Group and energy velocities of electromagnetic waves in bianisotropic superlattices

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This paper proves the equality of the group velocity and the energy velocity of electromagnetic Bloch waves in bianisotropic nonabsorbing periodic superlattices of generic crystallographic symmetry.

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I. INTRODUCTION

The relation between the group velocity and energy velocity of electromagnetic waves is of fundamental interest and has been attracted researcher's attention for a long time [1]. The proofs of the fact that these two velocities of bulk waves coincide in anisotropic nonabsorbing homogeneous condensed media are given in [2–6]. The group and energy velocities of electromagnetic waves are also equal in plasma [7–9]. The equality holds true for waveguide modes [10]. In [11] it was shown that both velocities are equal for plasmons on half-infinite metals whose electromagnetic properties are described by a scalar local dielectric function. The group and energy velocities of plasmons still coincide if nonlocal effect are taken into account [12]. In [13–15] it was established that the group and energy velocities are equal in three-dimensional photonic crystals. The photonic crystal was described by a continuously periodic dielectric permittivity, which was assumed to be purely real and frequency dependent, whereas the magnetic permeability was considered to be a frequency-independent scalar constant. The energy velocity was defined as the ratio of the Umov-Poynting vector and energy averaged over the unit cell and time. Note that the correctness of the definition of the group and energy velocities in periodic structures was analyzed in [16].

One of the popular trends in modern optics is the investigation of electromagnetic waves in bianisotropic media [17–21]. In such media the electric displacement and magnetic induction depend on the strength of both the electric and magnetic fields. This cross dependence is attributed to the magnetoelectric effect and natural optical activity [22,23]. Significant progress has been made in understanding the effect of bianisotropic coupling on bulk wave propagation in homogeneous media [24–29] and surface wave propagation in half-infinite structures [30–35]. Much attention was paid to other manifestations of bianisotropy such as negative refraction [36–39], optical activity and circular dichroism [40–43], finding effective constants [22,44–46], and nonreciprocal propagation [47,48]. The equality of the group and energy velocities of bulk waves in homogeneous bianisotropic media was proved in [49,50].

In this paper the equality of the group velocity and energy velocity is proved for Bloch waves which freely propagate in bianisotropic and/or magneto-optically active periodic superlattices of generic crystallographic symmetry. It is assumed that there are no losses in the superlattice. We use a matrix representation of the Maxwell equations for the tangential components of the plane-wave electromagnetic field but do not calculate explicitly these components. In addition, the frequency dependence of the material constants is allowed for. In [51,52] the equality of the group and energy velocities was proved for nonbianisotropic magneto-optically inactive superlattices whose electromagnetic properties are fully characterized by purely real symmetric tensors of dielectric permittivity and magnetic permeability. The methods used in [51,52] differ significantly from the method used in the present work, namely, explicit calculations [51] and variational method [52]. In both [51,52] the frequency dispersion was ignored.

Our paper is organized as follows. Section II contains a number of general relations. In Sec. III an explicit expression of a matrix used subsequently is given. Section IV proves the equality of the group and energy velocities. Section V summarizes the results obtained.

II. MATRIX FORM OF MAXWELL'S EQUATIONS FOR PLANE WAVES

We assume that a superlattice is periodic along the Z axis and consider an electromagnetic Bloch wave

$$\begin{pmatrix}\mathbf{E}(\mathbf{r},t)\\ \mathbf{H}(\mathbf{r},t)\end{pmatrix}=\begin{pmatrix}\mathbf{E}(z)\\ \mathbf{H}(z)\end{pmatrix}e^{i(k_xx+k_yy-\omega t)}, \tag{1}$$

where k_x and k_y are the tangential components of the wave vector, ω is the frequency, the vector functions $\mathbf{E}(z)$ and $\mathbf{H}(z)$ describe the z dependence of the electric and magnetic fields, respectively, and $\mathbf{r}$ is the radius vector (Fig. 1). The tangential components $E_{x,y}$ and $H_{x,y}$ of $\mathbf{E}(z)$ and $\mathbf{H}(z)$ can be found by solving the system of equations

$$\frac{1}{i}\frac{d\boldsymbol{\xi}}{dz}=\hat{\mathbf{N}}\boldsymbol{\xi}, \tag{2}$$

where $\boldsymbol{\xi}$ is a four-component vector column constructed of $E_{x,y}$ and $H_{x,y}$ and $\hat{\mathbf{N}}$ is a 4×4 matrix which depends on ω, $k_{x,y}$, and material constants. The expression of $\hat{\mathbf{N}}$ depends on the order of $E_{x,y}$ and $H_{x,y}$ in $\boldsymbol{\xi}$ [53–57]. Following our

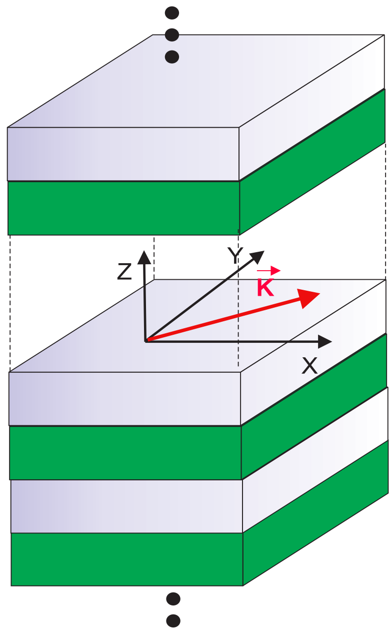

FIG. 1. Infinite periodic superlattice formed by two alternating layers. The layers filling the space between the bottom and top connected by dashed lines are not shown for convenience. The Z axis of the coordinate system XYZ is the stratification direction. The vector $\mathbf{K} = (k_x, k_y)$ indicates the direction of propagation along the layers, where k_x and k_y are the wave numbers of the Bloch wave (1). The motion of the wave along the Z axis is characterized by Bloch wave number k (see Sec. IV).

works [34,35,58,59], we set

$$\boldsymbol{\xi}(z) = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -E_y \\ H_y \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} H_x \\ E_x \end{pmatrix}. \tag{3}$$

In superlattices composed of discrete layers, $\hat{\mathbf{N}}(z) = \hat{\mathbf{N}}_j$ for $z_j < z < z_{j+1}$, where $\hat{\mathbf{N}}_j$ is the $\hat{\mathbf{N}}$ matrix of the layer occupying the space $z_j < z < z_{j+1}$. The tangential components of the electromagnetic field are supposed to be continuous at all the interlayer boundaries, so $\boldsymbol{\xi}(z)$ will be a continuous function of z over the entire structure.

The components E_z and H_z as well as the electric displacement $\mathbf{D}$ and the magnetic induction $\mathbf{B}$ may be calculated using the constitutive relations. These relations for a bianisotropic nonabsorbing medium are given, e.g., in [20–23,33,44,48] and we write them in the form

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} = \hat{\boldsymbol{\Gamma}} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad \hat{\boldsymbol{\Gamma}} = \begin{pmatrix} \hat{\boldsymbol{\varepsilon}} & \hat{\boldsymbol{\kappa}} \\ \hat{\boldsymbol{\kappa}}^{\dagger} & \hat{\boldsymbol{\mu}} \end{pmatrix}, \tag{4}$$

where the tensors of dielectric permittivity and magnetic permeability $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\boldsymbol{\mu}}$, respectively, are assumed to be complex Hermitian in order to allow for the magneto-optical activity, $\hat{\boldsymbol{\kappa}}$ is a complex nonsymmetric pseudotensor describing the bianisotropic coupling, and $\mathrm{Re}(\hat{\boldsymbol{\kappa}})$ and $\mathrm{Im}(\hat{\boldsymbol{\kappa}})$ characterize the contribution of the magnetoelectric effect and natural optical activity, respectively. The dagger stands for Hermitian conjugation.

III. EXPLICIT EXPRESSION OF MATRIX $\hat{\mathbb{N}}$

When proving the equality of the group and energy velocities, we will use an explicit expression of the matrix

$$\hat{\mathbb{N}} = \hat{\mathbf{T}}\hat{\mathbf{N}}, \tag{5}$$

where $\hat{\mathbf{T}}$ is a 4×4 matrix

$$\hat{\mathbf{T}} = \begin{pmatrix} \hat{\mathbf{0}} & \hat{\mathbf{I}} \\ \hat{\mathbf{I}} & \hat{\mathbf{0}} \end{pmatrix}, \tag{6}$$

where $\hat{\mathbf{0}}$ and $\hat{\mathbf{I}}$ are zero and identity 2×2 matrices. In particular, we need to know the explicit dependence of $\hat{\mathbb{N}}$ on k_x and k_y. In our paper [34] an expression of $\hat{\mathbb{N}}$ was derived assuming $k_y = 0$. For $k_y \neq 0$ the matrix $\hat{\mathbb{N}}$ is found similarly. We insert (1) in the Maxwell equations and write the resulting six equations as

$$\frac{1}{i}\frac{d\boldsymbol{\xi}}{dz} = \hat{\mathbf{T}}(\omega\boldsymbol{\psi} + \hat{\mathbf{J}}\boldsymbol{\phi}), \tag{7}$$

$$-\hat{\mathbf{J}}^t\boldsymbol{\xi} = \omega\boldsymbol{\nu}, \tag{8}$$

where $\boldsymbol{\psi} = (-D_y\ B_y\ B_x\ D_x)^t$, the symbol ^{t} denotes transposition,

$$\hat{\mathbf{J}} = k_x\hat{\mathbf{J}}_x + k_y\hat{\mathbf{J}}_y, \tag{9}$$

and $\hat{\mathbf{J}}_x$ and $\hat{\mathbf{J}}_y$ are 4×2 matrices

$$\hat{\mathbf{J}}_x = \begin{pmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{0}} \end{pmatrix}, \quad \hat{\mathbf{J}}_y = \begin{pmatrix} \hat{\mathbf{0}} \\ \hat{\mathbf{K}} \end{pmatrix}, \tag{10}$$

$$\hat{\mathbf{K}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\phi} = \begin{pmatrix} H_z \\ E_z \end{pmatrix}, \quad \boldsymbol{\nu} = \begin{pmatrix} B_z \\ D_z \end{pmatrix}. \tag{11}$$

Next, by multiplying both sides of (4) by the relevant permutation matrix $\hat{\boldsymbol{\Delta}}$, we transform (4) to

$$\begin{pmatrix} \boldsymbol{\psi} \\ \boldsymbol{\nu} \end{pmatrix} = \hat{\boldsymbol{\Omega}} \begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\phi} \end{pmatrix}, \tag{12}$$

where

$$\hat{\boldsymbol{\Omega}} = \hat{\boldsymbol{\Delta}}\hat{\boldsymbol{\Gamma}}\hat{\boldsymbol{\Delta}}^{-1} \equiv \begin{pmatrix} \hat{\boldsymbol{\Omega}}_1 & \hat{\boldsymbol{\Omega}}_2 \\ \hat{\boldsymbol{\Omega}}_2^{\dagger} & \hat{\boldsymbol{\Omega}}_4 \end{pmatrix} = \hat{\boldsymbol{\Omega}}^{\dagger}, \tag{13}$$

$\hat{\boldsymbol{\Omega}}_1$ and $\hat{\boldsymbol{\Omega}}_4$ are the upper 4×4 and lower 2×2 diagonal blocks of $\hat{\boldsymbol{\Omega}}$, respectively, and $\hat{\boldsymbol{\Omega}}_2$ is a 4×2 matrix with elements $(\hat{\boldsymbol{\Omega}}_2)_{ij} = (\hat{\boldsymbol{\Omega}})_{i,j+4}$, $i = 1, \ldots, 4$, $j = 1, 2$. The equality $\hat{\boldsymbol{\Omega}} = \hat{\boldsymbol{\Omega}}^{\dagger}$ follows from $\hat{\boldsymbol{\Gamma}} = \hat{\boldsymbol{\Gamma}}^{\dagger}$ and $\hat{\boldsymbol{\Delta}}^{-1} = \hat{\boldsymbol{\Delta}}^t$, since $\hat{\boldsymbol{\Delta}}$ is a purely real orthogonal matrix.

Inserting (8) in (12) yields

$$\boldsymbol{\phi} = -\hat{\boldsymbol{\Omega}}_4^{-1}(\hat{\boldsymbol{\Omega}}_2^{\dagger} + \omega^{-1}\hat{\mathbf{J}}^t)\boldsymbol{\xi}. \tag{14}$$

We replace $\boldsymbol{\phi}$ by (14) in (12), express $\boldsymbol{\psi}$ in terms of $\boldsymbol{\xi}$, and insert the obtained expressions of $\boldsymbol{\psi}$ and $\boldsymbol{\phi}$ in (7). As a result, we obtain (2), where $\hat{\mathbf{N}} = \hat{\mathbf{T}}\hat{\mathbb{N}}$ and

$$\hat{\mathbb{N}} = \omega\hat{\mathbf{A}} - \hat{\mathbf{B}} - \omega^{-1}\hat{\mathbf{C}}, \tag{15}$$

$$\hat{\mathbf{A}} = \hat{\boldsymbol{\Omega}}_1 - \hat{\boldsymbol{\Omega}}_2\hat{\boldsymbol{\Omega}}_4^{-1}\hat{\boldsymbol{\Omega}}_2^{\dagger}, \tag{16}$$

$$\hat{\mathbf{B}} = \hat{\boldsymbol{\Omega}}_2\hat{\boldsymbol{\Omega}}_4^{-1}\hat{\mathbf{J}}^t + \hat{\mathbf{J}}\hat{\boldsymbol{\Omega}}_4^{-1}\hat{\boldsymbol{\Omega}}_2^{\dagger}, \quad \hat{\mathbf{C}} = \hat{\mathbf{J}}\hat{\boldsymbol{\Omega}}_4^{-1}\hat{\mathbf{J}}^t. \tag{17}$$

These expressions also hold true when material constants depend on z.

IV. ENERGY AND GROUP VELOCITIES

The components $V_{E,a}$, $a = x, y, z$, of the energy velocity $\mathbf{V}_E$ in superlattices are the ratios of the corresponding components P_a of the Umov-Poynting vector $\mathbf{P}$ averaged over the period to the energy density W averaged over the period l of the superlattice,

$$V_{E,a} = \frac{1}{l}\int_0^l P_a(z)dz \Big/ \frac{1}{l}\int_0^l W(z)dz = \frac{\overline{P}_a}{\overline{W}}. \quad (18)$$

Both P_a and W are also averaged over time.

The local energy density $W(z)$ in bianisotropic media with frequency dispersion may be expressed in terms of the derivative $\partial(\omega\hat{\mathbf{\Gamma}})/\partial\omega$ [49,50] or $\partial(\omega\hat{\mathbf{\Omega}})/\partial\omega$,

$$W(z) = \frac{1}{4}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}^\dagger \frac{\partial(\omega\hat{\mathbf{\Gamma}})}{\partial\omega}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \frac{1}{4}\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\phi}\end{pmatrix}^\dagger \frac{\partial(\omega\hat{\mathbf{\Omega}})}{\partial\omega}\begin{pmatrix}\boldsymbol{\xi}\\ \boldsymbol{\phi}\end{pmatrix}. \quad (19)$$

Excluding $\boldsymbol{\phi}$ from (19) with the help of (14) and taking into account (15)–(17), we find that (19) is reduced after proper grouping of terms to

$$W(z) = \frac{1}{4}\boldsymbol{\xi}^\dagger \frac{\partial\hat{\mathbf{N}}}{\partial\omega}\boldsymbol{\xi}. \quad (20)$$

The frequency derivative is taken with k_x and k_y constant. Matrices (16) and (17), which implicitly depend on ω through material constants, are to be differentiated. The contraction in (20) is purely real because $\hat{\mathbf{N}}$ is a Hermitian matrix. This ensues from the fact that the matrix $\hat{\mathbf{\Omega}}$ (13) is Hermitian and hence so are the matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$, and $\hat{\mathbf{C}}$. [The expression (20) was derived in our paper [34] for $k_y = 0$.]

By inserting the vector $\boldsymbol{\phi}$ (11) and

$$\begin{pmatrix}-E_y\\ H_y\end{pmatrix} = \hat{\mathbf{J}}_x\boldsymbol{\xi}, \quad \begin{pmatrix}-E_x\\ H_x\end{pmatrix} = \hat{\mathbf{J}}_y\boldsymbol{\xi} \quad (21)$$

in $P_x = \mathrm{Re}(E_yH_z^* - H_yE_z^*)/2$ and $P_y = \mathrm{Re}(E_zH_x^* - H_zE_x^*)/2$, we obtain

$$P_a = -\tfrac{1}{4}(\boldsymbol{\xi}^\dagger\hat{\mathbf{J}}_a\boldsymbol{\phi} + \boldsymbol{\phi}^\dagger\hat{\mathbf{J}}_a^t\boldsymbol{\xi}), \quad a = x, y. \quad (22)$$

The substitution of (14) for $\boldsymbol{\phi}$ in (22) yields

$$P_a = \tfrac{1}{4}\boldsymbol{\xi}^\dagger[\hat{\mathbf{J}}_a\hat{\mathbf{\Omega}}_4^{-1}(\hat{\mathbf{\Omega}}_2^\dagger + \omega^{-1}\hat{\mathbf{J}}^t) + (\hat{\mathbf{\Omega}}_2 + \omega^{-1}\hat{\mathbf{J}})\hat{\mathbf{\Omega}}_4^{-1}\hat{\mathbf{J}}_a^t]\boldsymbol{\xi} \quad (23)$$

and, taking into account expressions (9) and (15)–(17), it may be noticed that

$$P_a = -\frac{1}{4}\boldsymbol{\xi}^\dagger\frac{\partial\hat{\mathbf{N}}}{\partial k_a}\boldsymbol{\xi}, \quad a = x, y. \quad (24)$$

The lossless condition $\mathrm{div}\mathbf{P} = 0$ reduces to $dP_z/dz = 0$ because $P_{x,y}$ do not depend on x and y. Therefore, $P_z = \mathrm{const}$, so by writing $P_z = \mathrm{Re}(E_xH_y^* - H_xE_y^*)/2$ in terms of $\boldsymbol{\xi}$, we find that for an arbitrary $z = \mathrm{const}$,

$$P_z = \tfrac{1}{4}\boldsymbol{\xi}^\dagger\hat{\mathbf{T}}\boldsymbol{\xi}|_z = \tfrac{1}{4}\boldsymbol{\xi}^\dagger\hat{\mathbf{T}}\boldsymbol{\xi}|_{z=0}. \quad (25)$$

We consider that wave (1) propagates freely through a periodic superlattice. Apart from k_x, k_y, and ω, such a wave is characterized by a purely real Bloch wave number $k = \theta/l$, where θ is the phase of an eigenvalue $\gamma = e^{i\theta}$ of the 4×4 transfer matrix of unit cell $\hat{\mathbf{M}}$. The matrix $\hat{\mathbf{M}}$ relates the vectors $\boldsymbol{\xi}(0)$ and $\boldsymbol{\xi}(l)$ pertaining to the boundaries $z = 0$ and $z = l$ of the period (unit cell) $\boldsymbol{\xi}(l) = \hat{\mathbf{M}}\boldsymbol{\xi}(0)$ and is calculated taking into account the continuity of $\boldsymbol{\xi}(z)$ over the entire superlattice [34,51–53,56–58].

Note that in infinite superlattices one can choose as the period boundaries a sequence of planes $z = z_n = \mathrm{const}$, where $z_n = z_0 + nl$, z_0 is an arbitrary point of the Z axis, and the n are integers. The matrix $\hat{\mathbf{M}}$ and its eigenvectors change with the position of period boundaries, whereas the eigenvalues and hence the Bloch vectors do not depend on it. It may be checked that $\overline{\mathbf{P}}$ and $\overline{W}$ also do not depend on the choice of the period boundaries.

In the intervals $z_n < z < z_{n+1}$ the amplitude and phase of a Bloch wave change in a complex way. For example, if the superlattice is built of homogeneous layers then, in the general case, $\boldsymbol{\xi}(z)$ in the jth layer is a linear combination of the four partial solutions $\boldsymbol{\xi}_\alpha^{(j)}e^{ip_\alpha^{(j)}z}$ of Eq. (2), where $\boldsymbol{\xi}_\alpha^{(j)}$ and $p_\alpha^{(j)}$, $\alpha = 1, \ldots, 4$, are the eigenvectors and eigenvalues of the matrix $\hat{\mathbf{N}}_j$. However, setting $\boldsymbol{\xi}(z_0) = \boldsymbol{\zeta}$, where $\boldsymbol{\zeta}$ is the eigenvector of $\hat{\mathbf{M}}$ corresponding to the eigenvalue $\gamma = e^{ikl}$, we see that then $\boldsymbol{\xi}(z_n) = \boldsymbol{\zeta}e^{iknl}$, i.e., over the n periods the wave undergoes only a phase shift by $\theta_n = knl$. Therefore, the motion of the Bloch wave along the sequence of points z_n may be viewed as an analog of the propagation along the Z axis of a monochromatic wave in a homogeneous medium. Accordingly, the components $V_{g,a}$, $a = x, y, z$, of the group velocity $\mathbf{V}_g$ are defined as $V_{g,a} = \partial\omega/\partial k_a$, where k_z is the Bloch wave number k [51,52].

To express $V_{g,a}$ in terms of the mean energy flux and energy, we take advantage of the equality

$$-i\frac{d}{dz}\left(\boldsymbol{\xi}^\dagger\hat{\mathbf{T}}\frac{d\boldsymbol{\xi}}{dg}\right) = \boldsymbol{\xi}^\dagger\frac{d\hat{\mathbf{N}}}{dg}\boldsymbol{\xi}, \quad (26)$$

where d/dg stands for a differential operator. The right-hand side of this equality is obtained by inserting the derivatives $d\boldsymbol{\xi}^\dagger/dz$ and $d^2\boldsymbol{\xi}/dzdg$ found via (2) on the left-hand side and using the identity $\hat{\mathbf{N}} = \hat{\mathbf{N}}^\dagger$.

Below it is assumed that the period is the interval $0 \leqslant z \leqslant l$. With the Bloch wave number k constant, the integration of the left-hand side of (26) over the period yields

$$I = -i\left(\boldsymbol{\xi}^\dagger\hat{\mathbf{T}}\frac{d\boldsymbol{\xi}}{dg}\Big|_{z=l} - \boldsymbol{\zeta}^\dagger\hat{\mathbf{T}}\frac{d\boldsymbol{\zeta}}{dg}\right) = 0, \quad (27)$$

since $\boldsymbol{\xi}(l) = \gamma\boldsymbol{\zeta}$ and $d\gamma/dg = 0$, so in this case

$$\int_0^l \boldsymbol{\xi}^\dagger\frac{d\hat{\mathbf{N}}}{dg}\boldsymbol{\xi}\,dz = 0. \quad (28)$$

In layered superlattices an integral over the period is the sum of the integrals over the thickness of each layer within the period. The continuity of $\boldsymbol{\xi}$ at the interlayer interfaces is also taken into account.

We fix k and k_y and insert $\frac{d}{dg} = \frac{\partial}{\partial\omega} + \frac{\partial k_x}{\partial\omega}\frac{\partial}{\partial k_x}$ in (28). Afterward we set $\frac{d}{dg} = \frac{\partial}{\partial\omega} + \frac{\partial k_y}{\partial\omega}\frac{\partial}{\partial k_y}$ with k and k_x constant. This yields

$$V_{g,a} = -\int_0^l \boldsymbol{\xi}^\dagger\frac{\partial\hat{\mathbf{N}}}{\partial k_a}\boldsymbol{\xi}\,dz \Big/ \int_0^l \boldsymbol{\xi}^\dagger\frac{\partial\hat{\mathbf{N}}}{\partial\omega}\boldsymbol{\xi}\,dz, \quad a = x, y. \quad (29)$$

Finally, we fix k_x and k_y and set $\frac{d}{dg} = \frac{\partial}{\partial\omega}$, so now

$$I = l\frac{\partial k}{\partial\omega}\boldsymbol{\zeta}^{\dagger}\hat{\mathbf{T}}\boldsymbol{\zeta} = \int_0^l \boldsymbol{\xi}^{\dagger}\frac{\partial\hat{\mathbf{N}}}{\partial\omega}\boldsymbol{\xi}\, dz \tag{30}$$

and

$$V_{g,z} = \boldsymbol{\zeta}^{\dagger}\hat{\mathbf{T}}\boldsymbol{\zeta}\bigg/\frac{1}{l}\int_0^l \boldsymbol{\xi}^{\dagger}\frac{\partial\hat{\mathbf{N}}}{\partial\omega}\boldsymbol{\xi}\, dz. \tag{31}$$

Thus, from (20), (24), (25), (29), and (31) and the fact that $\boldsymbol{\zeta} = \boldsymbol{\xi}(0)$ it follows that

$$V_{g,a} = V_{E,a}, \quad a = x, y, z, \tag{32}$$

which completes the proof.

V. CONCLUSION

We have shown that the group velocity of harmonic electromagnetic waves in bianisotropic periodic superlattices is equal to the energy velocity. The equality holds true provided the absorbtion is disregarded, which is a widely used assumption confirmed by data on electromagnetic wave absorption. Otherwise the equality fails.

The equality of the group and energy velocities means that the planes of constant amplitudes of a monochromatic wave modulated by a sufficiently smooth envelope function move with the energy velocity of this monochromatic carrier wave and that the energy velocity of a monochromatic wave is directed along the normal to the surface of constant frequency [1,5,10]. Such an interpretation of the group velocity and its equality to the energy velocity defined as the ratio (18) is applicable to Bloch waves at continuous values of x and y but discrete values $z_n = z_0 + nl$. As it has already been mentioned, it is the successive "jumps" of the Bloch wave over points z_n at a distance equal to the period l of the superlattice that may be viewed as a wave motion similar to the propagation of bulk waves in a homogeneous medium. Accordingly, the period-averaged values of the energy flux and energy turn out to be suitable energy characteristics of the Bloch wave. At points z_n and $z_n + \Delta z$, $\Delta z \neq l$, the amplitudes of the Bloch wave of a given frequency differ without any relation with possible modulation, so the concept of a smooth wave packet loses its meaning for points other than points z_n of a sequence specified by the choice of z_0. Note that the local energy velocity $\mathbf{P}(z)/W(z)$ changes in the interval $z_n < z < z_{n+1}$ and naturally does not equal the group velocity.

Note also that in homogeneous media the projection of the group (energy) velocity on the direction of wave propagation equals the phase velocity if the material constants do not depend on the frequency [5], but this does not hold true in superlattices. Owing to the spatial changes of material constants, the nonlinear dependence of the frequency on the wave numbers arises irrespective of the frequency dispersion of material constants.

The equality $\mathbf{V}_g = \mathbf{V}_E$ implies that the corresponding components of the velocities have to vanish simultaneously. In particular, if a gap between allowed frequency zones exists at the boundary of the Brillouin zone $|k| = \pi/l$, then the component $V_{g,z}$ along the stratification direction vanishes at $|k| = \pi/l$ together with $V_{E,z} \propto \boldsymbol{\zeta}^{\dagger}\hat{\mathbf{T}}\boldsymbol{\zeta}$. However, it may be that the gap does not open at $|k| = \pi/l$ and two dispersion curves just intersect at the boundary of the Brillouin zone, so $V_{g,z} = V_{E,z} \neq 0$. The two options $\boldsymbol{\zeta}^{\dagger}\hat{\mathbf{T}}\boldsymbol{\zeta} = 0$ and $\boldsymbol{\zeta}^{\dagger}\hat{\mathbf{T}}\boldsymbol{\zeta} \neq 0$ are due to properties of the transfer matrix of unit cell $\hat{\mathbf{M}}$. Specifically, $\hat{\mathbf{M}}$ has a pair of coinciding eigenvalues at $|k| = \pi/l$, the eigenvector $\boldsymbol{\zeta}$ is associated with this pair of eigenvalues, $\hat{\mathbf{M}}$ fulfills the identity $\hat{\mathbf{M}}^{-1} = \hat{\mathbf{T}}\hat{\mathbf{M}}^{\dagger}\hat{\mathbf{T}}$, and $\hat{\mathbf{M}}$ is not a Hermitian matrix. In view of the latter fact, $\hat{\mathbf{M}}$ need not be diagonalizable when its eigenvalues coincide (see, e.g., [60]). If $\hat{\mathbf{M}}$ is not diagonalizable, then from $\hat{\mathbf{M}}^{-1} = \hat{\mathbf{T}}\hat{\mathbf{M}}^{\dagger}\hat{\mathbf{T}}$ it follows that $\boldsymbol{\zeta}^{\dagger}\hat{\mathbf{T}}\boldsymbol{\zeta} = 0$. Otherwise $\boldsymbol{\zeta}^{\dagger}\hat{\mathbf{T}}\boldsymbol{\zeta} \neq 0$. Analogous situations occur inside the Brillouin zone at the extreme points of dispersion curves and points of their intersections.

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