Detecting Einstein-Podolsky-Rosen steering in non-Gaussian spin states from conditional spin-squeezing parameters

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We present an experimentally practical method to reveal Einstein-Podolsky-Rosen (EPR) steering in non-Gaussian spin states by exploiting a connection to quantum metrology. Our criterion is based on the quantum Fisher information, and uses bounds derived from generalized spin-squeezing parameters that involve measurements of higher-order moments. This leads us to introduce the concept of conditional spin-squeezing parameters, which quantify the metrological advantage provided by conditional states, as well as detect the presence of an EPR paradox.

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I. INTRODUCTION

Einstein-Podolsky-Rosen (EPR) steering was first termed by Schrödinger [1] to describe the contradiction to local complementarity in the EPR paradox [2]. As an intermediate correlation, EPR steering is stronger than entanglement but not as general as Bell nonlocality [3]. Being easier to generate and detect than nonlocality renders EPR steering a valuable resource for a variety of quantum information tasks [4–6], such as quantum teleportation [7,8], one-side device-independent quantum key distribution (QKD) [9–12], quantum secret sharing [13–15], and assisted quantum metrology [16].

Typically, EPR steering is revealed from the violation of a criterion based on a local uncertainty relation [4,17]. For this reason, such criteria are often expressed in terms of variances of linear operators, and therefore best suited to reveal steering in Gaussian states, where the correlations are fully described by first- and second-order moments. Recently, non-Gaussian states were shown to provide more competitive advantages in several quantum information protocols [18–21]. However, their nontrivial correlations appear in higher-order moments of physical operators, leading to the failure of steering criteria limited to linear observables. To detect non-Gaussian steering, some approaches have taken higher-order moments into account [22–24]. For example, nonlinear correlations in a three-photon down-conversion process with a quadratic steerability index were considered in Ref. [25]. A steering criterion derived from Hillery and Zubairy's multimode entanglement criterion [26] has been investigated to detect steering in a multipartite scenario [27] and further extended to a higher-order version in a two-well Bose-Einstein condensate (BEC) ground state [28]. Nevertheless, these methods are specifically tailored to particular states, and a general steering criterion for non-Gaussian states is still highly desirable to further unlock their potential applications.

Nonclassical spin states are many-body quantum states of great interest for fundamental studies as well as for practical applications. For example, squeezed spin states have attracted increasing attention in quantum metrology for precision improvements to overcome the standard quantum limit and are nowadays routinely prepared in a variety of platforms, from solid state systems to atomic ensembles [29]. Recent studies have in particular explored the metrological potential of non-Gaussian spin states, both in theory [30,31] and experiment [32–36].

Methods derived from quantum metrology [37–41] already allow for the efficient detection and characterization of multiparticle entanglement without addressing individual spins. In particular, the quantum Fisher information (QFI) constitutes a powerful tool for capturing even strongly non-Gaussian features of quantum states by probing them for their sensitivity under small perturbations [32]. Very recently, the QFI was also used to formulate a criterion for EPR correlations [16], thus providing us with a powerful method for detecting EPR steering in non-Gaussian states.

However, accessing the QFI is often challenging. Determining the QFI of arbitrary mixed states requires full knowledge of the quantum state. On the other hand, efficient approximations based on the full counting statistics demand that a carefully chosen observable is measured with high resolution, which is also difficult in multipartite systems. Spin-squeezing parameters [37] have proven to be efficient

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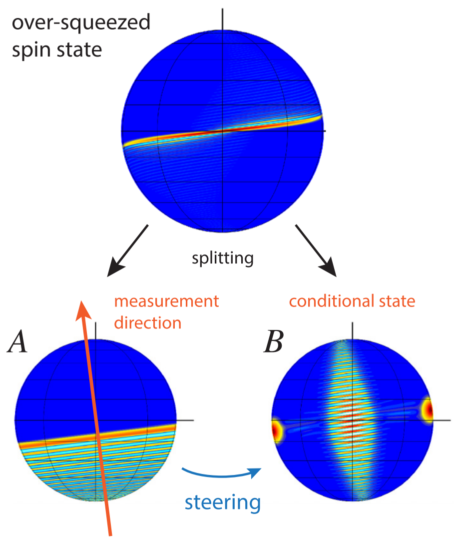

FIG. 1. Illustration of the investigated protocol. A non-Gaussian (oversqueezed) spin state is prepared in an ensemble of particles, that are then distributed to form subsystems A and B. Because of quantum correlations, a measurement on A projects B into one of several highly sensitive conditional states. The knowledge of A's measurement setting and result allow B to make the best use of his state by optimizing his local measurement. With the criteria we propose, steering between the two subsystems can be concluded.

alternatives with high practical relevance, especially for Gaussian spin systems; suitable generalizations are also able to capture non-Gaussian features from higher-order moments [30]. Apart from their metrological interpretation [29], they have also been widely used to detect and characterize multipartite entanglement [41,42], and Bell correlations [43,44]. However, so far the application of squeezing parameters has been limited to single many-spin ensemble and collective measurements.

Here, we introduce the concept of conditional spin-squeezing parameters, and based on that we propose a practical and convenient witness for EPR steering in split nonclassical spin states, see Fig. 1. For the purpose of detecting non-Gaussian steering, spin observables involving higher-order moments are taken into consideration. An optimization of the measurement within these accessible higher-order observable ensembles leads to conditional nonlinear spin-squeezing parameters, whose potential to detect steering in a wider class of non-Gaussian states is explored. We demonstrate that conditional spin-squeezing parameters approximate the conditional QFI criterion [16] and as we increase the order of the measured moments this approximation ultimately converges to the QFI criterion. In addition, we also prove that the conditional spin-squeezing parameter criterion detects a larger class of steerable correlations than Reid's criterion [4,17]. As a detailed study, we analyze their performance using analytical results for split one-axis-twisted states, where a hierarchy of criteria is clearly shown. Our work provides an experimentally practical tool to witness non-Gaussian steering, which helps to further investigate the quantum information of non-Gaussian spin states and paves a way to exploiting their promising potential.

II. PHASE ESTIMATION AND THE SPIN-SQUEEZING PARAMETER

In a typical parameter estimation protocol, an operator H imprints an unknown phase θ on quantum state ρ through the unitary evolution $\rho(\theta) = e^{-iH\theta}\rho e^{iH\theta}$. This evolution from the initial state $\rho = \rho(0)$ to $\rho(\theta)$ is usually referred to as the parameter encoding. Then, an observable M is measured on the probe state $\rho(\theta)$. This protocol is repeated m times, and an estimator θ_{est} for θ is constructed as a function of the measurement results. The variance of the estimator $\text{Var}[\theta_{\text{est}}]$ represents the error of the estimate θ_{est}. The goal of quantum phase estimation is to find an unbiased estimator that minimizes this variance. An estimator is called unbiased if it yields the true value of the parameter on average. A simple estimator known as the method of moments (since it is based on simple measurements of low moments of an observable) [29,30] is constructed from the average value of M and yields, in the limit $m \gg 1$, the phase uncertainty $\text{Var}[\theta_{\text{est}}] = \chi^2[\rho, H, M]/m$, where

$$\chi^2[\rho, H, M] := \frac{\text{Var}[\rho, M]}{|\langle [H, M]\rangle_\rho|^2} \tag{1}$$

is the squeezing parameter [30,37]. This parameter is widely used in spin systems and when H and M are collective spin observables, it coincides, up to a normalization factor, with the Wineland spin-squeezing parameter [37]. Intuitively, the phase uncertainty described by χ^2 is small when the expectation value of M depends strongly on variations of the parameter (leading to a large denominator) and when M has a small variance, i.e., exhibits "squeezing" (leading to a small numerator).

For an unbiased estimation, a fundamental limit to the sensitivity is given by the Cramér-Rao bound $\text{Var}[\theta_{\text{est}}] \geqslant (mF[\rho, H, M])^{-1}$, where $F[\rho, H, M]$ is the Fisher information (FI) [45,46]. By optimizing over all observables M, the maximum value of the FI defines the quantum Fisher information (QFI), i.e., $F_Q[\rho, H] = \max_M F[\rho, H, M]$, which determines the sensitivity of the probe state ρ to unitary evolutions generated by H [29,47,48]. Since the method of moments is not necessarily an optimal approach, the squeezing parameter yields a lower bound on the full sensitivity and we have $\chi^{-2}[\rho, H, M] \leqslant F[\rho, H, M] \leqslant F_Q[\rho, H]$ [40,48]. For practical experiments, the achievable sensitivity can be optimized by maximizing $\chi^{-2}[\rho, H, M]$ over a set of measurement operators M that can be realistically implemented [30]. Denoting with $\mathbf{M}$ a basis for such measurements, we can achieve the maximal sensitivity $\max_{M\in\text{span}(\mathbf{M})} \chi^{-2}[\rho, H, M]$.

III. ASSISTED PHASE ESTIMATION WITH CONDITIONAL SQUEEZING PARAMETER

In the previous section we have discussed the typical phase-estimation protocol, which considers a single system. Here, let us consider instead a metrological scenario involving two systems, Alice (A) and Bob (B). The latter acts as the "probe," on which θ is encoded by H, while Alice assists Bob in performing a better measurement. In fact, if the two parties share correlations, a local measurement performed by Alice can improve Bob's measurement sensitivity, if information about the measurement setting Y and result b is communicated to Bob [16,49]. This information allows Bob to choose a measurement observable $M \in \mathrm{span}(\mathbf{M})$ that is optimally tailored to the conditional state $\rho^B_{b|Y}$.

Based on this assisted phase-estimation protocol [16] and using the method of moments, Bob can achieve on average an estimation sensitivity given by the conditional spin-squeezing parameter [50]

$$(\chi^{-2})^{B|A}[\mathcal{A}, H, \mathbf{M}, Y] := \sum_b p(b|Y) \max_{M\in\mathrm{span}(\mathbf{M})} \chi^{-2}\big[\rho^B_{b|Y}, H, M\big]. \tag{2}$$

Here, we introduced the definition of assemblages $\mathcal{A}(b, Y) = p(b|Y)\rho^B_{b|Y}$, which are determined by the local probability distribution $p(b|Y)$ for results b conditioned on Alice's measurement observable Y and Bob's conditional state $\rho^B_{b|Y}$. Note that the ultimate limit for the precision that can be achieved on average by Bob in the assisted phase-estimation protocol for a specific measurement Y for Alice is expressed by the conditional Fisher information [16]

$$F^{B|A}[\mathcal{A}, H, Y] := \sum_b p(b|Y) F_Q\big[\rho^B_{b|Y}, H\big], \tag{3}$$

which corresponds to Bob performing an optimal measurement on each conditional state.

Since $F_Q[\rho, H] \geqslant \chi^{-2}[\rho, H, M]$ holds for arbitrary measurements M [40], we obtain that the conditional spin-squeezing parameter is a lower bound on the conditional Fisher information. Combining this with the Cauchy-Schwarz inequality yields the hierarchy of bounds

$$F^{B|A}[\mathcal{A}, H, Y] \geqslant \sum_b p(b|Y) \frac{\big|\langle [H, M]\rangle_{\rho^B_{b|Y}}\big|^2}{\mathrm{Var}\big[\rho^B_{b|Y}, M\big]} \geqslant \frac{|\langle [H, M]\rangle_{\rho^B}|^2}{\mathrm{Var}^{B|A}[\mathcal{A}, M, Y]}, \tag{4}$$

where $\langle [H, M]\rangle_{\rho^B} = \sum_b p(b|Y)\langle [H, M]\rangle_{\rho^B_{b|Y}}$ and we introduced the conditional variance [4]

$$\mathrm{Var}^{B|A}[\mathcal{A}, H, X] := \sum_a p(a|X)\mathrm{Var}\big[\rho^B_{a|X}, H\big]. \tag{5}$$

In the following, we will use this hierarchy of bounds in order to define and compare steering criteria with increasing potential to uncover steerable states.

IV. CONNECTION TO EPR STEERING

Besides the estimation of the phase θ, we could be interested in estimating its generator H. As θ and H are conjugate variables, the complementarity principle prevents their simultaneous knowledge with arbitrary precision [51]. However, by making use of EPR steering from Alice to Bob in the assisted phase-estimation protocol, an inference of these properties can be realized below the local uncertainty limit, corresponding to a violation of $\mathrm{Var}[\theta_{\mathrm{est}}]\mathrm{Var}[H_{\mathrm{est}}] \geqslant (4m)^{-1}$ [16]. Here, $\mathrm{Var}[H_{\mathrm{est}}]$ is the inference variance: Based on the additional information of Alice's measurement setting X and result a, Bob uses the estimator $h_{\mathrm{est}}(a)$ to predict the result h of his local measurement H with an error $\mathrm{Var}[H_{\mathrm{est}}] := \sum_{a,h} p(a, h|X, H)(h_{\mathrm{est}}(a) - h)^2$. We obtain $\mathrm{Var}[\theta_{\mathrm{est}}]$ from the assisted phase-estimation protocol with a different choice for Alice's measurement settings Y and Bob's measurement M. It is worth mentioning that inference variances are lower bounded by the conditional variances, such that, e.g., $\mathrm{Var}[H_{\mathrm{est}}] \geqslant \mathrm{Var}^{B|A}[\mathcal{A}, H, X]$ [4].

A quantitative criterion that reveals the EPR paradox was first introduced by Reid in 1989 based on the idea that inference variances of two noncommuting observables below the local uncertainty limit are incompatible with the local realism of Bob's system [17]. This idea can be improved by using a metrological complementarity relation between the sensitivity under unitary evolutions generated by H and measurements of H [16], which implies the above-mentioned phase-generator uncertainty relation between θ and H. In the context of quantum foundations, an EPR paradox, or equivalently the possibility of Alice to "steer" Bob's system into seemingly incompatible local quantum states, can be formalized in terms of local hidden state (LHS) models [3]. An EPR paradox is present when the assemblage cannot be explained in terms of a LHS model, $\mathcal{A}(a, X) = \sum_{a,\lambda} p(a|X, \lambda)p(\lambda)\sigma^B_\lambda$, for a classical random variable λ with probability distribution $p(\lambda)$, that determines both Alice's measurement results $p(a|X, \lambda)$ and Bob's LHS σ^B_λ.

Apart from fundamental studies, detecting EPR steering is of interest to certify the presence of necessary quantum resources for a number of quantum information tasks [6]. In Ref. [16], a metrological steering criterion based on the QFI is proposed, stating that for any assemblage $\mathcal{A}$ that admits a LHS model it holds

$$\Delta_1 := F^{B|A}[\mathcal{A}, H, Y] - 4\,\mathrm{Var}^{B|A}[\mathcal{A}, H, X] \leqslant 0, \tag{6}$$

independently of the choices of Alice's measurement.

As the conditional squeezing parameter provides a lower bound to the conditional FI (4), this allows us to formulate the following steering criterion: For any assemblage $\mathcal{A}$ that admits a LHS model we have

$$\Delta_2 := (\chi^{-2})^{B|A}[\mathcal{A}, H, \mathbf{M}, Y] - 4\,\mathrm{Var}^{B|A}[\mathcal{A}, H, X] \leqslant 0. \tag{7}$$

This criterion is one of the main results of this work. The violation of (7) reveals useful EPR steering in the assisted metrological protocol.

V. REDUCTION TO REID'S CRITERION

Reid's seminal criterion for EPR steering [4,17] is based on the uncertainty relation between two non-commuting observables H and M and reads in linearized form

$$\Delta_3 := \frac{|\langle [H, M]\rangle_{\rho^B}|^2}{\mathrm{Var}^{B|A}[\mathcal{A}, M, Y]} - 4\,\mathrm{Var}^{B|A}[\mathcal{A}, H, X] \leqslant 0. \tag{8}$$

The conditional variance (5) represents the average of individual variances for Bob's conditional states, and coincides with the minimal inference variance [4]. For linear observables, Reid's criterion is very powerful for Gaussian states; in a continuous variable setting, it has been shown to be necessary and sufficient for steering detection by Gaussian measurements [3], while it may fail to detect steering in non-Gaussian cases.

We observe from the hierarchy (4) that the conditional squeezing parameter yields a tighter approximation of the metrological steering criterion than Reid's criterion and therefore is able to detect more steerable states: We obtain $\Delta_1 \geqslant \Delta_2 \geqslant \Delta_3$. Interestingly, both criteria Δ_2 and Δ_3 are based on measurements of low moments of the same observable M; the difference between them consists in the way the results are processed.

A crucial advantage of Δ_2 over Reid's criterion Δ_3 is the possibility to adapt the measurement observable M to each conditional state $\rho^B_{b|Y}$ individually [see the maximization in Eq. (2)], while in (8) only a single M is used for the entire assemblage. As we will see below, this leads in particular to an increased potential to reveal non-Gaussian EPR steering in a wider class of states, especially when the set $\mathbf{M}$ contains higher-order moments of measurement operators.

VI. REDUCTION TO LINEAR-ESTIMATE REID'S CRITERION

If Bob's estimator $h_{\text{est}}(a)$ depends linearly on Alice's measurement result a and takes the form $h_{\text{est}}(a) = ga + d$, optimal estimates are obtained by minimizing the inference variance $\text{Var}[H_{\text{est}}]$. Based on that, a well-known linear-estimate Reid's criterion commonly used in experiments is

$$\Delta_4 := \frac{|\langle [H, M]\rangle_{\rho^B}|^2}{\text{Var}[Y + g'M]} - 4\text{Var}[X - gH] \leqslant 0. \tag{9}$$

Note that in a Gaussian system, where quantum correlations are well characterized with first- and second-order moments, the best estimator equals the optimized linear estimator [4]. In this case, we have $\text{Var}[X - gH] = \text{Var}^{B|A}[\mathcal{A}, H, X]$, which leads to $\Delta_3 = \Delta_4$ for Gaussian states.

We define the maximum value of the left-hand side of the above criteria as

$$\delta_i = \max_{H \in \text{span}(\mathbf{H}) M \in \text{span}(\mathbf{M})} \Delta_i, \tag{10}$$

respectively (see Secs. I B, II B, III C, and III D in the Supplemental Material [52]). As a result, a hierarchy of criteria reads

$$\delta_1 \geqslant \delta_2 \geqslant \delta_3 \geqslant \delta_4. \tag{11}$$

In the following, we will compare these criteria for a relevant experimental scenario.

Split spin-squeezed state. Squeezed states play a key role in measurement sensitivity enhancement, overcoming the standard quantum limit in quantum metrology [29,37,53–55]. Experimentally, these are routinely adopted in atom and optical interferometers, e.g., for Ramsey spectroscopy [29], atom clocks [56], and gravitational-wave detection [57]. Many experiments have demonstrated the preparation of spin squeezing in atomic systems [33,58–61]. Here, we focus on spin-squeezed states prepared via one-axis twisting (OAT) dynamics [62], one of the paramount approaches to generate squeezing via atomic collisions.

Initially, the atomic ensemble is prepared in a coherent spin state, which consists of N spins polarized along the x direction. Then, the state evolves according to the OAT Hamiltonian $H = \hbar\chi S_z^2$, where $S_z = \sum_i \sigma_z^{(i)}/2$ is the collective spin operator and χ the interaction strength. This gives

$$|\psi(\mu)\rangle = \frac{1}{\sqrt{2^N}} \sum_{k=0}^{N} \sqrt{\binom{N}{k}}\, e^{-i\frac{\mu}{2}(N/2-k)^2}|k\rangle, \tag{12}$$

where $\mu = 2\chi t$ parametrizes the time evolution, and k labels the basis of Dicke states $|k\rangle$, i.e., symmetric states with k spins up and $N - k$ spins down that are simultaneous eigenstates of S^2 and S_z. For short interaction times, OAT results in near-Gaussian spin-squeezed states, however, as time increases the state becomes oversqueezed and significantly non-Gaussian [30,40].

In order to use such states for assisted phase-estimation protocols, we distribute the N spins to the two parties A and B, see Fig. 1. In practice, this can be achieved by splitting the atomic ensemble into two spatially separated regions, as was done experimentally in Ref. [63]. From a theoretical point of view, this process can be described by a beam-splitter-like transformation on the state Eq. (12), that results in the split spin-squeezed state [64]

$$\begin{aligned}|\Phi(\mu)\rangle =& \frac{1}{2^N} \sum_{N_A=0}^{N} \sum_{k_A=0}^{N_A} \sum_{k_B=0}^{N-N_A} \sqrt{\binom{N}{N_A}\binom{N_A}{k_A}\binom{N-N_A}{k_B}} \\ &\times e^{-i\frac{\mu}{2}(N/2-k_A-k_B)^2}|k_A\rangle_{N_A}|k_B\rangle_{N-N_A}.\end{aligned} \tag{13}$$

Here, N_α is the number of particles in $\alpha \in \{A, B\}$, and $|k_\alpha\rangle_{N_\alpha}$ represents the Dicke state with k_α spins down and $N_\alpha - k_\alpha$ spins up along the z direction.

Relevant properties of split spin-squeezed states, such as polarization or entanglement, can be characterized by measurements of local collective spin observables [42,63,64]. These are defined as $\mathbf{S}^\alpha = \sum_{i\in\alpha} \boldsymbol{\sigma}^{(i)}/2$, where $\boldsymbol{\sigma}^{(i)}$ is the vector of Pauli matrices acting on particle i.

VII. MEASUREMENT OPTIMIZATION

The sharpest formulations of the above criteria are obtained by optimizing the measurement observables X, Y for Alice, and H, M for Bob, respectively. In principle, one would like to optimize over all possible measurements, but for practical purposes we are interested in measurements that are experimentally practical. In the simplest case of linear observables, $\mathbf{S}^{(1)} = (S_x, S_y, S_z)$ is used to describe all the local collective spin measurements for Alice and Bob and leads to steering criteria that contain the average values and variances of these operators. The measurements required to evaluate the linear criterion $\Delta_2^{(1)}$, e.g., $\langle S_x\rangle$ and $\langle S_z^2\rangle$, are routinely performed in experiments with atomic ensembles [29]. We consider the squeezing direction of the split spin-squeezed states to define the $\mathbf{z}$ axis, and the antisqueezing direction to define the $\mathbf{y}$ axis. Therefore, Alice's measurement settings X and Y can be also restricted to the yz plane. To optimize the measurement directions of Alice's and Bob's observables, we construct the moment matrix [30] and covariance matrix for Bob's conditional states (see Supplemental Material [52] Sec. II B for details).

For the metrological characterization of non-Gaussian spin states, higher-order moments of physical observables are of great importance [30]. Note that in Ref. [25], the criterion

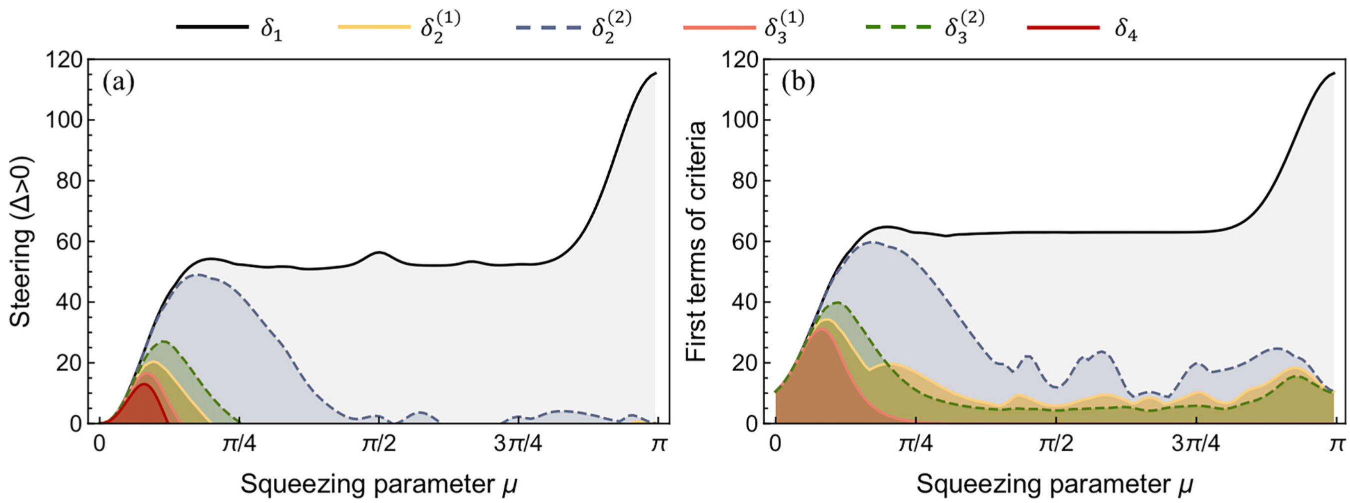

FIG. 2. Steering detection for split spin states $|\Phi(\mu)\rangle$ with total atom number $N = 21$. (a) A hierarchy of criteria with optimized measurement operators, where the dashed lines represent the optimized second-order criteria $\delta_2^{(2)}$ and $\delta_3^{(2)}$ involving the nonlinear spin operator ensemble $\mathbf{S}^{(2)}$. (b) Comparison among first terms of optimized criteria $\delta_{1,2,3}$, which is also the chain of inequalities in (4).

Δ_4 was first applied to measurements of nonlinear operators. Here, we also extend the criterion Δ_2 based from the conditional spin-squeezing parameter and Reid's criterion Δ_3 to nonlinear versions that can be further optimized by taking into account a set $\mathbf{M}$ of possible higher-order measurements for Bob (i.e., M). When M is a product of up to n linear spin observables, optimized nonlinear $\delta_{2,3}^{(n)}$ are still upper bounded by the Fisher criterion δ_1, but this bound becomes increasingly tight as n grows larger. To be concrete, let us start with the second-order criteria, $\mathbf{M} = \mathbf{S}^{(2)}$, where we choose the measurement observable M from an ensemble of linear and quadratic spin operators $\mathbf{S}^{(2)} = (S_x, S_y, S_z, S_x^2, S_y^2, S_z^2, \frac{1}{2}\{S_x, S_y\}, \frac{1}{2}\{S_x, S_z\}, \frac{1}{2}\{S_y, S_z\})$. We refer to Ref. [35] for an experimental measurement of such observables. Moreover, it was shown in Ref. [31] how such observables may become accessible by a second OAT evolution before the measurement of a linear spin observable. For our second-order criteria $\Delta_2^{(2)}$ and $\Delta_3^{(2)}$, Alice's measurements X, Y and Bob's generator H for the phase imprinting evolution are still linear, but the measurement operator for Bob takes into account second-order operators $M = \mathbf{m} \cdot \mathbf{S}^{(2)}$, with $\mathbf{m} \in \mathbb{R}^9$. Analogously, we can also obtain higher-order criteria, leading to another chain of inequalities

$$\delta_i^{(1)} \leqslant \delta_i^{(2)} \leqslant \delta_i^{(3)} \leqslant \cdots \leqslant \delta_1, \tag{14}$$

for both criteria $i = 2, 3$.

As illustrated in Fig. 2(a), for a split spin-squeezed state with $N = 21$ atoms, we obtain analytically optimized criteria δ_i evolved with the OAT squeezing parameter μ, where both the hierarchy relations (11) and (14) are shown clearly. At small squeezing levels μ, the evolution generates near-Gaussian split spin-squeezed states: All criteria detect steering and tend to converge. However, if only linear operators are considered, Reid's criteria δ_4 and $\delta_3^{(1)}$ decay soon in the non-Gaussian area for longer evolution times, while the conditional spin-squeezing parameter criterion $\delta_2^{(1)}$ reveals steering in a wider range of states. Furthermore, when it comes to the nonlinear version, the second-order $\delta_2^{(2)}$ shows significant advantages. The Fisher criterion δ_1 bounds all other criteria from above during the entire dynamics, but the conditional spin parameter criterion δ_2 is more practical experimentally. Moreover, as a consequence of the convergence of nonlinear spin-squeezing parameters to the quantum Fisher information for an optimal observable [30], it will tend towards the Fisher criterion as higher-order measurement operators are involved. All of these criteria compare the phase-estimation sensitivity [first term in Eqs. (6)–(9)] to the estimation variance for the generator (second term). While the second term hardly varies between the criteria, the hierarchy can be traced back to the chain of inequalities (4), which is reflected in Fig. 2(b).

VIII. CONCLUSIONS

We have proposed a non-Gaussian steering criterion based on conditional spin-squeezing parameters. By introducing nonlinear operators and optimizing measurement within accessible higher-order observables, the criterion shows an improved ability to reveal EPR steering with a larger range of non-Gaussian states. This approach is more powerful than Reid's criteria, and leads to a hierarchy of approximations to the most powerful metrological approach to steering. One key advantage of metrology-based steering criteria is the ability to adjust the measurement observable to each conditional state individually. Our steering criterion is experimentally feasible and constitutes a general method to reveal non-Gaussian steering in a bipartite scenario. This work provides a powerful approach to further investigate nonlinear EPR correlations in non-Gaussian states and takes a step further to unlock the promising applications for non-Gaussian systems.

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