Universality of isolated N-body resonances at large scattering length

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Non-Efimovian N-body resonances are investigated in the regime of a large two-body s-wave scattering length. In view of a universal description of low-energy bound and quasibound states, a contact model is introduced. The modeling requires two parameters in addition to the scattering length. Using a modified scalar product, the contact model provides a normalization of bound states, possibly not square integrable, that coincides with that of the corresponding finite range model.

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I. INTRODUCTION

The regime in which N-body quantum systems exhibit zero-energy resonances or shallow states is of great interest in various fields, from hadronic and nuclear physics to condensed matter and ultracold atoms. An interesting class of these phenomena, which can be called N-body resonance for short, occurs when the two-body s-wave scattering length (a_{sc}) between a fraction of the pairwise interacting particles is large relative to the characteristic range of the interactions (R_{pot}) [1–7]. This scale separation makes possible the existence of shallow bound states of large extension with respect to the microscopic details of the interaction and it is expected that universal laws can be found in this regime. For more than two particles, the Efimov effect is a fascinating example of such a regime, which is predicted up to five particles and is studied experimentally at the three-body level in ultracold atoms by using the Feshbach mechanism [8–10]. The reference Hamiltonians that describe this variety of systems are characterized by short-range potentials where the small distance details depend on the particular physics involved (pion exchange in nuclear physics, van der Waals and hard core interaction in atomic physics, etc.).

In this context, to emphasize the universal character of N-body resonances, it is intellectually appealing to replace the finite-range interactions of the reference model by pure contact interactions. In this approach, the wave function is a solution of the free Schrödinger equation almost everywhere with possibly external potentials, while all particle interactions are replaced by specific singularities at the contact of two or more particles. A remarkable feature is that in this regime, the N-body problem for contact interactions can be reduced to the study of a hyperradial two-dimensional (2D) Schrödinger equation with an effective long-range inverse square potential of the form s^2/ρ^2 for $\rho \ll |a_{\mathrm{sc}}|$. The Efimov effect occurs for imaginary values of s, where this potential is attractive, regardless of the microscopic details of the short-range interactions in the reference model. This regime has been studied in depth [2]. In what follows, one is interested in the non-Efimovian regime where s is real. In this regime, the N-body resonances do not have the same nature because of the repulsive hyperradial barrier. They occur only for a fine tuning of the finite-range interactions in the reference model and are therefore isolated resonances [11]. Despite the success of the universal Efimov theory, the development of the contact model for N-body resonances has been slowed down in the non-Efimovian domain by what can be called the normalization catastrophe, i.e., the fact that for $s \geqslant 1$, the contact bound states are not square integrable. This serious drawback may give the impression that N-body resonances in this regime do not have any universal character [11–16]. Nevertheless, the same problem was solved in the contact model for a 3D two-body system with a high partial wave resonance by introducing the notion of modified scalar product [17,18]. This suggests that it is possible to find an analogous method here [19–21].

Inspired by the modeling of two-body resonances, a contact model of N-body resonances is defined in this paper by using three parameters: the two-body scattering length a_{sc}, an effective radius R, and a detuning parameter ϵ. The main results obtained with this model are as follows: (i) At unitarity ($|a_{\mathrm{sc}}| = \infty$), for negative detuning $\epsilon < 0$ a single bound state always exists whereas for a positive detuning $\epsilon > 0$, a long-lived quasibound state exists only for $s \geqslant 1$. (ii) The states can be described using a contact model where the interactions are replaced by two- and N-body contact conditions; (iii) The equivalence between the contact model and the reference model is finalized by introducing a modified scalar product. For shallow bound states, even when they are not square integrable (i.e., for $s \geqslant 1$), this metric gives the same norm as that obtained in the reference model; (iv) The modified scalar product gives an upper bound for the effective radius parameter of the contact model. All the results can be qualitatively understood through a mapping to a 3D two-body l-wave resonant system, which will be denoted as the 3D mapping.

II. CONTACT MODEL

A. Separability region

One considers N particles labeled by i, of mass m_i and spatial coordinates $\mathbf{r}_i$, in their center-of-mass frame. The

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choice made for the definitions of the hyperradius vector $\boldsymbol{\rho}$, hyperradius ρ and of the hyperangle Ω for a given reference mass m_r is detailed in Appendix A.

In what follows, the contact model is constructed in four steps. First, one defines what is the contact state $|\Psi\rangle$ associated with a given reference state $|\Psi_{\text{ref}}\rangle$ of the reference model. For this purpose, the reference state is divided into an inner state $|\Psi_<\rangle$ and an outer state $|\Psi_>\rangle$:

$$|\Psi_{\text{ref}}\rangle = |\Psi_<\rangle + |\Psi_>\rangle. \tag{1}$$

The outer state lies in the outer region $\mathcal{D}_{\text{out}}$ where none of the pair (ij) of interacting particles is in the potential range ($r_{ij} < R_{\text{pot}}$), so that $\langle \boldsymbol{\rho}|\Psi_>\rangle = 0$ in the complementary region of $\mathcal{D}_{\text{out}}$. The inner state $|\Psi_<\rangle$ is associated with short distances behavior and cannot be described by the contact model: $\langle \boldsymbol{\rho}|\Psi_<\rangle = 0$ in the outer region. The shape of the inner state depends crucially on the details of the interaction potentials and the complementary region of $\mathcal{D}_{\text{out}}$ denoted as the inner region can be also qualified as the nonuniversal region. This is in deep contrast with the shape of the outer state, which is obtained from the free Schrödinger equation. Universal physics concerns only phenomena that occur predominantly in the outer region and thus linked to observables that are evaluated with the outer state. By definition, the contact state $|\Psi\rangle$ approximates the external state in the outer region $\mathcal{D}_{\text{out}}$: $\langle \boldsymbol{\rho}|\Psi\rangle \simeq \langle \boldsymbol{\rho}|\Psi_>\rangle$ and is also a solution of the free Schrödinger equation everywhere except at the contact of two or more interacting particles. Second, s-wave resonant two-body interactions are replaced by contact conditions. For a pair, say (12), interacting resonantly in an s wave the two-body contact condition for $r_{12} \to 0$ is

$$\langle \mathbf{r}_1 \dots \mathbf{r}_N|\Psi\rangle \propto \left(\frac{1}{a_{\text{sc}}} - \frac{1}{r_{12}}\right) + O(r_{12}). \tag{2}$$

Two-body contact conditions for the other interacting pairs are defined in the same manner. Third, one considers the region $\rho \ll |a_{\text{sc}}|$ where the contact state behaves as in the unitary limit $|a_{\text{sc}}| = \infty$. In this region, the set of equations deduced from the two-body contact condition of each interacting pair is scale invariant in the hyperradius. The contact state is thus separable in the hyperradius and the hyperangles:

$$\langle \boldsymbol{\rho}|\Psi\rangle = \rho^{\frac{5-3N}{2}} F(\rho)\Phi(\Omega), \tag{3}$$

where $\Phi(\Omega)$ is a normalized eigenstate ($\langle\Phi|\Phi\rangle = 1$) of the Laplacian Δ_Ω on the unit hypersphere: $\Delta_\Omega\Phi(\Omega) = -\Lambda\Phi(\Omega)$, with the boundary conditions obtained by the set of the two-body contact conditions of the form given by Eq. (2). One then introduces the notion of separability region where the reference state is also separable. This region is defined by the spatial domain $\mathcal{D}_{\text{out}}$ and the condition $\rho \ll |a_{\text{sc}}|$. There exists a minimal radius R_{sep} of the order of R_{pot} such that $R_{\text{sep}} < \rho \ll |a_{\text{sc}}|$ in the separability region. By construction, the contact and reference radial functions (almost) coincide in the separable region. The fourth step is to determine a boundary condition for the hyperradial function of the contact model to fix its behavior at a small hyperradius. This boundary condition will be given in Sec. II B or equivalently in Sec. IV.

B. Log-derivative condition

For $0 < \rho \ll |a_{\text{sc}}|$, the radial function $F(\rho)$ satisfies:

$$-\frac{\hbar^2}{2m_r}\left(\partial_\rho^2 + \frac{1}{\rho}\partial_\rho\right)F(\rho) + \frac{\hbar^2 s^2}{2m_r\rho^2}F(\rho) = EF(\rho), \tag{4}$$

where $s^2 = \Lambda + (\frac{3N-5}{2})^2$. At unitarity, Eq. (4) is formally equivalent to the radial equation of a 2D two-body problem with an angular momentum in the continuum (i.e., classical). The form of the N-body contact condition shared by all the contact wave functions will be deduced from the short distance behavior of the radial function in the unitary limit. One therefore focuses on this regime where Eq. (4) is always valid for $\rho > 0$. The bound-state solutions of Eq. (4) are given by the Macdonald function

$$F(\rho) = \mathcal{A}K_s(q\rho), \tag{5}$$

where $\mathcal{A}$ is the normalization constant. If there is no boundary condition at $\rho = 0$, all values of the binding wave number are possible, which is a physical nonsense. Following Efimov's seminal paper, the simplest contact condition can be deduced by imposing a specific value on the log derivative of the contact wave function considered at the effective radius R [22–24]:

$$\left.\frac{\partial_\rho F(\rho)}{F(\rho)}\right|_{\rho=R} = \frac{\epsilon - s}{R}. \tag{6}$$

From Eq. (5), a zero-energy N-body resonance occurs for a vanishing detuning parameter ϵ. The effective radius sets a high-energy scale E_R given by:

$$E_R = \frac{\hbar^2}{2m_r R^2}. \tag{7}$$

As in the Efimov's theory at unitarity, the log-derivative condition breaks the scale invariance related to the $1/\rho^2$ potential and gives rise to a quantum anomaly [25].

III. BOUND AND QUASIBOUND STATES

A. General equation for shallow states

The low-energy solutions (i.e., $|E| \ll E_R$ or $|\epsilon| \ll 1$) are deduced from Eq. (6) and a truncation of the expansion of the radial wave function in Eq. (5), considered as a function of the variable $z = q\rho$ for $z \to 0$ with

$$K_s(z) = \left[\left(\left(\frac{z}{2}\right)^{-s} \sum_{k=0}^{\infty} \frac{\Gamma(s-k)}{2k!}\left(\frac{-z^2}{4}\right)^k\right) + s \leftrightarrow -s\right]. \tag{8}$$

The truncated series must at least include the terms $z^{\pm s}$, which are the two possible zero-energy solutions of Eq. (4) for $z \to 0$. One obtains the equation verified by the energy E of a shallow state:

$$\sum_{k=0}^{k_{\max}} a(k, s, \epsilon)\left(\frac{E}{E_R}\right)^k = \frac{\pi}{\sin(\pi s)}\left(\frac{-E}{E_R}\right)^s \tag{9}$$

where

$$a(k, s, \epsilon) = 4^{s-k}\left(\frac{2k-\epsilon}{2s-\epsilon}\right)\frac{\Gamma(s+1)\Gamma(s-k)}{k!}. \tag{10}$$

The cutoff $k_{\max}$ in Eq. (9) has to be carefully chosen. Indeed, the term of order $2k \pm s$ in the expansion (8) of the modified Bessel function is proportional to the factor $\Gamma(\mp s - k)$, which provides an anomalously large contribution when $\mp s - k$ is in the vicinity of zero or of a negative integer value. When truncated, this expansion can thus lead to a very bad approximation. Nevertheless, one can verify that as s tends to n for a fixed value of z, the spurious singularity of the term in z^s is compensated by the one of the term in z^{-s+2n} [26]. To avoid also the next order spurious singularity when $n < s$ and $s \simeq n$, one introduces the small positive number η and the cutoff $k_{\max}$ is chosen as $k_{\max} = \lceil s - \eta \rceil$ with typically $\eta \simeq 0.2$.

B. Bound states

For small and negative values of the detuning (i.e., $\epsilon < 0$), a unique shallow bound state is always found. Near the threshold of the Efimov regime, i.e., in the limit $0 < s \ll 1$, the binding energy is

$$E \simeq -4E_R\left[\frac{-\epsilon s \sin(\pi s)\Gamma(s)^2}{(2s-\epsilon)\pi}\right]^{\frac{1}{s}}. \tag{11}$$

For $s = 0$, $E = -4E_R \exp(2/\epsilon - 2\gamma_E)$ where γ_E is the Euler's constant. For increasing values of s, the first-order term $k = 1$ in Eq. (9) cannot be neglected. At this order and for $|\epsilon| \ll 1$, Eq. (9) is approximated by:

$$-\epsilon + \frac{E}{2(s-1)E_R} = \frac{\pi(2s-\epsilon)4^{-s}}{s\Gamma(s)^2\sin(\pi s)}\left(\frac{-E}{E_R}\right)^s. \tag{12}$$

At $s = 1$, for a small negative detuning ($-\epsilon \ll 1$), the binding energy is

$$E \simeq -2E_R\epsilon/W_{-1}(x), \tag{13}$$

where $x = e^{2\gamma_E}\epsilon/2$ and W_{-1} is the lower branch of the Lambert function. In the limit of large values of s and again for a small detuning $|\epsilon| \ll 1$, except in the vicinity of integer values, summation of the terms $k \geqslant 2$ in Eq. (9) provides a negligible contribution. Consequently, one can safely neglect these terms and use Eq. (12) to find the limit solution for $s \gg 1$:

$$E \simeq 2(s-1)\epsilon E_R. \tag{14}$$

C. Quasibound states

For a small and positive detuning (i.e., $0 < \epsilon \ll 1$) the solutions of Eq. (9) are complex with a positive real part and an imaginary part that can be chosen negative: $E = E_{\mathrm{r}} - i\Gamma/2$. Long-lived quasibound states defined by a vanishing ratio Γ/E_r are found for $s \geqslant 1$. At the threshold $s = 1$, one finds in the limit $\epsilon \to 0$:

$$E_{\mathrm{r}} \simeq -\frac{2\epsilon E_R}{W_{-1}(-x)}, \quad \frac{\Gamma}{2E_{\mathrm{r}}} \simeq \frac{-\pi}{1+\ln x}. \tag{15}$$

For sufficiently large values of s, the quasibound state energy is given by

$$E_{\mathrm{r}} \simeq 2(s-1)\epsilon E_R, \quad \frac{\Gamma}{E_r} \simeq \frac{4^{1-s}\pi(s-1)}{[\Gamma(s)]^2}\left(\frac{E_r}{E_R}\right)^{s-1}. \tag{16}$$

The existence of the quasibound state for $s \geqslant 1$ is due to the effective centrifugal barrier s^2/ρ^2. In the limit of large values of s, the effective centrifugal barrier grows, explaining the reason why Eq. (16) predicts very long lived quasibound states with a ratio Γ/E_{r} that tends to zero. On the contrary, for $s < 1$, the barrier is not strong enough to support a long-lived quasibound state.

D. 3D mapping

The 3D mapping is the equivalence between a N-body resonance and a two-body resonance, which occurs for half-integer values of the index s. This mapping is described in the following lines, recalling also known results [27].

Substituting $F(\rho) = \sqrt{\rho}f(\rho)$ in the 2D effective radial equation of the N-body problem Eq. (4) gives

$$-\frac{\hbar^2}{2m_{\mathrm{r}}}\left(\partial_\rho^2 + \frac{2}{\rho}\partial_\rho\right)f(\rho) + \frac{\hbar^2\left(s^2 - \frac{1}{4}\right)}{2m_{\mathrm{r}}\rho^2}f(\rho) = Ef(\rho). \tag{17}$$

At unitarity for the reference model, i.e., in the limit $R_{\mathrm{pot}}/|a_{\mathrm{sc}}| = 0$, this equation is verified for all $\rho > R_{\mathrm{sep}}$ and is then equivalent to the radial equation of a two-body system in a l wave when

$$s = l + \frac{1}{2}. \tag{18}$$

The log-derivative condition in Eq. (6) is isotropic, implying that the corresponding two-body system experiences a symmetric l-wave resonance. The low-energy properties of this system can be studied in the effective range approximation of the partial wave amplitude:

$$f_l(k) = \frac{-k^{2l}}{\frac{1}{w_l} + \alpha_l k^2 + \cdots + ik^{2l+1}}. \tag{19}$$

The two scattering parameters w_l and α_l generalize the notion of scattering length and effective range used in the s-wave scattering. The bound and quasibound states correspond to the pole of the denominator and thus verify at the lowest order:

$$\frac{1}{w_l} + \alpha_l k^2 + ik^{2l+1} = 0. \tag{20}$$

This last equation is equivalent to the approximate equation in Eq. (12). By identifying Eq. (12) to Eq. (20), one finds the mapping between the two parameters of the log-derivative condition (ϵ, R) and the parameters of the effective range approximation:

$$\begin{aligned} \frac{1}{w_l} &= -\frac{\epsilon}{R^{2l+1}}[(2l-1)!!]^2, \\ \alpha_l &= R^{1-2l}(2l-1)!!(2l-3)!!. \end{aligned} \tag{21}$$

with the convention $(-1)!! = 1$ for $l = 1$.

As shown below, part of the results already derived from Eq. (12) can be understood qualitatively for $s \neq (l + 1/2)$ as standard two-body properties. For $s = 1/2$, the 3D mapping gives the s-wave resonant problem $l = 0$. In the present case, the effective range $\alpha_0 = R$ is negligible with respect to the scattering length $w_0 = -R/\epsilon$ and can be neglected (it is exactly zero if one considers the log-derivative condition without any further approximation [28]). In the usual terminology, this resonance is broad. There is no quasibound state and the pole of the scattering amplitude is at the binding wave number

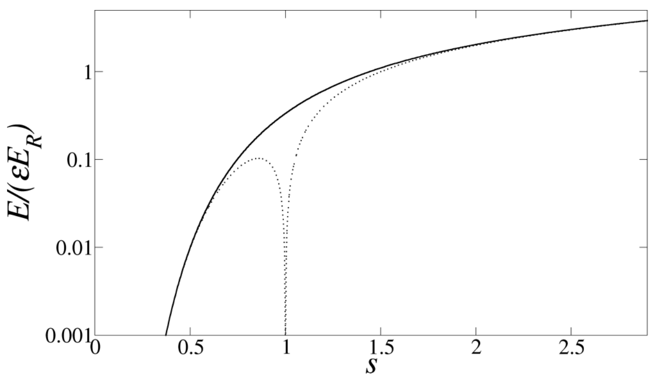

FIG. 1. Solid line: Ratio $E/(\epsilon E_R)$ $(\epsilon < 0)$ of the binding energy to the detuning deduced from Eq. (9) as a function of s for $\epsilon = -10^{-2}$. Dotted line: limit solutions given by Eq. (11) for $s < 1$ and Eq. (14) for $s > 1$.

$q = -ik = 1/w_0$, compatible with a bound state for $\epsilon < 0$. For larger half-integer values of s, i.e., in a high partial wave ($l \geqslant 1$), there is a shallow bound state for $w_l > 0$ ($\epsilon < 0$) of binding energy

$$E = -\frac{\hbar^2}{2m_r w_l \alpha_l} \tag{22}$$

and for $w_l < 0$, there is a low-energy quasibound state of energy $E_\mathrm{r} = \frac{\hbar^2}{2m_\mathrm{r}} k_\mathrm{r}^2$ and width Γ given by

$$E_\mathrm{r} = -\frac{\hbar^2}{2m_r w_l \alpha_l}, \quad \frac{\Gamma}{2E_\mathrm{r}} = \frac{k_\mathrm{r}^{2l-1}}{\alpha_l}. \tag{23}$$

The width of the resonance is inversely proportional to α_l. For this reason α_l can be denoted as the width parameter. When s is given by Eq. (18) then Eq. (22) and Eq. (23) coincide exactly with Eqs. (14) and (16).

To conclude this discussion about the 3D mapping it is not useless to attract the attention on the fact that there exists a spurious high-energy solution of Eq. (20) for odd partial wave with $q \simeq (\alpha_l)^{\frac{1}{2l-1}}$. This problem (already noticed in Refs. [17,18]) renders the contact model difficult to use in the effective range approximation as a tool for the few-body problem (for instance a spurious singularity appears in the analog of the Skorniakov Ter-Martirosian derived with this contact model). In contrast, the contact model based on the log-derivative condition in (6) or also the contact condition which will be given in Eq. (25) has no spurious bound state.

E. Comparison of the spectrum and of the limit spectrum

It is interesting to compare the solution of Eq. (9) with the limit solutions given by Eq. (11) for $s < 1$ and Eq. (14) for $s > 1$. This is done in Fig. 1 with the plot of $E/(\epsilon E_R)$ for $\epsilon = -0.01$. The approximation in Eq. (11) fails for values of s greater than about .7, a number compatible with the fact that in the 3D mapping, $s = 1/2$ corresponds to a s wave resonance with a single parameter model. Interestingly, the approximation for large values of s in Eq. (14) becomes relevant for s higher than, but of the order of, unity. Again this property is well understood with the 3D mapping where the binding energy of a shallow state for $l \geqslant 1$ is given accurately in the effective range approximation. For a vanishing value of ϵ, the spectrum tends to the limit solutions except at $s = 1$.

IV. CONTACT CONDITION

The N-body contact condition is such that it leads to the solutions of Eq. (6) in the low-energy limit. It corresponds to imposing a specific linear combination of the coefficients of the series of the radial wave function $F(\rho)$ for a vanishing value of the hyperradius $\rho \to 0$. For this purpose, it is convenient to introduce the operator

$$\lim_{\rho\to 0}]\rho^s, F(\rho)[, \tag{24}$$

which gives the coefficient of the term ρ^s in this series. The contact condition is obtained by establishing a mapping between the first terms of the series and the condition in Eq. (9). Using the behavior of the Macdonald function $K_s(z)$ when $z \to 0$ in Eq. (8), one finds:

$$\lim_{\rho\to 0} \left] R^s \rho^s + \sum_{k=0}^{k_{\max}} \left(\frac{2k-\epsilon}{2s-\epsilon}\right) (R\rho)^{2k-s}, F(\rho) \right[= 0. \tag{25}$$

This contact condition, which in the low-energy limit is equivalent to the log-derivative condition (6), can be used for any contact wave function in the regime of large scattering length ($|a_\mathrm{sc}| \gg R_\mathrm{pot}$).

It is worth pointing out that in the Efimov regime, the N-body contact condition can be written in terms of a single parameter $R^\star$ in the form

$$\lim_{\rho\to 0}](R^\star \rho)^{i|s|} + (R^\star \rho)^{-i|s|}, F(\rho)[= 0. \tag{26}$$

In this regime, $F(\rho)$ exhibits log-periodic oscillations in the separable region. Instead of this contact condition, the phase can be also imposed by using a nodal (and thus a one-parameter) condition to be compared with Eq. (6), which is a two-parameter condition. The nodal condition is $F(R_n) = 0$ where $R_n = R^\star e^{2n\pi/|s|}$ and n is chosen for having R_n in the separable region [29].

For integer values of $s = n$, the contact model of a N-body isolated resonance is formally equivalent to a contact model for the 2D two-body problem with a resonant interaction in the nth partial wave. In this case, the series of $F(\rho)$ contains logarithmic singularities of the form $\rho^n \ln(\rho q c_n)$. When $s = n > 0$ is an integer, one has in the limit $z \to 0$

$$K_n(z) = \left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{\Gamma(n-k)}{2k!} \left(\frac{-z^2}{4}\right)^k - \left(\frac{-z}{2}\right)^n \frac{\ln(z c_n)}{n!} + \dots \tag{27}$$

with $c_n = 1/2 \times \exp(\gamma_E - (1 + \dots 1/n)/2)$ and γ_E is the Euler's constant: $c_0 = \frac{1}{2} e^{\gamma_E}$, $c_1 = \frac{1}{2} e^{\gamma_E - 1/2}$, $c_2 = \frac{1}{2} e^{\gamma_E - 3/4}$, ... For $n \geqslant 1$, the contact condition in Eq. (25) is then modified by taking $k_{\max} = n - 1$ and by defining the action of the operator in Eq. (24) for such terms as:

$$\lim_{\rho\to 0}]\rho^n, \rho^n \ln(\alpha\rho)[= \ln(\alpha R) + 1/(2n - \epsilon). \tag{28}$$

For $s = 0$, the contact condition is given by imposing the behavior $F(\rho) = A \ln(\rho/l) + O(\rho^2)$ as $\rho \to 0$ where

$l = R\exp(-1/\epsilon)$ is analogous to a two-dimensional scattering length. An alternative way to impose the contact condition is to use the pseudopotential for a two-dimensional resonant s-wave interaction with the change $a_{2D} \to l$ [30].

V. ORTHOGONALITY AND NORMALIZATION

A. Modified scalar product and self-adjoint extension

In the domain defined by Eq. (6), two contact states of different energies are not mutually orthogonal and a bound state is not normalizable in the usual sense for $s \geqslant 1$. This problem is linked to the behavior of the contact radial function for $\rho < R_{\rm sep}$ and it is tempting to introduce a subtraction or a cutoff in the radial integral to avoid the unphysical divergence. As expected from the 3D mapping, the same problem exists in a contact model of a resonant interaction with $l \geqslant 1$ and the normalization catastrophe is solved by introducing a modified scalar product. The issue is to do again this type of regularization in an equivalent rigorous manner. To extend the method to the present situation, one can notice that at unitarity and using Eq. (4), two contact states of radial functions $F(\rho, E), F(\rho, E')$, with different energies $E' \neq E$, verify

$$\int_R^\infty \rho F(\rho, E')^* F(\rho, E) d\rho = \frac{\hbar^2 R}{2m_{\rm r}(E'-E)}[F(R,E)\partial_R F(R,E')^* - (E \leftrightarrow E')^*] \tag{29}$$

Using Eq. (6) in the right hand side of Eq. (29) proves that the integral is identically zero. It is thus natural to introduce the following modified scalar product in the center-of-mass frame

$$(\Psi'|\Psi)_0 \equiv \int_{\rho>R} d\mu \langle\Psi'|\boldsymbol{\rho}\rangle\langle\boldsymbol{\rho}|\Psi\rangle, \tag{30}$$

where $d\mu$ is the measure of integration in the $(3N-3)$-dimensional space. In the domain of functions defined by Eq. (6), one finds that using the modified scalar product, the contact model is a self-adjoint extension of the N-body kinetic operator H_0 [31]:

$$(\Psi'|H_0\Psi)_0 = (H_0\Psi'|\Psi)_0, \tag{31}$$

where the surface terms in the hyper-radial integration have been eliminated thanks to Eq. (6).

B. Equivalence between the contact and the reference model

An important feature of the modified scalar product is that in the low-energy limit, it gives a normalization of the contact states that coincides with the normalization of the reference state. In what follows, this property is derived for a bound state at unitarity. For this purpose, one considers two reference states $\{|\Psi_{\rm ref}(E)\rangle, |\Psi'_{\rm ref}(E')\rangle\}$ of arbitrary energies E and E' in the center-of-mass frame. From the stationary Schrödinger equation, one obtains :

$$\begin{aligned}\langle\boldsymbol{\rho}|\Psi'_{\rm ref}(E')\rangle^* H_0\langle\boldsymbol{\rho}|\Psi_{\rm ref}(E)\rangle - \langle\boldsymbol{\rho}|\Psi_{\rm ref}(E)\rangle H_0\langle\boldsymbol{\rho}|\Psi'_{\rm ref}(E')\rangle^* \\ = (E-E')\langle\boldsymbol{\rho}|\Psi'_{\rm ref}(E')\rangle^*\langle\boldsymbol{\rho}|\Psi_{\rm ref}(E)\rangle.\end{aligned} \tag{32}$$

The operator Δ_Ω is self-adjoint in the domain of the reference wave functions. Hence, integration of each side of Eq. (32) over the unit hypersphere and for hyperradii smaller than a given cutoff ρ_M, gives

$$\begin{aligned}\int_{\rho<\rho_M} d\mu\, (\langle\boldsymbol{\rho}|\Psi'_{\rm ref}(E')\rangle^* T_\rho\langle\boldsymbol{\rho}|\Psi_{\rm ref}(E)\rangle \\ - \langle\boldsymbol{\rho}|\Psi_{\rm ref}(E)\rangle T_\rho\langle\boldsymbol{\rho}|\Psi'_{\rm ref}(E')\rangle^*) \\ = (E-E')\int_{\rho=0}^{\rho=\rho_M} d\mu\, \langle\boldsymbol{\rho}|\Psi'_{\rm ref}(E')\rangle^*\langle\boldsymbol{\rho}|\Psi_{\rm ref}(E)\rangle,\end{aligned} \tag{33}$$

where the measure of integration in the $3(N-1)$ dimensional space is $d\mu = \rho^{3N-4}d\rho d\Omega$ and T_ρ is N-body radial part of the kinetic operator (its explicit expression is given in Eq. (A5) of the Appendix). For realistic potentials in the reference model, the reference wave function and its radial derivative vanish at the origin $\rho = 0$. Equation (33) can then be transformed into

$$\begin{aligned}\int_{\rho<\rho_M} d\mu\, \langle\Psi'_{\rm ref}(E')|\boldsymbol{\rho}\rangle\langle\boldsymbol{\rho}|\Psi_{\rm ref}(E)\rangle \\ = \frac{\hbar^2\rho_M^{3N-4}}{2m_r}\frac{\int d\Omega W[\langle\boldsymbol{\rho}|\Psi'_{\rm ref}(E')\rangle^*, \langle\boldsymbol{\rho}|\Psi_{\rm ref}(E)\rangle, \rho=\rho_M]}{E'-E}.\end{aligned} \tag{34}$$

The term $W[f, g, \rho = \rho_M] = f\partial_\rho g - g\partial_\rho f$ in Eq. (34) is the Wronskian of the functions f and g with respect to the variable ρ, considered at $\rho = \rho_M$. For ρ in the separable region, the reference wave functions are well approximated by their associated contact wave functions:

$$\langle\boldsymbol{\rho}|\Psi_{\rm ref}(E)\rangle = \rho^{\frac{5-3N}{2}}F(\rho, E)\Phi(\Omega) \tag{35}$$

$$\langle\boldsymbol{\rho}|\Psi'_{\rm ref}(E')\rangle = \rho^{\frac{5-3N}{2}}F(\rho, E')\Phi'(\Omega). \tag{36}$$

In what follows, one focuses on the case where $|\Phi'\rangle = |\Phi\rangle$ and without loss of generality $\langle\Phi|\Phi\rangle = 1$ as in Eq. (3). Then for $\rho_M > R_{\rm sep}$ Eq. (34) gives [32]

$$\begin{aligned}\int_{\rho<\rho_M} d\mu\langle\Psi_{\rm ref}(E')|\boldsymbol{\rho}\rangle\langle\boldsymbol{\rho}|\Psi_{\rm ref}(E)\rangle \\ = \frac{\hbar^2\rho_M}{2m_r}\frac{W[F(\rho,E')^*, F(\rho,E), \rho=\rho_M]}{E'-E}.\end{aligned} \tag{37}$$

Taking the limit $E' \to E$ in Eq. (37), where E is the energy of a bound state, one finds

$$\begin{aligned}\int_{\rho<\rho_M} d\mu|\langle\boldsymbol{\rho}|\Psi_{\rm ref}(E)\rangle|^2 \\ = \frac{\hbar^2\rho_M}{2m_r}W[\partial_E F(\rho,E)^*, F(\rho,E), \rho=\rho_M].\end{aligned} \tag{38}$$

Formally, one can consider solutions of the Schrödinger equation, $F(\rho, E)$ for arbitrary negative values of E. In the unitary limit, the general solution of the hyperradial problem is

$$F(\rho, E) = \mathcal{A}(E)K_s(q\rho) + \mathcal{B}(E)I_s(q\rho), \tag{39}$$

where I_s is the modified Bessel function of the first kind. The energy of a bound state is such that $\mathcal{B}(E) = 0$, and using the property

$$W[K_s(z), I_s(z), z] = \frac{1}{z}, \tag{40}$$

one has from Eq. (39)

$$\frac{\hbar^2\rho_M}{2m_r}W[\partial_E F(\rho,E)^*, F(\rho,E), \rho=\rho_M] = -\frac{\rho_M^2|\mathcal{A}(E)|^2}{2z_M}W[z\partial_z K_s, K_s, z=z_M] - \frac{\hbar^2}{2m_r}\partial_E\mathcal{B}(E)^*\mathcal{A}(E), \tag{41}$$

where $z_M = q\rho_M$. To find a relation for $\partial_E\mathcal{B}(E)$, one uses Eq. (6), which is verified by the contact radial function $F(\rho,E)$ and also by $F(\rho,E+dE)$. This gives

$$\partial_E\mathcal{B}(E) = \frac{m_r\mathcal{A}(E)}{\hbar^2 q^2}\frac{z\partial_z(z\partial_z K_s(z)) + (s-\epsilon)\partial_z K_s(z)}{z\partial_z I_s(z) + (s-\epsilon)I_s(z)}\Big|_{z=z_R} \tag{42}$$

where $z_R = qR$. In the limit $z_M \to \infty$, one has $W[z\partial_z K_s, K_s, z=z_M] = 0$, and thus

$$\langle\Psi_{\mathrm{ref}}(E)|\Psi_{\mathrm{ref}}(E)\rangle = \frac{-|\mathcal{A}(E)|^2}{2q^2}\frac{z\partial_z(z\partial_z K_s(z)) + (s-\epsilon)z\partial_z K_s(z)}{z\partial_z I_s(z) + (s-\epsilon)I_s(z)}\Big|_{z=z_R}. \tag{43}$$

Using the expression of $(\epsilon - s)$ deduced from Eq. (6), one obtains a crucial identity for the quantity $J(z_R)$ that appears in the right-hand side of Eq. (43):

$$J(z) \equiv -\frac{z\partial_z(z\partial_z K_s(z)) + (s-\epsilon)z\partial_z K_s(z)}{2z\partial_z I_s(z) + 2(s-\epsilon)I_s(z)} = \frac{z}{2}W[z\partial_z K_s(z), K_s(z), z]. \tag{44}$$

On the other hand

$$\langle\Psi_>(E)|\Psi_>(E)\rangle = \frac{|\mathcal{A}(E)|^2}{q^2}\int_{qR_{\mathrm{sep}}}^{\infty} uK_s(u)^2 du \tag{45}$$

and from Eq. (41) considered at $\rho_M = R_{\mathrm{sep}}$, the norm of the inner state is

$$\langle\Psi_<(E)|\Psi_<(E)\rangle = \frac{|\mathcal{A}(E)|^2}{q^2}[J(qR) - J(qR_{\mathrm{sep}})]. \tag{46}$$

Using the fact that

$$\langle\Psi_{\mathrm{ref}}(E)|\Psi_{\mathrm{ref}}(E)\rangle = \frac{|\mathcal{A}(E)|^2}{q^2}J(qR) = \langle\Psi_<(E)|\Psi_<(E)\rangle + \langle\Psi_>(E)|\Psi_>(E)\rangle \tag{47}$$

one can deduce *en passant* that

$$J(z) = \int_z^{\infty} uK_s(u)^2 du = \frac{z}{2}[zK_{s+1}(z)^2 - zK_s(z)^2 - 2sK_{s+1}(z)K_s(z)]. \tag{48}$$

Finally at the unitary limit of the s-wave interaction, one obtains:

$$\langle\Psi_{\mathrm{ref}}(E)|\Psi_{\mathrm{ref}}(E)\rangle = |\mathcal{A}(E)|^2\int_R^{\infty}\rho K_s(q\rho)^2 d\rho. \tag{49}$$

This last expression is the normalization obtained with the modified scalar product when it is applied to the contact state. One has thus the following crucial property: in the low-energy limit, the normalization of a contact bound state obtained from the modified scalar product $(\Psi|\Psi)_0 = 1$, coincides with the normalization of the associated reference state $|\Psi_{\mathrm{ref}}\rangle$ where one uses the standard scalar product $\langle\Psi_{\mathrm{ref}}|\Psi_{\mathrm{ref}}\rangle = 1$. Moreover, it has been shown in Sec. V A, that two contact states of different energies are orthogonal with respect to the modified scalar product. Hence, in the low-energy limit, for two contact states $|\Psi\rangle, |\Psi'\rangle$ associated with the reference states $|\Psi_{\mathrm{ref}}\rangle, |\Psi'_{\mathrm{ref}}\rangle$:

$$(\Psi'|\Psi)_0 = \langle\Psi'_{\mathrm{ref}}|\Psi_{\mathrm{ref}}\rangle. \tag{50}$$

Using the decomposition of the reference state in Eq. (1), one finds the norm of the inner state

$$\langle\Psi_<(E)|\Psi_<(E)\rangle = |\mathcal{A}(E)|^2\int_R^{R_{\mathrm{sep}}}\rho K_s(q\rho)^2 d\rho, \tag{51}$$

where $\mathcal{A}(E)$ is the normalization factor in the separable region and the contact wave function is given by Eqs. (3), (5).

Two important remarks are then in order. First, the positivity of the norm implies from Eq. (51)

$$R < R_{\mathrm{sep}}. \tag{52}$$

Using the 3D mapping with Eq. (21), this last inequality coincides exactly with the width-radius inequality

$$\alpha_l R_{\mathrm{sep}}^{2l-1} \gtrsim (2l-1)!!(2l-3)!! \tag{53}$$

already derived in Ref. [17] by using a modified scalar product associated with the contact model in the effective range approximation [see Eq. (64) in this latter reference where the notation R has to be replaced by R_{sep}]. Equation Eq. (52) is also reminiscent of the Wigner bound obtained for high partial waves [33].

Second, the expression in Eq. (51) has been derived in the unitary limit. However, it is only related to the behavior of the reference state in the separable region and thus it remains valid for finite but large values of the scattering length ($|a_{\mathrm{sc}}| \gg R_{\mathrm{sep}}$). On the other hand, the contribution of the external state in Eq. (30) is obtained using the standard scalar product. Therefore, the expression for the modified scalar product in Eq. (30) is also valid when the two-body scattering length is large.

C. Occupation of the inner region

Consider a bound state of energy E at the unitary limit of the s-wave interaction ($|a_{\mathrm{sc}}| = \infty$). The probability to find the particles in the inner (or nonuniversal) region $\rho < R_{\mathrm{sep}}$ is

$$\mathcal{P}_<(E) = \frac{\int_{R<\rho<R_{\mathrm{sep}}} d\mu\, |\langle\boldsymbol{\rho}|\Psi\rangle|^2}{(\Psi(E)|\Psi(E))_0}. \tag{54}$$

At the threshold of the resonance, using Eq. (48), one finds:

$$\lim_{E\to 0}\mathcal{P}_<(E) = \begin{cases} 1 - \left(\frac{R}{R_{\mathrm{sep}}}\right)^{2s-2} & \text{if } s > 1 \\ 0 & \text{otherwise} \end{cases}. \tag{55}$$

Hence, at the resonance threshold there is no sharp change in the occupation of the inner region at the critical value $s = 1$. Instead, one has a continuous increase of $\mathcal{P}_<$ for increasing

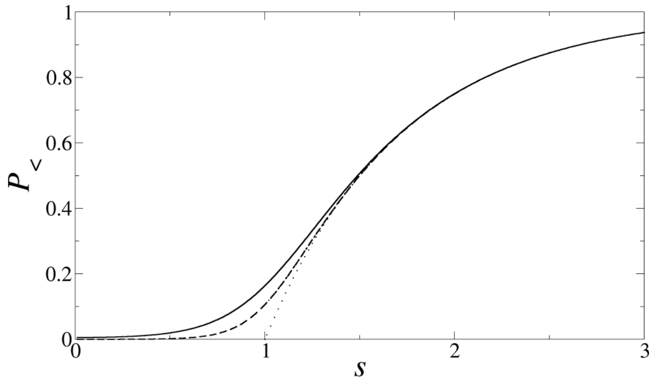

FIG. 2. Probability to find the particles in the inner region $\mathcal{P}_<$ at the unitary limit of the s-wave interaction for $R/R_{\rm sep} = 1/2$. Dotted line: $\mathcal{P}_<$ at the threshold; Dashed line: $\mathcal{P}_<$ at $\epsilon = -0.001$; Solid line: $\mathcal{P}_<$ at $\epsilon = -0.01$.

values of s. At a finite negative detuning, the transition between the regime $s < 1$ and the regime $s > 1$ is smoother. A plot of this probability is given in Fig. 2.

D. Usual scalar product used with the contact model

In the interval $0 < s < 1$ where the contact state is square integrable, it is interesting to consider the ratio of the modified norm to the usual norm

$$r = \frac{(\Psi|\Psi)_0}{\langle\Psi|\Psi\rangle} = \frac{\int_{qR}^{\infty} K_s(z)^2 z dz}{\int_0^{\infty} K_s(z)^2 z dz}. \tag{56}$$

This ratio is a decreasing function of s and qR, respectively. Notable deviations from unity occur in the vicinity of $s = 1$. For example, for $qR = 10^{-2}$ the ratio is plotted in Fig. 3. One finds $r = .9$ at $s \sim 0.72$, which shows that the norm of the internal part of the reference state cannot be neglected when s is close to unity. At the threshold $qR = 0$, $r = 1$ for $s < 1$, which means that the system is essentially in the outer domain $\mathcal{D}_{\rm out}$ as in a standard 3D two-body s-wave resonance. This property is lost for $s \geqslant 1$ [see the dotted line in Fig. 2].

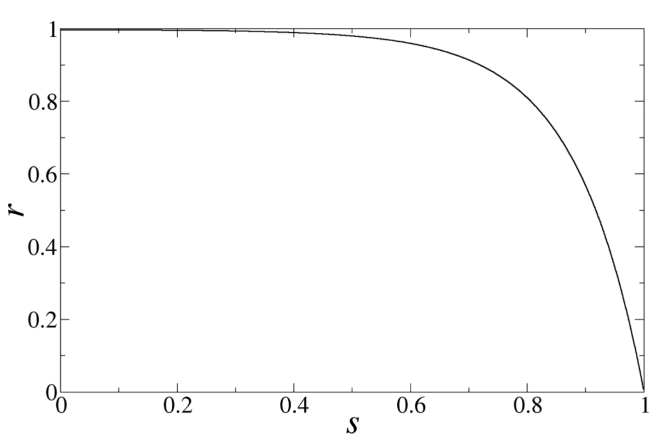

FIG. 3. Ratio between the norm calculated with the modified scalar product and the norm calculated with the standard scalar product for a contact state at $qR = .01$. See the discussion in Sec. V D.

The norm of the contact state calculated with the standard scalar product takes arbitrarily large values in the vicinity of the $s = 1$. In deep contrast, the norm calculated with the modified scalar product, which ensures the self-adjoint character of the contact Hamiltonian has no singularity at $s = 1$. This failure of the usual scalar product, if it is used with the present contact model, explains the behavior of the ratio when the index s tends to unity.

E. Finite values of the two-body *s*-wave scattering length

At finite values of the two-body scattering length $a_{\rm sc}$, the wave function is no longer separable and one is faced with coupled equations on the hyperangles and the hyperradius that depend on the nature of the system (number of particles, mass ratios, etc.). Interestingly, the case of two identical fermions interacting with an impurity has already been studied by using also a log-derivative condition for the three-body condition and an hyperspherical expansion [13]. The results of this last paper can be reinterpreted in the framework of a pure zero-range model, revealing their universal character, including the bound states already found for $s > 1$.

VI. POSSIBLE ACHIEVEMENT OF A TUNABLE THREE-BODY RESONANCE

Due to the large neutron-neutron scattering length, rich neutron nuclei in nuclear physics represent a possible application of the formalism, which remains to be explored. The possibility of tuning the interaction in ultracold atoms by use of the Feshbach mechanism open fascinating perspectives in the achievement of tunable three-body resonances.

The possibility of isolated three-body resonances was predicted for two identical fermions interacting resonantly in the s wave with a sufficiently massive impurity in Refs. [11,13,16,19]. This system offers the opportunity to tune the index s as a function of the mass ratio $x = M/m$ between the fermions of mass M and the impurity of mass m. For this system, the smallest values of s^2 are in the sector $J^\pi = 1^-$. For a given ratio, the index s is a solution (chosen positive in the convention adopted in the present paper) of Eq. [12]:

$$0 = \cos(t) - \frac{(s+1)\sin((s-1)t) - (s-1)\sin((s+1)t)}{2(s^2-1)\sin^2 t \sin(\pi s/2)}, \tag{57}$$

where $t = \arcsin(x/(1+x))$. For a given ratio, this last equation has several solutions. Considering the lowest branch, for increasing values of M/m, s^2 decreases from 4 to arbitrarily large negative values. For $M/m > 13.6\ldots$, s^2 becomes negative and the Efimov effect takes place. The plot of the index s as a function of the mass ratio in the non-Efimovian regime is given in Fig. 4. Importantly, unlike the Efimov effect, which occurs whatever the short-range details of the finite-range interaction potential (because the inverse square effective three-body potential is attractive in the separable region), isolated three-body resonances can occur only if the nonuniversal short-range interaction potentials are sufficiently attractive. Nevertheless, the strength of the attraction must be also not too large in order to get a shallow three-body bound state. Then, even if three-body resonances are possible in the interval $0 < M/m < 13.6\ldots$ near the s-wave unitarity, their

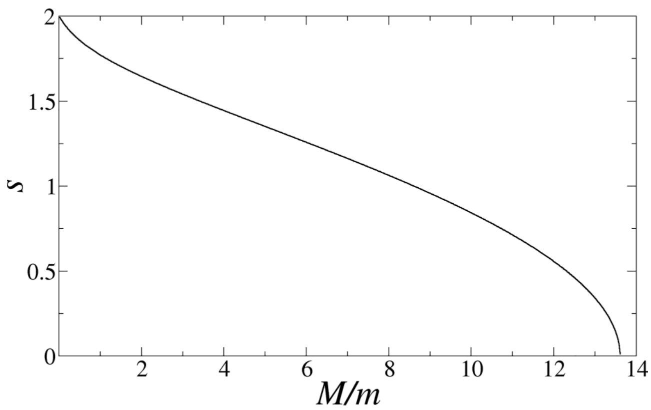

FIG. 4. Plot of the lowest solution of Eq. (57) in the non-Efimovian regime as a function of the mass ratio $x = M/m$.

occurrence depend in general on a fine tuning of the short-range part of the interaction potentials [11]. Interestingly, this three-body system has been considered recently with a pairwise short-range p-wave interaction between the two fermions in addition to the s-wave interaction with the impurity [4]. In this last reference, several interactions of different shapes were used and it has been shown that when the scattering volume is large, i.e., in the vicinity of a p-wave resonance, a shallow three-body bound Borromean state is always present in the vicinity of the s-wave unitary limit [34]. Due to the role of the p-wave interaction, these states were denoted as the p-wave induced states. If one is able to tune both the p-wave and the s-wave interaction, this system represents the opportunity to achieve isolated three-body resonances. A good candidate for such a study is given by using a cesium-ytterbium mixture. Optical p-wave Feshbach resonances (OFR) have been already achieved with ^{171}Yb isotope by using a coupling between the open channel and a purely long-range (PLR) molecular state, ensuring losses smaller than the one observed in magnetic p-wave Feshbach resonances [35,36]. Moreover, intraspecies magnetic s wave Feshbach resonance have been predicted by using realistic ytterbium-cesium interaction potentials [37]. For this system, from Eq. (57), the mass ratio gives the index $s \simeq 1.73$. Several ^{171}Yb-Cs s wave resonances have been predicted (see Table IV of Ref. [37]) for magnetic fields smaller than 80 G: an intensity compatible with a linear Zeeman shift of the energy of the PLR state (see Fig. 3 in Ref. [35]). It is thus possible in principle to achieve tunable isolated three-body resonances in this mixture. In a first step the unitary limit of the s-wave interaction can be reached at a given magnetic field. In a second step, the intensity of the detuned laser ensuring the OFR permits one to tune the scattering volume of the p-wave scattering. In this scheme, the binding energy of the three-body state and thus the detuning parameter ϵ is then a function of the laser intensity. The radius $R_{\rm sep}$, which gives also the order of magnitude of the effective Radius R is of the order of the largest van der Waals characteristic length $R_{\rm vdW} = \frac{1}{2}(2\mu C_6/\hbar^2)^{1/4}$ in the system where μ is the reduced mass of the considered atomic pair. One has $R_{\rm vdW} \simeq 86a_0$ for the Yt-Yt pair in the excited channel of the PLR state. Similarly to the first observations of Efimov states, one expects large three-body losses at the threshold of the three-body resonance. The occurrence of N-body resonances in fermionic mixtures (with a sufficiently small mass ratio to avoid again the Efimov effect) using this type of technique represents a challenging issue [38].

VII. CONCLUSIONS

To conclude, a standard log-derivative condition at finite hyperradius has been used to fix the small hyperradius behavior of the contact wave function. This condition is equivalent to a contact condition at vanishing hyperradius. A modified scalar product is introduced, which ensures the self-adjoint character of the contact model. In the low-energy limit, for a given bound state, the normalization obtained with the modified scalar product is equivalent to the standard normalization of the reference bound state (i.e., the bound state obtained with the finite-range model, which is modeled by the contact model). There is thus no normalization catastrophe for $s \geqslant 1$. The modified scalar product introduced in the present formalism is more simple to understand intuitively (the effective radius R plays the role of a cutoff) than the modified scalar products in Ref. [17,18] where δ distributions were used. At unitarity, due to the hyperradius/hyperangle separability and the continuous value that can be taken by the index s, the contact model for the radial part, can be viewed formally as a contact model for a continuous value of the angular momentum of a two-body 2D or 3D system. The 3D mapping (2D mapping) for half-integer (integer) values of s permits one to understand that the parameters ϵ and R for the resonant N-body problem have the same status as the two necessary parameters in the universal description of the two-body problem in high partial waves. Moreover, the 2D and 3D mappings give a qualitative way to understand the properties of the contact model (existence and energy of bound states or quasibound states). In this point of view, the N-body states for $s > 1$ have thus the same degree of universality as the high partial wave state in the two-body problem. The finite normalization for $s \geqslant 1$ of contact bound states obtained with the modified scalar product appears thus as a physically sound property. By fixing the log derivative at a finite hyperradius, this paper explores a natural self-adjoint extension of the N-body Laplacian and the properties of the inverse radius square potential that may be of interest in other fields.

APPENDIX: JACOBI AND HYPERSPHERICAL COORDINATES

In this section, the Jacobi variables are introduced for N the particles in the same manner as in Ref. [20]. Beginning with the relative coordinates of an interacting pair, say the pair (12), the other Jacobi coordinates are built iteratively by defining at the step n the relative particle formed by the particle n and the relative particle of the step $n-1$. One starts by defining the mass M_j and center of mass $\mathbf{C}_j$ of the set composed of the first j particles:

$$M_j = \sum_{i=1}^{j} m_i; \quad \mathbf{C}_j = \frac{1}{M_j}\sum_{i=1}^{j} m_i \mathbf{r}_i. \tag{A1}$$

The center of mass of the system is denoted by $\mathbf{C} = \mathbf{C}_N$. The reduced mass and the coordinates for the relative particle formed by the jth particle and the set composed of the first $j-1$ particles is

$$\mu_j = \frac{m_{j+1}M_j}{M_{j+1}}; \quad \boldsymbol{\eta}_j = \sqrt{\frac{\mu_j}{m_r}}(\mathbf{r}_{j+1} - \mathbf{C}_j), \tag{A2}$$

where $1 \leqslant j \leqslant N-1$ and m_r is an arbitrary reference mass. The $N-1$ vectors $\{\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \dots \boldsymbol{\eta}_{N-1}\}$ form a possible set of Jacobi coordinates. Other sets of Jacobi coordinates can be defined in the same manner by beginning the iteration with another interacting pair. This way, the two-body contact condition for the pair (ij) can be always written in terms of the variable η_1 of the set of Jacobi coordinates defined from the initial pair (ij). From the coordinates $\{\boldsymbol{\eta}_i\}$ one defines a $(3N-3)$-dimensional hyperradius vector $\boldsymbol{\rho}$, the hyperradius ρ

$$\boldsymbol{\rho} = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \dots \boldsymbol{\eta}_{N-1}), \quad \rho = \sqrt{\sum_{i=1}^{N-1} \eta_i^2}, \tag{A3}$$

and the set of angles Ω parameterized by the unit vector $(\frac{\boldsymbol{\eta}_1}{\rho}, \dots \frac{\boldsymbol{\eta}_{N-1}}{\rho})$. In configuration space the degrees of freedom can be then defined by the coordinates $(\mathbf{C}, \rho, \Omega)$. In the center-of-mass frame, the Hamiltonian H_0 of the N-body system reduces to the kinetic operator

$$H_0 = \sum_{i=1}^{N} \frac{-\hbar^2}{2m_i}\Delta_{\mathbf{r}_i} \equiv -\frac{\hbar^2}{2m_r}\sum_{i=1}^{N-1}\Delta_{\boldsymbol{\eta}_i} \equiv \frac{-\hbar^2}{2m_r}\Delta_{\boldsymbol{\rho}}. \tag{A4}$$

It can be expressed in terms of the hyperradial kinetic operator

$$T_\rho = -\frac{\hbar^2}{2m_r}\left(\partial_\rho^2 + \frac{3N-4}{\rho}\partial_\rho\right) \tag{A5}$$

and of the Laplacian Δ_Ω acting on the hypersphere of radius unity:

$$H_0 = T_\rho - \frac{\hbar^2}{2m_r}\frac{\Delta_\Omega}{\rho^2}. \tag{A6}$$

The expression of Δ_Ω is not useful in the present work. In the contact model, the stationary Schrödinger equation for a state $|\Psi\rangle$ of energy E can be then written as:

$$\left(T_\rho - \frac{\hbar^2}{2m_r}\frac{\Delta_\Omega}{\rho^2} - E\right)\langle\boldsymbol{\rho}|\Psi\rangle = 0. \tag{A7}$$

This equation is satisfied by the contact state everywhere except at the contact of two interacting particles or at $\rho = 0$ where the N-body contact condition is used.

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typical range of the two-body potential (i.e., $|\mathbf{r}_i - \mathbf{r}_j| < R_{\text{pot}}$). Excluding M-body resonances excepted for $M = 2$ with the s-wave resonance, these contributions can be neglected. Indeed, the probability of occupation of the sphere of radius R_{pot}) for an interacting pair in a resonant s wave is of the order of $R_{\text{pot}}/|a_{\text{sc}}| \ll 1$. It is thus reasonable for $\rho_M > R_{\text{sep}}$ to neglect all the contributions where the reference state is not separable in the right-hand side of Eq. (34).

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