Path distance of a quantum unitary evolution

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The Hamiltonian and evolution time of a quantum unitary system completely determine a unitary operator, which turns an initial state ρ_0 into its time-evolved state ρ_t and may also turn another state σ_0 into its time-evolved state σ_t. Although the two pairs of states (ρ_0, ρ_t) and (σ_0, σ_t) are governed by the same Hamiltonian and experience equal evolution periods, their state distances are not necessarily equal to each other. Specifically, the evolution operator contains information about the evolution other than that reflected by two specific states connected through it. As a matter of fact, two pairs of states with different state distances may have equal path distances. Here we propose a geometric measure for the path distance contained in a quantum unitary evolution, which depends only on the unitary operator itself and is independent of any pair of states connected through it. This path distance meets the basic requirements for a good measure of distance and is valid during the whole evolution process, no matter how long the evolution time is. This path distance is bounded by the path length of the evolution from above and by the Bures angle between the two states connected through it from below. It has potential applications in quantum information processing, such as providing another way to study the optimal control between two quantum states. We finally present the form of Hamiltonian to realize a unitary operator on demand.

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I. INTRODUCTION

The speed of quantum evolution has attracted a great deal of attention in recent years owing to its significance in quantum information processing. It is closely related to the computational capacity of quantum computation [1,2], the transferring rate of information [3–5], the measurement precision in quantum metrology [6–8], the entropy production rate in nonequilibrium quantum processes [9,10], the charging power of quantum batteries [11,12], and so on. Since the length of the geodesic path connecting two quantum states, usually defined as the angular distance between them, is no longer than the length of any other path, it could be used to investigate the speed limit of a quantum evolution [13,14].

Let us consider quantum evolution in a simple case. Given a quantum system governed by a time-independent Hamiltonian H, an initial state ρ_0 turns into its time-evolved state $\rho_t = U_t \rho_0 U_t^\dagger$, where the unitary operator U_t is completely determined by the Hamiltonian H and evolution time t through the relation $U_t = \exp(-iHt)$. If the quantum system is initially prepared in another state σ_0, it will turn into its time-evolved state $\sigma_t = U_t \sigma_0 U_t^\dagger$ under the same unitary operator U_t. Although the two initial states ρ_0 and σ_0 are connected to their time-evolved states ρ_t and σ_t through the same unitary operator, the two pairs of states (ρ_0, ρ_t) and (σ_0, σ_t) may have different state distances. For example, the two states $\rho_0 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\sigma_0 = \frac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ turn into $\rho_t = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\sigma_t = \sigma_0$ under the action of the unitary operator $U_t = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, respectively. However, the distance between the pairwise states (ρ_0, ρ_t), quantified by the Bures angle [15]

$$d_B(\rho_0, \rho_t) = \arccos \mathcal{F}(\rho_0, \rho_t), \tag{1}$$

with

$$\mathcal{F}(\rho_0, \rho_t) = \mathrm{Tr}\left(\sqrt{\sqrt{\rho_0}\rho_t\sqrt{\rho_0}}\right) \tag{2}$$

the Uhlmann fidelity [16,17], is not equal to the distance between the pairwise states (σ_0, σ_t), for $d_B(\rho_0, \rho_t) = \frac{\pi}{2}$ and $d_B(\sigma_0, \sigma_t) = 0$.

Since different quantum states may have different distance to their own time-evolved states under the action of the same unitary operator, we believe a unitary operator contains some common information about the evolution. In this paper we propose a measure for the path distance contained in a unitary evolution operator U, i.e., $d(U)$, which is determined by the unitary operator itself and is independent of the state of a quantum system. As is expected, the path distance proposed here is bounded from above by the path length of the evolution and bounded from below by the Bures angle between the two states connected through it. If a quantum state remains unchanged, the corresponding evolution operator is the identity operator I, and the path distance contained in a unitary evolution operator U proposed here could be regarded as the abbreviation of the distance between the evolution operator U and the identity operator I, i.e., $d(U) = d(I, U)$, which could be generalized to measure the distance between any two unitary operators. For example, if two unitary operators U_1 and U_2 are connected to each other through another unitary

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operator U_{12}, $U_2 = U_{12}U_1$, then the distance between the two unitary operators U_1 and U_2 is equal to the path distance contained in the unitary operator U_{12}, i.e., $d(U_1, U_2) = d(U_{12}) = d(U_2U_1^\dagger)$.

The tool of path distance could be used to investigate the issues on quantum evolution and has some potential applications in quantum information processing, for example, (i) defining the evolution speed as the changing rate of the path distance to reexam the quantum speed limit, (ii) minimizing the evolution time to reach a given path distance to study the optimal control of a quantum system, and (iii) preparing a quantum state in time. This paper is organized as follows. We introduce the path distance in Sec. II and discuss its properties and potential applications in Sec. III. In Sec. IV we present two examples used to compare the current path distance with the Bures angle and the Mandelstam-Tamm bound [13]. The main conclusions of this paper are summarized in Sec. V.

II. PATH DISTANCE

For the sake of simplicity, we consider in this paper the simple case in which the Hamiltonian H is time independent. The unitary operator U_t is then determined by H and the evolution time t through the relation $U_t = \exp(-iHt)$, where we have set the Planck constant $\hbar = 1$. Now we focus on the unitary operator U_t itself and apply it to a quantum pure state $|\psi_0\rangle$. The generalization to the case of mixed states will be presented in the next section.

We first make an eigenvalue decomposition on the unitary operator U_t, i.e., $U_t = V\Lambda V^\dagger$, where Λ is a diagonal matrix composed of the eigenvalues of the unitary operator U_t and V is formed of a set of orthogonal column vectors $\{|\psi_j\rangle\}$ $(j = 1, 2, \ldots, N)$ corresponding to the eigenvalues $\{\lambda_j\}$ in Λ. As is well known, every unitary operator satisfies the relation $UU^\dagger = I$, with I the identity operator, so the eigenvalues of the unitary operator U_t have the form of $\{\lambda_j = e^{i\phi_j}\}$ $(j = 1, 2, \ldots, N)$. Without loss of generality, we assume that the eigenvalues $\{\lambda_j\}$ are sorted in the diagonal matrix Λ based on the phases $\{\phi_j\}$ in ascending order, $\phi_1 \leqslant \phi_2 \leqslant \cdots \leqslant \phi_N$,

$$\Lambda = \begin{pmatrix} e^{i\phi_1} & 0 & \cdots & 0 \\ 0 & e^{i\phi_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\phi_N} \end{pmatrix}. \tag{3}$$

Here the phases $\{\phi_j\}$ are confined in the range $(-\pi, \pi]$, which is in fact the principal argument of the eigenvalues of the unitary operator U_t.

In the basis of the eigenvectors of U_t, i.e., $\{|\psi_j\rangle\}$, an initial pure state $|\psi_0\rangle = \sum_{j=1}^N c_j|\psi_j\rangle$, with the normalization condition $\sum_{j=1}^N |c_j|^2 = 1$, turns into its time-evolved state $|\psi_t\rangle = \sum_{j=1}^N c_j e^{i\phi_j}|\psi_j\rangle$ after the action of the unitary operator U_t, i.e., $|\psi_t\rangle = U_t|\psi_0\rangle$. The Uhlmann fidelity between the initial state $|\psi_0\rangle$ and its time-evolved state $|\psi_t\rangle$ is

$$\mathcal{F}(|\psi_0\rangle, |\psi_t\rangle) = \left|\sum_{j=1}^N |c_j|^2 e^{i\phi_j}\right|. \tag{4}$$

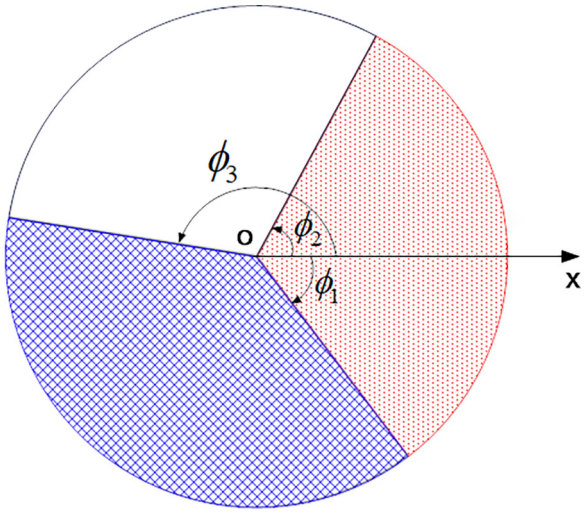

FIG. 1. Angular distance illustrated in a polar system. The absolute value of the difference between the two phases ϕ_1 and ϕ_2 is smaller than π, so the angular distance between them is $d(\phi_1, \phi_2) = |\phi_2 - \phi_1|$, corresponding to the sector angle filled with red dots. The absolute value of the difference between the two phases ϕ_1 and ϕ_3 is greater than π, so the angular distance between them is $d(\phi_1, \phi_3) = 2\pi - |\phi_3 - \phi_1|$, corresponding to the sector angle filled with blue crosshatching.

As a matter of fact, the Uhlmann fidelity is invariant under a global rotation of an arbitrary phase for the quantum states under consideration, so we could rewrite the above Uhlmann fidelity as $\mathcal{F}(|\psi_0\rangle, |\psi_t\rangle) = |e^{\phi_0}\sum_{j=1}^N |c_j|^2 e^{i(\phi_j - \phi_0)}| = |\sum_{j=1}^N |c_j|^2 e^{i(\phi_j - \phi_0)}|$, with the overall phase ϕ_0 confined in the range $(-\pi, \pi]$ for simplicity. Now we define the angular distance between the two phases ϕ_j and ϕ_0 as

$$d(\phi_j, \phi_0) = \begin{cases} |\phi_j - \phi_0| & \text{for } |\phi_j - \phi_0| \leqslant \pi \\ 2\pi - |\phi_j - \phi_0| & \text{for } |\phi_j - \phi_0| > \pi. \end{cases} \tag{5}$$

The angular distance $d(\phi_j, \phi_0)$ between the two phases ϕ_j and ϕ_0 is similar to the angular distance between two lines with polar angles ϕ_j and ϕ_0 in the polar coordinate system. For example, the absolute value of the difference between the two phases ϕ_1 and ϕ_2 is smaller than π in the polar coordinate system plotted in Fig. 1 and the angular distance between them is then $d(\phi_1, \phi_2) = |\phi_2 - \phi_1|$ (see the sector angle filled with red dots in Fig. 1). For the two phases ϕ_1 and ϕ_3, the absolute value of their difference is larger than π in Fig. 1, i.e., $|\phi_3 - \phi_1| > \pi$, so the angular distance between them is $d(\phi_1, \phi_3) = 2\pi - |\phi_3 - \phi_1|$ (see the sector angle filled with blue crosshatching in Fig. 1). The maximum angular distance among the phases $\{\phi_j\}$ with respect to the overall phase $\phi_0 \in (-\pi, \pi]$ is defined as $\mathcal{P}_{\phi_0} = \max_{\phi_j}[d(\phi_j, \phi_0)]$ hereafter, which plays an important role in the following discussion.

In the case of $\mathcal{P}_{\phi_0} \leqslant \frac{\pi}{2}$, the inequality $\cos(\phi_j - \phi_0) \geqslant \cos(\mathcal{P}_{\phi_0})$ is valid for all $j = 1, 2, \ldots, N$. Then the Uhlmann fidelity has a lower bound

$$\mathcal{F}(|\psi_0\rangle, |\psi_t\rangle) \geqslant \left|\sum_{j=1}^N |c_j|^2 \cos(\phi_j - \phi_0)\right| \geqslant \cos(\mathcal{P}_{\phi_0}). \tag{6}$$

The Uhlmann fidelity is used to quantify the closeness of two pure states, and its inverse-cosine-function value is regarded as the angular distance between the two quantum states, called the Bures angle [15]. Based on the Bures angle in Eq. (1), the above inequality is equivalent to

$$d_B(|\psi_0\rangle, |\psi_t\rangle) = \arccos \mathcal{F}(|\psi_0\rangle, |\psi_t\rangle) \leqslant \mathcal{P}_{\phi_0}, \tag{7}$$

which implies that the angular distance between the initial state $|\psi_0\rangle$ and its time-evolved state $|\psi_t\rangle$ is no larger than the maximum angular distance among the phases $\{\phi_j\} = \arg(U_t)$ with respect to an arbitrarily chosen overall phase ϕ_0. This conclusion is also valid in the case of $\mathcal{P}_{\phi_0} > \frac{\pi}{2}$, because the angular distance between two quantum states is inside the range $[0, \frac{\pi}{2}]$, i.e., $0 \leqslant d_B(|\psi_0\rangle, |\psi_t\rangle) \leqslant \frac{\pi}{2}$.

Since no other constraint is set for the overall phase ϕ_0 except for the range $-\pi < \phi_0 \leqslant \pi$, we could use the minimum value of $\mathcal{P}_{\phi_0}$ among all allowed overall phases ϕ_0 to define the path distance of the quantum evolution represented by the unitary operator U_t,

$$d(U_t) = \min_{\phi_0}\{\mathcal{P}_{\phi_0}\}. \tag{8}$$

Now we discuss how to find the target overall phase ϕ_0 to minimize $\mathcal{P}_{\phi_0} = \max_{\phi_j}[d(\phi_j, \phi_0)]$. As mentioned above, the phases $\{\phi_j\} = \arg(U_t)$ are the principal argument of the eigenvalues of the unitary operator U_t sorted in ascending order. We first work out the phase differences between neighboring phases among $\{\phi_j\}$, i.e., $D_j = \phi_{j+1} - \phi_j$ for $j = 1, 2, \ldots, N-1$, and the last one in this series of phase differences is calculated by $D_N = 2\pi + \phi_1 - \phi_N$. Suppose D_k is the largest one among $\{D_j\}$; then the two neighboring phases related to D_k, ϕ_k and ϕ_{k+1} (ϕ_N and ϕ_1 for D_N), are used to determine the target overall phase through the relation $\phi_0 = \frac{1}{2}(\phi_k + \phi_{k+1}) \pm \pi$. Here the alternative choice of $\pm$ is to make the obtained value of ϕ_0 inside the range $(-\pi, \pi]$. In this case, the overall phase ϕ_0 has the same angular distance to the two phases ϕ_k and ϕ_{k+1}, $d(\phi_k, \phi_0) = d(\phi_{k+1}, \phi_0) = \pi - \frac{1}{2}|\phi_{k+1} - \phi_k|$ or $d(\phi_N, \phi_0) = d(\phi_1, \phi_0) = \frac{1}{2}(\phi_N - \phi_1)$ for the largest phase difference of ϕ_N and ϕ_1, which is the largest one among the angular distances of the overall phase ϕ_0 to the series of phases $\{\phi_j\}$. At the same time, this angular distance is the smallest one among all allowed phases ϕ_0, because it is related to the two neighboring phases with the largest phase difference among $\{\phi_j\}$. Then the path distance of the quantum evolution represented by the unitary operator U_t in Eq. (8) is finally simplified as

$$d(U_t) = \pi - \tfrac{1}{2}\max\{D_j\}, \tag{9a}$$

with

$$D_j = \begin{cases} \phi_{j+1} - \phi_j & \text{for } j = 1, 2, \ldots, N-1 \\ 2\pi + \phi_1 - \phi_N & \text{for } j = N. \end{cases} \tag{9b}$$

Here $\{\phi_j\}$ are the principal arguments of the eigenvalues of U_t in ascending order and $\{D_j\}$ could be regarded as the angular difference between neighboring eigenvalues of U_t. The path distance defined in Eq. (9a) is one of the main conclusions in this paper. By combining Eqs. (7)–(9), we obtain the relation

$$d_B(|\psi_0\rangle, |\psi_t\rangle) \leqslant d(U_t). \tag{10}$$

This relation has a clear physical meaning, i.e., the distance between two pure states, quantified by the Bures angle, is a lower bound for the distance of the evolution path connecting the two states.

III. PROPERTIES OF PATH DISTANCE

The path distance defined above is determined through the principal argument of the eigenvalues of the unitary operator U_t only and is independent of any specific quantum state, which is much different from previous measures of distance between two quantum states. Because the eigenvalues of the unitary operator U_t keep invariant under a unitary transformation, the path distance defined here is a basis-free invariant. Furthermore, this path distance satisfies the basic requirements for a good measure of distance [18], which are

$$d(U) \geqslant 0, \tag{11a}$$

$$d(U) = 0 \quad \text{if and only if } U = e^{i\phi} I, \tag{11b}$$

$$d(U_1 U_2) \leqslant d(U_1) + d(U_2). \tag{11c}$$

The requirements (11a) and (11b) are satisfied obviously by the path distance defined above. In the following, we give a brief proof for the third requirement (11c), usually called the triangle relation.

We denote the principal arguments of the eigenvalues of the unitary operators U_1 and U_2 by $\{\phi_j^{(1)}\}$ and $\{\phi_j^{(2)}\}$, respectively, which are sorted in ascending order. By applying the distribution of $\{\phi_j^{(1)}\}$ and $\{\phi_j^{(2)}\}$ into Eq. (9), we could figure out the path distance contained in U_1 and U_2, denoted by $d(U_1)$ and $d(U_2)$, respectively. If $d(U_1) + d(U_2) > \pi$, then the triangle relation $d(U_1U_2) \leqslant d(U_1) + d(U_2)$ is satisfied automatically, because the path distance contained in an arbitrary unitary operator, including $d(U_1U_2)$, is no larger than π. More details about this conclusion are given in Sec. III B. So we only need to consider the case of $d(U_1) + d(U_2) \leqslant \pi$, which implies that $0 \leqslant d(U_1) \leqslant \pi$ and $0 \leqslant d(U_2) \leqslant \pi$.

Now we add an overall phase to the unitary operator U_1 (or U_2) to get $e^{i\varphi_1}U_1$ ($e^{i\varphi_2}U_2$) and they contain the equal path distance $d(e^{i\varphi_1}U_1) = d(U_1)$ [$d(e^{i\varphi_2}U_2) = d(U_2)$] no matter what value the added overall phase φ_1 (or φ_2) is. In the case of $0 \leqslant d(U_1) \leqslant \pi$ [$0 \leqslant d(U_2) \leqslant \pi$], we could always find at least one appropriate overall phase φ_1 (φ_2) so that the principal arguments of the eigenvalues of the unitary operators $e^{i\varphi_1}U_1$ (or $e^{i\varphi_2}U_2$) are confined in the range $\phi_j'^{(1)} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ ($\phi_j'^{(2)} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$). Then the distance contained in the unitary operator $e^{i\varphi_1}U_1$ ($e^{i\varphi_2}U_2$) has a simple formula, which is equal to half the difference between the maximum and minimum values among the principal arguments of $\{\phi_j'^{(1)}\}$ ($\{\phi_j'^{(2)}\}$), i.e., $d(e^{i\varphi_1}U_1) = \frac{1}{2}(\max\{\phi_j'^{(1)}\} - \min\{\phi_j'^{(1)}\})$ [$d(e^{i\varphi_2}U_2) = \frac{1}{2}(\max\{\phi_j'^{(2)}\} - \min\{\phi_j'^{(2)}\})$]. See Proposition 2 in Sec. III B for a brief interpretation of this result.

Based on the eigenvalue decomposition of the two unitary operators $e^{i\varphi_1}U_1 = V_1\Lambda_1V_1^\dagger$ and $e^{i\varphi_2}U_2 = V_2\Lambda_2V_2^\dagger$, we have $(e^{i\varphi_1}U_1)(e^{i\varphi_2}U_2) = V_1\Lambda_1V_1^\dagger V_2\Lambda_2V_2^\dagger$, which has the same eigenvalues as the matrix $\Lambda_1V_{12}\Lambda_2V_{12}^\dagger$ with $V_{12} = V_1^\dagger V_2$. The phases $\{\phi_j'^{(1)}\}$ ($\{\phi_j'^{(2)}\}$) mentioned above are just the principal arguments of the diagonal elements of the diagonal matrix Λ_1 (Λ_2). The unitary transformation $V_{12}\Lambda_2V_{12}^\dagger$ could

be regarded as the rotation of the diagonal matrix Λ_2, which does not change the eigenvalues. If this rotation makes the eigenvector $|\psi'^{(2)}_{\max}\rangle$ ($|\psi'^{(2)}_{\min}\rangle$) of the matrix $V_{12}\Lambda_2 V_{12}^\dagger$, which is associated with the maximum (minimum) principal argument of its eigenvalues, meet the eigenvector $|\psi^{(1)}_{\max}\rangle$ ($|\psi^{(1)}_{\min}\rangle$) of the matrix Λ_1 exactly, which is associated with the maximum (minimum) principal argument of the eigenvalues of Λ_1, the difference between the maximum and minimum values among the principal arguments of $\Lambda_1 V_{12}\Lambda_2 V_{12}^\dagger$, denoted by $\{\phi_j^{(12)}\}$, reaches its maximum value. Equivalently, the difference between the maximum and minimum values among the principal arguments of $(e^{i\varphi_1}U_1)(e^{i\varphi_2}U_2)$ satisfies

$$\begin{aligned}&\max\left\{\phi_j^{(12)}\right\} - \min\left\{\phi_j^{(12)}\right\}\\ &\leqslant \left(\max\left\{\phi_j'^{(1)}\right\} + \max\left\{\phi_j'^{(2)}\right\}\right) - \left(\min\left\{\phi_j'^{(1)}\right\}\right.\\ &\quad \left.+ \min\left\{\phi_j'^{(2)}\right\}\right)\\ &= 2[d(e^{i\varphi_1}U_1) + d(e^{i\varphi_2}U_2)] = 2[d(U_1) + d(U_2)]. \quad (12)\end{aligned}$$

According to Proposition 1 presented in Sec. III B, the path distance contained in the unitary operator $(e^{i\varphi_1}U_1)(e^{i\varphi_2}U_2)$ is bounded from above by half the difference between the maximum and minimum values, $\max\{\phi_j^{(12)}\}$ and $\min\{\phi_j^{(12)}\}$, among the principal arguments of $(e^{i\varphi_1}U_1)(e^{i\varphi_2}U_2)$. So we finally derive the above triangle relation

$$\begin{aligned}d(U_1U_2) &= d[(e^{i\varphi_1}U_1)(e^{i\varphi_2}U_2)]\\ &\leqslant \tfrac{1}{2}(\max\{\phi_j^{(12)}\} - \min\{\phi_j^{(12)}\}) \leqslant d(U_1) + d(U_2).\end{aligned} \quad (13)$$

In the following, we introduce some other properties of the path distance defined above.

A. Path distance bounded by the state distance from below

In an N-dimensional quantum unitary system, an initial state ρ_0 turns to its time-evolved state ρ_t through the action of a unitary operator U_t, $\rho_t = U_t\rho_0 U_t^\dagger$, which is why we believe the unitary operator U_t contains certain path information during the evolution, including the path distance $d(U_t)$. Given a Hamiltonian H for a quantum unitary evolution, the unitary operator $U_{t_{12}} = e^{-iH(t_2-t_1)}$ represents the evolution from time t_1 to t_2, which connects the two unitary operators $U_{t_1} = e^{-iHt_1}$ and $U_{t_2} = e^{-iHt_2}$ through the relation $U_{t_{12}} = U_{t_2}U_{t_1}^\dagger$. Just as we mentioned in the Introduction, the path distance $d(U_{t_{12}})$ is in fact the abbreviation of $d(U_{t_1}, U_{t_2})$, which could be regarded as the distance between the two points representing the two operators U_{t_1} and U_{t_2} in the space of unitary operators SU(N). The path distance between two points is no larger than the path length between the two points,

$$d(U_{t_{12}}) = d(U_{t_1}, U_{t_2}) \leqslant \int_{t_1}^{t_2}\left(\frac{ds}{dt}\right)dt, \quad (14)$$

where ds is the path distance contained in the operator $U_{dt} = e^{-iHdt}$, $ds = \lim_{dt\to 0} d(U_{dt}) = \lim_{dt\to 0} d(U_t, U_{t+dt})$. The derivative $\frac{ds}{dt}$ could be regarded as the evolution speed of the path distance, which depends on the Hamiltonian H only, $\frac{ds}{dt} = d(e^{-iH})$. The right-hand side of the above inequality is similar to the quantum complexity of a unitary operator, which is an operational definition for quantifying the cost of simulating this unitary operator [19–22]. The difference lies in that an optimization over all possible evolution paths is involved in the definition of quantum complexity and the path length defined on the right-hand side of the inequality (14) is based on a specific evolution path governed by the Hamiltonian H. The relation (14) is in fact a natural result of the triangle relation presented in Eq. (11). In Eq. (10) we showed that the path distance of a quantum evolution is bounded from below by the state distance measured by the Bures angle in the pure-state case. In the following, we will prove this conclusion is also valid for mixed states.

As shown in Eqs. (1) and (2) in the Introduction, the state distance of two mixed states ρ_0 and ρ_t is usually quantified by the Bures angle $d_B(\rho_0, \rho_t) = \arccos\mathcal{F}(\rho_0, \rho_t)$, where the Uhlmann fidelity between the two mixed states is actually equal to $\mathcal{F}(\rho_0, \rho_t) = \mathrm{Tr}|\sqrt{\rho_0}\sqrt{\rho_t}|$ [23], where $\mathrm{Tr}|\cdot|$ represents the absolute-value sum of the eigenvalues. In a unitary system, the initial state ρ_0 and the time-evolved state ρ_t have the same eigenvalue spectrum

$$\Lambda_0 = \begin{pmatrix}\nu_1 & 0 & \cdots & 0\\ 0 & \nu_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \nu_N\end{pmatrix}, \quad \sum_{j=1}^{N}\nu_j = 1. \quad (15)$$

Given the unitary operator U_t and the initial state $\rho_0 = M_0\Lambda_0 M_0^\dagger$, with $M_0 = (\psi_0^{(1)}, \psi_0^{(2)}, \ldots, \psi_0^{(N)})$ composed of its column eigenvectors, the time evolution of the quantum system is $\rho_t = U_t\rho_0 U_t^\dagger = U_t M_0\Lambda_0 M_0^\dagger U_t^\dagger$. The above Uhlmann fidelity is rewritten as

$$\mathcal{F}(\rho_0, \rho_t) = \mathrm{Tr}|M_0\sqrt{\Lambda_0}M_0^\dagger U_t M_0\sqrt{\Lambda_0}M_0^\dagger U_t^\dagger|. \quad (16)$$

Since a unitary operator would not change the absolute value of the eigenvalues of a matrix, we have

$$\begin{aligned}\mathcal{F}(\rho_0, \rho_t) &= \mathrm{Tr}|\sqrt{\Lambda_0}M_0^\dagger U_t M_0\sqrt{\Lambda_0}|\\ &\geqslant |\mathrm{Tr}(\sqrt{\Lambda_0}M_0^\dagger U_t M_0\sqrt{\Lambda_0})| = |\mathrm{Tr}(\Lambda_0 M_0^\dagger U_t M_0)|\\ &= \left|\sum_{j=1}^{N}\nu_j\langle\psi_0^{(j)}|\psi_t^{(j)}\rangle\right|. \quad (17)\end{aligned}$$

The lower bound of the Uhlmann fidelity is finally simplified as $\mathcal{F}(\rho_0, \rho_t) \geqslant |\sum_{j=1}^{N}\nu_j e^{i\phi_j}|$. Here $\{e^{i\phi_j}\}$ are the eigenvalues of the unitary operators U_t.

Following the same proof as in the previous section, where the Uhlmann fidelity between a pure state and its time-evolved state is expressed in Eq. (4), i.e., $\mathcal{F}(|\psi_0\rangle, |\psi_t\rangle) = |\sum_{j=1}^{N}|c_j|^2 e^{i\phi_j}|$, with $\sum_{j=1}^{N}|c_j|^2 = 1$, we could also obtain $\mathcal{F}(\rho_0, \rho_t) \geqslant \cos(\mathcal{P}_{\phi_0})$. Here $\mathcal{P}_{\phi_0}$ is the maximum angular distance, defined in Eq. (5), among the phases $\{\phi_j\}$ with respect to the overall phase ϕ_0. Based on the definition (8), we could obtain the path distance for the evolution of a quantum mixed state by finding the minimum value of $\mathcal{P}_{\phi_0}$ through the same proof. The path distance for the evolution of a quantum mixed state is also determined by the principal argument of the eigenvalues of the time-evolution unitary operator U_t, which has the same expression as the path distance for the evolution of a quantum pure state [see Eq. (9)]. Similar to the relation

between the state distance and path distance for the evolution of a pure state, presented in Eq. (10), we also have

$$d_B(\rho_0, \rho_t) \leqslant d(U_t). \tag{18}$$

So we conclude that the path distance contained in a unitary operator is no smaller than the state distance between two quantum states connected to each other through this unitary operator, regardless of whether the initial state is a pure state or a mixed state. This conclusion is valid for all possible evolution paths, given the initial state and target state.

As is well known, the Uhlmann fidelity between a quantum initial state and its time-evolved state in a unitary system has another lower bound, called the Mandelstam-Tamm bound [13], $\mathcal{F}(\rho_0, \rho_t) \geqslant \cos(\Delta E t)$. Here $\Delta E \equiv \sqrt{\langle H^2\rangle - \langle H\rangle^2}$ is the energy spread of the system, which remains constant for a quantum system undergoing a unitary evolution. The state distance measured by the Bures angle between the quantum initial state and its time-evolved state is accordingly bounded from above by

$$d_B(\rho_0, \rho_t) \leqslant \Delta E t. \tag{19}$$

In the following, we call $d_{\mathrm{MT}}(\rho_0, \rho_t) = \Delta E t$ the Mandelstam-Tamm bound of the state distance between an initial state and its time-evolved state. Here the evolution time t should be confined in the range $[0, \frac{\pi}{2\Delta E}]$; otherwise $\Delta E t > \frac{\pi}{2}$ becomes a trivial upper bound, because the Bures angle between two quantum states is always smaller than or equal to $\frac{\pi}{2}$. On the contrary, the path distance defined above depends only on the evolution operator U_t, where no constraint is imposed on the evolution time t, so the path distance and the related conclusions mentioned above are valid during the whole evolution.

B. Value range of the path distance

In order to figure out the value range of the path distance defined above, we give a brief interpretation of it. First, we use N radii in a circle to indicate N principal arguments $\{\phi_j\}$ of the eigenvalues of the time-evolution unitary operator U_t, where the jth radius is at an angle ϕ_j with respect to a fixed axis, e.g., the x axis. Second, we find the smallest sector in the circle to cover all N radii simultaneously. Then the path distance of the evolution represented by U_t is equal to half of the sector angle. We have two useful propositions related to this conclusion, which already have been used in the above proof of the triangle relation.

Proposition 1. The path distance contained in a unitary operator U is bounded from above by half the difference between the maximum and minimum values $\max\{\phi_j\}$ and $\min\{\phi_j\}$ among the principal arguments of U, i.e., $d(U) \leqslant \frac{1}{2}(\max\{\phi_j\} - \min\{\phi_j\})$. The equals sign holds when the difference D_N between the phases of $\min\{\phi_j\}$ and $\max\{\phi_j\}$, defined in Eq. (9b), is the largest one between neighboring phases $\{D_j\}$ among the principal arguments of U, where $\min\{\phi_j\}$ and $\max\{\phi_j\}$ are just the ϕ_1 and ϕ_N in Eq. (9b), respectively. Let us reconsider the example plotted in Fig. 1. Among the three phases $\{\phi_1, \phi_2, \phi_3\}$, with $\min\{\phi_j\} = \phi_1$ and $\max\{\phi_j\} = \phi_3$, we have $D_1 = \phi_2 - \phi_1$, $D_2 = \phi_3 - \phi_2$, and $D_3 = 2\pi + \phi_1 - \phi_3$. Here D_1 and D_3 are just the central angles of the two sectors filled with red dots and blue crosshatching, respectively. Based on Eq. (9a), the path distance contained in U satisfies

$$\begin{aligned} d(U) &= \pi - \tfrac{1}{2}\max\{D_1, D_2, D_3\} \leqslant \pi - \tfrac{1}{2}D_3 \\ &= \tfrac{1}{2}(\phi_3 - \phi_1) = \tfrac{1}{2}(\max\{\phi_j\} - \min\{\phi_j\}). \end{aligned} \tag{20}$$

Proposition 2. If all principal arguments of the eigenvalues of the unitary operators U are confined in the range $\phi_j \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the path distance contained in the unitary operators U is equal to $d(U) = \frac{1}{2}(\max\{\phi_j\} - \min\{\phi_j\})$, because the difference $D_N = 2\pi + \phi_1 - \phi_N$ between the phases of $\min\{\phi_j\} = \phi_1$ and $\max\{\phi_j\} = \phi_N$ is greater than π and thus becomes automatically the largest one among $\{D_j\}$ in this situation.

In a two-dimensional system, we need two radii in the circle to indicate the two principal arguments of the eigenvalues of the time-evolution unitary operator U_t. If the two radii form a diameter of the circle, the smallest sector required to cover the two radii is the semicircle and the path distance is $\frac{\pi}{2}$ in this case. In a general case of an N-dimensional system, N radii of the circle are required to indicate the N principal arguments of the eigenvalues of the unitary operator U_t. When the N radii are distributed uniformly in the circle, a sector with a central angle $\frac{2(N-1)\pi}{N}$ is enough to cover all these radii and the path distance is $\frac{(N-1)\pi}{N}$ in this case. This is the largest sector required to cover N radii in a circle, so we conclude that the value range of the path distance of a quantum evolution in an N-dimensional system is $[0, \frac{(N-1)\pi}{N}]$. Note that the angle between two vectors is in the range $[0, \pi)$.

IV. EXAMPLES AND DISCUSSION

Let us consider the simplest single-qubit example. In the particular basis composed of the eigenvectors of the time-independent Hamiltonian H, this Hamiltonian has a diagonal form $H = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, which leads to a unitary operator $U_t = \begin{pmatrix} e^{-iat} & 0 \\ 0 & e^{-ibt} \end{pmatrix}$ at the evolution time t. An initial state $\rho_0 = \begin{pmatrix} \frac{1}{2}+\mu & \eta^* \\ \eta & \frac{1}{2}-\mu \end{pmatrix}$ evolves to a state $\rho_t = \begin{pmatrix} \frac{1}{2}+\mu & \eta^* e^{i(b-a)t} \\ \eta e^{i(a-b)t} & \frac{1}{2}-\mu \end{pmatrix}$ at the evolution time t with $\rho_t = U_t \rho_0 U_t^\dagger$. Here we see that the time evolution of the quantum state depends on the difference between the two parameters a and b. To ensure the nonnegativity of the density matrix ρ_0, the two parameters μ and η should satisfy the conditions $\mu \in [-\frac{1}{2}, \frac{1}{2}]$ and $|\eta|^2 \leqslant \frac{1}{4} - \mu^2$. The two quantum states ρ_0 and ρ_t have the same energy spread $\Delta E = \sqrt{\frac{1}{4} - \mu^2}|a - b|$. According to the definition in Eq. (9), the path distance of the evolution represented by the above unitary operator U_t is $d(U_t) = \min[\frac{1}{2}A_t, \pi - \frac{1}{2}A_t]$, where $A_t = |a - b|t \bmod 2\pi$ is in the range $[0, 2\pi)$ and $|a - b|t$ is the phase difference between the eigenvalues of U_t. In Fig. 2 we plot the evolution of the state distance $d_B(\rho_0, \rho_t)$, the path distance $d(U_t)$, and the Mandelstam-Tamm bound $d_{\mathrm{MT}}(\rho_0, \rho_t)$ between the initial state ρ_0 and its time-evolved state ρ_t.

In the special case of $\mu = 0$ and $\eta = \frac{1}{2}$, the initial state ρ_0 and its time-evolved state ρ_t are in fact pure states $|\psi_0\rangle = \frac{\sqrt{2}}{2}(1\ 1)^T$ and $|\psi_t\rangle = \frac{\sqrt{2}}{2}(e^{-iat}\ \ e^{-ibt})^T$, respectively. The state distance d_B, path distance $d(U_t)$, and Mandelstam-Tamm bound d_{MT} between the initial state $|\psi_0\rangle$ and its

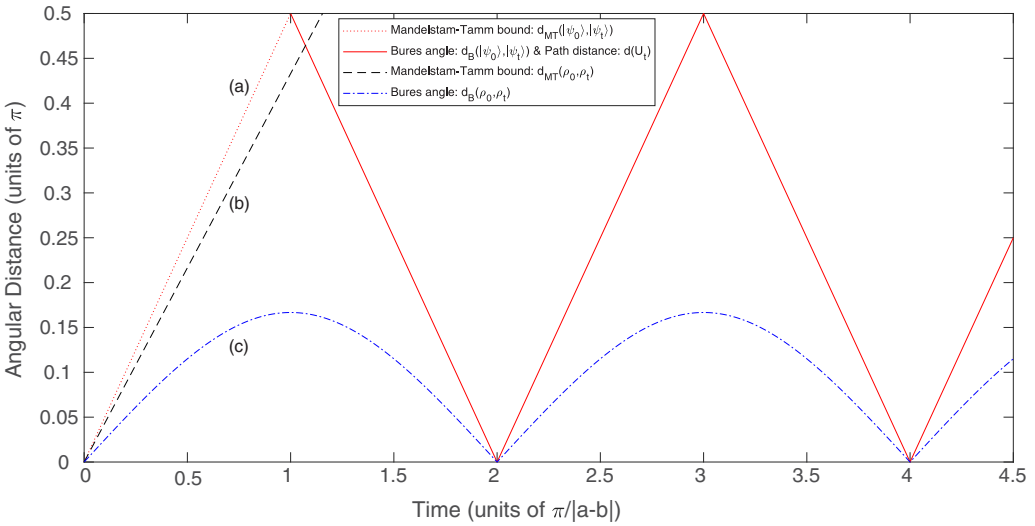

FIG. 2. Under the action of the Hamiltonian $H = \mathrm{diag}(a\, b)$, the single-qubit pure state $|\psi_0\rangle = \frac{\sqrt{2}}{2}(1\ 1)^T$ evolves to its time-evolved state $|\psi_t\rangle = \frac{\sqrt{2}}{2}(e^{-iat}\ e^{-ibt})^T$ at time t. The Mandelstam-Tamm bound $d_{\mathrm{MT}}(|\psi_0\rangle, |\psi_t\rangle)$ and the Bures angle $d_B(|\psi_0\rangle, |\psi_t\rangle)$ are both equal to the path distance $d(U_t)$ in the time period $t \in [0, \frac{\pi}{|a-b|}]$, with the evolution operator $U_t = \mathrm{diag}(e^{-iat}\ e^{-ibt})$ [see the dotted part of the red zigzag (a)]. In the time period $t > \frac{\pi}{|a-b|}$, the Mandelstam-Tamm bound $d_{\mathrm{MT}}(|\psi_0\rangle, |\psi_t\rangle)$ is invalid, but the Bures angle $d_B(|\psi_0\rangle, |\psi_t\rangle)$ and the path distance $d(U_t)$ are still equal to each other in this period, illustrated as the solid part of the red zigzag (a). For the evolution of the single-qubit mixed state ρ_0 governed by the same Hamiltonian H, the Bures angle between ρ_0 and its time-evolved state ρ_t, $d_B(\rho_0, \rho_t)$, is plotted as the blue dash-dotted line (c), which is below the red zigzag (a) of the path distance $d(U_t)$ during the whole evolution. In this particular case, the Bures angle $d_B(\rho_0, \rho_t)$ and the path distance $d(U_t)$ change synchronously with time and they reduce to zero periodically and simultaneously. The Mandelstam-Tamm bound between the initial state ρ_0 and its time-evolved state ρ_t, $d_{\mathrm{MT}}(\rho_0, \rho_t)$ [see the black dashed line (b)], is also an upper bound for the Bures angle $d_B(\rho_0, \rho_t)$ [blue dash-dotted line (c)], but the Mandelstam-Tamm bound $d_{\mathrm{MT}}(\rho_0, \rho_t)$ is valid only in the time period $[0, \frac{\pi}{2\Delta E}]$.

time-evolved state $|\psi_t\rangle$ could be obtained analytically, which are $d_B(|\psi_0\rangle, |\psi_t\rangle) = \arccos|\cos\frac{(a-b)t}{2}|$, $d(U_t) = \min[\frac{1}{2}A_t, \pi - \frac{1}{2}A_t]$, and $d_{\mathrm{MT}}(|\psi_0\rangle, |\psi_t\rangle) = \frac{1}{2}|a-b|t$, respectively. In the time period $t \in [0, \frac{\pi}{|a-b|}]$, the state distance d_B, path distance $d(U_t)$, and Mandelstam-Tamm bound d_{MT} are equal to each other [see the dotted part of the red zigzag (a) in Fig. 2, corresponding to the evolution from the initial pure state $|\psi_0\rangle = \frac{\sqrt{2}}{2}(1\ 1)^T$ to its orthogonal state $|\psi_\tau\rangle = \frac{\sqrt{2}}{2}(1\ -1)^T$]. It has been proven that such an evolution from a single-qubit pure state (or a higher-dimensional state equivalent to a single-qubit pure state) to its orthogonal state is the only case undergoing the fastest evolution at the quantum speed limit of the Mandelstam-Tamm bound [24]. In the time period $t > \frac{\pi}{|a-b|}$, the Bures angle d_B and path distance $d(U_t)$ oscillate over time, but still meet with each other during the evolution [see the solid part of the red zigzag (a)]. The Mandelstam-Tamm bound d_{MT} plays no role in the time period $t > \frac{\pi}{|a-b|}$, because it will present a trivial result of $d_{\mathrm{MT}}(|\psi_0\rangle, |\psi_t\rangle) > \frac{\pi}{2}$ in this situation.

In the mixed-state case of $|\eta|^2 < \frac{1}{4} - \mu^2$, the Bures angle $d_B(\rho_0, \rho_t)$ is bounded from above by both the path distance $d(U_t)$ and the Mandelstam-Tamm bound $d_{\mathrm{MT}}(\rho_0, \rho_t)$. As an example, we set the two parameters $\mu = \eta = \frac{1}{4}$; then the initial mixed state $\rho_0 = \frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ evolves to the mixed state $\rho_t = \frac{1}{4}\begin{pmatrix} 3 & e^{i(b-a)t} \\ e^{i(a-b)t} & 1 \end{pmatrix}$ at the evolution time t under the action by the same Hamiltonian used above, and the evolution operator U_t is the same as shown in the above pure-state case. The evolution of the Mandelstam-Tamm bound $d_{\mathrm{MT}}(\rho_0, \rho_t)$ and the Bures angle $d_B(\rho_0, \rho_t)$ between the initial mixed state ρ_0 and its time-evolved state ρ_t are numerically plotted as the black dashed line (b) and blue dash-dotted line (c) in Fig. 2, respectively. Here we see that the path distance $d(U_t)$ [red zigzag (a)] and the Mandelstam-Tamm bound $d_{\mathrm{MT}}(\rho_0, \rho_t)$ [black dashed line (b)] is larger than the Bures angle $d_B(\rho_0, \rho_t)$ [blue dash-dotted curve (c)] during the evolution. However, the Mandelstam-Tamm bound is valid only in the period $t \leqslant \frac{\pi}{2\Delta E}$, with the energy spread $\Delta E = \frac{\sqrt{3}}{4}|a-b|$ in the current mixed-state case. The path distance $d(U_t)$ is an upper bound of the Bures angle $d_B(\rho_0, \rho_t)$ during the whole evolution, no matter what evolution time is considered.

The pure state $|\psi_0\rangle$ and the mixed state ρ_0 are governed by the same Hamiltonian in the current case and their evolution is described by the same unitary operator U_t. Two different quantum states under the same evolution operator have the same path distance to their time-evolved states, but their fidelity or Bures angle with their own time evolution may be different.

We finally introduce how to construct a time-independent Hamiltonian to realize a unitary operator on demand, where the quantum initial state would turn into a target state at a fixed evolution speed. In a quantum unitary system, the initial state and its time-evolved state share the same eigenvalue spectrum; then the eigenvalue decompositions of the initial state and the target state could be expressed as $\rho_0 = M_0 \Lambda_0 M_0^\dagger$ and $\rho_\tau = M_\tau \Lambda_0 M_\tau^\dagger$, respectively. The unitary operators M_0

and M_τ are formed of the eigenvectors of ρ_0 and ρ_τ, and each eigenvector has a deterministic form up to an overall phase. The diagonal matrix Λ_0 with the form in Eq. (15) is composed of the eigenvalues $\{\nu_j\}$ of ρ_0 or ρ_τ. For simplicity, here we consider only the nondegenerate case with $\nu_j \neq \nu_k$ for $j \neq k$. Then the evolution operator U_τ connecting ρ_0 and ρ_τ could be expressed as $U_\tau = M_\tau M_0^\dagger$ so that the relation $\rho_\tau = U_\tau \rho_0 U_\tau^\dagger$ is satisfied.

We further make an eigenvalue decomposition of the unitary operator U_τ, that is, $M_\tau M_0^\dagger = V \Lambda V^\dagger$, where the diagonal matrix Λ has the form of Eq. (3) with the principal argument $\{\phi_j\}$ of the eigenvalues of the unitary operator U_τ being confined in the range $(-\pi, \pi]$. Then the Hamiltonian could be expressed as

$$H = \omega V[i \ln(\Lambda)]V^\dagger + R = i\omega \ln(U_\tau) + R = i\omega \ln(M_\tau M_0^\dagger) + R, \tag{21}$$

where the real parameter ω could be used to control the energy spread of the system so that it could be confined in a restricted range. The real number R, which only induces an overall phase to the time evolution and has no effect on any observable, could be used to control the average energy of the system. For example, we can always assume that the Hamiltonian is represented by a traceless operator or has a vanishing expectation value, and so on.

Based on the relation between the evolution operator and the time-independent Hamiltonian $U_t = e^{-iHt}$, we could obtain the evolution operator induced by the above Hamiltonian,

$$U_t = e^{-iRt} V(\Lambda^{\omega t}) V^\dagger. \tag{22}$$

The target state is achieved when the evolution time is equal to $\tau = 1/\omega$ and the corresponding evolution operator is $U_\tau = e^{-iR/\omega} V \Lambda V^\dagger = e^{-iR/\omega} M_\tau M_0^\dagger$. With the above Hamiltonian, the parameter ω is the only factor affecting the evolution time required to achieve a target state. The term $e^{-iR/\omega}$ introduces an overall phase to the target state. In the time period $t \in [0, 1/\omega]$, the path distance contained in the evolution operator (22) is $d(U_t) = d(\Lambda^{\omega t}) = t d(e^{-iH})$. Here we see that the path distance increases at a fixed speed $\frac{d}{dt} d(U_t) = d(e^{-iH})$. Equivalently, the initial state ρ_0 approaches the target state ρ_τ linearly with time under the action of the above Hamiltonian.

Let us look at a specific example of a two-qubit entangler gate

$$U_\tau = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \tag{23}$$

which turns the four tensor-product two-qubit states $|\alpha\rangle = |00\rangle$, $|\beta\rangle = |01\rangle$, $|\gamma\rangle = |10\rangle$, and $|\eta\rangle = |11\rangle$ into the four maximally entangled Bell states $|\phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$, respectively. Accordingly, the mixture of the four states $\rho_0 = \lambda_1|\alpha\rangle\langle\alpha| + \lambda_2|\beta\rangle\langle\beta| + \lambda_3|\gamma\rangle\langle\gamma| + \lambda_4|\eta\rangle\langle\eta|$ would turn into the mixture $\rho_\tau = \lambda_1|\phi^+\rangle\langle\phi^+| + \lambda_2|\phi^-\rangle\langle\phi^-| + \lambda_3|\psi^+\rangle\langle\psi^+| + \lambda_4|\psi^-\rangle\langle\psi^-|$ under the action of the above entangler gate U_τ. By substituting U_τ into the Hamiltonian in Eq. (21), we obtain the time-independent Hamiltonian to realize it,

$$H = \omega' \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} + R'. \tag{24}$$

The unitary operator induced by the above Hamiltonian could be described as

$$U_t = \begin{pmatrix} c_t - \frac{i}{\sqrt{2}} s_t & 0 & 0 & -\frac{i}{\sqrt{2}} s_t \\ 0 & c_t - \frac{i}{\sqrt{2}} s_t & -\frac{i}{\sqrt{2}} s_t & 0 \\ 0 & -\frac{i}{\sqrt{2}} s_t & c_t + \frac{i}{\sqrt{2}} s_t & 0 \\ -\frac{i}{\sqrt{2}} s_t & 0 & 0 & c_t + \frac{i}{\sqrt{2}} s_t \end{pmatrix}, \tag{25}$$

with $c_t = \cos(\sqrt{2}\omega' t)$ and $s_t = \sin(\sqrt{2}\omega' t)$. The path distance contained in the above time-evolution operator U_t is $d(U_t) = \sqrt{2}\omega' t$ in the period $t \in [0, \frac{\pi}{2\sqrt{2}\omega'}]$, which increases linearly with time t, until the target operator in Eq. (23) is achieved at time $t = \frac{\pi}{2\sqrt{2}\omega'}$.

V. CONCLUSION

To summarize, the Hamiltonian in a quantum unitary system completely determines a unitary operator, which contains the main evolution information of the system. Since a unitary operator connecting two quantum states could be used to quantify the evolution distance between the two states, we proposed here a geometric measure for the path distance contained in a quantum unitary evolution, which relies only on the unitary operator itself and is independent of the quantum state. The path distance defined here meets the basic requirements for a good measure of distance and is valid during the whole evolution process. This path distance between two quantum states is bounded from below by the state distance of the Bures angle between them, so the path distance provides a feasible way for the estimation of fidelity between a quantum state and its time-evolved state, no matter what evolution time is under consideration.

The path distance defined here is to map a unitary operator to a non-negative value, which has some potential applications in quantum information processing. For example, the quantification of evolution speed is usually based on the state distance between the initial state and its time-evolved state in the issue of quantum optimal control [25–30], and it will be very interesting to study the quantum optimal control from the perspective of path distance. By using the path distance introduced here to replace the state distance, we could also investigate several other issues in quantum information science, such as the generalization of a quantum speed limit [31–35] and estimation of the evolution time required to prepare a target state. Quantum complexity is also a map from unitary operators to non-negative values, which provides an operational definition for quantifying the cost of simulating a unitary operator. As is well known, quantum complexity is widely used in several topics such as distinguishing chaotic systems from integrable ones [36], studying aspects of black holes and quantum chaos [37], and quantifying state

complexity in continuous many-body quantum systems [38]. Since the path distance introduced here is easy to compute, it will be very interesting to apply the path distance introduced here to the relevant research.

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