Operational interpretation and estimation of quantum trace-norm asymmetry based on weak-value measurement and some bounds

Agung Budiyono [*]

Research Center for Quantum Physics, National Research and Innovation Agency, South Tangerang 15314, Indonesia; Research Center for Nanoscience and Nanotechnology, Bandung Institute of Technology, Bandung 40132, Indonesia; and Department of Engineering Physics, Bandung Institute of Technology, Bandung 40132, Indonesia

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The asymmetry of a quantum state relative to a translational group is a central concept in many areas of quantum science and technology. An important and geometrically intuitive measure of translational asymmetry of a state is given by the trace-norm asymmetry, which is defined as the trace norm of the commutator between the state and the generator of the translation group. While trace-norm asymmetry satisfies all the requirements for a bona fide measure of translational asymmetry of a state within the quantum resource theoretical framework, its meaning in terms of laboratory operations is still missing. Here, we first show that the trace-norm asymmetry is equal to the average absolute imaginary part of the weak value of the generator of the translation group optimized over all possible orthonormal bases of the Hilbert space. Hence, it can be estimated via the measurement of weak value combined with a classical optimization in the fashion of quantum variational circuit which may be implemented using the near-term quantum hardware. We then use the link between the trace-norm asymmetry and the nonreal weak value to derive the relation between the trace-norm asymmetry with other basic concepts in quantum statistics. We further obtain trade-off relations for the trace-norm asymmetry and quantum Fisher information, having analogous forms to the Kennard-Weyl-Robertson uncertainty relation.

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I. INTRODUCTION

Quantum information theory promises novel schemes of information processing technology that are exceedingly far more efficient and secure than those based on classical means. This is achieved by harnessing various nonclassical aspects of quantum mechanics. One such aspect, which has garnered a lot of attention lately, is the concept of quantum asymmetry [1,2]. Quantum asymmetry captures the behavior of a quantum state under (the unitary representation of) a certain group of transformation. Like other nonclassical aspects of quantum mechanics, it originates from the quantum superposition principle. Significant efforts over the past decade have shown that this manifestation of quantum superposition as asymmetry is a prerequisite for quantum frame alignment [3–5] and quantum metrology [6,7], and it is a key concept in the study of quantum speed limit [8,9] and quantum thermodynamics [4,10–14]. It is thus important to be able to characterize and quantify the asymmetry of an unknown quantum state using well-defined operations in the laboratory.

Consider a Hermitian operator K on a finite-dimensional Hilbert space generating a one-parameter group of translation unitaries: $\{U_{K,\theta} = e^{-iK\theta}, \theta \in \mathbb{R}\}$. A quantum state represented by a density operator ϱ on the Hilbert space is symmetric relative to the translation group generated by K if it is invariant under the translation unitaries, i.e., $e^{-iK\theta}\varrho e^{iK\theta} = \varrho$ for all $\theta \in \mathbb{R}$. All other states are asymmetric relative to the translation group. For a quantum state ϱ to be symmetric relative to the translation group generated by K, the state must therefore commute with the generator of the translation, i.e., $[K, \varrho] := K\varrho - \varrho K = 0$, so that they are jointly diagonalizable. Assuming that K is nondegenerate, and denoting the eigenstates of K as $\{|k\rangle\}$, we thus have $\varrho = \sum_k p_k |k\rangle \langle k|$, where $\{p_k\}$ are the real and nonnegative eigenvalues of ϱ satisfying $\sum_k p_k = 1$. Hence, a symmetric state relative to a translation group is a convex combination or a classical mixture of the eigenstates $\{|k\rangle\}$ of the generator K of the translation. This means that a translationally asymmetric state is a superposition of some elements of the eigenstates $\{|k\rangle\}$ of K. Namely, it is coherent with respect to the orthonormal basis $\{|k\rangle\}$ [2].

The asymmetry of a quantum state relative to a group of translation is better understood by regarding it as a resource in some information processing tasks. This insight has led to the application of the rigorous mathematical framework of quantum resource theory [15,16] to characterize, quantify, and manipulate the translational asymmetry [3,17,18]. In the general framework of quantum resource theory, permissible quantum operations are restricted to those which can be easily implemented reflecting certain physical and/or operational constraints. Such operations are regarded as free. Accordingly, quantum states are divided into those that can be prepared by the set of free operations, called free states, and those that cannot be prepared using any free operation and free state, which are regarded as resourceful states. In the resource theory of translational asymmetry, the free operations are given naturally by the set of translationally covariant quantum operations, i.e., those which commute with the translation

[*]agungbymlati@gmail.com

unitaries [2]. Such operations map symmetric states onto symmetric states, and they cannot create asymmetric states from symmetric states relative to the translation group. Hence, symmetric states are regarded as free, and asymmetric states are resourceful.

An important measure of translational asymmetry obtained within the framework of quantum resource theory is the trace-norm asymmetry [2,8]. The trace-norm asymmetry of a state ϱ relative to a translation group generated by a Hermitian operator K is defined as

$$A_{\mathrm{Tr}}(\varrho;K) := \tfrac{1}{2}\|[\varrho,K]\|_1, \tag{1}$$

where $\|O\|_1 := \mathrm{Tr}\sqrt{OO^\dagger}$ is the trace norm (also called 1-norm) of the operator O. The trace-norm asymmetry is not only geometrically intuitive, it also satisfies certain plausible requirements for a bona fide measure of asymmetry within the resource theoretical framework. Most importantly, it satisfies (i) faithfulness, i.e., it vanishes if and only if the state is symmetric, and (ii) monotonicity, i.e., it is nonincreasing under the translationally covariant operations $A_{\mathrm{Tr}}(\Phi(\varrho);K) \leqslant A_{\mathrm{Tr}}(\varrho;K)$, where $\Phi(\cdot)$ is a completely positive trace-nonincreasing linear map satisfying the translationally covariant condition $\Phi(e^{-iK\theta}\varrho e^{iK\theta}) = e^{-iK\theta}\Phi(\varrho)e^{iK\theta}$. While the trace-norm asymmetry offers a closed formula, its meaning in terms of laboratory operations is not clear except when the state is pure, $\varrho = |\psi\rangle\langle\psi|$, in the case of which the trace-norm asymmetry can be expressed as

$$A_{\mathrm{Tr}}(|\psi\rangle\langle\psi|;K) = \Delta_K(|\psi\rangle\langle\psi|), \tag{2}$$

where, for a generic state ϱ, $\Delta_K(\varrho)^2 := \mathrm{Tr}(K^2\varrho) - [\mathrm{Tr}(K\varrho)]^2$ is the quantum variance of the outcomes of the measurement of observable K over the state ϱ. A better understanding on the operational meaning of the trace-norm asymmetry may suggest a fresh insight into its application to characterize certain quantum (information) protocols, and its estimation in the laboratory. It may also reveal the relation between the trace-norm asymmetry and other measures of asymmetry and quantum coherence, and between the trace-norm asymmetry and other basic concepts in quantum statistics.

In the present work, we first show that the trace-norm asymmetry relative to a translation group is equal to the average absolute nonreal part of the weak value [19–23] of the generator of the translation, maximized over all possible orthonormal bases of the Hilbert space. Hence, it can be estimated in experiment directly, i.e., without recoursing to full state tomography, through the measurement of the weak value [19,20,22,24–32] combined with a classical optimization procedure, in the fashion of variational quantum circuit [33]. These estimation schemes should be realizable using the presently available NISQ (noisy intermediate-scale quantum) hardware [34]. Moreover, they lend themselves to the operational interpretation of the trace-norm asymmetry. Using the mathematical link between the trace-norm asymmetry and the nonreal part of the weak value, we then derive upper bounds for the trace-norm asymmetry in terms of quantum standard deviation, quantum Fisher information [35–39], nonreal (nonclassical) values of the Kirkwood-Dirac (KD) quasiprobability [40–42], l_1-norm coherence [43], and purity of the quantum state. We also obtain a lower bound for the trace-norm asymmetry and the quantum Fisher information in terms of the maximum average noncommutativity between the generator of the translation group and any other bounded Hermitian operators on the Hilbert space. This leads to the derivation of trade-off relations for the trace-norm asymmetry and quantum Fisher information similar to the Kennard-Weyl-Robertson uncertainty relation, suggesting an interpretation as the trade-off relation for the genuine quantum part of the uncertainty. Analytical computations of the results for the case of a single qubit are given in Appendix A.

II. OPERATIONAL INTERPRETATION AND ESTIMATION OF TRACE-NORM ASYMMETRY VIA WEAK VALUE MEASUREMENT

Let us first summarize the concept of weak value whose statistics we will use to characterize the trace-norm asymmetry defined in Eq. (1).

Definition 1. The weak value associated with a Hermitian operator K on a Hilbert space $\mathcal{H}$ with a preselected state represented by a density operator ϱ on $\mathcal{H}$ and a postselected pure state represented by a ray $|\phi\rangle$ in $\mathcal{H}$ is defined as follows [19–23]:

$$K_{\mathrm{w}}(\Pi_\phi|\varrho) := \frac{\mathrm{Tr}(\Pi_\phi K\varrho)}{\mathrm{Tr}(\Pi_\phi\varrho)}, \tag{3}$$

where $\Pi_\phi := |\phi\rangle\langle\phi|$ is a projector over a subset of the Hilbert space spanned by $|\phi\rangle$, and we have assumed $\mathrm{Tr}(\Pi_\phi\varrho) \neq 0$.

Note that the weak value is in general a complex number. Moreover, its real part may lie outside of the eigenvalues spectrum of K. Such complex weak values, and weak values with real part lying outside of the spectrum of K, are called strange weak values, and have been used to prove quantum contextuality [44–47]. In the past decade, there has been a surge of interest in the concept of strange weak values, in particular for its close relation with the anomalous values of the KD quasiprobability [40–42], in broad fields of quantum science and technology: quantum state tomography [48–50], quantum thermodynamics [47,51,52], quantum metrology [47,53,54], quantum information scrambling or quantum chaos in many-body systems [55,56], and the characterization of different forms of quantum fluctuations [28]. It has also been very recently used to characterize coherence and asymmetry [57,58]. Remarkably, the real and imaginary parts of the weak value can be measured or estimated in experiment via a number of methods [19,20,22,24–32]. Below we shall be concerned specifically with the imaginary part of the weak value.

We show that, combined with a classical optimization procedure, the measurement of weak value can be used to estimate the trace-norm asymmetry of an unknown quantum state in the laboratory, without first recoursing to quantum state tomography. Hereon, we shall consider quantum systems with a finite-dimensional Hilbert space.

First, let us define the following quantity which we introduced earlier in Ref. [58].

Definition 2. Let $\mathcal{B}_o(\mathcal{H})$ denote the set of all the orthonormal bases of a Hilbert space $\mathcal{H}$. Then, given a state ϱ and a Hermitian operator K on $\mathcal{H}$, we define a real-valued nonnegative quantity $A_{\mathrm{w}}(\varrho;K)$ as the average of the absolute imaginary part of the weak value $K_{\mathrm{w}}(\Pi_x|\varrho)$ defined in Eq. (3)

over the probability $\Pr(x|\varrho) = \mathrm{Tr}(\Pi_x\varrho)$ to get x in the measurement described by a projection-valued measure $\{\Pi_x\}$, maximized over all the orthonormal bases $\{|x\rangle\}$ of the Hilbert space, i.e.,

$$\begin{aligned} A_{\mathrm{w}}(\varrho; K) &:= \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_x |\mathrm{Im} K_{\mathrm{w}}(\Pi_x|\varrho)| \Pr(x|\varrho) \\ &= \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_x |\mathrm{Im}\, \langle x|K\varrho|x\rangle\, | \\ &= \frac{1}{2} \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_x |\, \langle x|[K,\varrho]|x\rangle\, |. \end{aligned} \tag{4}$$

It is clear from the last line of Eq. (4) that $A_{\mathrm{w}}(\varrho; K)$ captures the maximum noncommutativity between the state ϱ and the Hermitian operator K over all possible orthonormal bases $\{|x\rangle\} \in \mathcal{B}_o(\mathcal{H})$ under the l_1 norm. We argued in Ref. [58] that it can be used to quantify the asymmetry of ϱ relative to the translation group generated by K fulfilling certain plausible requirements. Below, we show that $A_{\mathrm{w}}(\varrho; K)$ is in fact equal to the trace-norm asymmetry defined in Eq. (1).

Proposition 1. The trace-norm asymmetry of a state ϱ relative to a translation group generated by a Hermitian operator K defined in Eq. (1) can be expressed in terms of the imaginary part of the weak value of K as

$$A_{\mathrm{Tr}}(\varrho; K) = A_{\mathrm{w}}(\varrho; K), \tag{5}$$

where $A_{\mathrm{w}}(\varrho; K)$ is defined in Eq. (4).

Proof. First, recall that the trace norm of an operator O is given by the total sum of the singular values or the eigenvalues modulus of the operator, i.e., $\|O\|_1 = \sum_i |o_i|$, where $\{o_i\}$ is the set of eigenvalues of O. Next, note that $[K, \varrho]$ is a skew Hermitian operator. Hence, it has the following spectral decomposition $[K, \varrho] = \sum_i \lambda_i |\lambda_i\rangle \langle\lambda_i|$, where $\{\lambda_i\}$ is the set of purely imaginary eigenvalues of $[K, \varrho]$ with the corresponding orthonormal set of eigenvectors $\{|\lambda_i\rangle\}$. We thus have, upon inserting this into the right-hand side of Eq. (4),

$$\begin{aligned} A_{\mathrm{w}}(\varrho; K) &= \frac{1}{2} \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_x \left| \sum_i \lambda_i \langle x|\lambda_i\rangle \langle\lambda_i|x\rangle \right| \\ &= \frac{1}{2} \sum_{x_*} \left| \sum_i \lambda_i \langle x_*|\lambda_i\rangle \langle\lambda_i|x_*\rangle \right| &(6) \\ &\leqslant \frac{1}{2} \sum_i |\lambda_i| \sum_{x_*} |\, \langle x_*|\lambda_i\rangle\, |^2 &(7) \\ &= \frac{1}{2} \sum_i |\lambda_i| = \frac{1}{2} \|[\varrho, K]\|_1 &(8) \\ &= A_{\mathrm{Tr}}(\varrho; K). &(9) \end{aligned}$$

Here, $\{|x_*\rangle\}$ in Eq. (6) is an orthonormal basis which reaches the supremum, the inequality in Eq. (7) is due to the triangle inequality, and we have used the completeness relation for $\{|x_*\rangle\}$ to obtain Eq. (8). On the other hand, one can see in Eq. (6) that the equality, i.e., the upper bound, is always attained by choosing $\{|x_*\rangle\} = \{|\lambda_j\rangle\}$ so that we get Eq. (5). ■

Proposition 1 thus reveals a link between two seemingly different basic concepts of quantum mechanics: the trace-norm asymmetry which quantifies the amount of asymmetry of the state relative to a translation group, and the anomalous nonreal part of the weak value of the generator of the translation group. A concrete analytical computation of the equality in Eq. (5) for a single qubit with arbitrary state and generator of translation group is given in Appendix A 1. Note that the computation of the left-hand side of Eq. (5) is equivalent to finding a basis of the Hilbert space which diagonalizes $[K, \varrho]$. By contrast, to compute the right-hand side of Eq. (5), we have to find a basis which optimizes Eq. (4).

As mentioned earlier, the weak value can be experimentally obtained via a number of methods [19,20,22,24–32]. Proposition 1 thus offers a scheme to experimentally estimate the trace-norm asymmetry of an unknown quantum state relative to a translation group, directly, i.e., without recoursing to full state tomograpy. To do this, one first makes the measurement of the weak value $K_{\mathrm{w}}(\Pi_x|\varrho)$ of the generator of the translation, averages its absolute imaginary part over the probability of outcomes $\Pr(x|\varrho) = \mathrm{Tr}(\Pi_x\varrho)$ of projective von Neumann measurement $\{\Pi_x\}$, and maximizes over all possible orthonormal bases $\{|x(\vec{\lambda})\rangle\} \in \mathcal{B}_o(\mathcal{H})$ of the Hilbert space $\mathcal{H}$, where $\vec{\lambda}$ is the parameters whose variation over their ranges of values scans all the orthonormal bases of the Hilbert space. The optimization over $\vec{\lambda}$ is carried out using some classical methods. This estimation scheme of the trace-norm asymmetry therefore requires the ability to implement a parametrized unitary circuit $V_{\vec{\lambda}}$ which prepares all the orthonormal bases $\{|x(\vec{\lambda})\rangle\}$ of the Hilbert space from the standard basis. Hence, we have a hybrid quantum-classical scheme in the fashion of quantum variational circuit [33] which can be realized using the NISQ hardware [34]. This scheme of estimation of the trace-norm asymmetry thus provides an operational interpretation. In contrast to this, the estimation of the trace-norm asymmetry based on full state tomography followed by diagonalization clearly does not offer operational interpretation of the trace-norm asymmetry.

The equality in Eq. (5) suggests that the trace-norm asymmetry can be given physical and statistical interpretation in terms of those of the weak values. For example, within the scheme of the estimation of the weak value based on weak measurement with postselection [19–23], $A_{\mathrm{w}}(\varrho; K)$, and thus the trace-norm asymmetry $A_{\mathrm{Tr}}(\varrho; K)$ of the state ϱ relative to the translation group generated by K, can be interpreted as the maximal disturbance of the state due to the translation unitary generated by K [59,60]. This goes in line with the fact that the trace-norm asymmetry indeed gives the rate of change of the state under the translation unitary generated by K as $\|U_{K,\delta\theta}\varrho U_{K,\delta\theta}^{\dagger} - \varrho\|_1 = \|[\varrho, K]\|_1\delta\theta + o(\delta\theta^2)$. This is also the reason why larger trace-norm asymmetry is desirable in quantum parameter estimation, as will be corroborated in the next section. On the other hand, within the scheme of the measurement of weak value based on two sequences of strong measurement [26,27], $A_{\mathrm{w}}(\varrho; K)$ can be interpreted as the maximal disturbance of the state ϱ due to the nonselective measurement of $\{\Pi_k\}$, the eigenbasis of K. Moreover, following the statistical interpretation of weak value developed in Refs. [61–63], $A_{\mathrm{w}}(\varrho; K)$ can also be interpreted as the maximum absolute error in the optimal estimation of K based on the outcomes $\{x\}$ of the projective measurement described by $\{\Pi_x\}$. Finally, we note that the idea that the imaginary part of the weak value captures the quantum fluctuations has been put forward in Refs. [63,64].

III. UPPER AND LOWER BOUNDS FOR TRACE-NORM ASYMMETRY AND UNCERTAINTY RELATIONS

As an immediate application of the mathematical equality of Eq. (5) connecting the trace-norm asymmetry relative to a translation group and the nonreal part of the weak value of the generator of the translation, we may obtain results for the former by studying the statistics of the latter. Using this approach, in this section, we derive some relations between the trace-norm asymmetry and certain important concepts in quantum statistics.

A. Upper bounds: Quantum standard deviation, quantum Fisher information, l_1-norm coherence and purity

First, we have the following proposition.

Proposition 2. The trace-norm asymmetry of a state ϱ relative to a translation group generated by a Hermitian operator K is bounded from above by the quantum standard deviation of K over ϱ, i.e.,

$$A_{\rm Tr}(\varrho;K) \leqslant \Delta_K(\varrho), \tag{10}$$

where equality is reached for all pure states.

Proof. First, as shown in Appendix B, $A_{\rm w}(\varrho;K)$ defined in Eq. (4) is upper bounded by the quantum standard deviation of K over ϱ as

$$A_{\rm w}(\varrho;K) \leqslant \Delta_K(\varrho). \tag{11}$$

Combining this with Eq. (5), we thus obtain Eq. (10). For pure states, as mentioned in Eq. (2), the trace-norm asymmetry of $\varrho = |\psi\rangle\langle\psi|$ relative to the translation group generated by K is exactly equal to the quantum standard deviation of K over $\varrho = |\psi\rangle\langle\psi|$ so that the inequality in Eq. (10) becomes equality. The case of a single qubit is given in Appendix A 2. ■

Proposition 2 thus generalizes Theorem 1 of Ref. [58] where we have derived Eq. (11) and showed that equality is obtained for a pure state single qubit. A different sketch of a proof of Proposition 2 based on state purification and the fact that $A_{\rm Tr}(\varrho;K) = \|[\varrho,K]\|_1/2$ is a monotonic measure of asymmetry and $\Delta_K(\varrho)$ is not is suggested in Ref. [8]. Here we have proven it using a decomposition of the trace-norm asymmetry in terms of the average absolute imaginary part of the weak value of the generator of the translation relative to which the asymmetry is defined.

Note further that unlike the trace-norm asymmetry (or any other measures of coherence), the upper bound $\Delta_K(\varrho)$ in Eq. (10), i.e., the quantum standard deviation, is not sensitive to whether the state is pure or mixed. To see this, consider, for instance, two extreme cases of maximally coherent state $|\psi_{\rm mc}\rangle = \frac{1}{\sqrt{d}}\sum_k e^{i\theta_k}|k\rangle$, $\theta_k \in \mathbb{R}$ and maximally mixed state $\varrho_{\rm mm} = \mathbb{I}/d$, where d is the dimension of the Hilbert space. Then, in both cases, the upper bound in Eq. (10) yields the same value: $\Delta_K(|\psi_{\rm mc}\rangle\langle\psi_{\rm mc}|) = \Delta_K(\varrho_{\rm mm})$. Hence, quantum standard deviation cannot distinguish the maximally coherent state from the maximally mixed state. This is because the quantum standard deviation does not only capture the genuine quantum uncertainty arising from the noncommutativity between the state ϱ and the generator K, it also counts the uncertainty arising from the (classical) statistical mixing in the preparation of the state ϱ when it is not pure. It is desirable to have an upper bound which depends on the purity of the state. Such a bound will be given later.

Proposition 2 thus suggests that $A_{\rm Tr}(\varrho;K) = A_{\rm w}(\varrho;K)$ can be seen as to capture the genuine quantum part of the uncertainty arising in the measurement of the observable K over the quantum system prepared in a state ϱ which originates from their noncommutativity. Namely, $A_{\rm Tr}(\varrho;K) = A_{\rm w}(\varrho;K)$ satisfies the following plausible requirements for any quantity which quantifies the genuine quantum uncertainty of K in ϱ [65–68]: (i) it vanishes if and only if K and ϱ commute; (ii) it is convex, i.e., $A_{\rm w}(\sum_j p_j\varrho_j;K) \leqslant \sum_j p_j A_{\rm w}(\varrho_j;K)$, where $\{p_j\}$, $\sum_j p_j = 1$, are probabilities of preparing the system in the states $\{\varrho_j\}$; and (iii) it is upper bounded by the quantum standard deviation and they are equal for all pure states. The property (ii) of convexity of $A_{\rm w}(\varrho;K)$ can be seen directly from the definition of $A_{\rm w}(\varrho;K)$ in Eq. (4) due to the triangle inequality. The above observation also suggests to interpret the difference between the quantum standard deviation of K over ϱ and the trace-norm asymmetry of ϱ relative to a translation group generated by K, i.e., $\Delta_K(\varrho) - A_{\rm w}(\varrho;K)$, as the classical part of the measurement uncertainty.

Let us proceed to use the equality in Eq. (5) to explore a connection between the trace-norm asymmetry and quantum Fisher information [35–39]. Consider an imprinting of a scalar parameter θ to the quantum state of a probe via a quantum process: $\varrho_\theta = \Phi_\theta(\varrho)$, where Φ_θ is a completely positive trace-preserving map, and ϱ is the initial quantum state of the probe. Let $\mathcal{M}_{\rm POVM}(\mathcal{H})$ denote the set of all POVMs (positive-operator-valued measures): $\{M_x\}$, $M_x \geqslant 0$, $\sum_x M_x = \mathbb{I}$, describing the most general measurement allowed by quantum mechanics with the outcomes $\{x\}$, when the postmeasurement states are not of concern.

Definition 3. The quantum Fisher information about the parameter θ encoded in ϱ_θ is defined as [37–39]

$$\mathcal{J}_\theta(\Phi_\theta(\varrho)) := \sup_{\{M_x\}\in\mathcal{M}_{\rm POVM}(\mathcal{H})} \sum_x [\partial_\theta \ln {\rm Pr}(x|\varrho_\theta)]^2 {\rm Pr}(x|\varrho_\theta), \tag{12}$$

where ${\rm Pr}(x|\varrho_\theta) = {\rm Tr}(M_x\varrho_\theta)$ is the probability to get x in the measurement described by a POVM $\{M_x\}$.

Quantum Fisher information is a central quantity in quantum metrology based on quantum parameter estimation [35,36,69], wherein one wishes to estimate the value of the parameter θ encoded in the quantum state of the probe ϱ_θ via some measurement $\{M_x\}$. It characterizes the optimal precision of such parameter estimation based on the quantum Cramér-Rao inequality [37–39]. Below, we are interested in the case where the state of the probe ϱ_θ is obtained using a translation unitary generated by a Hermitian operator K, i.e., $\varrho_\theta = \Phi_\theta(\varrho) = e^{-iK\theta}\varrho e^{iK\theta}$, and denote the associated quantum Fisher information as $\mathcal{J}_\theta(\varrho_\theta;K)$. It is known that, in this case, the quantum Fisher information is independent of the parameter θ, i.e., $\mathcal{J}_\theta(\varrho_\theta;K) = \mathcal{J}_\theta(\varrho;K)$. Moreover, it has also been shown that the quantum Fisher information $\mathcal{J}_\theta(\varrho;K)$ is a faithful and monotonic measure of the asymmetry of the state ϱ relative to the translation group generated by K [2]. It is thus instructive to study the relation between the quantum Fisher information and the trace-norm asymmetry.

We obtain the following result.

Proposition 3. The trace-norm asymmetry of ϱ relative to a translation group generated by a Hermitian operator K is upper bounded by the quantum Fisher information about a parameter θ contained in the state ϱ_θ obtained via a unitary imprinting generated by K as

$$A_{\rm Tr}(\varrho;K)^2 \leqslant \mathcal{J}_\theta(\varrho;K)/4. \quad (13)$$

Moreover, for pure states, the inequality becomes equality and the supremum in Eq. (12) is obtained by a measurement described by a projection-valued measure.

Proof. First, from Eq. (12) and noting the fact that the set $\mathcal{M}_{\rm POVM}(\mathcal{H})$ of measurements described by POVM $\{M_x\}$ includes the set $\mathcal{M}_{\rm PVM}(\mathcal{H})$ of measurements described by projection-valued measure $\{\Pi_x\}$, we have

$$\mathcal{J}_\theta(\varrho;K) = \mathcal{J}_\theta(\varrho_\theta;K) \geqslant \sup_{\{\Pi_x\}\in\mathcal{M}_{\rm PVM}(\mathcal{H})} \sum_x [\partial_\theta \ln {\rm Pr}(x|\varrho_\theta)]^2 {\rm Pr}(x|\varrho_\theta). \quad (14)$$

On the other hand, from the unitary imprinting $\varrho_\theta = e^{-iK\theta}\varrho e^{iK\theta}$, we have $\partial_\theta\varrho_\theta = -i[K,\varrho_\theta]$, so that noting ${\rm Pr}(x|\varrho_\theta) = {\rm Tr}(\Pi_x\varrho_\theta)$, the imaginary part of the weak value of K with the preselected state ϱ_θ and postselected state $|x\rangle$ can be expressed as

$${\rm Im}K_{\rm w}(\Pi_x|\varrho_\theta) = \frac{1}{2i}\frac{\langle x|[K,\varrho_\theta]|x\rangle}{\langle x|\varrho_\theta|x\rangle} = \frac{1}{2}\frac{\partial_\theta {\rm Pr}(x|\varrho_\theta)}{{\rm Pr}(x|\varrho_\theta)}. \quad (15)$$

Using this relation in Eq. (14), we thus obtain

$$\begin{aligned}\mathcal{J}_\theta(\varrho;K) &\geqslant 4 \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_x [{\rm Im}K_{\rm w}(\Pi_x|\varrho_\theta)]^2 {\rm Tr}(\Pi_x\varrho_\theta) \\ &\geqslant 4\Big[\sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_x |{\rm Im}K_{\rm w}(\Pi_x|\varrho_\theta)| {\rm Tr}(\Pi_x\varrho_\theta)\Big]^2 \quad (16)\\ &= 4A_{\rm w}(\varrho_\theta;K)^2 = 4A_{\rm w}(\varrho;K)^2 \quad (17)\\ &= 4A_{\rm Tr}(\varrho;K)^2. \quad (18)\end{aligned}$$

Here, Eq. (16) holds due to the Jensen inequality, and to get Eq. (17) we have used Eq. (4) and the fact that $A_{\rm w}(\varrho;K)$ is invariant under translation unitary $U_{K,\theta} = e^{-iK\theta}$, i.e., $A_{\rm w}(\varrho_\theta;K) = A_{\rm w}(U_{K,\theta}\varrho U_{K,\theta}^\dagger;K) = A_{\rm w}(\varrho;K)$, which can be proven directly from the definition [58]. Finally Eq. (18) is just Eq. (5).

For pure states, $\varrho = |\psi\rangle\langle\psi|$, from Proposition 2 we have $A_{\rm Tr}(|\psi\rangle\langle\psi|;K)^2 = \Delta_K^2[|\psi\rangle\langle\psi|]$. On the other hand, it is known that for pure states with the unitary imprinting $|\psi_\theta\rangle = e^{-iK\theta}|\psi\rangle$, we also have $\mathcal{J}_\theta[|\psi\rangle\langle\psi|;K] = 4\Delta_K^2[|\psi\rangle\langle\psi|]$ [39]. From these two equalities for pure states, we thus obtain Eq. (13) with inequality replaced by equality, i.e., $A_{\rm Tr}(\varrho;K)^2 = \mathcal{J}_\theta(\varrho;K)/4$. Note that as shown in Appendix A 3, for a single qubit, this equality applies even for arbitrary mixed states. Hence, for pure states $\varrho = |\psi\rangle\langle\psi|$, the quantum Fisher information can be expressed as, using Eqs. (5) and (4),

$$\begin{aligned}\mathcal{J}_\theta(|\psi\rangle\langle\psi|;K) &= 4A_{\rm Tr}(\varrho;K)^2 = 4A_{\rm w}(\varrho;K)^2 \\ &= 4 \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})}\Big[\sum_x |{\rm Im}K_{\rm w}(\Pi_x|\varrho)|{\rm Pr}(x|\varrho)\Big]^2 \\ &= \sup_{\{\Pi_x\}\in\mathcal{M}_{\rm PVM}(\mathcal{H})}\Big[\sum_x \Big|\frac{\partial_\theta{\rm Pr}(x|\varrho)}{{\rm Pr}(x|\varrho)}\Big|{\rm Pr}(x|\varrho)\Big]^2, \quad (19)\end{aligned}$$

where we have again used Eq. (15). Comparing Eq. (19) to Eq. (12), for pure states, the supremum in Eq. (12) is thus obtained for measurement described by a projection-valued measure. ■

Equation (13) of Proposition 3 in particular shows that a quantum state ϱ with larger trace-norm asymmetry relative to a translation group generated by a Hermitian operator K is sufficient for a larger quantum Fisher information about θ conjugate to K. In view of the quantum Cramér-Rao inequality, such a state is thus desirable, i.e., it may lead to a better precision, in quantum parameter estimation of θ.

Next, we use the result of Eq. (5) to connect the trace-norm asymmetry to an apparently different concept of nonclassicality captured by the nonclassical values of KD quasiprobability.

Definition 4. The KD quasiprobability associated with a state ϱ on a Hilbert space $\mathcal{H}$ over a pair of orthonormal bases $\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})$ and $\{|k\rangle\}\in\mathcal{B}_o(\mathcal{H})$ is defined as [40–42]

$${\rm Pr}_{\rm KD}(k,x|\varrho) := {\rm Tr}(\Pi_x\Pi_k\varrho). \quad (20)$$

KD quasiprobability gives correct marginal probabilities, i.e., $\sum_x {\rm Pr}_{\rm KD}(k,x|\varrho) = {\rm Tr}(\Pi_k\varrho) = {\rm Pr}(k|\varrho)$ and $\sum_k {\rm Pr}_{\rm KD}(k,x|\varrho) = {\rm Tr}(\Pi_x\varrho) = {\rm Pr}(x|\varrho)$, where ${\rm Pr}(\cdot)$ is the classical, i.e., real and nonnegative, probability. However, because of the noncommutativity among the state and the projection valued measures $\{\Pi_x\}$ and $\{\Pi_k\}$ corresponding to the two defining orthonormal bases, unlike the Kolmogorovian classical probability, KD quasiprobability may assume complex values and its real part may be negative. In this sense, the nonreality and/or the negativity of the KD quasiprobability therefore captures a form of nonclassicality. Remarkably, the nonreality and/or the negativity of KD quasiprobability, a.k.a. KD nonclassicality, has been shown to be tighter than noncommutativity [70,71]. Moreover, recent works showed that KD nonclassicality plays crucial roles in various areas of quantum science [28,44–58].

We further introduce the following normalized trace-norm asymmetry.

Definition 5. The normalized trace-norm asymmetry of ϱ relative to the translation group generated by a Hermitian operator K is defined as

$$\tilde{A}_{\rm Tr}(\varrho;K) := A_{\rm Tr}(\varrho;K)/\|K\|_{\rm max} = A_{\rm Tr}(\varrho;\tilde{K}), \quad (21)$$

where for any bounded Hermitian operator O on finite-dimensional Hilbert space, $\tilde{O}$ is defined as $\tilde{O} := O/\|O\|_{\rm max}$ with $\|O\|_{\rm max}$ the spectral radius, i.e., the maximum singular value, of O.

We then have the following result.

Proposition 4. The normalized trace-norm asymmetry of ϱ relative to the translation group generated by a Hermitian operator K is upper bounded by the total sum of the absolute imaginary part of the KD quasiprobability defined over the eigenbasis $\{|k\rangle\}$ of K, and a second orthonormal basis of the Hilbert space, maximized over all possible choices of the latter as

$$\tilde{A}_{\rm Tr}(\varrho;K) \leqslant \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_{k,x} |{\rm Im}{\rm Pr}_{\rm KD}(k,x|\varrho)| := C_{\rm KD}(\varrho;\{|k\rangle\}). \quad (22)$$

Proof. Using Eqs. (5), (4), and (20), we first have the following relation:

$$A_{\rm Tr}(\varrho;K) = \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_x \Big| \sum_k k {\rm Im}{\rm Tr}(\Pi_x \Pi_k \varrho)\Big| \leqslant \|K\|_{\max} \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_{k,x} |{\rm Im}{\rm Pr}_{\rm KD}(k,x|\varrho)|, \quad (23)$$

where we have used the spectral decomposition $K = \sum_k k\,|k\rangle\langle k|$. Dividing both sides with the spectral radius of K, i.e., $\|K\|_{\max}$, and noting Eq. (21), we obtain Eq. (22). For a single qubit, the inequality in Eq. (22) can be checked analytically, as shown in Appendix A 4. ∎

Next, it was shown in Ref. [57] that the right-hand side of Eq. (22) gives a lower bound to the l_1-norm coherence [43] of the state ϱ relative to the orthonormal basis $\{|k\rangle\}$ defined as $C_{l_1}(\varrho;\{|k\rangle\}) := \sum_{k\neq k'} |\langle k|\varrho|k'\rangle|$, i.e.,

$$C_{\rm KD}(\varrho;\{|k\rangle\}) \leqslant C_{l_1}(\varrho;\{|k\rangle\}). \quad (24)$$

Moreover, the inequality becomes equality for an arbitrary state of a single qubit. Noting this, we thus obtain the first corollary of Proposition 4.

Corollary 1. The normalized trace-norm asymmetry of a state ϱ relative to a translation group generated by a Hermitian operator K is upper bounded by the l_1-norm coherence of ϱ relative to the orthonormal basis $\{|k\rangle\}$ of K, i.e.,

$$\tilde{A}_{\rm Tr}(\varrho;K) \leqslant C_{l_1}(\varrho;\{|k\rangle\}). \quad (25)$$

Moreover, for a single qubit, assuming the eigenvalues of K are $\{1,-1\}$, the above inequality becomes an equality.

Proof. First, the inequality is obtained by chaining the inequalities of Eqs. (22) and (24). To prove the second half of the corollary, we note that for the case of a single qubit with K having the spectrum of eigenvalues $\{1,-1\}$, we have, as shown in Appendix A 1 [see Eq. (A4)], $\tilde{A}_{\rm Tr}(\varrho;K) = 2|\langle k_+|\varrho|k_-\rangle|$, where $|k_\pm\rangle$ is the eigenvectors of K belonging to the eigenvalues ± 1. On the other hand, for a single qubit with arbitrary state ϱ and orthonormal basis $\{|k\rangle\}$ we have $C_{l_1}(\varrho;\{|k\rangle\}) = 2|\langle k_+|\varrho|k_-\rangle|$. Combining these two equalities, we obtain Eq. (25), with the equality replaced by an equality. ∎

Now, consider the set $\Lambda_{\{k\}}$ of all bounded Hermitian operators K on a Hilbert space with a fixed nontrivial spectrum of eigenvalues $\{k\}$. By nontrivial we mean that not all the eigenvalues are equal, so that $K \neq k_0\mathbb{I}$ for some $k_0 \in \mathbb{R}$. Then, we obtain the following corollary of Proposition 4.

Corollary 2. The maximum normalized trace-norm asymmetry of a state ϱ relative to the translation groups generated by all $K \in \Lambda_{\{k\}}$ having a fixed nontrivial spectrum $\{k\}$ is bounded from above by the maximum total sum of the absolute imaginary part of the KD quasiprobability associated with ϱ over all possible pairs of the defining orthonormal bases of the Hilbert space:

$$\sup_{K\in\Lambda_{\{k\}}} \tilde{A}_{\rm Tr}(\varrho;K) \leqslant \sup_{\{|k\rangle\}\in\mathcal{B}_o(\mathcal{H}),\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_{k,x} |{\rm Im}{\rm Pr}_{\rm KD}(k,x|\varrho)|. \quad (26)$$

Conversely, Eq. (26) can be read as follows. Given a quantum state ϱ, the maximum total nonreality of the associated KD quasiprobability over all possible pairs of the defining orthonormal bases of the Hilbert space is bounded from below by the maximum normalized trace-norm asymmetry relative to the translation groups generated by all Hermitian operators $K \in \Lambda_{\{k\}}$. From this viewpoint, and noting the fact that the right-hand side of Eq. (26) is independent of the eigenvalues $\{k\}$ of K, the inequality in Eq. (26) can be strengthened as follows.

Corollary 3. The maximum total sum of the imaginary part of the KD quasiprobability associated with ϱ over all possible pairs of the defining orthonormal bases is never less than the maximum normalized trace-norm asymmetry of the state ϱ relative to the translation groups generated by all bounded Hermitian operators K on the Hilbert space:

$$\sup_{\{|k\rangle\}\in\mathcal{B}_o(\mathcal{H}),\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_{k,x} |{\rm Im}{\rm Pr}_{\rm KD}(k,x|\varrho)| \geqslant \sup_{K\in\mathcal{O}(\mathcal{H})} \tilde{A}_{\rm Tr}(\varrho;K), \quad (27)$$

where $\mathcal{O}(\mathcal{H})$ is the set of all bounded Hermitian operators on the Hilbert space $\mathcal{H}$.

We show in Appendix A 4 that for a single qubit with arbitrary state ϱ, the generator K of the translation group which reaches the equality in Eq. (27) has the form $K_* = k_+|k_+\rangle\langle k_+| + k_-|k_-\rangle\langle k_-|$, where $\{k_+,k_-\}$ are the real eigenvalues of K corresponding to the eigenvectors $\{|k_+\rangle, |k_-\rangle\}$ satisfying $k_+ = -k_-$. This is the case, e.g., when $K_* = \vec{n}\cdot\vec{\sigma}$, where $\vec{n}$ is a unit vector, and $\vec{\sigma} = (\sigma_x,\sigma_y,\sigma_z)^{\rm T}$ with σ_x, σ_y, and σ_z Pauli operators, so that $k_+ = -k_- = 1$.

These results show that the nonclassical aspect of quantum mechanics captured by the concept of asymmetry relative to a translation group is related to the nonclassicality captured by the imaginary part of the KD quasiprobability where one of the defining bases is given by the eigenbasis of the generator of the translation group. It thus suggests that the translational asymmetry of a quantum state may be a key quantum ingredient in diverse quantum phenomena where the nonclassicality captured by the anomalous KD quasiprobability has been shown to play an important role, and vice versa.

We further obtain an upper bound for the trace-norm asymmetry in terms of state purity.

Proposition 5. The normalized trace-norm asymmetry of a state ϱ on d-dimensional Hilbert space relative to a translation group generated by a Hermitian operator K is bounded from above by the purity of the state, i.e., ${\rm Tr}(\varrho^2)$, as

$$\tilde{A}_{\rm Tr}(\varrho;K) \leqslant \sqrt{(d-1)}[d{\rm Tr}(\varrho^2)-1]^{1/2}. \quad (28)$$

Proof. Using the relation between the trace-norm asymmetry and the KD quasiprobability of Eq. (22), we have

$$\tilde{A}_{\rm Tr}(\varrho;K)$$

$$\leqslant \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_{k,x} \left| \sum_{k'\neq k} \mathrm{Im}(\langle x|k\rangle \langle k|\varrho|k'\rangle \langle k'|x\rangle) \right| \tag{29}$$

$$\leqslant \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_{k} \sum_{x,k'\neq k} |\langle x|k\rangle \langle k|\varrho|k'\rangle \langle k'|x\rangle|$$

$$\leqslant \sum_k \left(\sum_{k'\neq k, x_*} |\langle x_*|k\rangle \langle k|\varrho|k'\rangle|^2 \sum_{k''\neq k, x'_*} |\langle k''|x'_*\rangle|^2 \right)^{1/2} \tag{30}$$

$$= \sum_k \sqrt{d-1} \left(\sum_{k',x_*} |\langle x_*|k\rangle \langle k|\varrho|k'\rangle|^2 - \langle k|\varrho|k\rangle^2 \right)^{1/2} \tag{31}$$

$$= \sum_k \sqrt{d-1}(\langle k|\varrho^2|k\rangle - \langle k|\varrho|k\rangle^2)^{1/2} \tag{32}$$

$$\leqslant \sqrt{(d-1)d} \left(\sum_k \langle k|\varrho^2|k\rangle - \langle k|\varrho|k\rangle^2 \right)^{1/2} \tag{33}$$

$$= \sqrt{(d-1)d} \left(\mathrm{Tr}(\varrho^2) - \sum_k \langle k|\varrho|k\rangle^2 \right)^{1/2}. \tag{34}$$

Here, to get Eq. (29) we have inserted an identity $\sum_{k'} |k'\rangle \langle k'| = \mathbb{I}$ and noted the fact that the diagonal terms $k = k'$ are real, in Eq. (30) $\{|x_*\rangle\}$ is a basis which achieves the supremum and we have used the Cauchy-Schwartz inequality, to get Eq. (31) we have used the completeness relation and completed the sum over k' (to include also the case $k' = k$), to get Eq. (32) we have used the completeness relation, and to get Eq. (33) we have again used the Cauchy-Schwartz inequality. Finally, noting that $\sum_{k=1}^{d} \langle k|\varrho|k\rangle^2 \geqslant 1/d$ in Eq. (34), we get Eq. (28). ■

Notice that for the maximally mixed state, $\varrho_{\rm mm} = \mathbb{I}/d$, the upper bound in Eq. (28) is indeed vanishing, as desired since translational asymmetry can be seen as a form of coherence. By contrast, for pure states, the upper bound is given by $d-1$.

B. Lower bounds: Maximum average noncommutativity and uncertainty relations

We first derive a lower bound for the trace-norm asymmetry.

Lemma 1. Consider the set $\Lambda_{\{x\}}$ of all bounded Hermitian operators X on a Hilbert space with a fixed nontrivial spectrum of eigenvalues $\{x\}$. Then, the trace-norm asymmetry of ϱ relative to a translation group generated by a Hermitian operator K can be bounded from below as

$$\tilde{A}_{\rm Tr}(\varrho;K) \geqslant \sup_{X\in\Lambda_{\{x\}}} |\mathrm{Tr}([\tilde{X},\tilde{K}]\varrho)|/2. \tag{35}$$

Proof. Using Eqs. (21), (5), and (4), we directly have

$$\tilde{A}_{\rm Tr}(\varrho;K)$$

$$= \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \frac{1}{\|X\|_{\max}\|K\|_{\max}} \sum_x \|X\|_{\max} |\mathrm{Im}\mathrm{Tr}(\Pi_x K\varrho)|$$

$$\geqslant \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \frac{1}{\|X\|_{\max}\|K\|_{\max}} |\mathrm{Im}\mathrm{Tr}(XK\varrho)|$$

$$= \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} |\mathrm{Im}\mathrm{Tr}(\tilde{X}\tilde{K}\varrho)|$$

$$= \sup_{X\in\Lambda_{\{x\}}} |\mathrm{Tr}([\tilde{X},\tilde{K}]\varrho)|/2, \tag{36}$$

where we have used $X := \sum_x x\Pi_x$. For a single qubit, it can be again checked analytically as shown in Appendix A 5. ■

Hence, the trace-norm asymmetry of a state ϱ relative to a translation group generated by K is lower bounded by the maximum average noncommutativity between K and any other possible bounded Hermitian operators $X \in \Lambda_{\{x\}}$ whose eigenbasis spans the Hilbert space, divided by their spectral radiuses. Notice that the lower bound takes a form similar to the lower bound of the Kennard-Weyl-Robertson uncertainty relation.

Furthermore, noting the fact that the left-hand side of the inequality in Eq. (35) does not depend on the eigenvalues $\{x\}$ of X, the inequality can be further tightened as follows.

Corollary 4. The trace-norm asymmetry of ϱ relative to a translation group generated by a Hermitian operator K can be bounded from below as

$$\tilde{A}_{\rm Tr}(\varrho;K) \geqslant \sup_{X\in\mathcal{O}(\mathcal{H})} |\mathrm{Tr}([\tilde{X},\tilde{K}]\varrho)|/2, \tag{37}$$

where the supremum is taken over the set $\mathcal{O}(\mathcal{H})$ of all bounded Hermitian operators on the Hilbert space $\mathcal{H}$.

We show in Appendix A 5 that for a single qubit, denoting the eigenvalues of X as $\{x\} = \{x_+, x_-\}$, $x_+, x_- \in \mathbb{R}$, the inequality in Eq. (37) becomes equality by choosing $x_+ = -x_-$. For example, when $x_+ = 1$, we may take $X_* = \vec{n}\cdot\vec{\sigma}$, where $\vec{n}$ is a unit vector and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^{\rm T}$.

Combining Corollaries 3 and 4, and taking the supremum to both sides of Eq. (37) over all bounded Hermitian operators $K \in \mathcal{O}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$, we thus obtain the following ordering of quantities:

$$\sup_{\{|k\rangle\}\in\mathcal{B}_o(\mathcal{H}),\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_{k,x} |\mathrm{Im}[\mathrm{Pr}_{\rm KD}(k,x|\varrho)]|$$

$$\geqslant \sup_{K\in\mathcal{O}(\mathcal{H})} \tilde{A}_{\rm Tr}(\varrho;K) \geqslant \sup_{K\in\mathcal{O}(\mathcal{H})} \sup_{X\in\mathcal{O}(\mathcal{H})} |\mathrm{Tr}([\tilde{X},\tilde{K}]\varrho)|/2. \tag{38}$$

Now, using Lemma 1, we obtain the following proposition.

Proposition 6. The trace-norm asymmetries of a state ϱ relative to groups of translation generated by Hermitian operators K and X satisfy the following trade-off relation:

$$\tilde{A}_{\rm Tr}(\varrho;K)\tilde{A}_{\rm Tr}(\varrho;X) \geqslant \tfrac{1}{4}|\mathrm{Tr}([\tilde{K},\tilde{X}]\varrho)|^2. \tag{39}$$

Proof. First, exchanging the role of K and X in Eq. (37), we have

$$\tilde{A}_{\rm Tr}(\varrho;X) \geqslant \sup_{K\in\mathcal{O}(\mathcal{H})} |\mathrm{Tr}([\tilde{X},\tilde{K}]\varrho)|/2, \tag{40}$$

where the supremum is taken over all bounded Hermitian operators $K \in \mathcal{O}(\mathcal{H})$ on the Hilbert space $\mathcal{H}$. Multiplying

Eqs. (37) and (40), we finally obtain

$$\begin{aligned}\tilde{A}_{\rm Tr}(\varrho;K)\tilde{A}_{\rm Tr}(\varrho;X) &\geqslant |{\rm Tr}([\tilde{X}_*,\tilde{K}]\varrho)||{\rm Tr}([\tilde{X},\tilde{K}_*]\varrho)|/4\\ &\geqslant |{\rm Tr}([\tilde{X},\tilde{K}]\varrho)|^2/4,\end{aligned} \tag{41}$$

where X_* and K_* are the Hermitian operators which respectively achieve the supremum in Eqs. (37) and (40). ■

Proposition 6 clarifies the intuition that when the expectation value of the commutator between the Hermitian operators K and X over ϱ is nonvanishing, then the state ϱ must be asymmetric relative to both the translation group generated by K and that generated by X. Moreover, the associated trace-norm asymmetries satisfy the trade-off relation of Eq. (39). Let us translate this trade-off relation in the language of coherence. Suppose that the lower bound in Eq. (39) is nonvanishing. Then, the state ϱ cannot be commuting with all the eigenprojectors $\{\Pi_x\}$ of X and with all the eigenprojectors $\{\Pi_k\}$ of K. This means that the state is coherent relative to both the orthonormal eigenbases $\{|x\rangle\}$ and $\{|k\rangle\}$. Moreover, the amount of respective coherences that are quantified by the trace-norm asymmetries satisfy the trade-off relation of Eq. (39). Finally, recall that $A_{\rm Tr}(\varrho;K)$ can be seen as the genuine quantum part of the uncertainty of the outcomes of the measurement of the observable K when the system is prepared in the state ϱ. Equation (39) can thus be seen as the trade-off relation between the genuine quantum part of the uncertainty in measurement of two noncommuting observables [65–68]. Let us mention that a similar trade-off relation is suggested in Ref. [66], wherein the genuine quantum uncertainty associated with the measurement of K over ϱ is identified by the Wigner-Yanase skew information defined as $I_{\rm WY}(\varrho;K)=-\frac{1}{2}{\rm Tr}([\sqrt{\varrho},K]^2)$ [72].

Next, combining Eqs. (13) and (37), we obtain the following corollary.

Corollary 5. Consider a setting whereby a parameter θ is encoded into the quantum state of a system via a translation unitary generated by K as $\varrho_\theta=e^{-iK\theta}\varrho e^{iK\theta}$. Then the quantum Fisher information about θ contained in ϱ_θ is bounded from below as

$$\tilde{\mathcal{J}}_\theta(\varrho;K)^{1/2}\geqslant \sup_{X\in\mathcal{O}(\mathcal{H})}|{\rm Tr}([\tilde{X},\tilde{K}]\varrho)|, \tag{42}$$

where $\tilde{\mathcal{J}}_\theta(\varrho;K)$ is a normalized quantum Fisher information about θ in ϱ_θ defined as $\tilde{\mathcal{J}}_\theta(\varrho;K):=\mathcal{J}_\theta(\varrho;K)/\|K\|^2_{\rm max}$.

The case of a single qubit is discussed in Appendix A 6, where equality in Eq. (42) is obtained when the spectrum of X, i.e., $\{x\}=\{x_+,x_-\}$, $x_+,x_-\in\mathbb{R}$, satisfies $x_-=-x_+$. Corollary 5 shows that the optimal sensitivity of the state ϱ relative to the translation unitary generated by K, or equivalently, the optimal sensitivity in the quantum parameter estimation of θ conjugate to K, is lower bounded by the maximum average noncommutativity between K and any other Hermitian operators $X\in\mathcal{O}(\mathcal{H})$ whose eigenbasis spans the Hilbert space $\mathcal{H}$.

We thus obtain the following result.

Proposition 7. Consider two Hermitian operators K and X, so that they generate unitary imprinting of scalar parameters to the quantum state of the probe in the protocol of quantum parameter estimation, respectively, as $\varrho_{\theta_K}=e^{-iK\theta_K}\varrho e^{iK\theta_K}$ and $\varrho_{\theta_X}=e^{-iX\theta_X}\varrho e^{iX\theta_X}$, where θ_K is the scalar parameter conjugate to K and θ_X is to X. Then, the normalized quantum Fisher information about θ_K in ϱ_{θ_K} and about θ_X in ϱ_{θ_X} satisfy the following trade-off relation:

$$\tilde{\mathcal{J}}_{\theta_K}(\varrho;K)^{1/2}\tilde{\mathcal{J}}_{\theta_X}(\varrho;X)^{1/2}\geqslant |{\rm Tr}([\tilde{X},\tilde{K}]\varrho)|^2. \tag{43}$$

Proof. Equation (43) can be directly obtained from Eq. (42) by following similarly the proof of Proposition 6. ■

Since the quantum Fisher information is a monotonic measure of asymmetry as coherence, Eq. (43) admits a similar interpretation as the uncertainty relation of Eq. (39) for trace-norm asymmetry. Moreover, Eq. (43) shows that when the quantum expectation value of the noncommutativity between the Hermitian operators K and X over the state ϱ is not vanishing, then the state must be sensitive relative to the translation unitaries generated by K and by X, and their sensitivities as quantified by the quantum Fisher information satisfy the trade-off relation (43).

Finally, from Eqs. (40) and (42) and following again similarly the proof of Proposition 6, we obtain the following result relating the sensitivity of the state relative to the translation unitary generated by K quantified by the quantum Fisher information, and the coherence of the state relative to the eigenbasis of X quantified by the trace-norm asymmetry.

Proposition 8. The trace-norm asymmetry of ϱ relative to a translation generated by a Hermitian operator X and the quantum Fisher information about θ in the state ϱ_θ obtained via a unitary imprinting generated by a Hermitian operator K satisfy the following trade-off relation:

$$\tilde{\mathcal{J}}_\theta(\varrho;K)^{1/2}\tilde{A}_{\rm Tr}(\varrho;X)\geqslant |{\rm Tr}([\tilde{X},\tilde{K}]\varrho)|^2/2. \tag{44}$$

As an implication of the Proposition 8 we have the following corollary.

Corollary 6. Consider a quantum state ϱ and two Hermitian operators K and X. Then, the quantum Fisher information about θ contained in ϱ_θ obtained via a translation unitary generated by K and the l_1-norm coherence of the state ϱ relative to the eigenbasis $\{|x\rangle\}$ of X satisfy the following trade-off relation:

$$\tilde{\mathcal{J}}_\theta(\varrho;K)^{1/2}C_{l_1}(\varrho;\{|x\rangle\})\geqslant |{\rm Tr}([\tilde{X},\tilde{K}]\varrho)|^2/2. \tag{45}$$

Proof. The trade-off relation of Eq. (45) can be obtained directly by imposing the inequality (25) of Corollary 1 to Eq. (44). ■

IV. CONCLUSION AND REMARKS

To conclude, we first showed that the trace-norm asymmetry of a state relative to a translation group is equal to the average absolute imaginary part of the weak value of the generator of the translation, maximized over all possible orthonormal bases of the Hilbert space. Hence, the trace-norm asymmetry of an unknown quantum state can be estimated in experiment using a number of methods for measuring the weak value proposed in the literatures, combined with a classical optimization procedure, in the fashion of a hybrid quantum-classical variational circuit which should be implementable using the presently available NISQ hardware. It also suggests the physical and statistical interpretation of the trace-norm asymmetry in terms of the interpretations of the imaginary part of the weak value.

Using the mathematical link between the trace-norm asymmetry and the nonreal weak value, we then derived upper bounds for the trace-norm asymmetry relative to a translation group in terms of the quantum uncertainty of the generator of the translation, the quantum Fisher information about a parameter imprinted via the translation unitary, the imaginary part of the corresponding KD quasiprobability, the l_1-norm coherence relative to the eigenbasis of the generator of the translation, and the purity of the state. We also obtain a lower bound in terms of the maximum average noncommutativity between the generator of the translation and any other bounded Hermitian operator on the Hilbert space. We then derived trade-off relations for the trace-norm asymmetry and the quantum Fisher information associated with two noncommuting generators of the translation unitary, with a lower bound reminiscent of that for the Kennard-Weyl-Robertson uncertainty relation.

We hope that by expressing the geometrical trace-norm asymmetry in terms of the operationally well-defined imaginary part of the weak value and KD quasiprobability, it may shed light on the applications of trace-norm asymmetry in a plethora of fields in which the strange weak values and the nonclassical anomalous values of KD quasiprobability have played crucial roles [28]. Conversely, our results may provide insight in promoting the concept of strange weak values and the nonclassical values of KD quasiprobability, which have played important roles in quantum foundation, as useful tools to access the nonclassicality captured by the concepts of asymmetry, coherence, nonclassical correlation, and entanglement, which are the key resources for quantum information processing and quantum technology. It is also interesting to extend the present approach to study the asymmetry relative to general quantum channels [73].

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APPENDIX A: SOME ANALYTICAL COMPUTATIONS FOR A SINGLE QUBIT

1. The equality of Eq. (5) for a single qubit

Assume first that the Hermitian generator of the translation group takes the form $K = k_0\,|0\rangle\langle 0| + k_1\,|1\rangle\langle 1|$, $k_0, k_1 \in \mathbb{R}$, where $\{|0\rangle, |1\rangle\}$ are the eigenvectors of the Pauli z-spin operator σ_z. Then, computing the trace-norm asymmetry, one directly gets

$$A_{\rm Tr}(\varrho;K) = \|[\varrho,K]\|_1/2 = |k_0-k_1|\,|\langle 0|\varrho|1\rangle\,|. \quad \text{(A1)}$$

On the other hand, to compute $A_{\rm w}(\varrho;K)$ defined in Eq. (4), we need to parametrize the whole orthonormal bases $\{|x(\vec{\lambda})\rangle\} \in \mathcal{B}_o(\mathcal{H})$ of the Hilbert space $\mathcal{H}$, so that varying the parameters $\vec{\lambda} = (\lambda_1,\dots,\lambda_N)^{\rm T}$ over their ranges of values will scan all the orthonormal bases of the Hilbert space over which we make the optimization. For the two-dimensional Hilbert space of interest, let us use the parametrization of the whole orthonormal bases $\{|x\rangle\} = \{|x_+\rangle, |x_-\rangle\} \in \mathcal{B}_o(\mathbb{C}^2)$ based on the Bloch sphere as

$$|x_+(\alpha,\beta)\rangle := \cos\frac{\alpha}{2}\,|0\rangle + e^{i\beta}\sin\frac{\alpha}{2}\,|1\rangle,$$
$$|x_-(\alpha,\beta)\rangle := \sin\frac{\alpha}{2}\,|0\rangle - e^{i\beta}\cos\frac{\alpha}{2}\,|1\rangle, \quad \text{(A2)}$$

$\alpha \in [0,\pi]$, $\beta \in [0,2\pi)$. Hence, one can scan all the possible orthonormal bases of the two-dimensional Hilbert space by varying the angular parameters α and β over their ranges of values. Using this expression for the defining basis in Eq. (4), we directly get

$$\begin{aligned} A_{\rm w}(\varrho;K) &= \sup_{\{|x(\alpha,\beta)\rangle\}\in\mathcal{B}_o(\mathbb{C}^2)} \sum_{x=\{x_+,x_-\}} |{\rm Im}\,\langle x(\alpha,\beta)|K\varrho|x(\alpha,\beta)\rangle\,| \\ &= \max_{(\alpha,\beta)\in[0,\pi]\times[0,2\pi)} |k_0-k_1|\,|\langle 0|\varrho|1\rangle\,|\,|\sin\alpha|\,|\sin(\beta+\phi_{01})| \\ &= |k_0-k_1|\,|\langle 0|\varrho|1\rangle\,| = A_{\rm Tr}(\varrho;K), \end{aligned} \quad \text{(A3)}$$

where $\phi_{01} = \arg\langle 0|\varrho|1\rangle$ and the last equality is just Eq. (A1). Note that the maximum is obtained for the basis of the form (A2) with $\alpha = \pi/2$ and $\beta = \pi/2 - \phi_{01}$.

The above result can be generalized to an arbitrary Hermitian operator generating a translation unitary to the state on two-dimensional Hilbert space: $K = k_+\,|k_+\rangle\langle k_+| + k_-\,|k_-\rangle\langle k_-|$, with the eigenvalues $k_+, k_- \in \mathbb{R}$, and the corresponding orthonormal eigenvectors $\{|k_+\rangle, |k_-\rangle\}$. First, the trace-norm asymmetry can be computed directly to get, noting Eq. (A1),

$$A_{\rm Tr}(\varrho;K) = \|[\varrho,K]\|_1/2 = |k_+-k_-|\,|\langle k_+|\varrho|k_-\rangle\,|. \quad \text{(A4)}$$

Let us show that $A_{\rm w}(\varrho;K)$ defined in Eq. (4) also yields the same value in accord with Proposition 1. To do this, we first show that for arbitrary state ϱ and Hermitian operator K on finite-dimensional Hilbert space, $A_{\rm w}(\varrho;K)$ is unitarily covariant. Namely, for any unitary transformation V, we have

$$\begin{aligned} A_{\rm w}(V\varrho V^\dagger; VKV^\dagger) &= \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_x |{\rm Im}\,\langle x|VKV^\dagger V\varrho V^\dagger|x\rangle\,| \\ &= \sup_{\{|x'\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_{x'} |{\rm Im}\,\langle x'|K\varrho|x'\rangle\,| \quad \text{(A5)} \\ &= A_{\rm w}(\varrho;K), \quad \text{(A6)} \end{aligned}$$

where we have defined a new orthonormal basis $\{|x'\rangle\} = \{V^\dagger|x\rangle\}$ to get Eq. (A5), and Eq. (A6) holds since the set of the new orthonormal bases $\{|x'\rangle\}$ is the same as the set of the old orthonormal bases $\{|x\rangle\}$ given by set $\mathcal{B}_o(\mathcal{H})$ of all the orthonormal bases of the same Hilbert space $\mathcal{H}$, so that $\sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})}(\cdot) = \sup_{\{|x'\rangle\}\in\mathcal{B}_o(\mathcal{H})}(\cdot)$.

Now, for the case of a single qubit of interest, let us choose the following unitary transformation: $V = |0\rangle\langle k_+| + |1\rangle\langle k_-|$, so that we have $VKV^\dagger = k_+\,|0\rangle\langle 0| + k_-\,|1\rangle\langle 1|$, and $V\varrho V^\dagger = \langle k_+|\varrho|k_+\rangle\,|0\rangle\langle 0| + \langle k_+|\varrho|k_-\rangle\,|0\rangle\langle 1| + \langle k_-|\varrho|k_+\rangle\,|1\rangle\langle 0| + \langle k_-|\varrho|k_-\rangle\,|1\rangle\langle 1|$. Noting these facts and using Eq. (A3), we thus obtain

$$A_{\rm w}(\varrho;K)=A_{\rm w}(V\varrho V^{\dagger};VKV^{\dagger}) = |k_+-k_-||\langle k_+|\varrho|k_-\rangle|=A_{\rm Tr}(\varrho;K), \quad (A7)$$

where the last equality is just Eq. (A4).

2. Trace-norm asymmetry vs quantum standard deviation of Eq. (10) for a single qubit

Assume first, as in Appendix A 1, the following form of generator of translation group: $K=k_0|0\rangle\langle 0|+k_1|1\rangle\langle 1|$, $k_0,k_1\in\mathbb{R}$. For our purpose, it is convenient to write the state as $\varrho=(\mathbb{I}+r_x\sigma_x+r_y\sigma_y+r_z\sigma_z)/2$, where (r_x,r_y,r_z) are real numbers satisfying $r_x^2+r_y^2+r_z^2=r^2\leqslant 1$, and $(\sigma_x,\sigma_y,\sigma_z)$ are the three Pauli operators. Then one directly obtains

$$\begin{aligned}\Delta_K(\varrho)&=\frac{1}{2}|k_0-k_1|\sqrt{(1-r_z^2)}\\ &\geqslant\frac{1}{2}|k_0-k_1|\sqrt{(r^2-r_z^2)}\\ &=\frac{1}{2}|k_0-k_1||r_x-ir_y|\\ &=|k_0-k_1||\langle 0|\varrho|1\rangle|\\ &=A_{\rm Tr}(\varrho;K),\end{aligned}\quad (A8)$$

where the last equality is just Eq. (A1). Equality is reached when $r=1$, i.e., for pure states as expected. The above result can be generalized to an arbitrary Hermitian operator on two-dimensional Hilbert space $K=k_+|k_+\rangle\langle k_+|+k_-|k_-\rangle\langle k_-|$, $k_+,k_-\in\mathbb{R}$ and arbitrary state ϱ, by first noting that $\Delta_K(\varrho)$, like $A_{\rm Tr}(\varrho;K)$, is unitarily covariant, i.e., $\Delta_{VKV^{\dagger}}(V\varrho V^{\dagger})=\Delta_K(\varrho)$ for arbitrary unitary transformation V, and by choosing a unitary transformation $V=|0\rangle\langle k_+|+|1\rangle\langle k_-|$, and noting further the fact that the unitary transformation conserves the state purity.

3. Trace-norm asymmetry vs quantum Fisher information of Eq. (13) for a single qubit

Let us write the density operator in terms of its spectral decomposition: $\varrho=\lambda_1|\lambda_1\rangle\langle\lambda_1|+\lambda_2|\lambda_2\rangle\langle\lambda_2|$, $\lambda_1,\lambda_2\in\mathbb{R}^+$. Then, assuming ϱ_θ is obtained via a unitary imprinting generated by K, i.e., $\varrho_\theta=U_{K,\theta}\varrho U_{K,\theta}^{\dagger}$, one has [38]

$$\begin{aligned}\mathcal{J}_\theta(\varrho;K)&=\mathcal{J}_\theta(\varrho_\theta;K)\\ &=4\frac{|\lambda_1-\lambda_2|^2}{\lambda_1+\lambda_2}|\langle\lambda_1|U_{K,\theta}^{\dagger}KU_{K,\theta}|\lambda_2\rangle|^2\\ &=4|\lambda_1-\lambda_2|^2|\langle\lambda_1|K|\lambda_2\rangle|^2,\end{aligned}\quad (A9)$$

where we have used the fact that $\lambda_1+\lambda_2=1$ and $U_{K,\theta}^{\dagger}KU_{K,\theta}=K$. On the other hand, one can directly compute the trace-norm asymmetry of ϱ relative to the translation group generated by K in the basis $\{|\lambda_1\rangle,|\lambda_2\rangle\}$ to get

$$\begin{aligned}4A_{\rm Tr}(K;\varrho)^2&=\|[\varrho,K]\|_1^2\\ &=4|\lambda_1-\lambda_2|^2|\langle\lambda_1|K|\lambda_2\rangle|^2\\ &=\mathcal{J}_\theta(\varrho;K),\end{aligned}\quad (A10)$$

where the last equality is just Eq. (A9). Hence, the inequality in Eq. (13) is saturated for arbitrary state of a single qubit.

4. Normalized trace-norm asymmetry vs maximum nonreality of KD quasiprobability of Eq. (22) for a single qubit

The Hermitian generator of the translation group can be in general written as $K=k_+|k_+\rangle\langle k_+|+k_-|k_-\rangle\langle k_-|$, $k_+,k_-\in\mathbb{R}$. Assume without loss of generality $|k_+|>|k_-|$, so that $\|K\|_{\max}=|k_+|$. Then, noting Eq. (A4), we have

$$\begin{aligned}\tilde{A}_{\rm Tr}(\varrho;K)&=\frac{|k_+-k_-|}{\|K\|_{\max}}|\langle k_+|\varrho|k_-\rangle|\\ &=\frac{|k_+-k_-|}{|k_+|}|\langle k_+|\varrho|k_-\rangle|.\end{aligned}\quad (A11)$$

On the other hand, for a single qubit, we have

$$\sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathbb{C}^2)}\sum_{k,x}\left|{\rm ImPr}_{\rm KD}(k,x|\varrho)\right|=2|\langle k_+|\varrho|k_-\rangle|. \quad (A12)$$

See Ref. [57] for a proof. Since we also have

$$\frac{|k_+-k_-|}{|k_+|}\leqslant 2, \quad (A13)$$

Eqs. (A11) and (A12) satisfy the inequality of Eq. (22) of Proposition 4:

$$\tilde{A}_{\rm Tr}(\varrho;K)\leqslant\sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathbb{C}^2)}\sum_{k,x}\left|{\rm Im}[{\rm Pr}_{\rm KD}(k,x|\varrho)]\right|. \quad (A14)$$

Next, notice that the equality in Eq. (A13) and thus the equality in Eq. (A14) are attained when $|k_+-k_-|=2|k_+|$, which is the case when $k_-=-k_+$. For example, assume that $|k_+|=1$. Then, we may take $K=\vec{n}\cdot\vec{\sigma}$, where $\vec{n}$ is a unit vector and $\vec{\sigma}=(\sigma_x,\sigma_y,\sigma_z)^{\rm T}$ is the vector of the three Pauli operators.

5. Normalized trace-norm asymmetry vs maximum average noncommutativity of Eq. (35) for a single qubit

Without loss of generality, we can assume that the generator of the translation group in the case of a single qubit has the following spectral decomposition: $K=k_0|0\rangle\langle 0|+k_1|1\rangle\langle 1|$, $k_0,k_1\in\mathbb{R}$. Now, let us denote the eigenvalues of X as $\{x\}=\{x_+,x_-\}$, $x_+,x_-\in\mathbb{R}$, so that

$$\begin{aligned}X(\alpha,\beta)&=x_+|x_+(\alpha,\beta)\rangle\langle x_+(\alpha,\beta)|\\ &\quad+x_-|x_-(\alpha,\beta)\rangle\langle x_-(\alpha,\beta)|,\end{aligned}\quad (A15)$$

where the eigenvectors $\{|x_+(\alpha,\beta)\rangle,|x_-(\alpha,\beta)\rangle\}$ are expressed using the Bloch sphere parametrization as in Eq. (A2). Furthermore, assume that $|k_0|>|k_1|$ and $|x_+|>|x_-|$, so that $\|K\|_{\max}=|k_0|$ and $\|X\|_{\max}=|x_+|$. Then, computing the average noncommutativity between $\tilde{K}$ and $\tilde{X}(\alpha,\beta)$ over the state ϱ, and taking the supremum over $(\alpha,\beta)\in[0,\pi]\times[0,2\pi)$, we obtain

$$\begin{aligned}&\sup_{X(\alpha,\beta)\in\Lambda_{\{x_0,x_1\}}}|{\rm Tr}([\tilde{X}(\alpha,\beta),\tilde{K}]\varrho)|/2\\ &=\max_{(\alpha,\beta)\in[0,\pi]\times[0,2\pi)}\frac{|x_+-x_-|}{2|x_+|}\frac{|k_0-k_1|}{|k_0|}|\langle 1|\varrho|0\rangle|\\ &\quad\times|\sin\alpha\sin(\beta+\phi_{01})|\\ &=\frac{|x_+-x_-|}{2|x_+|}\frac{|k_0-k_1|}{|k_0|}|\langle 0|\varrho|1\rangle|\end{aligned}$$

$$\leqslant \frac{|k_0 - k_1|}{|k_0|} |\langle 0|\varrho|1\rangle| = \tilde{A}_{\rm Tr}(\varrho; K), \tag{A16}$$

in accord with Eq. (35) of Lemma 1. Here, $\phi_{01} = \arg\langle 0|\varrho|1\rangle$, the maximum is obtained for $X(\alpha, \beta)$ having the form of (A15) with $\alpha = \pi/2$ and $\beta = \pi/2 - \phi_{01}$, the inequality is due to the fact that $|x_+ - x_-|/2|x_+| \leqslant 1$, and the last equality is due to Eq. (A1). Again, the equality in Eq. (A16) is attained when $x_- = -x_+$.

6. Normalized quantum Fisher information vs maximum average noncommutativity of Eq. (42) for a single qubit

Writing the density operator in terms of its spectral decomposition, i.e., $\varrho = \lambda_1 |\lambda_1\rangle\langle\lambda_1| + \lambda_2 |\lambda_2\rangle\langle\lambda_2|$, $\lambda_1, \lambda_2 \in \mathbb{R}^+$, $\lambda_1 + \lambda_2 = 1$, we have

$$\tilde{\mathcal{J}}_\theta(\varrho, K) = \mathcal{J}_\theta(\varrho; K)/\|K\|^2_{\max} = 4|\lambda_1 - \lambda_2|^2 |\langle\lambda_1|K|\lambda_2\rangle|^2/\|K\|^2_{\max}, \tag{A17}$$

where we have used Eq. (A9). For the case of a single qubit, let us express the Hermitian operator $X(\alpha, \beta)$ as in Eq. (A15), where $\alpha \in [0, \pi]$, $\beta \in [0, 2\pi)$ are the angular parameters of the Bloch sphere. Then, the maximum average noncommutativity between $\tilde{X}(\alpha, \beta)$ and $\tilde{K}$ in ϱ over $(\alpha, \beta) \in [0, \pi] \times [0, 2\pi)$ on the right-hand side of Eq. (42) can be computed directly, in the basis given by the complete set of orthonormal eigenvectors of ϱ, to obtain

$$\begin{aligned}\sup_{X(\alpha,\beta)\in\Lambda_{\{x\}}} &|\mathrm{Tr}([\tilde{X}, \tilde{K}]\varrho)| \\ &= \max_{(\alpha,\beta)\in[0,\pi]\times[0,2\pi)} \frac{|x_+ - x_-|}{|x_+|}|\lambda_2 - \lambda_1|\frac{|\langle\lambda_2|K|\lambda_1\rangle|}{\|K\|_{\max}} \\ &\quad\times |\sin\alpha||\sin(\beta - \varphi_{12})| \\ &= \frac{|x_+ - x_-|}{|x_+|}|\lambda_2 - \lambda_1||\langle\lambda_2|K|\lambda_1\rangle|/\|K\|_{\max} \\ &\leqslant 2|\lambda_2 - \lambda_1||\langle\lambda_2|K|\lambda_1\rangle|/\|K\|_{\max} \\ &= \sqrt{\tilde{\mathcal{J}}_\theta(\varrho, K)}.\end{aligned} \tag{A18}$$

Here, without loss of generality, we have assumed $\|X\|_{\max} = |x_+|$, $\varphi_{12} = \arg\langle\lambda_1|K|\lambda_2\rangle$, and the inequality is due to $|x_+ - x_-|/|x_+| \leqslant 2$. Notice that equality in Eq. (A18) is again attained when $x_+ = -x_-$.

APPENDIX B: PROOF OF EQ. (11)

First, from the definition of Eq. (4) we have

$$\begin{aligned}A_{\rm w}(\varrho; K) &= \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \sum_x |\mathrm{Im}K_{\rm w}(\Pi_x|\varrho)|\mathrm{Tr}(\Pi_x\varrho) \\ &\leqslant \sup_{\{|x\rangle\}\in\mathcal{B}_o(\mathcal{H})} \left(\sum_x [\mathrm{Im}K_{\rm w}(\Pi_x|\varrho)]^2 \mathrm{Tr}(\Pi_x\varrho)\right)^{1/2},\end{aligned} \tag{B1}$$

where we have made use of the Jensen inequality. Next, noting that $[\mathrm{Im}K_{\rm w}(\Pi_x|\varrho)]^2 = |K_{\rm w}(\Pi_x|\varrho)|^2 - [\mathrm{Re}K_{\rm w}(\Pi_x|\varrho)]^2$, and inserting into Eq. (B1), we get

$$A_{\rm w}(\varrho; K) \leqslant \left(\sum_{x_*}\left(\left|\frac{\mathrm{Tr}(\Pi_{x_*}K\varrho)}{\mathrm{Tr}(\Pi_{x_*}\varrho)}\right|^2 - \mathrm{Re}\left(\frac{\mathrm{Tr}(\Pi_{x_*}K\varrho)}{\mathrm{Tr}(\Pi_{x_*}\varrho)}\right)^2\right) \times \mathrm{Tr}(\Pi_{x_*}\varrho)\right)^{1/2} \tag{B2}$$

$$\leqslant \left(\sum_{x_*}\frac{|\mathrm{Tr}(\Pi_{x_*}K\varrho)|^2}{\mathrm{Tr}(\Pi_{x_*}\varrho)} - \left[\sum_{x_*}\mathrm{Re}\mathrm{Tr}(\Pi_{x_*}K\varrho)\right]^2\right)^{1/2}, \tag{B3}$$

where we have used the definition of the weak value of Eq. (3) to get Eq. (B2), $\{|x_*\rangle\}$ is a basis which achieves the supremum, and to get Eq. (B3) we have applied the Jensen inequality, i.e., $[\sum_{x_*}\mathrm{Re}\mathrm{Tr}(\Pi_{x_*}K\varrho)]^2 = [\sum_{x_*}\frac{\mathrm{Re}\mathrm{Tr}(\Pi_{x_*}K\varrho)}{\mathrm{Tr}(\Pi_{x^*}\varrho)}\mathrm{Tr}(\Pi_{x^*}\varrho)]^2 \leqslant \sum_{x_*}(\frac{\mathrm{Re}\mathrm{Tr}(\Pi_{x_*}K\varrho)}{\mathrm{Tr}(\Pi_{x_*}\varrho)})^2\mathrm{Tr}(\Pi_{x_*}\varrho)$. Finally, applying the Cauchy-Schwartz inequality to the numerator in the first term on the right-hand side of Eq. (B3), i.e., $|\mathrm{Tr}(\Pi_{x_*}K\varrho)|^2 = |\mathrm{Tr}[(\Pi_{x_*}^{1/2}K\varrho^{1/2})(\varrho^{1/2}\Pi_{x_*}^{1/2})]|^2 \leqslant \mathrm{Tr}(\Pi_{x_*}K\varrho K)\mathrm{Tr}(\Pi_{x_*}\varrho)$, and using the completeness relation $\sum_{x_*}\Pi_{x_*} = \mathbb{I}$, we obtain

$$A_{\rm w}(\varrho; K) \leqslant (\mathrm{Tr}(K^2\varrho) - [\mathrm{Tr}(K\varrho)]^2)^{1/2} = \Delta_K(\varrho), \tag{B4}$$

as claimed. ■

[1] S. D. Bartlett, T. Rudolph, and R. W. Spekkens, Rev. Mod. Phys. **79**, 555 (2007).
[2] I. Marvian and R. W. Spekkens, Phys. Rev. A **94**, 052324 (2016).
[3] G. Gour and R. W. Spekkens, New J. Phys. **10**, 033023 (2008).
[4] J. A. Vaccaro, F. Anselmi, H. M. Wiseman, and K. Jacobs, Phys. Rev. A **77**, 032114 (2008).
[5] I. Marvian and R. W. Spekkens, New J. Phys. **15**, 033001 (2013).
[6] I. Marvian and R. W. Spekkens, Nat. Commun. **5**, 3821 (2014).
[7] I. Marvian and R. W. Spekkens, Phys. Rev. A **90**, 062110 (2014).
[8] I. Marvian, R. W. Spekkens, and P. Zanardi, Phys. Rev. A **93**, 052331 (2016).
[9] D. Mondal, C. Datta, and S. Sazim, Phys. Lett. A **380**, 689 (2016).
[10] J. Åberg, Phys. Rev. Lett. **113**, 150402 (2014).

[11] M. Lostaglio, K. Korzekwa, D. Jennings, and T. Rudolph, Phys. Rev. X **5**, 021001 (2015).
[12] M. Lostaglio, D. Jennings, and T. Rudolph, Nat. Commun. **6**, 6383 (2015).
[13] P. Ćwikliński, M. Studziński, M. Horodecki, and J. Oppenheim, Phys. Rev. Lett. **115**, 210403 (2015).
[14] G. Chiribella and Y. Yang, Phys. Rev. A **96**, 022327 (2017).
[15] M. Horodecki and J. Oppenheim, Int. J. Mod. Phys. B **27**, 1345019 (2013).
[16] E. Chitambar and G. Gour, Rev. Mod. Phys. **91**, 025001 (2019).
[17] G. Gour, I. Marvian, and R. W. Spekkens, Phys. Rev. A **80**, 012307 (2009).
[18] I. Marvian, Symmetry, Asymmetry and Quantum Information, Ph.D. Thesis, University of Waterloo, 2012, https://uwspace.uwaterloo.ca/handle/10012/7088.
[19] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
[20] H. M. Wiseman, Phys. Rev. A **65**, 032111 (2002).
[21] Y. Aharonov and D. Rohrlich, *Quantum Paradoxes: Quantum Theory for the Perplexed* (Wiley-VCH, New York, 2005).
[22] J. Dressel, M. Malik, F. M. Miatto, A. N. Jordan, and R. W. Boyd, Rev. Mod. Phys. **86**, 307 (2014).
[23] B. Tamir and E. Cohen, Quanta **2**, 7 (2013).
[24] J. S. Lundeen and K. J. Resch, Phys. Lett. A **334**, 337 (2005).
[25] R. Jozsa, Phys. Rev. A **76**, 044103 (2007).
[26] L. M. Johansen, Phys. Rev. A **76**, 012119 (2007).
[27] L. M. Johansen, Phys. Lett. A **366**, 374 (2007).
[28] M. Lostaglio, A. Belenchia, A. Levy, S. Hernández-Gómez, N. Fabbri, and S Gherardini, arXiv:2206.11783.
[29] E. Haapasalo, P. Lahti, and J. Schultz, Phys. Rev. A **84**, 052107 (2011).
[30] G. Vallone and D. Dequal, Phys. Rev. Lett. **116**, 040502 (2016).
[31] E. Cohen and E. Pollak, Phys. Rev. A **98**, 042112 (2018).
[32] R. Wagner, Z. Schwartzman-Nowik, I. L. Paiva, A. Te'eni, A. Ruiz-Molero, R. S. Barbosa, E. Cohen, and E. F. Galvão, arXiv:2302.00705.
[33] M. Cerezo, A. Arrasmith, R. Babbush, S. C. Benjamin, S. Endo, K. Fujii, J. R. McClean, K. Mitarai, X. Yuan, L. Cincio, and P. J. Coles, Nat. Rev. Phys. **3**, 625 (2021).
[34] J. Preskill, Quantum **2**, 79 (2018).
[35] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[36] A. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Edizioni della Normale, Pisa, 2011).
[37] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).
[38] S. L. Braunstein, C. M. Caves, and G. J. Milburn, Ann. Phys. **247**, 135 (1996).
[39] M. G. A. Paris, Int. J. Quantum Inf. **07**, 125 (2009).
[40] J. G. Kirkwood, Phys. Rev. **44**, 31 (1933).
[41] P. A. M. Dirac, Rev. Mod. Phys. **17**, 195 (1945).
[42] S. Chaturvedi, E. Ercolessi, G. Marmo, G. Morandi, N. Mukunda, and R. Simon, J. Phys. A: Math. Gen. **39**, 1405 (2006).
[43] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).
[44] M. F. Pusey, Phys. Rev. Lett. **113**, 200401 (2014).
[45] M. Lostaglio, Phys. Rev. Lett. **120**, 040602 (2018).
[46] R. Kunjwal, M. Lostaglio, and M. F. Pusey, Phys. Rev. A **100**, 042116 (2019).
[47] M. Lostaglio, Phys. Rev. Lett. **125**, 230603 (2020).
[48] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, Nature (London) **474**, 188 (2011).
[49] J. S. Lundeen and C. Bamber, Phys. Rev. Lett. **108**, 070402 (2012).
[50] L. Maccone and C. C. Rusconi, Phys. Rev. A **89**, 022122 (2014).
[51] A. E. Allahverdyan, Phys. Rev. E **90**, 032137 (2014).
[52] A. Levy and M. Lostaglio, PRX Quantum **1**, 010309 (2020).
[53] D. Arvidsson-Shukur, N. Y. Halpern, H. Lepage, A. Lasek, C. Barnes, and S. Lloyd, Nat. Commun. **11**, 3775 (2020).
[54] N. Lupu-Gladstein, Y. B. Yilmaz, D. R. M. Arvidsson-Shukur, A. Brodutch, A. O. T. Pang, A. M. Steinberg, and N. Y. Halpern, Phys. Rev. Lett. **128**, 220504 (2022).
[55] N. Y. Halpern, B. Swingle, and J. Dressel, Phys. Rev. A **97**, 042105 (2018).
[56] J. R. González Alonso, N. Y. Halpern, and J. Dressel, Phys. Rev. Lett. **122**, 040404 (2019).
[57] A. Budiyono and H. K. Dipojono, Phys. Rev. A **107**, 022408 (2023).
[58] A. Budiyono, M. K. Agusta, B. E. B. Nurhandoko, and H. K. Dipojono, J. Phys. A: Math. Theor. **56**, 235304 (2023).
[59] Y. Aharonov and A. Botero, Phys. Rev. A **72**, 052111 (2005).
[60] J. Dressel and A. N. Jordan, Phys. Rev. A **85**, 012107 (2012).
[61] L. M. Johansen, Phys. Lett. A **322**, 298 (2004).
[62] M. J. W. Hall, Phys. Rev. A **69**, 052113 (2004).
[63] S. Luo, Int. J. Theor. Phys. **41**, 1713 (2002).
[64] A. Budiyono and H. K. Dipojono, Phys. Rev. A **103**, 022215 (2021).
[65] S. Luo, Theor. Math. Phys. **143**, 681 (2005).
[66] S. Luo, Phys. Rev. A **72**, 042110 (2005).
[67] K. Korzekwa, M. Lostaglio, D. Jennings, and T. Rudolph, Phys. Rev. A **89**, 042122 (2014).
[68] M. J. W. Hall, Phys. Rev. A **107**, 062215 (2023).
[69] V. Giovannetti, S. Llyod, and L. Maccone, Nat. Photonics **5**, 222 (2011).
[70] D. R. M. Arvidsson-Shukur, J. C. Drori, and N. Y. Halpern, J. Phys. A: Math. Theor. **54**, 284001 (2021).
[71] S. De Bièvre, Phys. Rev. Lett. **127**, 190404 (2021).
[72] E. P. Wigner and M. M. Yanase, Proc. Natl. Acad. Sci. USA **49**, 910 (1963).
[73] S. Luo, Phys. Rev. A **98**, 012113 (2018).