Direct generation of one-way Einstein-Podolsky-Rosen steering with a self-phase-locked optical parametric oscillator

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(Received 28 August 2022; revised 26 May 2023; accepted 6 July 2023; published 25 July 2023)

Einstein-Podolsky-Rosen (EPR) steering, that is, an intermediate form of quantum correlations with asymmetry in between entanglement and Bell nonlocality, plays an essential role in quantum information processing, such as one-sided device-independent quantum cryptography tasks. Here we theoretically study the direct generation of EPR steering in $\chi^{(2)}$ nonlinear optical processes with cascaded crystals in a self-phase-locked optical parametric oscillator. Based on a standard linearization method, we analyze different conditions for generating EPR steering in Gaussian states with vanishing or nonzero mean fields, and also show bistable regions for different types of EPR steering. In particular, such system can intrinsically generate asymmetry one-way EPR steering, without introducing any postoperation. Our results suggest that the self-phase-locked optical parametric oscillator is a good candidate to generate different types of EPR steering at will by varying the cavity detunings, photon loss, or input pump power.

DOI: 10.1103/PhysRevA.108.012430

I. INTRODUCTION

The nonlocality of entangled quantum states was first pointed out by Einstein, Podolsky, and Rosen (EPR) [1]. Schrödinger mentioned in his article [2] in 1935 an interesting quantum phenomenon that can be described as follows: for two subsystems, Alice and Bob, that are very far apart, the local measurements performed on Alice seem to immediately affect the state of another distant Bob. In this article, he called this nonlocal effect embodied in the EPR paradox as "steering." Different from Bell nonlocality [3] and entanglement [4] where the roles of the involved parties are symmetric, steering is a directional form of nonlocality for which the losses or noises can act asymmetrically [5–8]. For continuous-variable systems, the two subsystems sharing entangled states share quantum correlations in phase and amplitude quadrature, to test such phenomena. To quantify the EPR steering, Reid introduced an experimental criterion based on the inferred Heisenberg uncertainty relation in 1989 [9]. Later Wiseman mathematically formalized the concept of steering by violating the local hidden state model [10]. EPR steering is an intermediate form of entanglement with asymmetry [5,6,11–16] between the entanglement [4] indistinguishability of quantum states and Bell nonlocality [3]; that is, EPR steering allows one to verify entanglement shared between Alice and Bob without the assumptions of the full trust of their devices [17,18], which can be used for one-sided device-independent quantum cryptography tasks [19–26]. Recently, many efforts have been made in generating versatile EPR steering. Händchen *et al.* generated one-way EPR steering via a squeezed light and vacuum setting [27]. Armstrong *et al.* [28] and Deng *et al.* [29] produced multipartite EPR steering by mixing squeezing light via beam-splitter networks, and Cai *et al.* generate a large-scale EPR steering within a quantum frequency comb [30]. Four-wave mixing processes of atoms were also proposed to generate multipartite and collective EPR steering [7,8,31]. Moreover, there are great prospects for applications in quantum information technology, e.g., quantum key distribution [19,20,22,23,25,26], secure quantum teleportation [22,23,26], quantum cryptography [19,20,32,33], and subchannel discrimination [24,34,35].

However, generally, if one wants to govern the EPR steering, passive postoperation of the linear optics is required, which can involve the use of devices like beam splitters and phase shifters [27]. It still lacks a quantum source that can intrinsically generate asymmetry EPR steering with active control within the generation process. Conventionally, an entangled optical state of a continuous variable system is generated in a parametric down-conversion (PDC) process inside an optical parametric oscillator, on condition that the photon energy conservation as well as the fixed initial-phase relationship among the pump and the generated fields are fulfilled [36]. However, the initial phases of generated fields can take on any value, as long as their sum is fixed. It is therefore a challenging task to acquire high-quality quantum correlations in experiments [37] without locking the phases. The difficulty is even more prominent for a nondegenerate optical parametric oscillator. With the utilization of an additional nonlinear interaction from another crystal in a self-phase-locked optical parametric oscillator (SPLOPO), the initial phases can be self-locked in a passive manner [38,39],

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and the phase diffusion should be small [40]. Therefore, apart from adding an extra nonlinear crystal, this self-locking technique does not introduce any additional complexity or cost to the experimental setup, making it highly attractive for practical applications. Due to the robustness against phase diffusion inside SPLOPOs, the passive locking scheme has been widely used in frequency metrology [36,38,39], and some schemes for obtaining high-quality two-mode entanglement have been proposed [41,42] or demonstrated [36,37].

In a divide-by-3 SPLOPO [38], the initial phases can be locked to three possible discrete values [38,39], and the generation of robust two-mode entanglement has been theoretically investigated in [42]. The nonlinear interaction coming from the second crystal will give birth to the three types of effective interactions after linearization: the beam splitter (mixing different frequencies), single-mode squeezer, and two-mode squeezer, when the mean amplitudes are nonzero [see Eq. (6) below]. It may also be observed that introducing cavity detunings is equivalent to adding phase shifters inside to change the effective cavity lengths. Therefore, it is expected that all the linear operations required for generating EPR steering from a two-mode entangled state can be implemented inside the divide-by-3 SPLOPO. One is then led to an interesting question: can EPR steering, or even one-way EPR steering, be directly produced in the SPLOPO?

In this paper, we propose to use the SPLOPO system to generate one-way and two-way EPR steering directly, without the need for postoperation of the linear optics. We investigate the variation of the steering and the corresponding quantum properties via modifying the system parameters. Moreover, we also investigate the controllability of the one-way steering direction in the bistable region. The direction of the one-way steering can be tuned by adjusting the cavity detunings, the photon loss, and the input pump power. Our results show that one-way steering can be obtained in an active way, which is useful for the secure transmission of quantum information.

This paper is organized as follows. In Sec. II, we explain the model of the system that achieves self-phase locking and the existence of stable solutions in the system. In Sec. III, we introduce the covariance matrix that can be used in the later analysis of the steering and the EPR steering in the case of stable solutions. In Sec. IV, the EPR steering for pure states is analyzed, and the focus is on the bright-beam cases. The first important scenario is related to the nil-cavity-detunings condition, which is the most common situation in quantum optical experiments. There, one knows that the steering is symmetric, and that the amount of steering changes along with the correlations. Based on this information, one-way EPR steering can be produced by either introducing cavity detunings or adding losses. Then, one is led to the most important subsection of this paper, discussing steering for pure states for the bright-beam case under nonvanishing cavity detunings. The most important result of our paper is that the one-way EPR steering can be produced *directly* by introducing cavity detunings. In Sec. V, we study the effect of decoherence effects on the steering when there is a photon-loss rate. Finally, we summarize in Sec. VI. For relevance, some helpful information can be found in the Appendices.

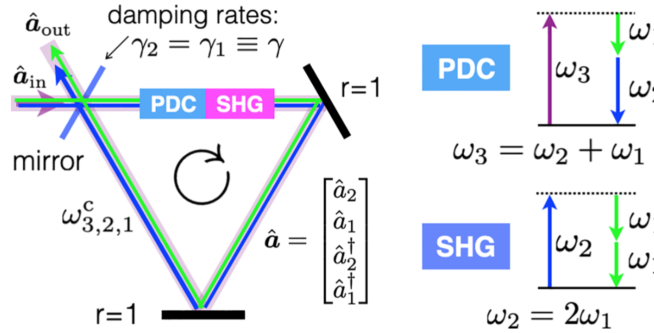

FIG. 1. Sketches of a SPLOPO with cascaded crystals, and the nonlinear optical processes in the PDC and the subharmonic generation crystals.

II. MODEL

Let us first make precise our model of the divide-by-3 SPLOPO sketched in Fig. 1. The ring cavity contains three mirrors, one leaky and the other two perfectly reflective. Inside the ring cavity, two crystals, called PDC and subharmonic generation (SHG), are concatenated on a path of beams. Coupled with an intense classical field of central frequency ω_3 at the leaky mirror, the nearest intracavity pump mode ω_{3c} is significantly excited. The pump detuning $\omega_3 - \omega_{3c}$ is small so that the intracavity pump can be effectively converted from the external one and treated as a classical wave with a strong amplitude α_3. In the PDC crystal, a pump photon can be split into a signal photon of frequency ω_2 near the cavity mode ω_{2c} and an idler photon of frequency ω_1 near the cavity mode ω_{1c}. In the SHG crystal, the pump is inactive, while the signal and idler satisfy the phase-matching conditions for the subharmonic generation process, i.e., a signal photon can break into two identical idler photons or vice versa, inducing a nonlinear self-injection locking of the signal and idler subharmonics. The crystals, therefore, are named after the involved nonlinear optical processes, PDC and SHG, respectively. Because nonlinear crystals produce nonuniform free spectral range, in any real experiments, the cavity should be compensated by some intracavity dispersive elements if necessary. We assume that the intracavity modes of pump, signal, and idler waves well meet the 3:2:1 frequency ratio conditions, i.e., $\omega_{3c} : \omega_{2c} : \omega_{1c} \simeq 3 : 2 : 1$, so that the pump, signal, and idler photons satisfy the exact 3:2:1 frequency ratios, $\omega_3 : \omega_2 : \omega_1 = 3 : 2 : 1$. Hereinafter, the subscripts 3, 2, and 1 designate the pump, signal, and idler waves, respectively, unless otherwise noted. We also call the signal and idler waves as the (frequency) mode 2ω and 1ω, respectively, when needed. The photons in the SPLOPO leak at the loss rates $\gamma_j (j = 3, 2, 1)$ of the pump, signal, and idler amplitude, respectively. We consider a nonresonant pump($\gamma_3 \gg \gamma_2, \gamma_1$), i.e., a doubly resonant optical parametric oscillator, in which only the two subharmonics are resonantly enhanced. The interaction Hamiltonian reads

$$\hat{V} = i\hbar\big(\chi \hat{A}_2^\dagger \hat{A}_1^\dagger + \kappa \hat{A}_2 \hat{A}_1^{\dagger 2}/2\big) + \text{H.c.}, \tag{1}$$

where χ and κ are two positive coupling coefficients, and χ is proportional to the pump field amplitude. $\hat{A}_2$ and $\hat{A}_1$ are field annihilation operators of the signal and idler, respectively, satisfying the canonical equal-time commutation relations, of which the nonvanishing ones are

$$[\hat{A}_2(t), \hat{A}_2^\dagger(t)] = [\hat{A}_1(t), \hat{A}_1^\dagger(t)] = 1. \tag{2}$$

The field operators $\hat{A}_2$ and $\hat{A}_1$ are governed by the Heisenberg-Langevin equation [42]:

$$\dot{\hat{A}}_1 + d_1\hat{A}_1 = \chi\hat{A}_2^\dagger + \kappa\hat{A}_1^\dagger\hat{A}_2 + \sqrt{2\gamma_1}\hat{a}_{\text{in},1}, \tag{3a}$$

$$\dot{\hat{A}}_2 + d_2\hat{A}_2 = \chi\hat{A}_1^\dagger - \kappa\hat{A}_1^2/2 + \sqrt{2\gamma_2}\hat{a}_{\text{in},2}, \tag{3b}$$

where $d_j = \gamma_j - i\Delta_j$ are complex loss rates, and $\Delta_j = \omega_j - \omega_{jc}$ are the cavity detunings, for $j = 1, 2$. The Langevin noise forces, $\hat{a}_{\text{in}}$, preserve the canonical equal-time commutation relations of $\hat{A}_{2,1}$ and have vanishing values of the mean and second-order moments except $\langle\hat{a}_{\text{in},2}(t)\hat{a}_{\text{in},2}^\dagger(t')\rangle = \langle\hat{a}_{\text{in},1}(t)\hat{a}_{\text{in},1}^\dagger(t')\rangle = \delta(t-t')$. For theoretical simplicity, we take $\gamma_1 = \gamma_2 = \gamma$.

Stable properties of the SPLOPO at a long time are of great interest, as the mean amplitudes become time independent, known as the steady solution, and any small deviations from the *stable* steady solution should converge to zero. To this end, let us divide the field operators $\hat{A}_j(t) = A_j + \hat{a}_j(t)$ with $j = 1, 2$ into two parts: the steady mean fields A_j and the pure quantum fluctuations $\hat{a}_j(t)$. The steady mean fields satisfy

$$d_1A_1 = \chi A_2^* + \kappa A_1^* A_2, \tag{4a}$$

$$d_2A_2 = \chi A_1^* - \kappa A_1^2/2. \tag{4b}$$

There exist three possible discrete phases, which are evenly spaced, of the mean fields [42], and without loss of generality we assume both the phases are in the range between $-\pi/3$ and $\pi/3$. If both sides of Eqs. (4) are multiplied by κ, then the new equations of κA_1 and κA_2 become independent of the parameter κ, i.e., any acceptable solution $(\kappa A_1, \kappa A_2)$ should be independent of κ. The nondepletion assumption of the pump light requires the coupling strength κ not to be too small, as it is valid when the down-converted pump photons should be much less than the pump photons converted from the external pump light. But this restriction is weak, because we have assumed that γ_3 is so great that the external pump photons can be easily converted into the intracavity ones. For instance, when κ is comparable to the coupling strength χ/α_3 of the three waves in the PDC crystal, the assumption can be safely applied.

If the quantum fluctuations are denoted by an operator-valued vector $\hat{\boldsymbol{a}} = \text{col}(\hat{a}_2, \hat{a}_1, \hat{a}_2^\dagger, \hat{a}_1^\dagger)$, and the Langevin input noises are denoted by another one $\hat{\boldsymbol{a}}_{\text{in}} = \text{col}(\hat{a}_{\text{in},2}, \hat{a}_{\text{in},1}, \hat{a}_{\text{in},2}^\dagger, \hat{a}_{\text{in},1}^\dagger)$, the linearized Heisenberg-Langevin equations turn out to be

$$\dot{\hat{\boldsymbol{a}}} = \mathbb{L}\hat{\boldsymbol{a}} + \sqrt{2\gamma}\hat{\boldsymbol{a}}_{\text{in}}, \quad \mathbb{L} = \begin{bmatrix} -d_2 & -\kappa A_1 & 0 & \chi \\ \kappa A_1^* & -d_1 & \chi & \kappa A_2 \\ 0 & \chi & -d_2^* & -\kappa A_1^* \\ \chi & \kappa A_2^* & \kappa A_1 & -d_1^* \end{bmatrix}. \tag{5}$$

It suggests that the effective interaction Hamiltonian in the two crystals turns out to be

$$\hat{V}_{\text{eff}} = i\hbar[\chi\hat{a}_2^\dagger\hat{a}_1^\dagger + \kappa(A_2\hat{a}_1^{\dagger 2}/2 + A_1^*\hat{a}_2\hat{a}_1^\dagger)] + h.c., \tag{6}$$

which is a mixture of a two-mode squeezer, a single-mode squeezer, and a beam-splitter-like interaction. The latter two terms are adjustable because of A_1 and A_2, and the role of the beam splitter can be noticeable. It should be emphasized that single-mode squeezing and two-mode squeezing can be interconverted in the SPLOPO, indicating that entanglement between two modes originates not only from two-mode squeezing but also indirectly from the conversion of single-mode squeezing, according to the effective Hamiltonian $\hat{V}_{\text{eff}}$ in Eq. (6).

Equation (4b) suggests that the mean field A_2 has a simple solution in terms of A_1. Because κ and χ are positive parameters, if (A_1, A_2) is a solution of Eqs. (4) for given detunings (Δ_1, Δ_2), then (A_1^*, A_2^*) should be a solution of Eqs. (4) with detunings $(-\Delta_1, -\Delta_2)$. Thus, when the detunings are both zero, the mean fields A_1 and A_2 are both real. The stability condition for the mean fields (A_1, A_2) is that the real parts of all eigenvalues belonging to matrix $\mathbb{L}$ should be less than zero: max Re eig$(\mathbb{L}) < 0$. So, if the mean fields are stable for the parameters $(\chi, \Delta_1, \Delta_2)$, it should be equally stable for the new parameters $(\chi, -\Delta_1, -\Delta_2)$.

The mean field $A_1 = A_2 = 0$, called the dark-beam solution, is stable only when [42]

$$\chi^2 < N_{\text{osc}} = \gamma^2 + (\Delta_1 + \Delta_2)^2/4. \tag{7}$$

Here, N_{osc} is the upper bound of χ^2 (which is proportional to the pump power) derived from the stability condition max Re eig$(\mathbb{L}) < 0$ for the dark-beam solution. According to [42], when $\chi^2 \geqslant N_{\text{br}}$ with

$$N_{\text{br}} = \frac{4}{9}\Big[\gamma^2 - 9\Delta_1\Delta_2 + \sqrt{(\gamma^2 + 9\Delta_1^2)(\gamma^2 + 9\Delta_2^2)}\Big], \tag{8}$$

the SPLOPO allows for another solution, which can also be stable. With

$$\mathcal{B} = \chi[1 - d_2/(2d_2^*)], \quad \mathcal{C} = d_2(d_1 - \chi^2/d_2^*), \tag{9}$$

the solution of A_1 reads [42]

$$\kappa|A_1| = \sqrt{|\mathcal{B}|^2 + |\mathcal{C}| - \text{Re}\,\mathcal{C}} + \sqrt{|\mathcal{B}|^2 - |\mathcal{C}| - \text{Re}\,\mathcal{C}}, \tag{10a}$$

$$\arg A_1 = \frac{1}{3}\arg\left(\frac{\mathcal{B}}{\kappa^2|A_1|^2 + 2\mathcal{C}}\right). \tag{10b}$$

Because its elements A_1 and A_2 are nonvanishing, the new solution is called the bright-beam solution. According to the definition in Eq. (8), $N_{\text{br}} \geqslant 4\gamma^2/9 > 0$; thus, the bright-beam solution exists only when the pump field is sufficiently intense $(\chi^2 \geqslant N_{\text{br}})$. In other words, N_{br} is related to the lower bound of the pump power needed to observe the bright beams. Figure 2 show two three-dimensional domains (colored) of parameters $(\Delta_2, \Delta_1, \chi)$ for the existence of the stable steady solution to Eq. (4): Fig. 2(a) is for the dark-beam solution while Fig. 2(b) is for the bright-beam solution. When $N_{\text{osc}} > \chi^2 \geqslant N_{\text{br}}$, the SPLOPO may allow for the existence of both the two solutions and, therefore, the existence of the bistable region [42].

The quantum properties measured at the output of the leaky mirror can be completely characterized by the covariance matrix of the field quadratures [43–45]:

$$\text{col}(\hat{X}_2, \hat{X}_1, \hat{P}_2, \hat{P}_1) \equiv \mathbb{M}\hat{\boldsymbol{a}}, \quad \mathbb{M} = \begin{bmatrix} \mathbb{I}_2 & \mathbb{I}_2 \\ -i\mathbb{I}_2 & i\mathbb{I}_2 \end{bmatrix} \tag{11}$$

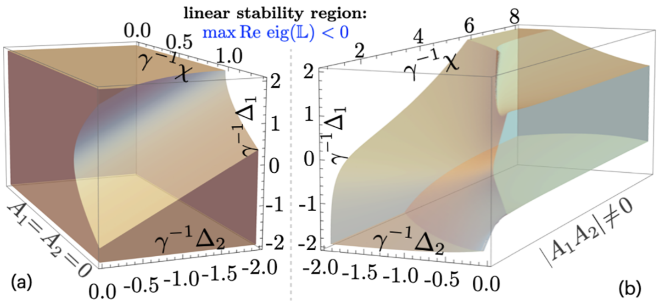

FIG. 2. Regions for admissible steady solution with respect to two cavity detunings Δ_1 and Δ_2 and the coupling strength χ: (a) for dark-beam solution and (b) for bright-beam solution.

with $\mathbb{O}_n$ and $\mathbb{I}_n$ being the null and identity matrices of $n \times n$ type, respectively. For long-time behaviors, the covariance matrix at zero Fourier frequency reads [42]

$$\mathbb{C}_{\text{out}} = 2\mathbb{M}\left(\gamma\mathbb{L}^{-1} + \frac{\mathbb{I}_4}{2}\right)\begin{bmatrix} \mathbb{O}_2 & \mathbb{I}_2 \\ \mathbb{I}_2 & \mathbb{O}_2 \end{bmatrix}\left(\gamma\mathbb{L}^{-1} + \frac{\mathbb{I}_4}{2}\right)^{\mathsf{T}}\mathbb{M}^{\mathsf{T}}. \tag{12}$$

Recall that the parameter κ always comes with the mean amplitudes A_1 and A_2 [see Eqs. (5)], and κA_1 and κA_2 are independent of κ; the matrix $\mathbb{L}$ and therefore the covariance matrix $\mathbb{C}_{\text{out}}$ are determined by parameters such as the cavity detunings, but should be independent of κ. One then concludes that the coupling strength κ of the SHG crystal does not have a direct contribution to the quantum properties of the system under the linearization approximation.

III. COVARIANCE MATRIX AND QUANTUM PROPERTIES

Before proceeding with the discussion, let us briefly introduce the definition of the EPR steering of a Gaussian state and how it relates to the covariance matrix.

A generic covariance matrix $\mathbb{C}$ can be partitioned as

$$\mathbb{C} = \begin{bmatrix} \mathbb{X} & \mathbb{C}_{XP} \\ \mathbb{C}_{XP}^{\mathsf{T}} & \mathbb{P} \end{bmatrix}, \tag{13}$$

where submatrices $\mathbb{X}$ and $\mathbb{P}$ represent the correlations of the $\hat{X}$ quadratures and the $\hat{P}$ quadratures, respectively, and submatrix $\mathbb{C}_{XP}$ stands for the correlations between a $\hat{X}$ quadrature and a $\hat{P}$ quadrature.

Due to the existence of bipartite correlations, one can make an estimate of a local measurement $\hat{\mathcal{O}}_j$ on the mode $j\omega$ based on the results from another local measurement $\hat{\mathcal{O}}_{3-j}$ on the remaining subsystem. The optimal inference variance is given by

$$\Delta_{\text{inf}}\hat{\mathcal{O}}_j = V_{\hat{\mathcal{O}}_j,\hat{\mathcal{O}}_j} - V^{-1}_{\hat{\mathcal{O}}_{3-j},\hat{\mathcal{O}}_{3-j}} V^2_{\hat{\mathcal{O}}_j,\hat{\mathcal{O}}_{3-j}}, \tag{14}$$

where $V_{\hat{A},\hat{B}} = \langle\hat{A}\hat{B} + \hat{B}\hat{A}\rangle/2 - \langle\hat{A}\rangle\langle\hat{B}\rangle$ is the correlation between $\hat{A}$ and $\hat{B}$. In this paper, we only consider field quadratures $\hat{X}_1, \hat{X}_2, \hat{P}_1$, and $\hat{P}_2$. The inferred variances for the field quadratures read (for j=1,2)

$$\Delta_{\text{inf}}\hat{X}_j = V_{\hat{X}_j,\hat{X}_j} - V^{-1}_{\hat{X}_{3-j},\hat{X}_{3-j}} V^2_{\hat{X}_j,\hat{X}_{3-j}}, \tag{15a}$$

$$\Delta_{\text{inf}}\hat{P}_j = V_{\hat{P}_j,\hat{P}_j} - V^{-1}_{\hat{P}_{3-j},\hat{P}_{3-j}} V^2_{\hat{P}_j,\hat{P}_{3-j}}. \tag{15b}$$

The two inferred variances can be reexpressed as

$$\Delta_{\text{inf}}\hat{X}_j = \frac{\det\mathbb{X}}{V_{\hat{X}_{3-j},\hat{X}_{3-j}}}, \quad \Delta_{\text{inf}}\hat{P}_j = \frac{\det\mathbb{P}}{V_{\hat{P}_{3-j},\hat{P}_{3-j}}}. \tag{16}$$

According to Reid's criterion [9,46], when the amount of EPR steering

$$\mathcal{E}_{j|3-j} = \Delta_{\text{inf}}\hat{X}_j\Delta_{\text{inf}}\hat{P}_j = \frac{\det\mathbb{X}\det\mathbb{P}}{V_{\hat{X}_{3-j},\hat{X}_{3-j}}V_{\hat{P}_{3-j},\hat{P}_{3-j}}}, \tag{17}$$

the product of the two inferred variances is less than 1, the mode $j\omega$ can be steered by measurements performed on the remaining mode, and the EPR paradox is demonstrated. For the subsystem of the frequency mode $j\omega$ ($j = 1, 2$), the determinant of the reduced covariance matrix $\mathbb{C}_j$ is $\det\mathbb{C}_j = V_{\hat{X}_j,\hat{X}_j}V_{\hat{P}_j,\hat{P}_j} - V^2_{\hat{X}_j,\hat{P}_j}$, and then the amount of steering can be reexpressed as

$$\mathcal{E}_{j|3-j} = \det\mathbb{X}\det\mathbb{P}/\big(\det\mathbb{C}_{3-j} + V^2_{\hat{X}_{3-j},\hat{P}_{3-j}}\big). \tag{18}$$

The monotone for the amount of EPR steering is guaranteed for two-mode states [47]. So, the ability to steer the quantum lights in two different ways (1|2 and 2|1) should be different, if

$$V_{\hat{X}_1,\hat{X}_1}V_{\hat{P}_1,\hat{P}_1} \neq V_{\hat{X}_2,\hat{X}_2}V_{\hat{P}_2,\hat{P}_2}. \tag{19}$$

This inequality or its variants is the starting point for all the discussions on symmetric or asymmetric steering. For example, in the pure two-mode state cases, the condition (19) becomes $V_{\hat{X}_2,\hat{P}_2} \neq \pm V_{\hat{X}_1,\hat{P}_1}$, because the purity of the two subsystems should be the same, the fact of which imposes that $\det\mathbb{C}_1 = \det\mathbb{C}_2$.

The quantum state and its correlations can be very simple and easy to manipulate if the covariance matrix is block diagonal, $\mathbb{C} = \mathbb{X} \oplus \mathbb{P}$; then the amount of EPR steering becomes

$$\mathcal{E}_{j|3-j} = \frac{\det\mathbb{C}}{\det\mathbb{C}_{3-j}}, \quad j = 1, 2. \tag{20}$$

Reid's criterion suggests that when a subsystem has poorer purity than the total state has, then the remaining subsystem can be steered. Because of the identity $\det(ABC) = \det A\det B\det C$, for any symplectic transformation $\mathbb{S}$, one should have $\det(\mathbb{S}\mathbb{C}\mathbb{S}^{\mathsf{T}}) = \det\mathbb{C}$. Similarly, the purity of the reduced systems, or $\det\mathbb{C}_j(j = 1, 2)$, is invariant under any local symplectic transformations. Then, the amount of EPR steering (20) is invariant under any local symplectic transformation provided that the covariance matrix after transformation is still a block-diagonal one.

IV. PURE TWO-MODE GAUSSIAN STATES

To be present, the lights generated from the SPLOPO should be in pure two-mode Gaussian states, since no additional loss scheme is introduced. The focus in this section, therefore, is about the quantum properties of pure states, especially the EPR steering.

A. EPR steering: Stable dark-beam case

When the mean field belongs to the stable dark-beam solution, its corresponding covariance matrix is already given in [42], and the two related submatrices read

$$\mathbb{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12} & X_{11} \end{bmatrix}, \quad \mathbb{P} = \begin{bmatrix} X_{11} & -X_{12} \\ -X_{12} & X_{11} \end{bmatrix}, \tag{21a}$$

$$X_{11} = 1 + 8\frac{\gamma^2\chi^2}{\mathcal{D}}, \quad X_{12} = 4\gamma\chi\frac{\chi^2+\gamma^2-\Delta_1\Delta_2}{\mathcal{D}}, \tag{21b}$$

$$\mathcal{D} = \gamma^2(\Delta_1-\Delta_2)^2 + (\gamma^2+\Delta_1\Delta_2-\chi^2)^2. \tag{21c}$$

Subsequently, the amounts of EPR steering yield $\mathcal{E}_{2|1} = \mathcal{E}_{1|2} = \mathcal{E}$:

$$\mathcal{E} = \left(X_{11} - X_{12}^2/X_{11}\right)^2. \tag{22}$$

B. EPR steering: Stable bright-beam case

1. Zero cavity detuning condition

Let us now consider the simplest situation that the cavity detunings are zero for the signal and idler. In this case, the SPLOPO allows for the stable dark-beam solution $A_1{=}A_2{=}0$ when $\chi < \gamma$, and the stable bright-beam solution for $\chi \geqslant \sqrt{8}\gamma/3$,

$$A_1 = \frac{\chi + \sqrt{9\chi^2 - 8\gamma^2}}{2\kappa}, \quad A_2 = \frac{\gamma}{\kappa}\frac{\chi+\sqrt{9\chi^2-8\gamma^2}}{3\chi+\sqrt{9\chi^2-8\gamma^2}}, \tag{23}$$

then a bistable region for $\gamma > \chi \geqslant \sqrt{8}\gamma/3$ [42].

When $\Delta_2 = \Delta_1 = 0$, the matrix $\mathbb{L}$ defined in Eq. (5) is filled with real elements. Then, it begins to have the symmetry $\mathbb{L} = (\sigma_1 \otimes \mathbb{I}_2)\mathbb{L}(\sigma_1 \otimes \mathbb{I}_2)$, where σ_1 is the first Pauli matrix. So do $\mathbb{L}^{-1}$ and $(\mathbb{L}^{-1})^{\intercal}$, because $(\sigma_1 \otimes \mathbb{I}_2)^2 = \mathbb{I}_4$. According to Eq. (12), the covariance matrices under the zero-cavity-detuning condition can always be written in the block-diagonal form $\mathbb{C}_{\text{out}} = \mathbb{X} \oplus \mathbb{P}$ due to the symmetry of $\mathbb{L}$, regardless of which solution set the mean fields (A_2, A_1) belong to. See more mathematical details in Appendix B. After direct calculations, one should have $\det \mathbb{C}_{\text{out}} = 1$, which reveals the fact that the Gaussian states are pure. It is not surprising, as no decoherence schemes are considered to be present.

Now that the covariance matrix is block diagonal, the amount of single-mode squeezing SMS_j in the mode $j\omega$ and the amount of two-mode squeezing TMS can be defined (see more information in Appendix A) as

$$\text{SMS}_j = 5\log_{10}(V_{\hat{X}_j,\hat{X}_j}/V_{\hat{P}_j,\hat{P}_j}), \quad j = 1, 2; \tag{24a}$$

$$\text{TMS} = 10\log_{10}\left(\sqrt{\det \mathbb{C}_{\text{out},1}} + \sqrt{-V_{\hat{X}_1,\hat{X}_2}V_{\hat{P}_1,\hat{P}_2}}\right). \tag{24b}$$

When the amount of single-mode squeezing $\text{SMS}_{j=1,2}$ is positive (negative), the single-mode squeezer of the mode $j\omega$ squeezes the variance of the $\hat{P}_j$ $(\hat{X}_j)$ quadrature. The three curves of squeezing are shown in Fig. 3(a), and one observes that the single-mode squeezing of the idler is always stronger than the one of the signal, and the two never vanish simultaneously, indicating that the corresponding Gaussian state is not symmetric. This property is important for the discussion in Sec. V.

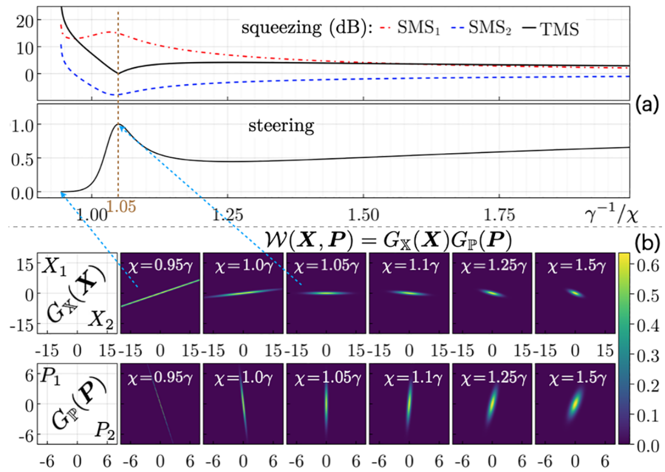

FIG. 3. (a) Squeezing and steering at different input pump field amplitudes. (b) Rotation of Wigner functions in phase space. One observes that the EPR steering is strong at $\chi = 0.95\gamma$, and is vanishing at $\chi = 1.05\gamma$.

As shown in Eq. (5), the effective interaction Hamiltonian consists of a "beam splitter," a single-mode squeezer, and a two-mode squeezer. It should be natural to observe contributions from the single-mode and two-mode squeezers in covariance matrices by making the Bloch-Messiah reduction [48]. A necessary condition for a pure bipartite Gaussian state is that the purities of the two reduced density matrices are the same, i.e., $\det \mathbb{C}_{\text{out},1} = \det \mathbb{C}_{\text{out},2}$ with $\mathbb{C}_{\text{out},j} (j = 1, 2)$ being the reduced covariance matrix of the mode $j\omega$. Because the determinants of reduced covariance matrices $\det \mathbb{C}_{\text{out},j} = V_{\hat{X}_j,\hat{X}_j}V_{\hat{P}_j,\hat{P}_j}$ (for $j = 1, 2$), the necessary condition implies that

$$V_{\hat{X}_1,\hat{X}_1}V_{\hat{P}_1,\hat{P}_1} = V_{\hat{X}_2,\hat{X}_2}V_{\hat{P}_2,\hat{P}_2}. \tag{25}$$

Similarly, because of the following two identities, $\det \mathbb{C}_{\text{out}} = \det \mathbb{X} \det \mathbb{P} = 1$, as well as $\det \mathbb{C}_{\text{out},j} = V_{\hat{X}_j,\hat{X}_j}V_{\hat{P}_j,\hat{P}_j} (j = 1, 2)$, the amount of EPR steering (17) turns out to be

$$\mathcal{E}_{j|3-j} = \frac{1}{\det \mathbb{C}_{\text{out},3-j}}, \tag{26}$$

which is the square of the purity of the mode $(3-j)\omega$. Thus, one arrives at the following conclusion: the amounts of EPR steering for the two different ways share the same value:

$$\mathcal{E}_{1|2} = \mathcal{E}_{2|1} \equiv \mathcal{E}, \tag{27}$$

due to the fact that $\det \mathbb{C}_{\text{out},1} = \det \mathbb{C}_{\text{out},2}$; the subsystems are not steerable only when the two-mode system is separable, $\det \mathbb{C}_{\text{out},1} = \det \mathbb{C}_{\text{out},2} = 1$, or equivalently, $V_{\hat{X}_1,\hat{X}_2} = V_{\hat{P}_1,\hat{P}_2} = 0$.

For pure Gaussian states whose covariance matrix is in block-diagonal form, its displaced Wigner distribution $\mathcal{W}(\boldsymbol{X},\boldsymbol{P}) = G_{\mathbb{X}}(\boldsymbol{X})G_{\mathbb{P}}(\boldsymbol{P})$ can be decomposed as a product of two Gaussian distributions $G_{\mathbb{X}}(\boldsymbol{X})$ and $G_{\mathbb{P}}(\boldsymbol{P})$ (see Appendix A). The Gaussian distributions $G_{\mathbb{X}}(\boldsymbol{X})$ and $G_{\mathbb{P}}(\boldsymbol{P})$ show correlations between X quadratures and the ones between P quadratures, respectively. Thus, the two Gaussians are helpful to show how quantum properties of the states evolve with the parameters.

It is known in the literature that by mixing a squeezed state and a vacuum at a beam splitter, the two output beams should be squeezed in the same direction, while for a two-mode squeezed state after a rotation in the phase space by a beam splitter the two squeezed directions should be perpendicular with each other. Therefore, according to Fig. 3(b), the two-mode state can be essentially obtained from a two-mode squeezed state rotated by a beam splitter, and the two-mode squeezing is sensitive to the rotation of the displaced Wigner distribution. When χ is close to the start point $\sqrt{8}\gamma/3(\approx 0.94\gamma)$, the two-mode squeezing dominates in the quantum fluctuations, and the strong correlations in both the X quadratures and the P quadratures [see Fig. 3(b)] enable the SPLOPO a very strong steerability.

According to the expressions in Appendix B2, the conditions $V_{\hat{X}_1,\hat{X}_2} = V_{\hat{P}_1,\hat{P}_2} = 0$ are equivalent to

$$\Omega = \gamma\kappa^2 A_1 A_2 + \chi\left(\kappa^2 A_1^2 - \gamma^2 - \chi^2\right) = 0, \tag{28}$$

which has a single root

$$\chi_0 = \gamma\sqrt{\frac{35}{27} - \frac{2\sqrt{577}}{27}\cos\left(\frac{\pi}{3} + \frac{1}{3}\arctan\frac{216\sqrt{3687}}{4481}\right)}, \tag{29}$$

and is approximately equal to 1.05γ. When this condition is satisfied, as shown in Fig. 3(a), at $\chi = 1.05\gamma$, the amount of two-mode squeezing is zero, and the amount of EPR steering is 1, i.e., entanglement and correlations between the two modes disappear, and the SPLOPO loses its ability to steer the quantum states. The physical origin is that the displaced Wigner distribution rotates in phase space because of the existence of the beam-splitter interaction Hamiltonian [see Fig. 3(b)]. The SPLOPO goes from being strongly capable of steering to a complete loss of the ability in a very small range: $0.95\gamma < \chi < 1.05\gamma$.

The beam-splitter interaction Hamiltonian mainly plays two roles: (1) to convert the photons of the two modes from each other, including all possible components of the mean fields and the fluctuations, and (2) to rotate the displaced Wigner distribution. Because the interaction strength of the beam splitter is proportional to the mean field amplitude A_1, with the increasing of the pump power ($\propto \chi^2$), the mean field amplitude A_1 increases accordingly, and in a general sense, the squeezing in individual modes as well as correlations between modes attenuate and, finally, disappear [see Fig. 3(a)]. In more detail, when $\sqrt{8}\gamma/3(\sim 0.94\gamma) < \chi < 0.967\gamma$, the beam splitter plays a more important role in converting the single-mode squeezing from the mode ω, and the effective single-mode squeezers squeeze the variances in the same direction; when $\chi \simeq 0.967\gamma$, the squeezing converted from the single-mode squeezing from the mode ω and the anti-squeezing from the two-mode squeezing are balanced, and thus no squeezing in mode 2ω; when $1.05\gamma \geqslant \chi > 0.967\gamma$, the two-mode squeezing is gradually converted to enhance the single-mode squeezing in the two modes, so the amounts of the single-mode squeezing are improved; when $\chi \geqslant 1.05\gamma$, the correlations (two-mode squeezing) as well as the ability of steering improve noticeably because the single-mode squeezing in the two modes is converted back into the two-mode squeezing, and then worsens simultaneously with the attenuation of the single-mode squeezing (see Fig. 3).

2. Nonzero cavity detunings

In the presence of cavity detunings, the bright-beam solution of the mean fields (A_1, A_2) is a pair of complex functions. In this case, the stability of the mean fields relies on the values of the parameters, and the region of the parameters does not have a simple shape in general (see the right panel of Fig. 2). However, from the right panel of Fig. 2, one observes that when $|\Delta_1| < \gamma$, the mean fields (A_1, A_2) are stable in general, while the stability is weakly dependent with the cavity detuning of the signal Δ_2, because the mean field A_2 quickly saturates to a small complex number as the pump power increases [42]. In this paper, we tacitly assume that the mean fields with cavity detunings have been verified to be stable.

Recall that when the detunings both take opposite values, $(\Delta_1, \Delta_2) \to (-\Delta_1, -\Delta_2)$, the mean fields (A_1, A_2) change into (A_1^*, A_2^*). Subsequently, the matrix $\mathbb{L} \to \mathbb{L}^*$, and the covariance matrix at the output port (12) yields

$$\mathbb{C}_{\text{out}} \to \begin{bmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{bmatrix} \mathbb{C}_{\text{out}} \begin{bmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & -\mathbb{I} \end{bmatrix}. \tag{30}$$

In other words, the correlations in $\mathbb{X}$ and $\mathbb{P}$ remain the same, while elements of the XP correlations become their opposite numbers, respectively. So, the amount of EPR steering (17) remains when the cavity detunings both take opposite values.

One may also notice that the mean value $A_j + A_j^* \to A_j^* + A_j$ is invariant, while $i(A_j^* - A_j) \to i(A_j - A_j^*)$, which is its opposite, for $j = 1, 2$. Together with Eq. (30), one knows that taking opposite values of both detunings is equivalent to making a "mirror reflection" in the phase space, which in fact reveals the property of time reversal [49], because the new covariance matrix obtained under time-reversal operation follows the same transformation (30). When the quantum lights and the cavity resonate, $\Delta_1 = \Delta_2 = 0$, the right- and left-hand sides of Eq. (30) should be equal, suggesting that all XP correlations disappear. Under this condition, the covariance matrix is invariant under time-reversal operation, as $\langle \hat{X}_j \hat{X}_k \rangle = \langle (+\hat{X}_j)(+\hat{X}_k) \rangle$, $\langle \hat{P}_j \hat{P}_k \rangle = \langle (-\hat{P}_j)(-\hat{P}_k) \rangle = \langle \hat{P}_j \hat{P}_k \rangle$, as well as $\langle \hat{X}_j \hat{P}_k \rangle = 0 \Leftrightarrow \langle \hat{X}_j \hat{P}_k \rangle = \langle (+\hat{X}_j)(-\hat{P}_k) \rangle = -\langle \hat{X}_j \hat{P}_k \rangle$, for $j, k = 1, 2$. Thus, whether a covariance matrix is invariant under the time-reversal operation should be tightly linked to the existence of detunings. In a time-reversal process starting at an infinite future, the mean fields $A_j \to A_j^* (j = 1, 2)$, which is not invariant in the presence of cavity detunings, nor are the XP correlations: $\langle \hat{X}_j \hat{P}_k \rangle \neq \langle \hat{X}_j (-\hat{P}_k) \rangle$ and $\langle \hat{X}_j \hat{P}_k \rangle \neq [\langle \hat{X}_j \hat{P}_k \rangle + \langle \hat{X}_j (-\hat{P}_k) \rangle]/2 = 0$, for $j, k = 1, 2$. In other words, the XP correlations of a subsystem are not vanishing due to the presence of cavity detunings.

To some extent, introducing a cavity detuning is equivalent to introducing a phase shifter in the cavity (changing the effective cavity length), which effectively introduces an extra symplectic matrix in the Williamson decomposition of a covariance matrix, and leads to the appearance of the XP correlations. For a bipartite pure state, the purities of two subsystems are equal, thus $\det \mathbb{C}_1 = \det \mathbb{C}_2$. Given that

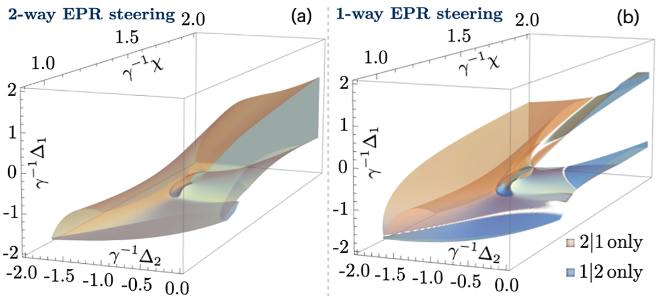

FIG. 4. Regions for different types of EPR steering with respect to the three parameters Δ_1, Δ_2, and χ (normalized by γ). (a) Region for two-way EPR steering. (b) Two regions colored differently for the two exclusive types of EPR steerings.

$\det \mathbb{C}_j = V_{\hat{X}_j,\hat{X}_j} V_{\hat{P}_j,\hat{P}_j} - V^2_{\hat{X}_j,\hat{P}_j}$ for $j = 1, 2$, one knows that adjusting the values of $V_{\hat{X}_j,\hat{P}_j}$ will cause corresponding changes of $V_{\hat{X}_j,\hat{X}_j} V_{\hat{P}_j,\hat{P}_j}$. Since the XP correlations can be easily manipulated by adjusting the cavity detunings, the condition (19) can be easily realized, and it should be promising for the direct generation of one-way EPR steering.

Indeed, one observes from the left panel of Fig. 4 that in the colored domain the EPR steering can be performed in both two ways, while from the right panel one also observes the existence of one-way EPR steering, though the region allowing for one-way EPR steering $\mathcal{E}_{2|1}$ only is much larger than the one for one-way EPR steering $\mathcal{E}_{1|2}$ only. The result is very interesting: Even when the whole state is pure, the SPLOPO enables one to perform one-way EPR steering in the presence of cavity detunings. For a pure two-mode state, according to the Schmidt decomposition, the two quantities of the purity of the reduced system should be the same. If there are no XP correlations in each mode, for a given reduced covariance matrix, the product of the field quadrature variances is exactly equal to the determinant; thus, one-way EPR steering in a pure state should not exist. One then concludes that the physical origin of one-way EPR steering of pure states from the SPLOPO must be the asymmetric XP correlations in each mode, $V^2_{\hat{X}_1,\hat{P}_1} \neq V^2_{\hat{X}_2,\hat{P}_2}$.

Figure 5 shows the amounts of EPR steering at five sets of cavity detunings. When detunings are both zeros, the amounts of EPR steering are the same, and in a small range $0.95\gamma < \chi < 1.00\gamma$ the two-way EPR steerings for bright and dark beams are both allowed, and such a situation is known as bistable. Under this situation, no matter which solution the mean fields belong to, the amounts of steering in both directions are the same. However, because of cavity detunings, even when the values are small, the amounts should become different in a bright-beam case, which may give birth to one-way steering [see Fig. 5(b)]. When $\gamma^{-1}(\chi, \Delta_1, \Delta_2) \simeq (1.16, -1.03, -0.21)$, $\mathcal{E}_{1|2} \simeq 1.0, \mathcal{E}_{2|1} \simeq 0.32$, and when $\gamma^{-1}(\chi, \Delta_1, \Delta_2) \simeq (1.16, -2.00, -0.76)$, $\mathcal{E}_{1|2} \simeq 0.71, \mathcal{E}_{2|1} \simeq 1.0$ [see Figs. 5(c) and 5(d)]. So, the special quantum correlations, one-way EPR steering, are noticeable.

The last diagram of Fig. 5 is of great interest since there exists a bistable region, in which the one-way steering $\mathcal{E}_{2|1}$ can be realized by manipulating the frequency mode 1ω for the

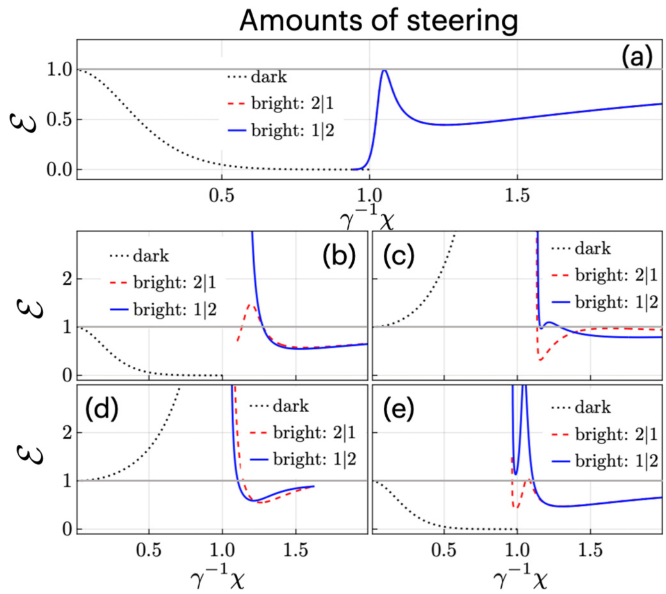

FIG. 5. Amounts of EPR steering with respect to the coupling strength at different detunings. (a) $\Delta_2 = \Delta_1 = 0$. (b) $\Delta_2 = -0.2\gamma, \Delta_1 = 0.2\gamma$. (c) $\Delta_2 = -0.21\gamma, \Delta_1 = -1.03\gamma$. (d) $\Delta_2 = -0.76\gamma, \Delta_1 = -2.0\gamma$. (e) $\Delta_2 = -0.07\gamma, \Delta_1 = 0.07\gamma$. In panels (b)–(d), noticeable one-way EPR steerings can be observed for the cases associated with the bright-beam solutions.

bright-beam case, while for the dark-beam case the two ways are symmetric, having the same amounts of EPR steering. In other words, the SPLOPO enables one to precisely control the ability to steer the subsystems, one way or two way, by increasing or decreasing the pump power to access the bistable region.

V. DECOHERENCE EFFECTS: PHENOMENOLOGICAL DESCRIPTION

In the previous section, we arrive at the important conclusions for pure Gaussian states that one-way EPR steering can be directly generated by introducing detunings, and further that the manners of the one-way or two-way EPR steering can be precisely controlled by the pump power in a bistable region. The assumption of a pure Gaussian state is too limited. It is natural to consider a more practical scenario: the case of a mixed Gaussian state. On the other hand, by introducing additional vacuum noises, or equivalently photon losses in the two modes, the symmetric EPR steering condition $V_{\hat{X}_1,\hat{X}_1} V_{\hat{P}_1,\hat{P}_1} = V_{\hat{X}_2,\hat{X}_2} V_{\hat{P}_2,\hat{P}_2}$ can also be violated, even when the detunings are zeros. This fact suggests that photon loss can also play a role in producing one-way EPR steering. So, we consider the generation of EPR steering in the presence of losses in this section.

Losses in an experiment, such as intracavity losses and detection loss, should contribute to the covariance matrix. For simplicity, we treat all the lossy channels together as a beam splitter after the SPLOPO, and assume that the reservoirs are vacua; thus, the detected covariance matrix for ideal measurement reads [42]

$$\mathbb{C}_{\rm de} = \sqrt{\mathbb{I}_4 - \mathbb{T}}\,\mathbb{C}_{\rm out}\sqrt{\mathbb{I}_4 - \mathbb{T}} + \mathbb{T}, \tag{31}$$

where $\mathbb{T} = \mathrm{diag}(\eta_2, \eta_1, \eta_2, \eta_1)$, with η_2 and η_1 being photon-loss coefficients, $0 \leqslant \eta_1, \eta_2 < 1$. One then gets the following relations:

$$V_{\hat{X}_j,\hat{X}_j} \to (1-\eta_j)V_{\hat{X}_j,\hat{X}_j} + \eta_j,\ \ j = 1, 2; \tag{32a}$$

$$V_{\hat{P}_j,\hat{P}_j} \to (1-\eta_j)V_{\hat{P}_j,\hat{P}_j} + \eta_j,\ \ j = 1, 2; \tag{32b}$$

$$V_{\hat{X}_1,\hat{X}_2} \to \sqrt{(1-\eta_1)(1-\eta_2)}V_{\hat{X}_1,\hat{X}_2}, \tag{32c}$$

$$V_{\hat{P}_1,\hat{P}_2} \to \sqrt{(1-\eta_1)(1-\eta_2)}V_{\hat{P}_1,\hat{P}_2}. \tag{32d}$$

One may verify that the amount of EPR steering $\mathcal{E}_{3-j|j}$ is a nondecreasing function of η_j. In general, when $V_{\hat{X}_1,\hat{X}_2} = V_{\hat{P}_1,\hat{P}_2} = 0$, the SPLOPO should have no ability to steer the quantum state. So, when $V_{\hat{X}_1,\hat{X}_2}$ and $V_{\hat{P}_1,\hat{P}_2}$ are not both equal to zero, the best EPR steering should be performed at $\eta_j = 0 (j = 1, 2)$. It is comprehensive, because the purer the mode $j\omega$ is, the more information from the cross correlations can be retrieved.

A. Dark-beam case

For the stable dark-beam case, the amounts of EPR steering explicitly read

$$\mathcal{E}_{2|1} = \left[(1-\eta_2)X_{11} + \eta_2 - \frac{(1-\eta_1)(1-\eta_2)X_{12}^2}{(1-\eta_1)X_{11}+\eta_1}\right]^2 \tag{33a}$$

$$\mathcal{E}_{1|2} = \left[(1-\eta_1)X_{11} + \eta_1 - \frac{(1-\eta_1)(1-\eta_2)X_{12}^2}{(1-\eta_2)X_{11}+\eta_2}\right]^2 \tag{33b}$$

The amounts of EPR steering are expected to be different unless the loss rates are the same. The expressions in the brackets are positive; Reid's criterion $\mathcal{E}_{3-j|j} < 1$ is equivalent to $\sqrt{\mathcal{E}_{3-j|j}} < 1$ for $j = 1, 2$. One can verify directly that $\sqrt{\mathcal{E}_{3-j|j}} - 1 = -(1-\eta_{3-j})\partial_{\eta_{3-j}}\sqrt{\mathcal{E}_{3-j|j}}$. As a result, the criterion is equivalent to

$$\partial_{\eta_{3-j}}\sqrt{\mathcal{E}_{3-j|j}} > 0, \tag{34}$$

whose left-hand side is monotone decreasing with η_j.

With respect to the loss coefficient η_{3-j} ($j = 1, 2$), the amount of EPR steering $\mathcal{E}_{3-j|j}$ takes its extreme value under the condition $\partial_{\eta_{3-j}}\sqrt{\mathcal{E}_{3-j|j}} = 0$ at

$$\eta_j = \tilde{\eta} \equiv \frac{1}{2}\left[1 - \frac{\gamma^2(\Delta_1+\Delta_2)^2}{(\chi^2-\gamma^2-\Delta_1\Delta_2)^2 - 4\gamma^2\Delta_1\Delta_2}\right]. \tag{35}$$

Because $0 \leqslant \eta_j < 1$, the extreme value condition $\eta_j = \tilde{\eta}$ is not always attainable. In any case, $\tilde{\eta}$ must be greater than zero for the existence of EPR steering, i.e.,

$$(\chi^2 - \Delta_1\Delta_2 - \gamma^2)^2 - 4\gamma^2\Delta_1\Delta_2 > \gamma^2(\Delta_1+\Delta_2)^2. \tag{36}$$

Then, Reid's criterion $\mathcal{E}_{3-j|j} < 1$ imposes $\eta_j < \tilde{\eta}$.

When the two conditions are satisfied, the SPLOPO has the ability to steer the mode $(3-j)\omega$ by manipulating the mode $j\omega$. Thus, $\tilde{\eta}$ in fact defines an upper bound of photon-loss coefficients for EPR steering, and when the SPLOPO enables one to steer the mode $(3-j)\omega$, by increasing the photon-loss coefficient η_{3-j}, the steerability should be weakened. Providing that the condition (36) is already satisfied, one-way EPR steering can be observed when only one of η_1 and η_2 is less than $\tilde{\eta}$; the two ways of EPR steering are allowable when both η_1 and η_2 are less than $\tilde{\eta}$. Otherwise, the SPLOPO has no ability to steer the quantum state.

The value of the upper bound $\tilde{\eta}$ can be small, even close to zero when the inequality (36) almost saturates. Then, the loss of a small amount of photons will make the SPLOPO lose its ability to steer quantum states. In other words, the correlations are poor, and the steerability is not robust. When $\tilde{\eta}$ takes a larger value, the system needs to lose more photons within a given period to disable the steerability. In this sense, the conditions for large $\tilde{\eta}$ should be of interest.

According to its definition in Eq. (35), the upper bound $\tilde{\eta}$ reaches its maximum value $1/2$, when $\Delta_1 + \Delta_2 = 0$. The bound $1/2$ of photon loss for steering in two-mode squeezed vacuum states was pointed out in [18] when the cavity detunings are vanishing, $\Delta_1 = \Delta_2 = 0$. It is a special case of our result. What we need to emphasize is that $1/2$ is the maximum value of the upper bound $\tilde{\eta}$, which means that in general the upper bound given by $\tilde{\eta}$ should be better. The value of $\tilde{\eta}$ should be quite large when the detunings are small, in which cases EPR steering should also be robust to be demonstrated. In order to generate robust one-way EPR steering with the SPLOPO in the dark-beam case, a good experimental condition is that the two detunings are taken as opposite values, and only one of the photon-loss coefficients is no less than $1/2$.

B. Bright-beam case: Zero cavity detunings

When the cavity detunings vanish, the covariance matrix $\mathbb{C}_{\text{out}}$, subsequently the detected covariance matrix $\mathbb{C}_{\text{de}}$, should be in the block-diagonal form $\mathbb{X} \oplus \mathbb{P}$. Then, the amounts of EPR steering read

$$\mathcal{E}_{3-j|j} = \frac{\det \mathbb{C}_{\text{de}}}{\det \mathbb{C}_{\text{de},j}}, \quad j = 1, 2. \tag{37}$$

One can verify Reid's criterion $\mathcal{E}_{3-j|j} < 1$ by checking the sign of $\det \mathbb{C}_{\text{de}} - \det \mathbb{C}_{\text{de},j}$. Then, by removing the common positive factors $16\gamma^2(1-\eta_{3-j})/(\kappa^2A_1^2 + \gamma\kappa A_2 + \gamma^2 - \chi^2)^2(\kappa^2A_1^2 - \gamma\kappa A_2 + \gamma^2 - \chi^2)^2$, as well as introducing the following two positive expressions,

$$\Upsilon_j = \frac{j\chi A_1 + (3-j)\gamma A_2}{4\kappa^{-1}\chi^2}, \quad j = 1, 2, \tag{38}$$

Reid's criterion $\mathcal{E}_{3-j|j} < 1$ is equivalent to

$$\eta_{3-j}\left[\Upsilon_j^2 - \left(\eta_j - \frac{1}{2}\right)^2\right] < \frac{1-2\eta_j}{16\gamma^2\chi^4}\Omega^2, \tag{39}$$

where the symbol Ω is defined in Eq. (28). If $(1-2\eta_j)\Omega^2 \leqslant 0$, i.e., $\eta_j \geqslant 1/2$ or $\chi = \chi_0$ [the single root of $\Omega = 0$, see Eq. (29)], inequality (39) requires that $\eta_j > 1/2 + \Upsilon_j$ or $\eta_j < 1/2 - \Upsilon_j$. When $\chi = \chi_0$, $\Upsilon_1 \simeq 0.533$, $\Upsilon_2 \simeq 0.701$, i.e., $1/2 + \Upsilon_j > 1$ and $1/2 - \Upsilon_j < 0$ for any $j = 1, 2$; thus, the conditions (39) break down for both $j = 1, 2$. It is expected, as the SPLOPO already has no steerability for pure states at $\chi = \chi_0$. Likewise, the inequality (39) further implies that $\eta_j < 1/2$, because no (η_1, η_2) can simultaneously satisfy the inequalities (39) as well as $\eta_j > 1/2 + \Upsilon_j$.

Together with the steerability conditions for the dark-beam case, one concludes the following: *the mode $(3-j)\omega$ cannot be steered by manipulating the mode $j\omega$, if the photon loss of the mode $j\omega$ exceeds $1/2$,* when both detunings vanish.

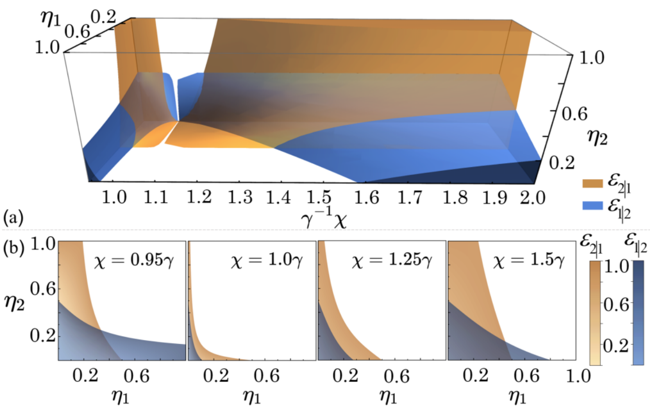

FIG. 6. (a) The amounts of EPR steering for the bright-beam case at zero detunings vs photon-loss coefficients. (b) The slices at different pump amplitudes. One observes that the regions of two-way and one-way EPR steerings change from slice to slice, and the boundaries are not sufficiently simple. However, if its photon-loss coefficient exceeds 0.5, the subsystem loses its ability to steer the other one.

The condition $0 \leqslant \eta_j \leqslant 1/2 - \Upsilon_j$ with $j = 1, 2$ imposes that

$$\chi \leqslant \chi_j \equiv \gamma\delta_{j1} + \sqrt{(9+5\sqrt{5})/22}\gamma\delta_{j2} \tag{40}$$

with δ_{jk} the Kronecker delta function; then the left-hand side of (39) is negative, and the condition of whether the mode $(3-j)\omega$ is steerable is very tolerant for the photon-loss coefficient, $0 \leqslant \eta_{3-j} < 1$. Note that $\chi_1 > \chi_2$; this is because the squeezing in the frequency mode 1ω is stronger than the one in the frequency mode 2ω in the absence of photon losses.

When the inequalities (39) for both $j = 1, 2$ are satisfied simultaneously, the SPLOPO enables one to perform EPR steering in two ways; if only one inequality is satisfied, then the EPR steering can be performed only in one way; otherwise, the SPLOPO has no ability to steer quantum states. According to Fig. 6, by choosing proper quantities of photon-loss coefficients, one can easily perform the one-way and two-way steerability with the SPLOPO when cavity detunings are zeros.

If one photon-loss coefficient exceeds 0.5, while the other one remains small, the SPLOPO can allow for a bistable region for one-way EPR steering. For example, when $(\eta_1, \eta_2) = (0.1, 0.5)$, one observes a bistable region for one-way EPR steering $\mathcal{E}_{2|1}$ in the left panel of Fig. 7. Likewise, in the right panel, there exists a bistable region for one-way EPR steering $\mathcal{E}_{1|2}$ when $(\eta_1, \eta_2) = (0.5, 0.1)$. By increasing or decreasing the pump power around and inside the interval for bistability, one also observes the hysteresis cycles in the two panels of Fig. 7. What is more important, the amounts of EPR steering in the bistable regions can be small; therefore, the SPLOPO is a good device to directly generate one-way EPR steering in both dark-beam and bright-beam cases.

The maximum value of the upper bound is once again 1/2, as the case in the previous subsection. The bound 1/2 of photon loss for steering in [18] was derived from a two-mode squeezed vacuum state, which is symmetric. Because the quantum properties are completely determined by the covariance matrix of a Gaussian state, their conclusion can be easily extended to the cases above threshold, provided that the states are symmetric. So, the conclusion in this subsection is better than the value 1/2 obtained by an analogy from [18]. On the other hand, from Fig. 3(a), one should observe that the single-mode squeezings never vanish simultaneously, so the Gaussian states above threshold are not symmetric when the detunings are both vanishing. Mathematically speaking, when the asymmetry among the diagonal elements of the covariance matrix in a Gaussian state tends to be unnoticeable, the state gradually turns into a symmetric Gaussian one. The Gaussian states above threshold here are more complicated, so their applicability should be more general.

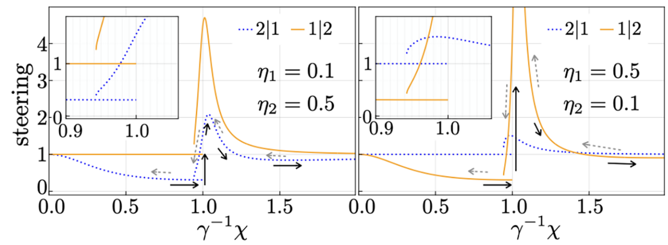

FIG. 7. The amounts of EPR steering at different photon losses. In each panel, for given line style, the left curve represents the steering for the dark-beam case, while the right one represents the steering for the bright-beam case. The arrows show that the amount of steering goes with the increase or decrease of χ, and one observes the hysteresis cycles in the bistable regions. For plots, $\Delta_1 = \Delta_2 = 0$.

C. Bright-beam case: Nonzero cavity detunings

Recall that our SPLOPO enables one to directly generate one-way EPR steering in the bright-beam case when the cavity detunings are not both vanishing, due to the existence of XP correlations. Naturally, it is expected that the steerability should be almost the same when the photon-loss coefficients are small. For example, when $\eta_1 = \eta_2 = 0.05$, if $\gamma^{-1}(\chi, \Delta_1, \Delta_2) = (1.36, 1.53, 0.14)$, one then should observe the one-way EPR steering by manipulating the 1ω mode, since $\mathcal{E}_{2|1} \simeq 0.39, \mathcal{E}_{1|2} \simeq 1.11$; if $\gamma^{-1}(\chi, \Delta_1, \Delta_2) = (1.98, 0.69, -0.22)$, one then should observe another one-way EPR steering by manipulating the 2ω mode, as $\mathcal{E}_{2|1} \simeq 1.00, \mathcal{E}_{1|2} \simeq 0.77$. So, one can directly generate the one-way EPR steering for small photon-loss coefficients.

When a photon-loss coefficient becomes large, the quantum correlations should be seriously waned. However, one can still observe the one-way EPR steering; e.g., when $\gamma^{-1}(\chi, \Delta_1, \Delta_2) = (1.16, -1.03, -0.21)$, $\eta_2 = 0.5$, $\eta_1 = 0.1$, the detected covariance matrix reads

$$\mathbb{C}_{\rm de} \simeq \begin{bmatrix} 2.2969 & -9.0301 & -1.3365 & -0.87501 \\ -9.0301 & 55.284 & 10.268 & 6.6190 \\ -1.3365 & 10.268 & 3.5636 & 1.6863 \\ -0.87501 & 6.6190 & 1.6863 & 1.1122 \end{bmatrix}, \tag{41}$$

and subsequently, the amounts of EPR steering are $\mathcal{E}_{2|1} \simeq 0.83, \mathcal{E}_{1|2} \simeq 6.22$.

Due to the photon loss and the existence of XP correlations, it should be true in general that the SPLOPO loses its ability to steer the $(3-j)\omega$ mode if $\eta_j \geqslant 0.5$, for $j = 1, 2$. The upper bound 0.5 should be lowered in principle. So, a first attempt to directly generate one-way EPR steering could make one photon-loss coefficient greater than 0.5.

From all the discussions about the role of loss in this subsection one concludes that the subsystem should definitely lose its ability to steer the other subsystem if the photon-loss coefficient of a subsystem is larger than 0.5. However, it is still unknown whether the other subsystem has the steerability. The edges of the one-way EPR steering are quite complicated in general (see Fig. 6 for example). Furthermore, if the photon-loss coefficient of one subsystem, say the mode 1ω, is larger than 0.5, the subsystem cannot steer the mode 2ω, and very likely, the mode 2ω has the ability to steer the mode 1ω when its photon-loss coefficient is small (e.g., 0.1, see Fig. 6). In order to observe the one-way EPR steering, it is a good starting point to make the photon-loss coefficient of one subsystem greater than 0.5.

Finally, one arrives at the following conclusions from the previous discussions in Sec. V.

(1) One-way EPR steering due to cavity detunings can be still observed in the presence of photon loss such as intracavity and detection loss.

(2) For both dark-beam and bright-beam cases, the two types of EPR steering, one way and two way, can be directly produced in the SPLOPO in the presence of photon loss from detection inefficiency or intracavity loss, whether the cavity is detuned or not.

The upper bound to lose steerability for photon-loss coefficients is less than $1/2$ in general. Even when the maximum value $1/2$ is reached, the applicability is more general than the one in [18], since the Gaussian states involved here are not symmetric in general.

VI. CONCLUSION

In summary, we have theoretically studied the direct generations of one-way EPR steering using the SPLOPO, taking into consideration that all Gaussian operations required to produce EPR steering can be manipulated in the SPLOPO. We first analyze the relationship between the steering and the input pump amplitude (or the coupling strength χ) in the absence of cavity detunings and photon losses. For the dark-beam case, the two ways of EPR steering are symmetric. In the bright-beam case, we show that two-mode squeezing and single-mode squeezing are convertible into each other, hence they are both quantum resources for generating steering. We then find that the amounts of EPR steering for both ways are the same, and the value can be dramatically changed by rotations of the Wigner distributions from the beam-splitter-like Hamiltonian. In the presence of cavity detunings, we arrive at our main conclusion that one-way EPR steering in a pure Gaussian state can be directly generated, because of nonzero XP correlations.

We then investigate Reid's criterion for EPR steering in mixed Gaussian states (in the presence of photon loss such as intracavity loss and detection loss), in both dark-beam and bright-beam cases with and without cavity detunings. We show one-way EPR steering can be directly generated due to cavity detunings even with photon loss, and obtain the upper bound for the photon-loss coefficient when one subsystem loses its ability to steer the other. The maximum value of the upper bound is $1/2$. Generally speaking, one-way EPR steering can be directly generated if one photon-loss coefficient exceeds its upper bound, or even $1/2$, while the other one remains small.

Regarding the upper bound of photon-loss coefficients with a maximum value of $1/2$, it can be achieved when the two detunings are opposite in the dark-beam case, as well as when the quantum state is asymmetric in the bright-beam case. Therefore, the applicability conditions of this maximum value of $1/2$ have been extended compared to those in [18]. We also show that it is possible to find a bistable region where one-way EPR steering can be performed both for the dark-beam and the bright-beam cases. Therefore, by controlling the cavity detunings or/and the photon loss η, or changing the input pump power, the SPLOPO enables one to directly generate one-way and two-way EPR steering at will.

Because the phases of the quantum lights are frozen by the SPLOPO, direct generation of EPR steering in this system should be quite robust against phase diffusion. Thus, the SPLOPO should be a good candidate for directly generating EPR steering, especially the one-way EPR steering.

ACKNOWLEDGMENTS

We acknowledge Jean-Jacques Zondy for his kind guidance on the subject of SPLOPO. This work is supported by National Natural Science Foundation of China (Grants No. 61727821, No. 11904279, and No. 12174302) and Key Scientific and Technological Innovation Team of Shaanxi Province (Grant No. 2021TD-56).

APPENDIX A: GAUSSIAN STATE AND COVARIANCE MATRIX

A state ρ is called Gaussian, when its Wigner function or distribution is Gaussian. The corresponding Wigner distribution [43–45] of the Gaussian state ρ reads

$$\mathcal{W}(\boldsymbol{X}, \boldsymbol{P}) = \frac{\exp[-(\boldsymbol{X}^{\intercal}, \boldsymbol{P}^{\intercal})\mathbb{C}^{-1}\mathrm{col}(\boldsymbol{X}, \boldsymbol{P})/2]}{\pi^2\sqrt{\det\mathbb{C}}/4}, \quad \text{(A1a)}$$

$$\boldsymbol{X} = \mathrm{col}(X_2 - \langle\hat{X}_2\rangle, X_1 - \langle\hat{X}_1\rangle), \quad \text{(A1b)}$$

$$\boldsymbol{P} = \mathrm{col}(P_2 - \langle\hat{P}_2\rangle, P_1 - \langle\hat{P}_1\rangle), \quad \text{(A1c)}$$

in terms of the X and P quadratures of the phase space. Here, $\mathbb{C}$ is the covariance matrix, and $\langle\hat{X}_j\rangle$ and $\langle\hat{P}_j\rangle$ (with $j = 1, 2$) are means of the field quadrature operators defined in Eq. (11). Following the standard linearization procedure, the steady solutions of the mean amplitudes can be obtained independently. The quantum properties of a Gaussian state ρ are completely determined by its covariance matrix $\mathbb{C}$ [43–45], e.g., the purity of a Gaussian state is $\mathrm{tr}\rho^2 = 1/\sqrt{\det\mathbb{C}}$. So in this respect, the Wigner distribution can always be displaced to the origin for visualization, therefore the name of the displaced Wigner distribution.

If the covariance matrix is block diagonal, $\mathbb{C} = \mathbb{X} \oplus \mathbb{P}$, then the displaced Wigner distribution can be decomposed as $\mathcal{W}(\boldsymbol{X}, \boldsymbol{P}) = G_{\mathbb{X}}(\boldsymbol{X}) G_{\mathbb{P}}(\boldsymbol{P})/\sqrt{\det \mathbb{C}}$, where

$$G_{\mathbb{X}}(\boldsymbol{X}) = \frac{2}{\pi} \exp\left(-\frac{1}{2}\boldsymbol{X}^{\mathsf{T}}\mathbb{X}^{-1}\boldsymbol{X}\right), \tag{A2a}$$

$$G_{\mathbb{P}}(\boldsymbol{P}) = \frac{2}{\pi} \exp\left(-\frac{1}{2}\boldsymbol{P}^{\mathsf{T}}\mathbb{P}^{-1}\boldsymbol{P}\right). \tag{A2b}$$

The correlations of X quadratures can be observed from the Gaussian function $G_{\mathbb{X}}(\boldsymbol{X})$, and the correlations of P quadratures can be observed from $G_{\mathbb{P}}(\boldsymbol{P})$.

If the elements in the vector of field quadratures are arranged in the order (X_2, P_2, X_1, P_1), then the new form of the covariance matrix $\mathbb{V} = \mathbb{Z}\mathbb{C}\mathbb{Z}$ with $\mathbb{Z} = 1 \oplus \sigma_1 \oplus 1$, and can be partitioned as

$$\mathbb{V} = \begin{bmatrix} \mathbb{C}_2 & \mathbb{V}_{12} \\ \mathbb{V}_{12}^{\mathsf{T}} & \mathbb{C}_1 \end{bmatrix}. \tag{A3}$$

Here, $\mathbb{C}_2$ and $\mathbb{C}_1$ are covariance matrices of the reduced systems, the signal and idler, respectively, and $\mathbb{V}_{12}$ reveals correlations of the two modes. By performing the Williamson decomposition to the 2×2 matrices $\mathbb{C}_j$ [43],

$$\mathbb{C}_j = \nu_j S_j S_j^{\mathsf{T}}, \quad j = 1, 2, \tag{A4}$$

one gets two local symplectic transformation matrices S_2 and S_1 and two symplectic eigenvalues ν_2 and ν_1. The matrix $\mathbb{V}$ under the local symplectic transformation $(S_2 \oplus S_1)^{\mathsf{T}}$ turns out to be

$$\mathbb{V} \to \begin{bmatrix} \nu_2 \mathbb{I}_2 & S_2^{\mathsf{T}}\mathbb{V}_{12}S_1 \\ S_1^{\mathsf{T}}\mathbb{V}_{12}^{\mathsf{T}}S_2 & \nu_1\mathbb{I}_2 \end{bmatrix}. \tag{A5}$$

The singular value decomposition yields

$$S_2^{\mathsf{T}}\mathbb{V}_{12}S_1 = U_2 \Lambda U_1^{\mathsf{T}}, \tag{A6}$$

where U_2 and U_1 are unitary matrices, and $\Lambda = \mathrm{diag}(\lambda_2, \lambda_1)$ is a diagonal one. At this point, one can introduce a local symplectic transformation matrix $S_{\mathrm{lo}} = (U_2 S_2) \oplus (U_1 S_1)$, such that

$$S_{\mathrm{lo}}^{\mathsf{T}}\mathbb{V}S_{\mathrm{lo}} = \begin{bmatrix} \nu_2\mathbb{I}_2 & \Lambda \\ \Lambda & \nu_1\mathbb{I}_2 \end{bmatrix}. \tag{A7}$$

The form of the covariance matrix on the right-hand side of Eq. (A7) is called the normal form [43] or the standard form I [50]. When $\nu_2 = \nu_1$, it is called symmetric. For pure states, the purities of the two reduced systems should be the same, then $\nu_1 = \nu_2 \equiv \nu$. Thus, $|\lambda_2| = |\lambda_1| = \sqrt{\nu^2 - 1}$ [50], as det$\mathbb{C}$=1. Furthermore, the fact that $[\begin{smallmatrix} \nu & \lambda_1 \\ \lambda_1 & \nu \end{smallmatrix}] = [\begin{smallmatrix} \nu & \lambda_2 \\ \lambda_2 & \nu \end{smallmatrix}]^{-1}$ for pure Gaussian states [51] implies that $\lambda_2 = -\lambda_1$. A covariance matrix of such a form represents a pure two-mode squeezed state. At this point, one can define the amount of the two-mode squeezing as $10\log_{10}(\nu + \sqrt{\nu^2 - 1})$.

In [50], the standard form II is defined as

$$\begin{bmatrix} n_1 & & c_1 & \\ & n_2 & & c_2 \\ c_1 & & m_1 & \\ & c_2 & & m_2 \end{bmatrix}, \tag{A8}$$

whose elements can be different in general. If one introduces two local squeeze operators with r_2 and r_1 being the squeezing parameters, then the covariance matrix in the normal form turns out to be

$$\begin{bmatrix} \mathrm{diag}(r_2\nu_2, \nu_2/r_2) & \mathrm{diag}(\sqrt{r_1 r_2}\lambda_2, \lambda_1/\sqrt{r_1 r_2}) \\ \mathrm{diag}(\sqrt{r_1 r_2}\lambda_2, \lambda_1/\sqrt{r_1 r_2}) & \mathrm{diag}(r_1\nu_1, \nu_1/r_1) \end{bmatrix},$$

whose form, in fact, is the standard form II. If a covariance matrix is given in the standard form II, then one should have

$$r_2 = \sqrt{n_1/n_2}, \quad r_1 = \sqrt{m_1/m_2}, \tag{A9}$$

which can be used to characterize single-mode squeezing in two-mode pure states.

The standard form I is a special case of the standard form II. If a covariance matrix is in the standard form II (there exists no XP correlation), then the covariance matrix should be in the block-diagonal form when the field quadratures are arranged in the order (X_2, X_1, P_2, P_1), and vice versa.

APPENDIX B: CONDITIONS FOR NO XP CORRELATIONS

In this Appendix, we derive some general conditions on the absence of the XP correlations by doing some linear algebra.

Let us consider a generic block matrix $\mathbb{K} = [\begin{smallmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{smallmatrix}]$, then one can verify that

$$\mathbb{M}\mathbb{K}\mathbb{M}^{\dagger} = \begin{bmatrix} \mathbb{A} + \mathbb{B} + \mathbb{C} + \mathbb{D} & i(\mathbb{A} - \mathbb{B} + \mathbb{C} - \mathbb{D}) \\ i(\mathbb{C} + \mathbb{D} - \mathbb{A} - \mathbb{B}) & \mathbb{A} - \mathbb{B} - \mathbb{C} + \mathbb{D} \end{bmatrix}, \tag{B1}$$

where the matrix $\mathbb{M}$ is defined in Eq. (11). If the matrix $\mathbb{K}$ can be block diagonalized by the matrix $\mathbb{M}/\sqrt{2}$, then one should have $\mathbb{A} - \mathbb{B} + \mathbb{C} - \mathbb{D} = \mathbb{C} + \mathbb{D} - \mathbb{A} - \mathbb{B} = \mathbb{O}$, which are equivalent to $\mathbb{B} = \mathbb{C}$, and $\mathbb{A} = \mathbb{D}$. Then, the block-diagonalizable matrix $\mathbb{K}$ has the symmetry $\mathbb{K} = (\sigma_1 \otimes \mathbb{I}_2)\mathbb{K}(\sigma_1 \otimes \mathbb{I}_2)$. The unitary matrix $(\sigma_1 \otimes \mathbb{I}_2)$ is transpose invariant $(\sigma_1 \otimes \mathbb{I}_2)^{\mathsf{T}} = \sigma_1 \otimes \mathbb{I}_2$, and satisfies $(\sigma_1 \otimes \mathbb{I}_2)^2 = \mathbb{I}_4$.

For simplicity, let us introduce the following matrix:

$$\mathbb{F} = 2\left(\gamma\mathbb{L}^{-1} + \frac{\mathbb{I}_4}{2}\right)(\sigma_1 \otimes \mathbb{I}_2)\left(\gamma\mathbb{L}^{-1} + \frac{\mathbb{I}_4}{2}\right)^{\mathsf{T}}. \tag{B2}$$

Because of the relation $(\sigma_1 \otimes \mathbb{I}_2)\mathbb{M}^{\mathsf{T}} = \mathbb{M}^{\dagger}$, the covariance matrix in Eq. (12) can be reexpressed as

$$\mathbb{C}_{\mathrm{out}} = \mathbb{M}\mathbb{F}\mathbb{M}^{\mathsf{T}} = \mathbb{M}\mathbb{F}(\sigma_1 \otimes \mathbb{I}_2)^2\mathbb{M}^{\mathsf{T}} = \mathbb{M}\mathbb{F}(\sigma_1 \otimes \mathbb{I}_2)\mathbb{M}^{\dagger}. \tag{B3}$$

1. Covariance matrix for $\Delta_1 = \Delta_2 = 0$

Let us consider a special condition when the cavity detunings are zero. In this case, the stable steady solutions A_2 and A_1 are real, and the matrix $\mathbb{L}$ turns out to be

$$\mathbb{L} = \begin{bmatrix} L_1 & L_2 \\ L_2 & L_1 \end{bmatrix}, \quad L_1 = \begin{bmatrix} -\gamma & -\kappa A_1 \\ \kappa A_1 & -\gamma \end{bmatrix}, \quad L_2 = \begin{bmatrix} 0 & \chi \\ \chi & \kappa A_2 \end{bmatrix}. \tag{B4}$$

Now, the matrix $\mathbb{L}$ is filled with real elements.

Because $\sigma_1 \otimes \mathbb{I}_2$ is the permutation matrix for the blocks, one should get $\mathbb{L} = (\sigma_1 \otimes \mathbb{I}_2)\mathbb{L}(\sigma_1 \otimes \mathbb{I}_2)$. By taking the inverse operation, one gets $\mathbb{L}^{-1} = [(\sigma_1 \otimes \mathbb{I}_2)\mathbb{L}(\sigma_1 \otimes \mathbb{I}_2)]^{-1} = (\sigma_1 \otimes \mathbb{I}_2)\mathbb{L}^{-1}(\sigma_1 \otimes \mathbb{I}_2)$, as $(\sigma_1 \otimes \mathbb{I}_2)^{-1} = (\sigma_1 \otimes \mathbb{I}_2)$.

Similarly, $(\mathbb{L}^{-1})^\intercal = [(\sigma_1 \otimes \mathbb{I}_2)\mathbb{L}^{-1}(\sigma_1 \otimes \mathbb{I}_2)]^\intercal = (\sigma_1 \otimes \mathbb{I}_2)(\mathbb{L}^{-1})^\intercal(\sigma_1 \otimes \mathbb{I}_2)$. As a result, $(\sigma_1 \otimes \mathbb{I}_2)(\gamma\mathbb{L}^{-1} + \mathbb{I}_4/2)^\intercal = (\gamma\mathbb{L}^{-1} + \mathbb{I}_4/2)^\intercal(\sigma_1 \otimes \mathbb{I}_2)$, and $\mathbb{F}(\sigma_1 \otimes \mathbb{I}_2) = 2(\gamma\mathbb{L}^{-1} + \mathbb{I}_4/2)(\sigma_1 \otimes \mathbb{I}_2)(\gamma\mathbb{L}^{-1} + \mathbb{I}_4/2)^\intercal(\sigma_1 \otimes \mathbb{I}_2) = (\sigma_1 \otimes \mathbb{I}_2)2(\gamma\mathbb{L}^{-1} + \mathbb{I}_4/2)(\sigma_1 \otimes \mathbb{I}_2)(\gamma\mathbb{L}^{-1} + \mathbb{I}_4/2)^\intercal = (\sigma_1 \otimes \mathbb{I}_2)\mathbb{F} = (\sigma_1 \otimes \mathbb{I}_2)\mathbb{F}(\sigma_1 \otimes \mathbb{I}_2)^2$, which suggests the covariance matrix $\mathbb{C}_{\text{out}}$ is block diagonalized, and has no XP correlation.

2. Some elements of the covariance matrix

In this case, the covariance matrix is simple enough to calculate manually. After some algebras, one has

$$V_{\hat{X}_2,\hat{X}_2} = 1 + 4\gamma\frac{(\kappa A_1 - \chi)[\kappa^2 A_1 A_2 + \chi(\kappa A_2 - 2\gamma)]}{\left(\kappa^2 A_1^2 - \gamma\kappa A_2 + \gamma^2 - \chi^2\right)^2}, \tag{B5a}$$

$$V_{\hat{X}_1,\hat{X}_1} = 1 + 4\gamma^2\frac{2\chi(\kappa A_1 + \chi) + \gamma\kappa A_2}{\left(\kappa^2 A_1^2 - \gamma\kappa A_2 + \gamma^2 - \chi^2\right)^2}, \tag{B5b}$$

$$V_{\hat{P}_2,\hat{P}_2} = 1 + 4\gamma\frac{(\kappa A_1 + \chi)[\chi(\kappa A_2 + 2\gamma) - \kappa^2 A_1 A_2]}{\left(\kappa^2 A_1^2 + \gamma\kappa A_2 + \gamma^2 - \chi^2\right)^2}, \tag{B5c}$$

$$V_{\hat{P}_1,\hat{P}_1} = 1 - 4\gamma^2\frac{2\chi(\kappa A_1 - \chi) + \gamma\kappa A_2}{\left(\kappa^2 A_1^2 + \gamma\kappa A_2 + \gamma^2 - \chi^2\right)^2}, \tag{B5d}$$

$$V_{\hat{X}_1,\hat{X}_2} = -\frac{4\gamma\Omega}{\left(\kappa^2 A_1^2 - \gamma\kappa A_2 + \gamma^2 - \chi^2\right)^2}, \tag{B5e}$$

$$V_{\hat{P}_1,\hat{P}_2} = \frac{4\gamma\Omega}{\left(\kappa^2 A_1^2 + \gamma\kappa A_2 + \gamma^2 - \chi^2\right)^2}, \tag{B5f}$$

$$\Omega \equiv \kappa^2\gamma A_1 A_2 + \chi\left(\kappa^2 A_1^2 - \gamma^2 - \chi^2\right). \tag{B5g}$$

For pure states, $\mathbb{C}_{\text{out}} = \mathbb{X} \oplus \mathbb{X}^{-1}$ when $\Delta_1 = \Delta_2 = 0$ [51]. If $\Omega = 0$, then $V_{\hat{X}_1,\hat{X}_2} = V_{\hat{P}_1,\hat{P}_2} = 0$, and vice versa.

3. The inverse problem

Now, let us assume that there is no XP correlation in the covariance matrix $\mathbb{C}_{\text{out}}$, then $\mathbb{F}(\sigma_1 \otimes \mathbb{I}_2)$ should be invariant under the unitary transformation $\sigma_1 \otimes \mathbb{I}_2$,

$$\mathbb{F}(\sigma_1 \otimes \mathbb{I}_2) = (\sigma_1 \otimes \mathbb{I}_2)[\mathbb{F}(\sigma_1 \otimes \mathbb{I}_2)](\sigma_1 \otimes \mathbb{I}_2) = (\sigma_1 \otimes \mathbb{I}_2)\mathbb{F}, \tag{B6}$$

which imposes that $\mathbb{F}$ should also be invariant:

$$\mathbb{F} = \mathbb{F}(\sigma_1 \otimes \mathbb{I}_2)(\sigma_1 \otimes \mathbb{I}_2) = (\sigma_1 \otimes \mathbb{I}_2)\mathbb{F}(\sigma_1 \otimes \mathbb{I}_2). \tag{B7}$$

Because of the relations $\mathbb{M}^\intercal\mathbb{M} = 2\sigma_1 \otimes \mathbb{I}_2$, the matrix $\mathbb{F}$ can be decomposed as

$$\mathbb{F} = \left[\left(\gamma\mathbb{L}^{-1} + \frac{\mathbb{I}_4}{2}\right)\mathbb{M}^\intercal\right]\left[\left(\gamma\mathbb{L}^{-1} + \frac{\mathbb{I}_4}{2}\right)\mathbb{M}^\intercal\right]^\intercal. \tag{B8}$$

Taking into account the symmetry of $\mathbb{L}$ in Eq. (B7), and introducing a rotation matrix $\mathbb{R}(\mathbb{R}\mathbb{R}^\intercal = \mathbb{I}_4)$, one has

$$\left(\gamma\mathbb{L}^{-1} + \frac{\mathbb{I}_4}{2}\right)\mathbb{M}^\intercal\mathbb{R} = (\sigma_1 \otimes \mathbb{I}_2)\left(\gamma\mathbb{L}^{-1} + \frac{\mathbb{I}_4}{2}\right)\mathbb{M}^\intercal. \tag{B9}$$

Plugging the left product with $\sigma_1 \otimes \mathbb{I}$ and the right product with $\mathbb{R}$ into Eq. (B9), one has

$$(\sigma_1 \otimes \mathbb{I}_2)\left(\gamma\mathbb{L}^{-1} + \frac{\mathbb{I}_4}{2}\right)\mathbb{M}^\intercal\mathbb{R}^2 = \left(\gamma\mathbb{L}^{-1} + \frac{\mathbb{I}_4}{2}\right)\mathbb{M}^\intercal\mathbb{R}. \tag{B10}$$

Compared with Eq. (B9), one gets $\mathbb{R}^2 = \mathbb{I}_4$. As a result,

$$\mathbb{R} = \mathbb{R}(\mathbb{R}\mathbb{R}^\intercal) = \mathbb{R}^2\mathbb{R}^\intercal = \mathbb{R}^\intercal. \tag{B11}$$

The matrix $\mathbb{L}$ in Eq. (5) satisfies $\mathbb{L}^* = (\sigma_1 \otimes \mathbb{I}_2)\mathbb{L}(\sigma_1 \otimes \mathbb{I}_2)$. From Eq. (B9) and the relation $\mathbb{R} = \mathbb{R}^\intercal$, one gets $2\gamma\mathbb{L}^{-1*}\mathbb{L}^{-1\intercal} + \mathbb{L}^{-1\intercal} + \mathbb{L}^{-1*} = 2\gamma\mathbb{L}^{-1}\mathbb{L}^{-1\dagger} + \mathbb{L}^{-1\dagger} + \mathbb{L}^{-1}$. By comparing the coefficients, one has

$$\mathbb{L}^{-1*}\mathbb{L}^{-1\intercal} = \mathbb{L}^{-1}\mathbb{L}^{-1\dagger} \Leftrightarrow \mathbb{L}^\intercal\mathbb{L}^* = \mathbb{L}^\dagger\mathbb{L}, \tag{B12a}$$

$$\mathbb{L}^{-1\intercal} + \mathbb{L}^{-1*} = \mathbb{L}^{-1\dagger} + \mathbb{L}^{-1}. \tag{B12b}$$

Let $\mathbb{L}$ be partitioned as $[\begin{smallmatrix} L_1 & L_2 \\ L_2^* & L_1^* \end{smallmatrix}]$, then Eq. (B12a) implies

$$L_1^\dagger L_1 + L_2^\intercal L_2^* = L_2^\dagger L_2 + L_1^\intercal L_1^*, \tag{B13a}$$

$$L_1^\dagger L_2 + L_2^\intercal L_1^* = L_2^\dagger L_1 + L_1^\intercal L_2^*. \tag{B13b}$$

By substitution of explicit expressions of the blocks L_1 and L_2, one should have

$$A_1 = A_1^*, \tag{B14a}$$

$$d_1 A_2^* = d_1^* A_2, \tag{B14b}$$

$$2i(\Delta_2 + \Delta_1)A_1 = \chi(A_2 - A_2^*), \tag{B14c}$$

$$\kappa^2 A_1(A_2 - A_2^*) = 2i(\Delta_2 + \Delta_1)\chi. \tag{B14d}$$

For the dark-beam solution $A_1 = A_2 = 0$, Eq. (B14d) implies $\Delta_2 = -\Delta_1$. Then, *the corresponding covariance matrix can be block diagonalized only when* $\Delta_2 = -\Delta_1$.

For the bright-beam solution, Eqs. (B14c) and (B14d) suggest

$$(\Delta_1 + \Delta_2)\left(\kappa^2 A_1^2 - \chi^2\right) = 0. \tag{B15}$$

If $\kappa^2 A_1^2 = \chi^2$, together with Eqs. (4b) and (B14c), one finds that $d_1 d_2 = (d_1 d_2)^*$, which is equivalent to $\Delta_2 = -\Delta_1$. So, in any case, Eq. (B15) is equivalent to

$$\Delta_2 = -\Delta_1. \tag{B16}$$

Then, Eq. (B14c) suggests $A_2 = A_2^*$, and Eq. (B14b) suggests that $d_1 = d_1^*$, i.e., $\Delta_1 = 0$. One then concludes that *the covariance matrix can be block diagonalized only when* $\Delta_2 = \Delta_1 = 0$ *for the steady bright-beam solution*.

It may be observed that such a conclusion can also be obtained from the analysis in Sec. IV B 2 based on the properties of time reversal.

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