shift is proportional to the change in interaction energy between the charged particle and the external current source. As a result, these attempts argue that the AB phase change along a path does not depend on the gauge choice, and that the AB phase shift for a nonclosed path is in principle measurable. In this paper, we critically examine this claim and demonstrate that the phase obtained through this approach is actually gauge dependent and not an observable for a nonclosed path. We also provide a brief critical discussion of the proposed experiment for observing the AB phase shift of a nonclosed path.

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I. INTRODUCTION

The Aharonov-Bohm (AB) effect [1,2] is a quantum mechanical phenomenon in which a charged particle is influenced by the electromagnetic potential even in regions where the electromagnetic field can be neglected. For reviews, see, e.g., Refs. [3–6]. The existence of the AB effect was first experimentally demonstrated by Tonomura and co-workers using electron beam holography techniques [7,8].

In a typical scenario of the AB effect, we consider an external current source, such as a solenoid, and assume that a charged particle is moving in a region where the magnetic field generated by the current source is negligible but the corresponding vector potential is finite. In the standard interpretation of the AB effect, the Schrödinger equation for the charged particle includes a coupling term between the particle and the background vector potential generated by the external current. As a result, the charged particle acquires an extra phase even in regions where no magnetic field is present. This phase (the AB phase) is represented by a line integral of the background vector potential along the path of the particle. Therefore it is widely accepted that the AB phase is a gauge-dependent quantity and is not an observable unless the particle's path forms a closed loop.

However, there are some interesting unconventional approaches to understand the AB phase. For instance, Vaidman made an attempt to explain the AB phase without relying on gauge-dependent potentials [9]. According to this explanation, the AB phase is attributed to the local interaction between the charged particle's field and the potential source. For further discussions on this approach, refer to Refs. [10–13].

There have been yet other proposals of new attempts based on a quantum electrodynamics framework [14–17]. In this approach, it is considered that a charged particle and an external current source interact by the exchange of quantum electrodynamics photons, with the AB phase being directly proportional to the change in interaction energy. Since energy is commonly regarded as a gauge-invariant observable, it is asserted that the AB phase shift is also gauge invariant and can be measured even when the particle's path is not closed [15–17]. In this paper, however, we will disprove this claim by showing that the energy correction computed in the quantum electrodynamics framework generally depends on the choice of gauge.

In Sec. II, we first review the approach to the AB phase in the quantum electrodynamics framework. This approach primarily involves perturbations in both the charge of the particle, denoted as e, and the strength parameter of the external current, denoted as g. Next, we introduce a different scheme using the coherent state, where the assumption of g being small is no longer required. In this scheme, while the results remain unchanged, the calculations are simplified, providing us with a deeper understanding of how the energy correction depends on the gauge condition. The Coulomb gauge or the Lorenz gauge is commonly used in the quantum electrodynamics approach to the AB phase. In Sec. III, we adopt the axial gauge condition $A_3 = 0$ and demonstrate that the energy correction is a gauge-dependent quantity, implying that the AB phase for a nonclosed path is not an observable. Discussions including brief comments on the proposed experiment for measuring the AB phase for a nonclosed path are given in Sec. IV.

II. THE SYSTEM OF A CHARGED PARTICLE, QUANTUM ELECTROMAGNETIC FIELD, AND AN EXTERNAL CURRENT SOURCE

We consider the combined system of a charged particle, quantum electromagnetic field, and a static external current

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source such as a solenoid. The mass, the charge, and the coordinate of the particle are denoted as m, e, and q, respectively. The charged particle interacting with the electromagnetic field is also subject to the action of a certain potential $V(\boldsymbol{q})$. The quantum electromagnetic field also couples to the current $J_\mu(\boldsymbol{x}) = (0, \boldsymbol{J}(\boldsymbol{x}))$ produced by the static external current source. The equations of motion for the charged particle and the electromagnetic fields can be written as follows:

$$m\frac{d^2\boldsymbol{q}}{dt^2} = -\frac{\partial V}{\partial \boldsymbol{q}} + e(\boldsymbol{E}(\boldsymbol{q},t) + \dot{\boldsymbol{q}} \times \boldsymbol{B}(\boldsymbol{q},t)), \tag{1a}$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x},t) = j^0(\boldsymbol{x},t), \tag{1b}$$

$$\nabla \times \boldsymbol{B}(\boldsymbol{x},t) = \boldsymbol{j}(\boldsymbol{x},t) + g\boldsymbol{J}(\boldsymbol{x}) + \frac{\partial \boldsymbol{E}(\boldsymbol{x},t)}{\partial t}, \tag{1c}$$

where $j^\mu(x)$ is the current of the charged particle given by

$$j^0(\boldsymbol{x},t) = e\delta(\boldsymbol{x}-\boldsymbol{q}), \tag{2}$$

$$\boldsymbol{j}(\boldsymbol{x},t) = e\dot{\boldsymbol{q}}\delta(\boldsymbol{x}-\boldsymbol{q}), \tag{3}$$

and we introduced the strength parameter g for the external current $\boldsymbol{J}(\boldsymbol{x})$ for later convenience. Note that the static external current $\boldsymbol{J}(\boldsymbol{x})$ is conserved: $\nabla \cdot \boldsymbol{J} = 0$. Furthermore, we take into account that the external current is localized in space. As a result, we can safely assume that all the fields rapidly approach zero at infinity. Throughout this paper we employ the Heaviside-Lorentz (rationalized Gaussian) system of units with $\hbar = c = 1$.

We start by adopting the Coulomb gauge, where the vector potential $\boldsymbol{A}$ satisfies the transverse condition $\nabla \cdot \boldsymbol{A} = 0$, and for clarity the vector potential with this condition is denoted by $\boldsymbol{A}_\perp$. The Hamiltonian of this system is then given by

$$H_C = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A}_\perp(\boldsymbol{q}))^2 + V(\boldsymbol{q}) + \int d^3x \frac{\boldsymbol{E}_\perp^2 + \boldsymbol{B}^2}{2} - g\int d^3x \boldsymbol{J}(\boldsymbol{x})\cdot\boldsymbol{A}_\perp(\boldsymbol{x}), \tag{4}$$

where $\boldsymbol{E}_\perp$ is the transverse component of the electric field $\boldsymbol{E}$, and the longitudinal component $\boldsymbol{E}_\parallel$ and the time component A_0 are given by

$$\boldsymbol{E}_\parallel = -\nabla A_0, \quad A_0(\boldsymbol{x},t) = \frac{1}{4\pi}\frac{e}{|\boldsymbol{x}-\boldsymbol{q}|}. \tag{5}$$

We have the following equal-time commutation relations:

$$[q^i, p^j] = i\delta^{ij}, \tag{6}$$

$$[A_i^\perp(\boldsymbol{x},t), E_j^\perp(\boldsymbol{x}',t)] = -i\delta_{ij}^{tr}(\boldsymbol{x}-\boldsymbol{x}'), \tag{7}$$

for $i, j = 1, 2, 3$, and all the other commutation relations vanish. Here, $\delta_{ij}^{tr}(\boldsymbol{x}-\boldsymbol{x}')$ is the transverse delta function defined by

$$\delta_{ij}^{tr}(\boldsymbol{x}-\boldsymbol{x}') = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\left(\delta_{ij} - \frac{k_i k_j}{\boldsymbol{k}^2}\right). \tag{8}$$

The fields $\boldsymbol{A}_\perp$ and $\boldsymbol{E}_\perp$ are expanded in terms of the photon creation and annihilation operators as

$$\boldsymbol{A}_\perp(\boldsymbol{x}) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega}} \sum_{\lambda=1}^{2} \boldsymbol{e}(k,\lambda) \times (a(k,\lambda)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + a^\dagger(k,\lambda)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}), \tag{9}$$

$$\boldsymbol{E}_\perp(\boldsymbol{x}) = i\int \frac{d^3k}{\sqrt{(2\pi)^3}}\sqrt{\frac{\omega}{2}} \sum_{\lambda=1}^{2} \boldsymbol{e}(k,\lambda) \times (a(k,\lambda)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} - a^\dagger(k,\lambda)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}), \tag{10}$$

where $\omega = |\boldsymbol{k}|$ and $\boldsymbol{e}(k,\lambda)$ is the transverse polarization vector satisfying $\boldsymbol{e}(k,\lambda)\cdot\boldsymbol{k} = 0$ for $\lambda = 1, 2$.

A. The effective Hamiltonian for the charged particle: The second order in *g* and *e*

The system described by the Hamiltonian of Eq. (4) consists of the charged particle and the electromagnetic fields. To obtain an effective Hamiltonian for the charged particle, we need to eliminate the degrees of freedom of the electromagnetic fields. In this section we mainly follow the derivation by Saldanha [16]. See also Refs. [14,15,17].

We write the Hamiltonian H_C as

$$H_C = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{q}) + H_C', \tag{11}$$

$$H_C' = H_{\rm EM} + H_g + H_e + O(e^2), \tag{12}$$

where

$$H_{\rm EM} = \int d^3x \frac{\boldsymbol{E}_\perp^2 + \boldsymbol{B}^2}{2} = \sum_{\lambda=1}^{2}\int d^3k\, \omega a^\dagger(k,\lambda)a(k,\lambda),$$

$$H_g = -g\int d^3x \boldsymbol{J}(\boldsymbol{x})\cdot\boldsymbol{A}_\perp(\boldsymbol{x}), \quad H_e = -e\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{A}_\perp(\boldsymbol{q}).$$

In this section we assume that the parameters e and g are small, and we calculate the energy of H_C' up to the second order in e and g. The resultant energy depends on the dynamical variables of the charged particle, $\boldsymbol{q}$ and $\boldsymbol{p}$, and it contributes part of the effective Hamiltonian of the charged particle.

The unperturbed state is the photon vacuum $|0\rangle$ that is the ground state of $H_{\rm EM}$. It is evident that the first-order energy correction vanishes, as the expectation value of $\boldsymbol{A}$ in the vacuum $|0\rangle$ is 0. The second-order energy in parameters e and g is given by

$$\langle 0|\,(H_e + H_g)\frac{Q}{E_0 - H_{\rm EM}}(H_e + H_g)\,|0\rangle\,, \tag{13}$$

where $Q = \boldsymbol{1} - |0\rangle\langle 0|$ and E_0 is the unperturbed energy of the vacuum $|0\rangle$. Note that the terms of $O(e^2)$ and $O(g^2)$ are constants; they are independent of the dynamical variables of the charged particle. The relevant energy is therefore of $O(ge)$ and given by

$$\Delta\epsilon = \langle 0|\,H_g\frac{Q}{E_0 - H_{\rm EM}}H_e\,|0\rangle + \text{c.c.} \tag{14a}$$

$$= \int d^3k \sum_{\lambda=1}^{2}\langle 0|H_g|k,\lambda\rangle\,\frac{-1}{\omega}\,\langle k,\lambda|H_e|0\rangle + \text{c.c.}, \tag{14b}$$

with c.c. representing the complex conjugate of the terms preceding it. The intermediate state $|k,\lambda\rangle$ is the one-photon state of momentum k and polarization $\lambda = 1, 2$. Using the Fourier expansion of $\boldsymbol{A}_\perp$ of Eq. (9), we have

$$H_g = -g\sum_{\lambda=1}^{2}\int d^3k \frac{1}{\sqrt{2\omega}}\boldsymbol{e}(k,\lambda) \times (a(k,\lambda)\boldsymbol{J}_{\boldsymbol{k}}^* + a^\dagger(k,\lambda)\boldsymbol{J}_{\boldsymbol{k}})), \tag{15}$$

where $\boldsymbol{J_k}$ is the Fourier transform of the external current source $\boldsymbol{J}(\boldsymbol{x})$:

$$\boldsymbol{J_k} = \frac{1}{\sqrt{(2\pi)^3}}\int d^3x \boldsymbol{J}(\boldsymbol{x})e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}. \quad (16)$$

We can now calculate $\Delta\epsilon$ of Eq. (14). We find

$$\Delta\epsilon = -eg\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{q}), \quad (17)$$

where $\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{x})$ is defined to be

$$\begin{aligned}\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{x}) &= \frac{1}{\sqrt{(2\pi)^3}}\int d^3k\frac{\boldsymbol{J_k}}{\boldsymbol{k}^2}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\\ &= \frac{1}{4\pi}\int d^3x'\frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|},\end{aligned} \quad (18)$$

which is just the vector potential generated by the current source $\boldsymbol{J}(\boldsymbol{x})$ according to the Biot-Savart law. This $\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{x})$ satisfies the transverse condition $\nabla\cdot\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{x}) = 0$, which follows from the current conservation law $\nabla\cdot\boldsymbol{J}(\boldsymbol{x}) = 0$.

Thus the effective Hamiltonian for the charged particle is given by

$$h_C = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{q}) - eg\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{q}) + O(e^2). \quad (19)$$

Note that the h_C takes the gauge-covariant form

$$h_C = \frac{(\boldsymbol{p} - eg\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{q}))^2}{2m} + V(\boldsymbol{q}), \quad (20)$$

if the appropriate higher-order term of $O(g^2e^2)$ is added. This can be interpreted as representing the Hamiltonian of a charged particle moving under the influence of the external potential $\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{q})$.

In this approach, the coupling term in the effective Hamiltonian given by Eq. (17) is the second-order energy correction. Suppose that the charged particle moves along a path L: from $t = t_1$ to $t = t_2$. The AB phase shift acquired by the particle is then given by a time integral of $\Delta\epsilon$:

$$\Phi_{\text{AB}} = -eg\int_{t_1}^{t_2}dt\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{q}) = -eg\int_L d\boldsymbol{q}\cdot\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{q}),$$

while energy is generally considered to be a gauge-invariant observable. It is this observation that underlies the claim that the AB phase shift is a gauge-invariant and measurable observable even when the path L is not closed [15–17].

In the subsequent sections, we will refute this assertion by showing that the energy correction $\Delta\epsilon$ is generally gauge dependent. In the Lorenz gauge, the form of $\Delta\epsilon$ given in Eq. (14a) remains unaltered, although the intermediate states should include scalar and longitudinal photon states alongside the transverse ones. However, it is not difficult to see that the energy correction $\Delta\epsilon$ coincides with that in the Coulomb gauge. In Sec. III, we will choose the axial gauge to demonstrate that the energy correction $\Delta\epsilon$ is a gauge-dependent quantity. In Sec. II B, however, we present the scheme using the coherent state where the parameter g is not assumed to be small. We will see that this scheme gives the same result as the one presented here but provides more insight into how the energy correction depends on the gauge condition.

B. Coherent state scheme: The first order in *e*

In this section, we present the method that uses photon coherent states without assuming that the parameter g is small. The results we will obtain are the same as those in Sec. II A. However, the calculations are simplified in this method. It also provides better insight into the gauge dependence of the energy correction in question. In this method we treat H'_C of Eq. (12) in the first-order perturbation in e. The unperturbed Hamiltonian is then $H_{\text{EM}} + H_g$, which is expressed as

$$\begin{aligned}H_{\text{EM}} + H_g &= \sum_{\lambda=1}^{2}\int d^3k\omega a^{\dagger}(k,\lambda)a(k,\lambda)\\ &\quad - g\sum_{\lambda=1}^{2}\int\frac{d^3k}{\sqrt{2\omega}}\boldsymbol{e}(k,\lambda)\cdot(a(k,\lambda)\boldsymbol{J}^*_{\boldsymbol{k}} + a^{\dagger}(k,\lambda)\boldsymbol{J_k}).\end{aligned}$$

This Hamiltonian can be "diagonalized" in terms of the photon annihilation and creation operators, $\tilde{a}(k,\lambda)$ and $\tilde{a}^{\dagger}(k,\lambda)$, which are defined through

$$a(k,\lambda) = \tilde{a}(k,\lambda) + \alpha(k,\lambda), \quad (21)$$

with $\alpha(k,\lambda) = g/\sqrt{2\omega^3}\boldsymbol{e}(k,\lambda)\cdot\boldsymbol{J_k}$. Up to a constant the result is given by

$$H_{\text{EM}} + H_g = \sum_{\lambda=1}^{2}\int d^3k\,\omega\,\tilde{a}^{\dagger}(k,\lambda)\tilde{a}(k,\lambda). \quad (22)$$

The ground state of $H_{\text{EM}} + H_g$, denoted by $|\tilde{0}\rangle$, is then the state that is annihilated by $\tilde{a}(k,\lambda)$; that is, $\tilde{a}(k,\lambda)|\tilde{0}\rangle = 0$. The state $|\tilde{0}\rangle$ is therefore an eigenstate of the annihilation operator $a(k,\lambda)$, the coherent state of photons [18].

$$a(k,\lambda)|\tilde{0}\rangle = \alpha(k,\lambda)|\tilde{0}\rangle\,. \quad (23)$$

Note that the expectation value of $\boldsymbol{A}_{\perp}$ with respect to the coherent state $|\tilde{0}\rangle$ is not 0, but given by g times $\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{x})$ of Eq. (18).

$$\begin{aligned}\langle\tilde{0}|\boldsymbol{A}_{\perp}(\boldsymbol{x})|\tilde{0}\rangle &= \int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega}}\sum_{\lambda=1}^{2}\boldsymbol{e}(k,\lambda)\\ &\quad\times(\alpha(k,\lambda)e^{i\boldsymbol{k}\cdot\boldsymbol{x}} + \alpha^{\dagger}(k,\lambda)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}})\\ &= g\frac{1}{4\pi}\int d^3x'\frac{\boldsymbol{J}(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} = g\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{x}).\end{aligned} \quad (24)$$

The unperturbed energy $\tilde{E}_0$, which is a constant, can be taken to be 0. The first-order energy correction in e is simply given by the expectation value of H_e in the unperturbed state $|\tilde{0}\rangle$:

$$\begin{aligned}\Delta E = \langle\tilde{0}|H_e|\tilde{0}\rangle &= -e\frac{\boldsymbol{p}}{m}\cdot\langle\tilde{0}|\boldsymbol{A}_{\perp}(\boldsymbol{q})|\tilde{0}\rangle\\ &= -eg\frac{\boldsymbol{p}}{m}\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{q}).\end{aligned} \quad (25)$$

This is exactly equal to the energy correction $\Delta\epsilon$ of Eq. (17) that is the result of the second-order perturbation in e and g. Thus we have obtained the same effective particle Hamiltonian h_C as that given in Eq. (19).

A few remarks are now in order. First, the results obtained in Sec. II A hold true regardless of the magnitude of g. This is not a coincidence, but is due to the following reasons:

Equation (24) shows that the expectation value $\langle\tilde{0}|\boldsymbol{A}_\perp(\boldsymbol{x})|\tilde{0}\rangle$ is linear in g though the coherent state $|\tilde{0}\rangle$ itself contains higher-order terms in g. Suppose that we calculate the expectation value of $\boldsymbol{A}_\perp(\boldsymbol{x})$ in the state $|0_1\rangle$ that is the approximate ground state of $H_{\rm EM}+H_g$ in the first-order perturbation in g.

$$|0_1\rangle = |0\rangle + \frac{Q}{E_0 - H_{\rm EM}} H_g\,|0\rangle\,.$$

We then have

$$\begin{aligned}\langle\tilde{0}|\boldsymbol{A}_\perp(\boldsymbol{x})|\tilde{0}\rangle &= \langle 0_1|\boldsymbol{A}_\perp(\boldsymbol{x})|0_1\rangle + \delta \\ &= \langle 0|\,H_g\frac{Q}{E_0 - H_{\rm EM}}\boldsymbol{A}_\perp(\boldsymbol{x})\,|0\rangle + \text{c.c.} + \delta,\end{aligned}$$

where the error δ is $O(g^2)$. Notice that the left-hand side is order $O(g)$. This implies that δ should be exactly 0 as the terms other than δ on the right-hand side are order $O(g)$. Thus we find

$$\langle\tilde{0}|H_e|\tilde{0}\rangle = \langle 0|\,H_g\frac{Q}{E_0 - H_{\rm EM}}H_e\,|0\rangle + \text{c.c.}, \tag{26}$$

which is just the second-order energy correction of Eq. (14) obtained in Sec. II A.

Second, as can be seen in Eq. (25), the coupling term in the effective Hamiltonian, which is identified with the energy correction $\Delta E = \Delta\epsilon$, is expressed in terms of the expectation value of the vector potential in the coherent state. This expectation value should reflect the gauge condition imposed on the vector potential. This strongly suggests that the coupling term depends on the gauge that we choose. We will examine specifically the case of the axial gauge in Sec. III.

Third, this concerns the change in magnetic field energy due to the motion of the charged particle. Using classical electromagnetism, Boyer calculated this quantity and obtained the following result [19]:

$$\Delta\mathcal{E}_{\rm Boyer} = eg\frac{\boldsymbol{p}}{m}\boldsymbol{A}^\perp_{\rm ext}(\boldsymbol{q}), \tag{27}$$

which has the same form as the coupling term between the charged particle and the external potential $\boldsymbol{A}^\perp_{\rm ext}(\boldsymbol{q})$. However, no attention seems to have been paid to the fact that the signs are different. Where does the Boyer's energy (27) appear in our treatment, in which the energy correction of $O(e)$ is given by $\langle\tilde{0}|H_e|\tilde{0}\rangle$ only? The answer is that it is hidden as part of the expectation value of the unperturbed Hamiltonian. Let $|\tilde{\phi}_1\rangle$ be the first-order perturbed eigenstate of $H_{\rm EM}+H_g+H_e$.

$$|\tilde{\phi}_1\rangle = |\tilde{0}\rangle + \frac{\tilde{Q}}{\tilde{E}_0 - (H_{\rm EM}+H_g)}H_e\,|\tilde{0}\rangle\,, \tag{28}$$

with $\tilde{Q} = \mathbf{1} - |\tilde{0}\rangle\,\langle\tilde{0}|$. Now consider the expectation value of the unperturbed Hamiltonian in the state $|\tilde{\phi}_1\rangle$. The result should be given by

$$\langle\tilde{\phi}_1|H_{\rm EM}+H_g|\tilde{\phi}_1\rangle = \tilde{E}_0 + O(e^2), \tag{29}$$

with no terms of order $O(e)$. This, however, does not necessarily imply that neither $\langle\tilde{\phi}_1|H_{\rm EM}|\tilde{\phi}_1\rangle$ nor $\langle\tilde{\phi}_1|H_g|\tilde{\phi}_1\rangle$ contains a contribution of order $O(e)$. It turns out that $\langle\tilde{\phi}_1|H_{\rm EM}|\tilde{\phi}_1\rangle$ is given by $\Delta\mathcal{E}_{\rm Boyer}$ up to a constant but is canceled by the contribution from $\langle\tilde{\phi}_1|H_g|\tilde{\phi}_1\rangle$. See the Appendix for details.

III. THE AB PHASE OF A NONCLOSED PATH IS NOT OBSERVABLE

In this section we show that the coupling term in the effective Hamiltonian depends on the gauge. To do so, we treat the same system as that discussed in the preceding section by imposing the axial gauge condition, $A_3 = 0$. Note that this condition completely fixes the vector potential as we assume all the fields should approach zero at infinity. For canonical quantization of electromagnetic fields in the axial gauge, we refer the reader to Ref. [20]. The Hamiltonian is given by

$$\begin{aligned}H_X = &\frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{q}))^2 + V(\boldsymbol{q}) + \int d^3x\frac{\boldsymbol{E}^2+\boldsymbol{B}^2}{2} \\ &- g\int d^3x\boldsymbol{J}(\boldsymbol{x})\cdot\boldsymbol{A}(\boldsymbol{x}).\end{aligned} \tag{30}$$

Here, for $\boldsymbol{q}$, $\boldsymbol{p}$ and the x and y components of $\boldsymbol{A}$ and $\boldsymbol{E}$, the following typical canonical commutation relations are assumed:

$$\begin{aligned}&[q^k, q^l] = [p^k, p,^l] = 0, \quad [q^k, p^l] = i\delta_{kl}, \\ &[A_i(\boldsymbol{x}), A_j(\boldsymbol{x}')] = [E_i(\boldsymbol{x}), E_j(\boldsymbol{x}')] = 0, \\ &[A_i(\boldsymbol{x}), E_j(\boldsymbol{x}')] = -i\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{x}'), \\ &[q^k, A_i(\boldsymbol{x})] = [q^k, E_i(\boldsymbol{x})] = [p^k, A_i(\boldsymbol{x})] = [p^k, E_i(\boldsymbol{x})] = 0,\end{aligned}$$

for $k, l = 1, 2, 3$ and $i, j = 1, 2$. The other commutation relations are determined by the constraint conditions in the axial gauge. First, we have the gauge fixing condition: $A_3(\boldsymbol{x}) = 0$. Second, the z component of $\boldsymbol{E}$ is constrained to be

$$E_3(\boldsymbol{x}) = -\frac{1}{\partial_3}\left(\sum_{i=1}^{2}\partial_i E_i(\boldsymbol{x}) + j_0(\boldsymbol{x})\right), \tag{31}$$

so that Gauss's law $\boldsymbol{\nabla}\cdot\boldsymbol{E} = j_0$ is fulfilled. Third, the component A_0 is given by $A_0(\boldsymbol{x}) = \partial_3^{-1}E_3(\boldsymbol{x})$, which respects the relations $E_3 = \partial_3 A_0 - \partial_0 A_3$ with $A_3 = 0$. Under these commutation relations and constraints, one can verify that the Hamiltonian H_X correctly reproduces the equations of motion of Eq. (1).

The vector potential $\boldsymbol{A}(\boldsymbol{x})$ and the electric field $\boldsymbol{E}(\boldsymbol{x})$ in the axial gauge are related to the transverse vector potential $\boldsymbol{A}^\perp(\boldsymbol{x})$ and the transverse electric field $\boldsymbol{E}^\perp(\boldsymbol{x})$, respectively:

$$A_i(\boldsymbol{x}) = A_i^\perp(\boldsymbol{x}) - \frac{\partial_i}{\partial_3}A_3^\perp(\boldsymbol{x}), \quad i = 1, 2, 3, \tag{32}$$

$$E_i(\boldsymbol{x}) = E_i^\perp(\boldsymbol{x}) - \delta_{i3}\frac{1}{\partial_3}j_0(\boldsymbol{x}), \quad i = 1, 2, 3, \tag{33}$$

with the Fourier expansions of $\boldsymbol{A}^\perp(\boldsymbol{x})$ and $\boldsymbol{E}^\perp(\boldsymbol{x})$ given in Eqs. (9) and (10), respectively. Using these Fourier expansions, one can verify that the three aforementioned constraints in the axial gauge are satisfied and all the related commutation relations follow. The Hamiltonian of Eq. (30) can be written as

$$\begin{aligned}H_X = &\frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{q}))^2 + V(\boldsymbol{q}) + \int d^3k\,\omega\sum_{\lambda=1}^{2}a^\dagger(k,\lambda)a(k,\lambda) \\ &- e\frac{1}{(\partial_3)^2}\left(\sum_{i=1}^{2}\partial_i E_i(\boldsymbol{q})\right) - g\int d^3x\boldsymbol{J}(\boldsymbol{x})\cdot\boldsymbol{A}(\boldsymbol{x}).\end{aligned}$$

We are now ready to calculate the effective Hamiltonian for the charged particle in the axial gauge. As in the case of the Coulomb gauge, we write

$$H_X = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{q}) + H'_X, \tag{34}$$

$$H'_X = H_{\rm EM} + H_g + H_e^{(1)} + H_e^{(2)} + O(e^2), \tag{35}$$

where

$$\begin{aligned} H_{\rm EM} &= \sum_{\lambda=1}^{2}\int d^3k\omega a^\dagger(k,\lambda)a(k,\lambda),\\ H_g &= -g\int d^3x \boldsymbol{J}(\boldsymbol{x})\cdot\boldsymbol{A}(\boldsymbol{x}),\\ H_e^{(1)} &= -e\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{A}(\boldsymbol{q}),\\ H_e^{(2)} &= -e\frac{1}{(\partial_3)^2}\left(\sum_{i=1}^{2}\partial_i E_i(\boldsymbol{q})\right). \end{aligned}$$

We evaluate H'_X in terms of the first-order perturbation theory of e, as we did for the Coulomb gauge in Sec. II B. The Hamiltonian H_g is expanded as

$$\begin{aligned} H_g = &-g\sum_{\lambda=1}^{2}\int d^3k\frac{1}{\sqrt{2\omega}}\boldsymbol{e}^X(k,\lambda)\\ &\times\left(a(k,\lambda)\boldsymbol{J}_{\boldsymbol{k}}^* + a^\dagger(k,\lambda)\boldsymbol{J}_{\boldsymbol{k}}\right), \end{aligned} \tag{36}$$

where

$$e_i^X(k,\lambda) = e_i(k,\lambda) - \frac{k_i}{k_3}e_3(k,\lambda), \quad i=1,2,3, \tag{37}$$

is the polarization vector of the vector potential $\boldsymbol{A}(\boldsymbol{x})$ in the axial gauge. In the above expression of H_g, however, this polarization vector $\boldsymbol{e}^X(k,\lambda)$ can be replaced by the transverse polarization vector $\boldsymbol{e}(k,\lambda)$, as the current conservation of the external current, $\boldsymbol{k}\cdot\boldsymbol{J}_{\boldsymbol{k}} = 0$, implies $\boldsymbol{e}^X(k,\lambda)\cdot\boldsymbol{J}_{\boldsymbol{k}} = \boldsymbol{e}(k,\lambda)\cdot\boldsymbol{J}_{\boldsymbol{k}}$. Thus the unperturbed Hamiltonian $H_{\rm EM}+H_g$ is the same as that, given by Eq. (22), in the Coulomb gauge. The unperturbed ground state is therefore the coherent state $|\tilde{0}\rangle$ defined through $a(k,\lambda)\,|\tilde{0}\rangle = \alpha(k,\lambda)\,|\tilde{0}\rangle$ with $\alpha(k,\lambda) = g/\sqrt{2\omega^3}\boldsymbol{e}(k,\lambda)\cdot\boldsymbol{J}_{\boldsymbol{k}}$.

The expectation value of $\boldsymbol{A}(\boldsymbol{x})$ in the coherent state $|\tilde{0}\rangle$ can be calculated as

$$\begin{aligned} \langle\tilde{0}|\boldsymbol{A}(\boldsymbol{x})|\tilde{0}\rangle &= \langle\tilde{0}|A^\perp(\boldsymbol{x}) - \frac{\boldsymbol{\nabla}}{\partial_3}A_3^\perp(\boldsymbol{x})|\tilde{0}\rangle\\ &= g\left(\boldsymbol{A}^\perp_{\rm ext}(\boldsymbol{x}) - \frac{\boldsymbol{\nabla}}{\partial_3}A^3_{\perp,\rm ext}(\boldsymbol{x})\right)\\ &\equiv g\boldsymbol{A}^X_{\rm ext}(\boldsymbol{x}), \end{aligned} \tag{38}$$

where $\boldsymbol{A}^\perp_{\rm ext}(\boldsymbol{x})$ is defined in Eq. (18). Remember that $\boldsymbol{A}^\perp_{\rm ext}(\boldsymbol{x})$ is the transverse vector potential generated by the external current. This implies that $\boldsymbol{A}^X_{\rm ext}(\boldsymbol{x})$ is also the vector potential generated by the external current, but with the axial gauge condition $\boldsymbol{A}^{X,3}_{\rm ext}(\boldsymbol{x}) = 0$. As for the electric field $\boldsymbol{E}$, it can be readily checked that $\langle\tilde{0}|E_i(\boldsymbol{x})|\tilde{0}\rangle = 0$ for $i = 1, 2$.

The unperturbed energy is constant and taken to be 0, and the first-order energy correction in e is given by

$$\Delta E = \langle\tilde{0}|H_e^{(1)} + H_e^{(2)}|\tilde{0}\rangle = -eg\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{A}^X_{\rm ext}(\boldsymbol{q}), \tag{39}$$

which leads to the particle effective Hamiltonian:

$$h_X = \frac{1}{2m}\boldsymbol{p}^2 + V(\boldsymbol{q}) - eg\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{A}^X_{\rm ext}(\boldsymbol{q}) + O(e^2). \tag{40}$$

In general the vector potentials $\boldsymbol{A}^\perp_{\rm ext}(\boldsymbol{q})$ and $\boldsymbol{A}^X_{\rm ext}(\boldsymbol{q})$ are different for a given external current $\boldsymbol{J}(\boldsymbol{x})$, implying that the AB phase acquired by the charged particle along a nonclosed path depends on the gauge that we choose, and it is not an observable.

We obtained the effective Hamiltonian h_X by using the first-order perturbation theory of e, as we did in Sec. II B for the case of the Coulomb gauge. The result should be unchanged if we perform the second-order perturbation in g and e as the calculation described in Sec. II A. Before concluding this section, we will confirm this equivalence.

In the perturbation in g and e, the unperturbed state is $|0\rangle$ given by the ground state of $H_{\rm EM}$, and $H_g + H_e^{(1)} + H_e^{(2)}$ is the perturbation. Since the first-order energy correction is 0, we consider the energy correction of order $O(ge)$:

$$\Delta\epsilon = \langle 0|H_g\frac{1}{E_0 - H_{\rm EM}}\left(H_e^{(1)} + H_e^{(2)}\right)|0\rangle + \text{c.c.} \tag{41}$$

As in the case of the Coulomb gauge, this $\Delta\epsilon$ can be expressed as

$$\Delta\epsilon = \langle 0_1|H_e^{(1)} + H_e^{(2)}|0_1\rangle + O(g^2), \tag{42}$$

with $|0_1\rangle$ being the approximate ground state of $H_{\rm EM}+H_g$ in the first-order perturbation in g. In this way, we can see that the calculation of perturbation in g and e will eventually lead to the result of Eq. (39), but let us now analyze $\Delta\epsilon$ in Eq. (41) explicitly. First, after some involved calculation, we find

$$\begin{aligned} &\langle 0|H_g\frac{1}{E_0 - H_{\rm EM}}H_e^{(1)}|0\rangle\\ &\quad= -\frac{eg}{2m}\boldsymbol{p}\cdot\left(\boldsymbol{A}^\perp_{\rm ext}(\boldsymbol{q}) - \frac{\boldsymbol{\nabla}}{\partial_3}A^3_{\perp,\rm ext}(\boldsymbol{q})\right)\\ &\quad= -\frac{eg}{2m}\boldsymbol{p}\cdot\boldsymbol{A}^X_{\rm ext}(\boldsymbol{q}). \end{aligned} \tag{43}$$

Second, the following matrix element turns out to be purely imaginary:

$$\langle 0|H_g\frac{1}{E_0 - H_{\rm EM}}H_e^{(2)}|0\rangle\,. \tag{44}$$

Combining these results, we conclude that $\Delta\epsilon$ is eventually given by ΔE of Eq. (39), which is as expected from Eq. (42).

IV. DISCUSSION AND CONCLUSIONS

The purpose of this paper is to investigate whether the AB phase shift in the case of a nonclosed path is independent of the gauge and therefore measurable. In the recent approach [14–17], the coupling term between the charged particle and the electromagnetic potential is identified with the energy change ΔE resulting from the quantum mechanical electromagnetic interaction between the charged particle and the external current source. Since energy is generally thought to be gauge invariant, this leads to the claim that the AB phase shift is a gauge-invariant and measurable observable even for a nonclosed path [15–17]. We have disproved this claim by explicitly showing that the energy correction ΔE is generally gauge dependent.

One may still wonder why the energy change ΔE can be gauge dependent. Let H_C and H_X be the Hamiltonians in the Coulomb gauge and the axial gauge, respectively. As operators we know that $H_C \neq H_X$, though we believe that their eigenvalues are the same. However, we do not try to directly determine the eigenvalues of H_C or H_X. What we do in this approach is to eliminate the degrees of freedom of the electromagnetic field and obtain an effective Hamiltonian h for the charged particle:

$$h = \frac{1}{2m}\boldsymbol{p}^2 + V(\boldsymbol{q}) - eg\frac{\boldsymbol{p}}{m}\cdot\langle\tilde{0}|\boldsymbol{A}(\boldsymbol{q})|\tilde{0}\rangle + \cdots. \tag{45}$$

Starting from H_C yields h_C, and starting from H_X yields h_X. Generally, we have $h_C \neq h_X$, as $\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{x}) \neq \boldsymbol{A}^{X}_{\text{ext}}(\boldsymbol{x})$. To calculate the energy of the system, one must still determine the eigenvalues of h_C or h_X. They should be the same. In conclusion, it may be misleading to say that this approach "calculates the energy change of the system." It would be more appropriate to say that it "derives the effective Hamiltonian for the charged particle." From this perspective, it seems natural for the effective Hamiltonian to depend on the gauge choice.

We have found that the AB phase for a nonclosed path cannot be a measurable physical quantity. On the other hand, some experiments have been proposed [15–17] that appear to allow the measurement of the AB phase along a nonclosed path. This is a contradiction that necessitates an examination of the proposed experiments. Here we take the experiment proposed in Ref. [16] because it appears to be the simplest conceptually. Essentially, it involves abruptly cutting off the current in a solenoid before the charged particle completes a closed path. The assumption here is that this would cause the charged particle to remember its phase change at the time of the interruption, and the interference pattern at the closed path would reflect the phase change at the intermediate point in the path. The problem here, however, is that a time-varying magnetic field produces an electric field. If $A_\mu(x)$ is time dependent, then the closed curve drawn by the charged particle would be a closed curve in four-dimensional space-time. In other words, when the solenoid current changes, the charged particle also undergoes a phase change with a contribution of $\int_C A_0(x)dt$, and it does not retain the phase difference at the time when the solenoid current is suddenly cut. One way to demonstrate this would be to assume that the current $J_\mu(x)$ in the solenoid is time dependent and to show that the phase difference after drawing a closed curve in four-dimensional space-time is independent of the gauge, while the phase accumulated during the path is gauge dependent. For a more comprehensive discussion on the four-dimensional loop integral of the time-dependent vector potential, we refer the reader to Ref. [21].

Note added. Recently, Wakamatsu [22] showed that the AB phase depends on the residual gauge freedom in the Coulomb gauge within a quantum electrodynamics approach.

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APPENDIX A: THE BOYER'S ENERGY CHANGE

We first rewrite H_{EM} as

$$\begin{aligned} H_{\text{EM}} &= \int d^3x \frac{\boldsymbol{E}_\perp^2 + \boldsymbol{B}^2}{2} \\ &= \int d^3x \frac{\boldsymbol{E}_\perp^2 + \tilde{\boldsymbol{B}}^2}{2} + \int d^3x \tilde{\boldsymbol{B}}\cdot\boldsymbol{B}_{\text{ext}} + \text{const}, \end{aligned}$$

where $\boldsymbol{B}_{\text{ext}} = g\nabla\times\boldsymbol{A}^{\perp}_{\text{ext}}$ and $\tilde{\boldsymbol{B}} = \boldsymbol{B} - \boldsymbol{B}_{\text{ext}}$. Then we find that, up to a constant,

$$\langle\tilde{\phi}_1|H_{\text{EM}}|\tilde{\phi}_1\rangle = \int d^3x\,\langle\tilde{\phi}_1|\tilde{\boldsymbol{B}}|\tilde{\phi}_1\rangle\cdot\boldsymbol{B}_{\text{ext}},$$

with $|\tilde{\phi}_1\rangle$ defined in Eq. (28). After some calculation, we find that

$$\langle\tilde{\phi}_1|\tilde{\boldsymbol{B}}(\boldsymbol{x})|\tilde{\phi}_1\rangle = -\frac{\boldsymbol{p}}{m}\times\nabla\frac{1}{4\pi}\frac{e}{|\boldsymbol{x}-\boldsymbol{q}|} + O(e^2), \tag{A1}$$

which is the nonrelativistic expression of the magnetic field generated by the charged particle moving at velocity $\boldsymbol{p}/m$.

Using this expression, we have

$$\begin{aligned} \langle\tilde{\phi}_1|H_{\text{EM}}|\tilde{\phi}_1\rangle &= -\frac{e}{4\pi m}\int d^3x\left(\boldsymbol{p}\times\nabla\frac{1}{|\boldsymbol{x}-\boldsymbol{q}|}\right)\cdot\boldsymbol{B}_{\text{ext}}(\boldsymbol{x}) \\ &= \frac{e}{4\pi m}\boldsymbol{p}\cdot\int d^3x\frac{1}{|\boldsymbol{x}-\boldsymbol{q}|}\nabla\times\boldsymbol{B}_{\text{ext}}(\boldsymbol{x}) \\ &= \frac{eg}{4\pi m}\boldsymbol{p}\cdot\int d^3x\frac{\boldsymbol{J}(\boldsymbol{x})}{|\boldsymbol{x}-\boldsymbol{q}|} = eg\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{q}), \end{aligned}$$

where $\boldsymbol{A}^{\perp}_{\text{ext}}$ is defined in Eq. (18). This is the Boyer's energy change $\Delta\mathcal{E}_{\text{Boyer}}$ given in Eq. (27). We now observe that H_g can be rewritten as follows:

$$\begin{aligned} H_g = -g\int d^3x J\cdot A_\perp &= -\int d^3x \nabla\times B_{\text{ext}}\cdot A^\perp \\ &= -\int d^3x B_{\text{ext}}\cdot\nabla\times A^\perp \\ &= -\int d^3x B_{\text{ext}}\cdot\tilde{B} + \text{const}. \end{aligned}$$

Therefore, up to a constant, we conclude that

$$\langle\tilde{\phi}_1|H_g|\tilde{\phi}_1\rangle = -eg\frac{\boldsymbol{p}}{m}\cdot\boldsymbol{A}^{\perp}_{\text{ext}}(\boldsymbol{q}), \tag{A2}$$

which cancels the Boyer's energy $\langle\tilde{\phi}_1|H_{\text{EM}}|\tilde{\phi}_1\rangle$.

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