Sequential sharing of two-qudit entanglement based on the entropic uncertainty relation

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Entanglement and uncertainty relation are two focuses of quantum theory. We relate entanglement sharing to the entropic uncertainty relation in a $(d\times d)$-dimensional system via weak measurements with different pointers. We consider both the scenarios of one-sided sequential measurements in which the entangled pair is distributed to multiple Alices and one Bob and two-sided sequential measurements in which the entangled pair is distributed to multiple Alices and Bobs. It is found that the maximum number of observers sharing the entanglement strongly depends on the measurement scenarios, the pointer states of the apparatus, and the local dimension d of each subsystem, while the required minimum measurement precision to achieve entanglement sharing decreases to its asymptotic value with the increase of d. The maximum number of observers remain unaltered even when the state is not maximally entangled but has strong-enough entanglement.

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I. INTRODUCTION

Quantum entanglement is a central topic of quantum theory, and the distant parties sharing an entangled quantum state can establish strong correlations that have no classical analog [1]. Aside from its fundamental significance, it is also an essential physical resource for various quantum information processing tasks, such as quantum teleportation [2], dense coding [3], remote state preparation [4], and quantum key distribution [5]. It is also deeply connected to other characteristics of quantum states, including Bell nonlocality [6,7], Einstein-Podolsky-Rosen (EPR) steering [8,9], quantum discord [10], and quantum coherence [11,12].

Since quantum correlations (nonlocality, entanglement, EPR steering, etc.) are vital physical resources, starting from the perspective of quantum information processing, it would be desirable to demonstrate these characteristics between separated observers who perform local measurements on a shared quantum state. The usual projective (strong) measurement induces collapse of the initial state into one of the eigenbases of the measured observable, hence in this case the quantum correlations can be shared by not more than two observers. Silva *et al.* considered instead a different scenario in which one Alice and multiple Bobs (say, Bob_1, Bob_2, etc.) share an entangled pair, with the middle Bobs (Alice and the last Bob) performing weak (projective) measurements [13]. In this context, they showed that Bell nonlocality could be sequentially shared in the sense that a simultaneous violation of Clauser-Horne-Shimony-Holt (CHSH) inequalities between Alice-Bob_1 and Alice-Bob_2 was observed. Subsequently, a double violation of CHSH inequality with equal sharpness of measurements has also been theoretically predicted [14–16] and experimentally observed [17–19]. Further studies showed that an unbounded number of CHSH violations can be achieved using weak measurements with unequal sharpness [20,21]. The sequential sharing of tripartite Bell nonlocality [22], EPR steering [23–27], entanglement [28], and nonlocal advantage of quantum coherence [29,30] have also been studied via weak measurement, showing the advantage of considering this measurement strategy. While the above works are limited to one-sided measurements, the sharing of Bell nonlocality [31,32], EPR steering [33], two-qubit entanglement [34], as well as the quantum advantage in generating random access codes [35] under two-sided measurements for which an entangled pair is distributed sequentially to multiple Alices and Bobs attracts growing interest. The nonlocality sharing via trilateral sequential measurements [36] and network nonlocality sharing [37–40] have also been studied recently.

The sequential sharing of quantum correlations strongly depends on the measurement precision. The uncertainty principle sets a fundamental limit on the precise measurements of two incompatible observables on a particle [41], and hence can be related to entanglement sharing. As for its characterization, apart from the conventional one expressed in terms of the variance of two observables [42], the entropy is another preferred quantity and the entropic uncertainty relations (EURs) have been widely studied in the past decades [41,43]. In particular, by introducing a quantum memory entangled with the particle to be measured, Berta *et al.* [44] proved an entanglement-assisted EUR for two orthonormal measurements, which has been verified experimentally [45,46] and leads to applications including quantum cryptography [44], entanglement witness [44–46], and teleportation [47]. It also attracted growing interest in studying it from various aspects [41], especially its connections with quantum correlations [48–50] and quantum coherence [51,52]. The

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entanglement-assisted EUR also holds if one of the positive operator-valued measures (POVMs) has rank-one elements [53]. For arbitrary POVMs, a generalized EUR was obtained by Frank and Lieb [54].

In this paper, we address the question as to how many observers can sequentially and independently sharing the entanglement in a $(d\times d)$-dimensional (i.e., two-qudit) state. Different from the two-qubit case [28], we consider this problem from the perspective of EUR, and this enables us to explore the two-qudit states which include the two-qubit states as a special case. Moreover, it also enables us to explore entanglement sharing in the two-sided measurements scenario where the entangled qudit pair is distributed sequentially to multiple Alices and Bobs. Compared to the detection of entanglement using witness operators which, in general, needs to collect statistics of the joint measurements on two qudits to infer the expectation values of the witness operators, the advantage of using EUR also lies in that it is sufficient to estimate the probabilities of different outcomes of the local measurements on each of the qudits [44]. Although under specific circumstances, the witness operators can be decomposed as a sum of local operators for which the outcomes of local measurements are also sufficient to infer the expectation values of the witness observables [28,55], the EUR is still a feasible complement to the usual approach for detecting entanglement. In particular, the efficiency of different approaches may be different, and the construction and decomposition of a general witness operator for high-dimensional systems are not very easy tasks [56]. We will consider both the one- and two-sided weak measurements with different pointers and show that the number of observers sharing the entanglement strongly depends on the measurement scenarios, the pointer states of the apparatus, and the local dimension of each qudit. These results not only provide an alternative dimension for the investigation of entanglement sharing, but can also shed light on the interplay between entanglement and quantum measurements for high-dimensional systems.

II. ENTANGLEMENT-ASSISTED EUR FOR GENERAL POVMS

We consider two POVMs $\mathbb{X}$ with elements $\{\mathbb{X}_x\}$ and $\mathbb{Z}$ with elements $\{\mathbb{Z}_z\}$. Defining $X_x=\sqrt{\mathbb{X}_x}$ and $Z_z=\sqrt{\mathbb{Z}_z}$ the measurement operators corresponding to $\mathbb{X}_x$ and $\mathbb{Z}_z$, respectively, then for a two-qudit state ρ_{AB}, the entanglement-assisted EUR reads [54]

$$H(X|B)+H(Z|B)\geqslant \log_2\frac{1}{c''}+H(A|B), \tag{1}$$

where $c''=\max_{x,z}\operatorname{tr}(\mathbb{X}_x\mathbb{Z}_z)$ quantifies the incompatibility of $\mathbb{X}$ and $\mathbb{Z}$, the conditional entropy $H(A|B)=S(\rho_{AB})-S(\rho_B)$ with $S(\rho_{AB})=-\operatorname{tr}(\rho_{AB}\log_2\rho_{AB})$, and likewise for $S(\rho_B)$ of the reduced state $\rho_B=\operatorname{tr}_A\rho_{AB}$. Moreover, $H(X|B)$ and $H(Z|B)$ are given by [54]

$$\begin{aligned} H(X|B)&=-\sum_{x=0}^{d-1}\operatorname{tr}\big(\tilde\rho_{B|X_x}\log_2\tilde\rho_{B|X_x}\big)-S(\rho_B),\\ H(Z|B)&=-\sum_{z=0}^{d-1}\operatorname{tr}\big(\tilde\rho_{B|Z_z}\log_2\tilde\rho_{B|Z_z}\big)-S(\rho_B),\end{aligned} \tag{2}$$

where $\tilde\rho_{B|X_x}$ and $\tilde\rho_{B|Z_z}$ are the (unnormalized) postmeasurement states of qudit B given by

$$\tilde\rho_{B|X_x}=\operatorname{tr}_A(X_x\rho_{AB}X_x^\dagger),\quad \tilde\rho_{B|Z_z}=\operatorname{tr}_A(Z_z\rho_{AB}Z_z^\dagger), \tag{3}$$

where we omitted the identity operators whenever their presence is implied by context, e.g., $X_x\rho_{AB}X_x^\dagger$ should be understood as $(X_x\otimes\mathbb{1})\rho_{AB}(X_x^\dagger\otimes\mathbb{1})$. Note that $H(X|B)$ does not equal the conditional entropy of $\sum_x X_x\rho_{AB}X_x^\dagger$ (i.e., the output of $\mathbb{X}$ without postselection) if $\mathbb{X}$ is not a rank-one measurement, and likewise for $H(Z|B)$ [54]. In addition, for the rank-one orthogonal-projective measurements, Eq. (1) reduces to the entanglement-assisted EUR given by Berta *et al.* [44].

As $-H(A|B)$ gives the lower bound of the one-way distillable entanglement in state ρ_{AB} [57,58], one can see from Eq. (1) that whenever the uncertainty $U_{AB}^{\rm OS}:=H(X|B)+H(Z|B)$ estimated by the one-sided measurement on qudit A and state tomography on qudit B is smaller than $\log_2(1/c'')$, then ρ_{AB} is entangled. Moreover, as quantum measurements never decrease entropy, $U_{AB}^{\rm TS}:=H(X|X)+H(Z|Z)$, which corresponds to the uncertainty estimated by direct two-sided measurements and is favored for its ease of implementation, provides an upper bound of $U_{AB}^{\rm OS}$. As a consequence, $U_{AB}^{\rm TS}<\log_2(1/c'')$ can also be used for entanglement witness. Here, $H(X|X)$ and $H(Z|Z)$ can be obtained as follows [45]:

$$\begin{aligned} H(X|X)&=-\sum_{x_1,x_2=0}^{d-1}p_{x_1x_2}\log_2 p_{x_1x_2}+\sum_{x_2=0}^{d-1}p_{x_2}\log_2 p_{x_2},\\ H(Z|Z)&=-\sum_{z_1,z_2=0}^{d-1}q_{z_1z_2}\log_2 q_{z_1z_2}+\sum_{z_2=0}^{d-1}q_{z_2}\log_2 q_{z_2},\end{aligned} \tag{4}$$

where the probability distribution $p_{x_1x_2}=\operatorname{tr}[\rho_{AB}(\mathbb{X}_{x_1}\otimes\mathbb{X}_{x_2})]$, $p_{x_2}=\sum_{x_1}p_{x_1x_2}$, and likewise for $q_{z_1z_2}$ and q_{z_2}.

III. WEAK MEASUREMENTS ON TWO-QUDIT STATES

For a weak measurement, the system is weakly coupled to a pointer that serves as the measurement apparatus, thus it provides less information about the system while producing less disturbance [59–61]. Defining $\{\Pi_i\}$ the projectors in the basis $\{|i\rangle\}$, $|\phi\rangle$ the initial pointer state, and $\{|\phi_i\rangle\}$ the evolved pointer states associated with $\{|i\rangle\}$, then for the initial system state ρ_0, the system-pointer state after the coupling is given by the following map:

$$\mathcal{E}(\rho_0\otimes|\phi\rangle\langle\phi|)=\sum_{ij}\Pi_i\rho_0\Pi_j\otimes|\phi_i\rangle\langle\phi_j|, \tag{5}$$

and by tracing out the pointer one can obtain the nonselective postmeasurement state as $\rho=\sum_{ij}\Pi_i\rho_0\Pi_j\langle\phi_i|\phi_j\rangle$.

We consider the two-qudit state ρ_{AB}. Defining $F=\langle\phi_i|\phi_{j\neq i}\rangle$ (it depends on i and j in general, here for simplicity, supposing it is a constant) the quality factor measuring the disturbance of a measurement [17]. Furthermore, we denote by $\mathbb{E}$ the POVM defining the weak measurement. Then the nonselective postmeasurement states of the one-sided measurement on A and two-sided measurements on AB, after tracing out the pointer, can be obtained as [30]

$$\mathbb{E}(\rho_{AB}) = F\rho_{AB} + (1-F)\sum_i \Pi_i\rho_{AB}\Pi_i,$$
$$\mathbb{E}^{\otimes 2}(\rho_{AB}) = F\rho_{AB} + (1-F)\sum_{i,j}(\Pi_i\otimes\Pi_j)\rho_{AB}(\Pi_i\otimes\Pi_j) + (F^2-F)\sum_{i\neq k, j\neq l}(\Pi_i\otimes\Pi_j)\rho_{AB}(\Pi_k\otimes\Pi_l). \tag{6}$$

As the pointer states $\{|\phi_i\rangle\}$ are not perfectly distinguishable, we further choose a complete orthogonal set $\{|\varphi_i\rangle\}$ as the reading states. Then for the one-sided case, the probability p_i of getting outcome i and the associated (unnormalized) collapsed state $\tilde{\rho}_{AB|i} = \langle\varphi_i|\mathcal{E}(\rho_{AB}\otimes|\phi\rangle\langle\phi|)|\varphi_i\rangle$ are given by

$$p_i = \mathrm{tr}(\Pi_i\rho_{AB})|\langle\varphi_i|\phi_i\rangle|^2 + \sum_{j\neq i}\mathrm{tr}(\Pi_j\rho_{AB})|\langle\varphi_i|\phi_j\rangle|^2,$$
$$\tilde{\rho}_{AB|i} = \Pi_i\rho_{AB}\Pi_i|\langle\varphi_i|\phi_i\rangle|^2 + \sum_{j\neq i}\Pi_j\rho_{AB}\Pi_j|\langle\varphi_i|\phi_j\rangle|^2 + \sum_{k\neq l}\Pi_k\rho_{AB}\Pi_l\langle\varphi_i|\phi_k\rangle\langle\phi_l|\varphi_i\rangle, \tag{7}$$

and without loss of generality, we suppose the measurements are unbiased. Here, by saying the measurements are unbiased, we mean that the pointer states $\{|\phi_i\rangle\}$ and reading states $\{|\varphi_i\rangle\}$ yield the same $|\langle\varphi_i|\phi_i\rangle|^2$ $(\forall i)$ and the same $|\langle\varphi_i|\phi_{j\neq i}\rangle|^2$ $(\forall j\neq i)$, that is, $|\langle\varphi_i|\phi_i\rangle|^2$ is independent of i and $|\langle\varphi_i|\phi_{j\neq i}\rangle|^2$ is independent of $j\neq i$, then Eq. (7) can be rewritten as

$$p_i = G\mathrm{tr}(\Pi_i\rho_{AB}) + \frac{1-G}{d},$$
$$\tilde{\rho}_{AB|i} = \frac{\mathcal{F}}{d}\rho_{AB} + \frac{1+d_1G-\mathcal{F}}{d}\Pi_i\rho_{AB}\Pi_i + \frac{1-G-\mathcal{F}}{d}\left(\sum_{j\neq i}\Pi_j\rho_{AB}\Pi_j + \sum_{\substack{k\neq l\\ k,l\neq i}}\Pi_k\rho_{AB}\Pi_l\right), \tag{8}$$

where we defined $d_1 = d-1$, $\mathcal{F} = [(1+d_1G)(1-G)]^{1/2}$, and $G = 1-d|\langle\varphi_i|\phi_{j\neq i}\rangle|^2$ measures the precision (or equivalently, the information gain) of the weak measurement. From Eq. (8), one can obtain $\tilde{\rho}_{B|i} = \mathrm{tr}_A\tilde{\rho}_{AB|i}$ which will be used to calculate $H(X|B)$ and $H(Z|B)$.

Similarly, for the two-sided measurements, the probability distribution p_{ij} of the measurement outcomes i on A and j on B can be obtained as

$$p_{ij} = \mathrm{tr}[(\Pi_i\otimes\Pi_j)\rho_{AB}]|\langle\varphi_i|\phi_i\rangle|^2|\langle\varphi_j|\phi_j\rangle|^2 + \sum_{k\neq i}\mathrm{tr}[(\Pi_k\otimes\Pi_j)\rho_{AB}]|\langle\varphi_i|\phi_k\rangle|^2|\langle\varphi_j|\phi_j\rangle|^2 + \sum_{l\neq j}\mathrm{tr}[(\Pi_i\otimes\Pi_l)\rho_{AB}]|\langle\varphi_i|\phi_i\rangle|^2|\langle\varphi_j|\phi_l\rangle|^2 + \sum_{k\neq i, l\neq j}\mathrm{tr}[(\Pi_k\otimes\Pi_l)\rho_{AB}]|\langle\varphi_i|\phi_k\rangle|^2|\langle\varphi_j|\phi_l\rangle|^2, \tag{9}$$

then by defining $\tilde{\Pi}_i = \mathbb{1}-\Pi_i$ and using $\mathcal{F}$ and G defined above, Eq. (9) can be reformulated as

$$p_{ij} = \frac{(1+d_1G)^2}{d^2}\mathrm{tr}[(\Pi_i\otimes\Pi_j)\rho_{AB}] + \frac{(1-G)^2}{d^2}\mathrm{tr}[(\tilde{\Pi}_i\otimes\tilde{\Pi}_j)\rho_{AB}] + \frac{\mathcal{F}^2}{d^2}\mathrm{tr}[(\Pi_i\otimes\tilde{\Pi}_j + \tilde{\Pi}_i\otimes\Pi_j)\rho_{AB}], \tag{10}$$

and this will be used to calculate $H(X|X)$ and $H(Z|Z)$.

As we are interested in entanglement sharing, we consider $\Pi_x = |\psi_x^d\rangle\langle\psi_x^d|$ and $\Pi_z = |\psi_z^0\rangle\langle\psi_z^0|$ $(x, z = 0, 1, \ldots, d-1)$ the projectors associated with $\mathbb{X}$ and $\mathbb{Z}$ defining two weak measurements, where $\{|\psi_x^d\rangle\}$ and $\{|\psi_z^0\rangle\}$ are two sets of mutually unbiased bases (MUBs). For $d=2$, the three MUBs $\{|\psi_x^2\rangle\}$, $\{|\psi_y^1\rangle\}$, and $\{|\psi_z^0\rangle\}$ are eigenbases of the Pauli operators σ_1, σ_2, and σ_3, respectively. For $d>2$, we construct them via the following MUBs:

$$|\psi_x^d\rangle = \frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}e^{i\frac{2\pi}{d}xn}|n\rangle, \quad |\psi_z^0\rangle = \sum_{n=0}^{d-1}\delta_{zn}|n\rangle, \tag{11}$$

where δ_{zn} and i represent the Delta function and the imaginary unit, respectively. Moreover, for any prime $d\geqslant 3$, one can also construct $\mathbb{X}$ and $\mathbb{Z}$ via any pair of the MUBs. In addition to $\{|\psi_x^d\rangle\}$ and $\{|\psi_z^0\rangle\}$ in Eq. (11), the remaining $d-1$ MUBs $\{|\psi_a^r\rangle\}$ $(r = 1, \ldots, d-1)$ are given by [62,63]

$$|\psi_a^r\rangle = \frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}e^{i\frac{2\pi}{d}r(a+n)^2}|n\rangle \; (r = 1, \ldots, d-1), \tag{12}$$

where the index $a\in\{0, 1, \ldots, d-1\}$ for any given r.

For the weak measurements, there is a trade-off $F^2+G^2\leqslant 1$ for two-qubit systems [13]. For a two-qudit system, as F and G defined above are basis independent, one can write the reading and pointer states as $|\varphi_i\rangle = |i\rangle$ and $|\phi_i\rangle = u_i|i\rangle + \sum_{k\neq i}v_{ik}|k\rangle$, respectively. By combining this with the assumption that the measurements are unbiased, one has $|u_i|\equiv u$ $(\forall i)$, $|v_{ik}|\equiv v$ $(\forall k\neq i)$, and $u^2+d_1v^2 = 1$, which gives $F = |u_iv_{ji} + u_jv_{ij} + \sum_{k\neq i,j}v_{ik}v_{jk}|$ and $G = 1-dv^2$. Hence

$$F^2+G^2 \leqslant (2uv+d_2v^2)^2 + (1-dv^2)^2 \leqslant 1, \tag{13}$$

that is, one still has the trade-off $F^2+G^2\leqslant 1$. When the equality holds, the pointer is said to be optimal in the sense that the highest precision is achievable for a given quality factor, and the corresponding optimal pointer can be obtained by solving the equation $(2uv+d_2v^2)^2 + (1-dv^2)^2 = 1$. But for the general weak measurements (e.g., the biased case), whether or not the trade-off $F^2+G^2\leqslant 1$ holds still remains an open question.

In addition, there are other pointers, including the square and Gaussian pointers [13]. The unsharp measurement is another kind of weak measurement [64], and when it does not lead to confusion we also term it as the weak measurement with an unsharp pointer. The associated POVM elements of $\mathbb{X}$ and $\mathbb{Z}$ are given by [24]

$$\mathbb{X}_x = \lambda\Pi_x + \frac{1-\lambda}{d}\mathbb{1}, \quad \mathbb{Z}_z = \lambda\Pi_z + \frac{1-\lambda}{d}\mathbb{1}, \tag{14}$$

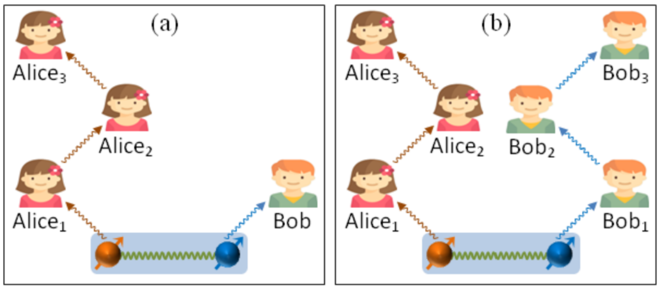

FIG. 1. Two scenarios of entanglement sharing via EUR. (a) Multiple Alices measure sequentially on their side and Bob checks whether or not there is shared entanglement via the uncertainties of Alices' outcomes estimated by his measurements. (b) Multiple Alices and Bobs measure sequentially on their respective sides, after which they check whether there is shared entanglement via the uncertainties estimated by their measurement statistics.

where $\lambda \in [0, 1]$ is the sharpness parameter. For $\mathbb{X}$ constructed by the MUB $\{\Pi_x\}$, one has [24]

$$X_x = \left(\sqrt{\frac{1+d_1\lambda}{d}} - \sqrt{\frac{1-\lambda}{d}}\right)\Pi_x + \sqrt{\frac{1-\lambda}{d}}\mathbb{1}, \quad (15)$$

and likewise for Z_z. In the language of weak measurements, the reading and pointer states of the unsharp measurement can be written as $|\varphi_i\rangle = |i\rangle$ and $|\phi_i\rangle = \sqrt{1-d_1u^2}|i\rangle + u\sum_{j\neq i}|j\rangle$ ($\forall i$), respectively, where $u = \sqrt{(1-\lambda)/d}$ [33]. In addition, by defining $d_2 = d-2$, the quality factor F and measurement precision G of the unsharp measurement can be obtained as

$$F = \frac{d_2(1-\lambda) + 2\sqrt{1+d_2\lambda - d_1\lambda^2}}{d}, \quad G = \lambda, \quad (16)$$

thus it is optimal for $d = 2$ and nonoptimal for $d \geqslant 3$.

Supposing that $\{\mathbb{E}_i\}$ are the elements of the POVM $\mathbb{E}$ defining the weak measurement, then one also has $p_i = \text{tr}(\mathbb{E}_i\rho_{AB})$. Combining this with Eq. (8) and taking $\rho_{AB} = \mathbb{1}\otimes\mathbb{1}/d^2$ yields $\text{tr}\mathbb{E}_i = 1$ ($\forall i$). Then for $\mathbb{X}$ and $\mathbb{Z}$ with the elements $\mathbb{X}_x = r_0\mathbb{1}/d + \sum_m r_{xm}|\psi_m^r\rangle\langle\psi_m^r|$ and $\mathbb{Z}_z = s_0\mathbb{1}/d + \sum_n s_{zn}|\psi_n^s\rangle\langle\psi_n^s|$ comprising of the identity operator $\mathbb{1}$ and the MUBs $\{|\psi_m^r\rangle\}$ and $\{|\psi_n^s\rangle\}$, one can obtain $\text{tr}(\mathbb{X}_x\mathbb{Z}_z) = 1/d$ ($\forall x, z$), thus $c'' = 1/d$. Here, the coefficients r_0 and r_{xm} satisfy $\sum_x \mathbb{X}_x = \mathbb{1}$ and $\text{tr}\mathbb{X}_x = 1$ ($\forall x$), and likewise for s_0 and s_{zn}.

In the following, we mainly concentrate on $\mathbb{X}$ and $\mathbb{Z}$ constructed by $\{|\psi_x^d\rangle\}$ and $\{|\psi_z^0\rangle\}$, respectively, as we aim at exploring the two-qudit state with d being a general positive integer. When d is a prime, we will also sketch the main results for the cases that $\mathbb{X}$ and $\mathbb{Z}$ are constructed via other MUBs at the end of this paper.

IV. ENTANGLEMENT SHARING OF TWO-QUDIT STATES

In this work, we consider entanglement sharing via both the one- and two-sided sequential weak measurements. As shown in Fig. 1(a), the one-sided scenario refers to the case in which multiple Alices (say, Alice$_1$, Alice$_2$, etc.) have access to half of an entangled pair and a single Bob has access to the other half. Alice$_1$ performs her randomly selected measurement ($\mathbb{X}$ or $\mathbb{Z}$) and records the outcome. She then passes her qudit to Alice$_2$ who also measures randomly $\mathbb{X}$ or $\mathbb{Z}$ on the received qudit, records the outcome, and passes it to Alice$_3$, and so on. For the two-sided scenario, as shown in Fig. 1(b), multiple Alices (Alice$_1$, Alice$_2$, etc.) have access to half of an entangled pair and multiple Bobs (Bob$_1$, Bob$_2$, etc.) have access to the other half. To proceed, Alice$_1$ and Bob$_1$ choose randomly the same POVM $\mathbb{X}$ or $\mathbb{Z}$, perform measurements and record their outcomes. They then pass their qudits to Alice$_2$ and Bob$_2$ who repeat this process again, and so on.

For each measurement scenario, we further consider two slightly different cases for which we illustrate through the one-sided scenario (they are similar for the two-sided scenario). In the first case, the classical information pertaining to each Alice's measurement choice and outcome is not shared, thus the state shared between Alice$_n$ ($n \geqslant 2$) and Bob is averaged over all the possible inputs and outputs of Alice$_{n-1}$, from which one arrives at Bob's uncertainty about Alice$_n$'s outcome. In the second case, however, each Alice informs the next Alice of her measurement choice but not the outcome, thus Bob's uncertainty has to be averaged over the uncertainty for each of their possible inputs. More specifically, for the first case one first obtain the average state of all the possible inputs and then arrives at the associated uncertainty, while for the second case one first obtain the uncertainty associated with each possible input and then arrives at their average effect. For convenience of later presentation, we term these two cases as scenario OS1 and scenario OS2, respectively. Similarly, the two cases associated with the two-sided scenario are termed as scenario TS1 and scenario TS2, respectively.

We would like to mention that when exploring nonlocality, EPR steering, and entanglement, both the scenarios OS1 [13–16,20,24,25,28] and TS1 [31–34] have been considered previously. Moreover, scenario OS2 is analogous to a scenario of entanglement-assisted EUR which is useful in cryptographic protocols such as quantum key distribution [41,44]. Due to the fact that the multiple Alices (Bobs) in each side can be spatially separated, these scenarios are expected to play a similar role in multipartite cryptographic tasks.

Having collected all the required tools for our analysis, we are now in a position to investigate sequential sharing of entanglement for the aforementioned scenarios. To set the stage, we start by indicating some notations that will be employed in the following discussion. We denote by $\mathbb{E}^{(0)} = \mathbb{1}$ the identity map and $\mathbb{E}^{(n)} \in \{\mathbb{X}, \mathbb{Z}\}$ ($n = 1, 2, \dots$) the POVMs defining the measurements of Alice$_n$ and Bob$_n$. Moreover, we denote by F_n (G_n) the quality factor (precision) of the corresponding weak measurement and $G_{n,c}^{\text{OS}}$ ($G_{n,c}^{\text{TS}}$) the critical value of G_n larger than which the entanglement could be shared by Alice$_n$ and Bob (Bob$_n$), provided that the previous Alices and Bob (Bobs) have already shared the entanglement. Finally, we denote by $|\Psi^+\rangle = \sum_{k=0}^{d-1}|kk\rangle/\sqrt{d}$ the two-qudit maximally entangled state shared between Alice$_1$ and Bob (one-sided case) or Alice$_1$ and Bob$_1$ (two-sided case).

A. One-sided scenario

We first consider the scenario OS1. If the initial state shared between Alice$_1$ and Bob is ρ_{A_1B}, then the state shared between

Alice$_n$ and Bob will be given by

$$\rho_{A_nB} = \frac{1}{2^{n-1}} \sum_{\mathbb{E}^{(0)},\ldots,\mathbb{E}^{(n-1)}} \mathbb{E}^{(n-1)} \circ \ldots \circ \mathbb{E}^{(0)}(\rho_{A_1B}), \quad (17)$$

where the factor $1/2^{n-1}$ is due to the fact that ρ_{A_nB} is an average state of the 2^{n-1} possible inputs of Alice$_n$, while $\mathbb{E}^{(0)}$ is introduced for ensuring that ρ_{A_nB} reduces to ρ_{A_1B} when $n = 1$. For the general weak measurement, $E^{(1)}(\rho_{A_1B})$ can be obtained from Eq. (6), and likewise for $\mathbb{E}^{(n-1)} \circ \ldots \circ \mathbb{E}^{(0)}(\rho_{A_1B})$. As the POVMs $\mathbb{E}^{(l)} \in \{\mathbb{X}, \mathbb{Z}\}$ $(l = 1, \ldots, n-1)$, when the elements $\{\mathbb{X}_x\}$ of $\mathbb{X}$ and $\{\mathbb{Z}_z\}$ of $\mathbb{Z}$ are known, by defining $X_x = \sqrt{\mathbb{X}_x}$ and $Z_z = \sqrt{\mathbb{Z}_z}$, we also have

$$\rho_{A_2B} = \frac{1}{2}\left(\sum_x X_x \rho_{A_1B} X_x^\dagger + \sum_z Z_z \rho_{A_1B} Z_z^\dagger\right), \quad (18)$$

and likewise for ρ_{A_nB} [65]. Moreover, ρ_{A_nB} in Eq. (17) is normalized. This is because $\mathbb{E}^{(n-1)} \circ \ldots \circ \mathbb{E}^{(0)}(\rho_{A_1B})$ is a nonselective postmeasurement state which is normalized and different from the selective postmeasurement state, e.g., for the POVM $\mathbb{X}$ and initial state ρ_{A_1B}, the postmeasurement state associated with the outcome x is $X_x \rho_{A_1B} X_x^\dagger / \mathrm{tr}(\mathbb{X}_x \rho_{A_1B})$ [65].

In the following, we choose $\rho_{A_1B} = |\Psi^+\rangle\langle\Psi^+|$. Then from Eqs. (17) and (8) we can obtain $\tilde{\rho}_{A_nB|\mathbb{X}_x}$ and $\tilde{\rho}_{A_nB|\mathbb{Z}_z}$. A direct calculation shows that for both $\tilde{\rho}_{B|\mathbb{X}_x} = \mathrm{tr}_{A_n}\tilde{\rho}_{A_nB|\mathbb{X}_x}$ $(\forall x)$ and $\tilde{\rho}_{B|\mathbb{Z}_z} = \mathrm{tr}_{A_n}\tilde{\rho}_{A_nB|\mathbb{Z}_z}$ $(\forall z)$, the eigenvalues are given by

$$\epsilon_0 = \frac{1 + d_1\mu_n}{d^2}\,(1), \quad \epsilon_1 = \frac{1 - \mu_n}{d^2}\,(d_1), \quad (19)$$

where the numbers in the brackets denote the degeneracy, and by defining $F_0 = 1$, the parameter μ_n can be written as

$$\mu_n = \frac{G_n \prod_{k=0}^{n-1}(1 + F_k)}{2^n}. \quad (20)$$

As $\rho_B = \mathbb{1}/d$, from Eqs. (2) and (19), we arrive at the uncertainty

$$U_{A_nB}^{\mathrm{OS}} = 2\left[H_2\left(\frac{1 + d_1\mu_n}{d}\right) + \frac{d_1(1 - \mu_n)}{d}\log_2 d_1\right], \quad (21)$$

where $H_2(\cdot)$ denotes the binary Shannon entropy function.

Based on Eq. (21), we can determine the maximum number of Alices sharing the entanglement with Bob by checking whether or not $U_{A_nB}^{\mathrm{OS}} < \log_2 d$ holds. As $U_{A_nB}^{\mathrm{OS}}$ is a monotonic decreasing function of G_n, the measurement precision must be larger than a critical value $G_{n,c}^{\mathrm{OS}}$ for achieving possible entanglement sharing. Of course, $G_{n,c}^{\mathrm{OS}}$ with $n \geqslant 2$ is not a constant as the actual precision G_l of Alice$_l$ $(l \leqslant n-1)$ may be larger than $G_{l,c}^{\mathrm{OS}}$. As for $G_{1,c}^{\mathrm{OS}}$, its value could be obtained by solving numerically the transcendental equation $U_{A_1B}^{\mathrm{OS}} = \log_2 d$, and it is obvious that it is independent of F_1. This is because $U_{A_1B}^{\mathrm{OS}}$ depends on the initial state ρ_{A_1B} and the measurement precision G_1 (i.e., the information gain) of Alice$_1$, while the disturbance of her measurement affects only the output state of the qudit which is passed on to Alice$_2$, and this will further affects $U_{A_2B}^{\mathrm{OS}}$ and $G_{2,c}^{\mathrm{OS}}$. As illustrated in Fig. 2, $G_{1,c}^{\mathrm{OS}}$ decreases with the increase of d, and for infinite large d, it approaches the asymptotic value $G_{1,c}^{\mathrm{OS}}(\infty) = 1/2$. This shows that the larger the local dimension d of each qudit, the lower the critical measurement precision is required for sharing the entanglement.

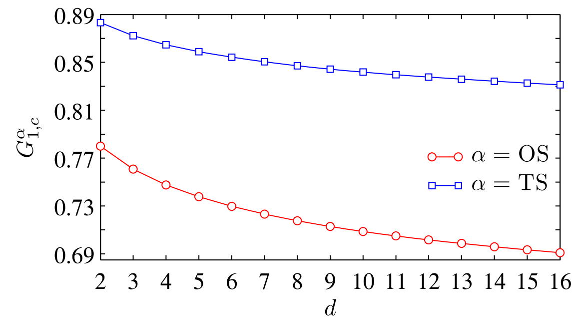

FIG. 2. The critical precision $G_{1,c}^{\alpha}$ (α = OS or TS) versus d for the weak measurement.

When the measured qudit is passed to the subsequent Alices, the uncertainties $U_{A_nB}^{\mathrm{OS}}$ will be dependent on the quality factors F_l $(l = 1, \ldots, n-1)$ of the measurements. Based on Eq. (21), one can see that the maximum number of Alices sharing the entanglement with Bob can be determined by checking whether the following inequalities hold

$$G_l \geqslant G_{l,c}^{\mathrm{OS}}\ (l = 1, \ldots, n-1), \quad G_{n,c}^{\mathrm{OS}} = \frac{2^n G_{1,c}^{\mathrm{OS}}}{\prod_{k=0}^{n-1}(1 + F_k)} \leqslant 1, \quad (22)$$

where the first inequality ensures that Alice$_l$ $(l = 1, \ldots, n-1)$ share the entanglement simultaneously. Moreover, as $U_{A_nB}^{\mathrm{OS}}$ is a decreasing function of μ_n and μ_n of Eq. (20) is an increasing function of G_n, one must have $\mu_{n,c}^{\mathrm{OS}} = \mu_{1,c}^{\mathrm{OS}}$ ($\mu_{n,c}^{\mathrm{OS}}$ is the critical μ_n obtained by replacing G_n with $G_{n,c}^{\mathrm{OS}}$) to ensure that Alice$_n$ can also share the entanglement. This, together with Eq. (20), gives the above equality. As for the second inequality, it is due to the fact that $G_n \leqslant 1$ $(\forall n)$ by its definition.

We now discuss whether or not the inequalities in Eq. (22) hold when the qudit is passed to Alice$_2$. We consider the weak measurements with unsharp, optimal, Gaussian, and square pointers.

(A1.1) For the unsharp pointer, the relation between F_1 and G_1 is in Eq. (16), by combining of which with Eq. (22) we can obtain that when $G_{1,c}^{\mathrm{OS}} \leqslant G_1 \leqslant 2\xi/d$, there exists valid $G_{2,c}^{\mathrm{OS}}$ for all $d \geqslant 2$, where the parameter ξ is given by

$$\xi = d_2\big(1 - G_{1,c}^{\mathrm{OS}}\big) + \sqrt{2\big(1 - G_{1,c}^{\mathrm{OS}}\big)\big(2d_1 G_{1,c}^{\mathrm{OS}} - d_2\big)}. \quad (23)$$

(A1.2) For the optimal pointer, $F_1^2 + G_1^2 = 1$, then from Eq. (22) we arrive at $G_{1,c}^{\mathrm{OS}} \leqslant G_1 \leqslant 2[G_{1,c}^{\mathrm{OS}}(1 - G_{1,c}^{\mathrm{OS}})]^{1/2}$, which holds if $G_{1,c}^{\mathrm{OS}} \leqslant 0.8$. As $G_{1,c}^{\mathrm{OS}} \simeq 0.7799$ for $d = 2$ and decreases with the increasing d, there exists valid $G_{2,c}^{\mathrm{OS}}$ for all $d \geqslant 2$.

(A1.3) For the Gaussian pointer, the relation of F_1 and G_1 can be obtained numerically [13]. We denote it by $\Lambda(F_1) = G_1$, i.e., for any given F_1, the map $\Lambda(F_1)$ gives the associated G_1. Then from Eq. (22) we can obtain $G_{1,c}^{\mathrm{OS}} \leqslant G_1 \leqslant \Lambda(2G_{1,c}^{\mathrm{OS}} - 1)$, which holds when $G_{1,c}^{\mathrm{OS}} \lesssim 0.7547$. By combining this with

$G_{1,c}^{\rm OS}$ obtained for different local dimension d (see Fig. 2), we can see that there exists valid $G_{2,c}^{\rm OS}$ for $d \geqslant 4$.

(A1.4) For the square pointer which is far from optimal, we have $F_1 + G_1 = 1$ [13], then from Eq. (22) we arrive at $G_{1,c}^{\rm OS} \leqslant G_1 \leqslant 2 - 2G_{1,c}^{\rm OS}$. This inequality holds when $G_{1,c}^{\rm OS} \leqslant 2/3$. By combining this with $G_{1,c}^{\rm OS}$ obtained for different d, we can see that there exists valid $G_{2,c}^{\rm OS}$ for $d \geqslant 34$.

The critical $G_{2,c}^{\rm OS}$ also decreases with the increase of d and approaches its asymptotic value $1/[1 + F_{1,c}^{\rm OS}(\infty)]$ when $d \to \infty$, where $F_{1,c}^{\rm OS}(\infty)$ is the quality factor associated with $G_{1,c}^{\rm OS}(\infty)$. Following the same line of reasoning as above, we can obtain that for the optimal pointer with $d \geqslant 34$ and the Gaussian pointer with $d \geqslant 141$, the entanglement could be sequentially shared by, at most, three Alices and Bob, and for large enough d, the maximum number of Alices could be further enhanced. In fact, a similar phenomenon for sequential sharing of EPR steering has also been observed previously [24]. But for the unsharp and square pointers, the maximum number of Alices remains 2. We explain this for the square pointer (the case is similar for the unsharp pointer). From Eq. (22) we can obtain that if there exists a valid $G_{3,c}^{\rm OS}$, then $(2 - G_1)(2 - G_2) \geqslant 4G_{1,c}^{\rm OS}$. However, even under $G_1 = G_{1,c}^{\rm OS}$ and $G_2 = G_{2,c}^{\rm OS}$, this inequality holds only when $G_{1,c}^{\rm OS} \leqslant 1/2$. As illustrated above, $G_{1,c}^{\rm OS} \geqslant 1/2$, and the equality holds when $d \to \infty$, thus the entanglement cannot be sequentially shared by three Alices and Bob.

Next, we turn to consider the scenario OS2 for which each Alice knows the measurement choice but not the outcome of the former Alices, and our aim is to determine whether or not the entanglement can be shared by multiple Alices and Bob. We will consider the average effect, that is, the shared entanglement between Alice$_n$ and Bob is averaged over the possible inputs and outputs of Alice$_{n-1}$, and likewise for Bob's uncertainty about Alice$_n$'s outcome. Clearly, this is different from the scenario OS1 for which the entanglement and uncertainty are obtained for the state averaged over the previous Alice's possible inputs and outputs. For this case, the possible states shared between Alice$_n$ and Bob are given by

$$\rho_{A_nB}^{\mathbb{E}^{(0)}\ldots\mathbb{E}^{(n-1)}} = \mathbb{E}^{(n-1)} \circ \ldots \circ \mathbb{E}^{(0)}(\rho_{A_1B}), \tag{24}$$

where the meanings of $\mathbb{E}^{(0)}$ and $\mathbb{E}^{(n)}$ $(n \geqslant 1)$ are the same as that in Eq. (17). For each possible input, from Eq. (8) we can obtain the unnormalized postmeasurement states $\tilde{\rho}_{A_nB|\mathbb{O}_o}^{\mathbb{E}^{(0)}\ldots\mathbb{E}^{(n-1)}}$, where $\mathbb{O}_o = \mathbb{X}_x$ or $\mathbb{Z}_z$. Then after a straightforward but somewhat complex calculation, we can obtain the eigenvalues of $\tilde{\rho}_{B|\mathbb{O}_o}^{\mathbb{E}^{(0)}\ldots\mathbb{E}^{(n-1)}} = {\rm tr}_{A_n}\tilde{\rho}_{A_nB|\mathbb{O}_o}^{\mathbb{E}^{(0)}\ldots\mathbb{E}^{(n-1)}}$ $(\forall o)$ as

$$\begin{aligned} \epsilon_0 &= \frac{1 + d_1 G_n \prod_{k=0}^{n-1} F_k^{\delta(\mathbb{E}^{(k)},\mathbb{O})}}{d^2}\,(1), \\ \epsilon_1 &= \frac{1 - G_n \prod_{k=0}^{n-1} F_k^{\delta(\mathbb{E}^{(k)},\mathbb{O})}}{d^2}\,(d_1), \end{aligned} \tag{25}$$

where we define the function $\delta(\mathbb{E}^{(k)}, \mathbb{O}) = 0$ if $\mathbb{E}^{(k)} = \mathbb{O}$ and $\delta(\mathbb{E}^{(k)}, \mathbb{O}) = 1$ if $\mathbb{E}^{(k)} \neq \mathbb{O}$, and the numbers in the brackets still denote the degeneracy. Combining this with Eq. (2) gives

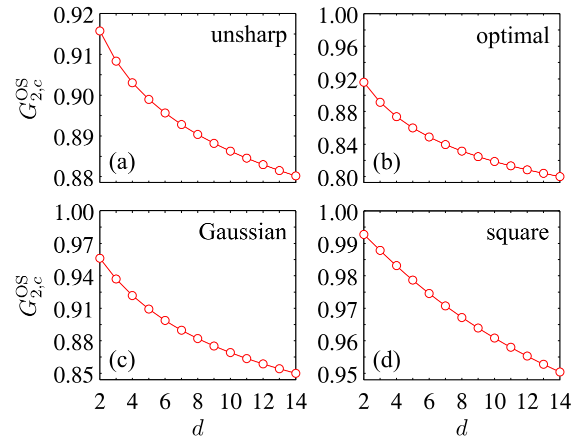

FIG. 3. The d dependence of the critical precision $G_{2,c}^{\rm OS}$ (obtained with $G_1 = G_{1,c}^{\rm OS}$) for the scenario OS2 with different pointers.

the average uncertainty

$$\begin{aligned} \mathcal{U}_{A_nB}^{\rm OS} &= \frac{1}{2^{n-1}} \sum_{\mathbb{E}^{(0)},\ldots,\mathbb{E}^{(n-1)}} U_{A_nB}^{\rm OS}\big(\rho_{A_nB}^{\mathbb{E}^{(0)}\ldots\mathbb{E}^{(n-1)}}\big) \\ &= \frac{1}{2^{n-2}} \Bigg\{ \sum_{i_0,\ldots,i_{n-1}} H_2\Bigg(\frac{1 + d_1 G_n \prod_{k=0}^{n-1} F_k^{i_k}}{d}\Bigg) \\ &\quad + \frac{d_1}{d}\Bigg[2^{n-1} - \frac{G_n}{2}\prod_{k=0}^{n-1}(1 + F_k)\Bigg]\log_2 d_1\Bigg\}, \end{aligned} \tag{26}$$

where $i_0 = 0$ and $i_{k|k\geqslant 1} \in \{0, 1\}$ are the power exponents.

Based on Eq. (26), we can determine the number of Alices sharing the average entanglement with one Bob. We focus on $n \geqslant 2$ as the scenario OS2 differentiates the scenario OS1 only for this case. From Eqs. (17) and (24) one can see that Alice$_n$'s input for the scenario OS1 can be recognized as an equal mixture of the possible inputs for the scenario OS2, thus the entanglement shared between Alice$_n$ and Bob for the scenario OS1 will be weaker than or equal to that for the scenario OS2 due to the convexity of entanglement [1]. As the measurement uncertainty strongly depends on the amount of entanglement [44], then intuitively, the maximum number of Alices sharing the entanglement for the scenario OS2 will be no less than that for the scenario OS1. In the following, we derive $G_{n,c}^{\rm OS}$ with $G_l = G_{l,c}^{\rm OS}$ $(l = 1, \ldots, n-1)$, for which the corresponding quality factor F_l can be obtained for the measurements with different pointers. By substituting F_l into Eq. (26) and then solving numerically the transcendental equation $\mathcal{U}_{A_nB}^{\rm OS} = \log_2 d$, we can obtain the critical $G_{n,c}^{\rm OS}$. For $n = 2$, the corresponding results are shown in Fig. 3.

(A2.1) For the unsharp pointer, $G_{2,c}^{\rm OS}$ approaches the asymptotic value of 2/3 when $d \to \infty$. Moreover, from Eq. (26) we can check that when the first two Alices share the average entanglement with Bob, $\mathcal{U}_{A_3B}^{\rm OS}$ is always larger than $\log_2 d$. Thus Alice$_3$ cannot share the average entanglement with Bob.

(A2.2) For the optimal pointer, $G_{2,c}^{\rm OS}$ decreases to $4 - 2\sqrt{3}$ when $d \to \infty$. For small d, the number of Alices sharing the entanglement with Bob remains 2. But with the increasing d, the number of Alices may be enhanced. We performed calculations based on Eq. (26) and it is found that for $d \geqslant 10$, 180, and 30 608, respectively, at most three, four, and five Alices can share the average entanglement with Bob, and the critical $G_{n,c}^{\mathrm{OS}}$ ($n = 3, 4, 5$) all decrease with the increase of d. For large enough d, it is reasonable to conjecture that the maximum number of Alices will be further enhanced.

(A2.3) For the Gaussian pointer, a comparison of Figs. 3(a) and 3(c) shows that for $d \geqslant 7$, $G_{2,c}^{\mathrm{OS}}$ becomes smaller than that for the unsharp pointer, that is, from the point of view of entanglement sharing, the Gaussian pointer performs better than the unsharp pointer for $d \geqslant 7$. A further calculation shows that for $d \geqslant 67$, the average entanglement can also be shared by Alice$_3$ and Bob, and the number of Alices can be further enhanced by increasing d. However, the critical d starting from which the average entanglement can be sequentially shared will be far larger than that with an optimal pointer.

(A2.4) For the square pointer, $G_{2,c}^{\mathrm{OS}}$ decreases to the same asymptotic value as that of the unsharp pointer, which can be explained from Eq. (16) as for the unsharp pointer, $F_1 + G_1 \to 1$ when $d \to \infty$. In addition, as its precision is worse than that of the unsharp pointer, the maximum number of Alices sharing the average entanglement with Bob remains two.

B. Two-sided scenario

In this subsection, we discuss sharing of entanglement via two-sided measurements, and for simplicity, we focus on the case that both F_n and G_n ($\forall n$) for Alice$_n$ equal to that for Bob$_n$. We begin with the scenario TS1 and suppose the state $\rho_{A_1B_1} = |\Psi^+\rangle\langle\Psi^+|$ is initially shared between Alice$_1$ and Bob$_1$, then the state $\rho_{A_nB_n}$ shared between Alice$_n$ and Bob$_n$ takes the similar form as that given in Eq. (17), with only $\mathbb{E}^{(k)}$ ($k = 0, 1, \ldots, n-1$) being replaced by $\mathbb{E}^{(k)} \otimes \mathbb{E}^{(k)}$. By combining this with Eq. (10) we can obtain the probability distributions $p_{x_1x_2}$ and $q_{z_1z_2}$ as

$$\begin{aligned} p_{x_1x_2|x\in S} &= \frac{1+d_1\nu_n}{d^2}, \quad p_{x_1x_2|x\notin S} = \frac{1-\nu_n}{d^2}, \\ q_{z_1z_1} &= \frac{1+d_1\nu_n}{d^2}, \quad q_{z_1z_2|z_1\neq z_2} = \frac{1-\nu_n}{d^2}, \end{aligned} \tag{27}$$

where we define $x = (x_1, x_2)$ and $S = \{(0, 0), (n, d-n)|n = 1, \ldots, d-1\}$ for simplifying the equations, and ν_n can be obtained by replacing G_n and F_k in Eq. (20) with G_n^2 and F_k^2, respectively.

From Eqs. (27) and (4) we can show that the uncertainty $U_{A_nB_n}^{\mathrm{TS}}$ has a similar form to $U_{A_nB}^{\mathrm{OS}}$ in Eq. (21), with only the parameter μ_n being replaced by ν_n. So the critical $G_{1,c}^{\mathrm{TS}}$ larger than which the entanglement can be shared by Alice$_1$ and Bob$_1$ equals the square root of $G_{1,c}^{\mathrm{OS}}$, i.e., $G_{1,c}^{\mathrm{TS}} = (G_{1,c}^{\mathrm{OS}})^{1/2}$, implying a similar behavior for their dependence on d, see Fig. 2.

Similar to the one-sided scenario, the maximum number of Alices and Bobs sharing the entanglement can be obtained by checking whether the following inequalities hold:

$$\begin{aligned} G_l &\geqslant G_{l,c}^{\mathrm{TS}}\,(l = 1, \ldots, n-1), \\ G_{n,c}^{\mathrm{TS}} &= \frac{2^{n/2}G_{1,c}^{\mathrm{TS}}}{\sqrt{\prod_{k=0}^{n-1}\left(1+F_k^2\right)}} \leqslant 1. \end{aligned} \tag{28}$$

When the two qudits are passed to Alice$_2$ and Bob$_2$, Eq. (28) reduces to $G_1^2 \geqslant G_{1,c}^{\mathrm{OS}}$ and $F_1^2 \geqslant 2G_{1,c}^{\mathrm{OS}} - 1$. We now analyze whether or not the two inequalities hold simultaneously for the weak measurements with different pointers.

(B1.1) For the unsharp pointer, the above two inequalities depend on d and it is hard to give an analytical analysis. The numerical calculation by using the trade-off between the quality factor F_1 and measurement precision G_1 given in Eq. (16) shows that they hold simultaneously when $G_{1,c}^{\mathrm{OS}} \lesssim 0.7321$ and $d \gtrsim 224526395$. This condition is difficult to meet in experiments, hence the unsharp measurement is not a good choice for achieving entanglement sharing.

(B1.2) For the optimal pointer, the above inequalities are equivalent to $G_{1,c}^{\mathrm{OS}} \leqslant G_1^2 \leqslant 2 - 2G_{1,c}^{\mathrm{OS}}$, which holds when $G_{1,c}^{\mathrm{OS}} \leqslant 2/3$. By combining this with the results of $G_{1,c}^{\mathrm{OS}}$ for different d, we can note that the entanglement can be sequentially shared by two Alices and Bobs for $d \geqslant 34$.

(B1.3) For the Gaussian pointer, based on $\Lambda(F_1) = G_1$, we can obtain $(G_{1,c}^{\mathrm{OS}})^{1/2} \leqslant G_1 \leqslant \Lambda[(2G_{1,c}^{\mathrm{OS}} - 1)^{1/2}]$, which holds for $G_{1,c}^{\mathrm{OS}} \lesssim 0.6102$, and this corresponds to $d \geqslant 404$.

(B1.4) For the square pointer, the above inequalities hold simultaneously when $G_{1,c}^{\mathrm{OS}} \leqslant 4 - 2\sqrt{3}$, which corresponds to $d \gtrsim 226154704$. Hence it is also not a good choice for achieving sequential sharing of entanglement.

When the two qudits are passed to Alice$_3$ and Bob$_3$, respectively, from Eq. (28) we can obtain that, if they can sequentially sharing the entanglement, then one must have $G_1 \geqslant G_{1,c}^{\mathrm{TS}}$, $G_2 \geqslant G_{2,c}^{\mathrm{TS}}$, and $(1+F_1^2)(1+F_2^2) \geqslant 4G_{1,c}^{\mathrm{OS}}$. For the optimal pointer, we can show that the three inequalities hold simultaneously when $G_{1,c}^{\mathrm{OS}} \leqslant 1/2$. As we showed previously, $G_{1,c}^{\mathrm{OS}}$ decreases to $1/2$ only when $d \to \infty$, indicating that the entanglement cannot be shared by Alice$_3$ and Bob$_3$. As the optimal pointer gives the highest measurement precision under the same disturbance, we can conclude that the entanglement can be sequentially shared by at most two Alices and Bobs via the two-sided weak measurements.

Next, we consider the scenario TS2. Now, the 2^{n-1} possible inputs for Alice$_n$ and Bob$_n$ can be obtained in the same way as that in Eq. (24), with, however, $\mathbb{E}^{(k)}$ ($k = 0, 1, \ldots, n-1$) being replaced by $\mathbb{E}^{(k)} \otimes \mathbb{E}^{(k)}$. For $\rho_{A_nB_n}^{\mathbb{E}^{(0)}\ldots\mathbb{E}^{(n-1)}}$, using Eq. (10) and after a straightforward but somewhat complex calculation, one can obtain the probability distribution $p_{x_1x_2}$ as

$$\begin{aligned} p_{x_1x_2|x\in S} &= \frac{1+d_1G_n^2\prod_{k=0}^{n-1}F_k^{2\delta(\mathbb{E}^{(k)},\mathbb{X})}}{d^2}, \\ p_{x_1x_2|x\notin S} &= \frac{1-G_n^2\prod_{k=0}^{n-1}F_k^{2\delta(\mathbb{E}^{(k)},\mathbb{X})}}{d^2}, \end{aligned} \tag{29}$$

while the probability $q_{z_1z_1}$ ($q_{z_1z_2|z_1\neq z_2}$) can be obtained directly by replacing $\mathbb{X}$ in $p_{x_1x_2|x\in S}$ ($p_{x_1x_2|x\notin S}$) with $\mathbb{Z}$. By combining this with Eq. (4) one can obtain the average uncertainty $\mathcal{U}_{A_nB_n}^{\mathrm{TS}}$, whose form is similar to $\mathcal{U}_{A_nB}^{\mathrm{OS}}$ in Eq. (26), with however the parameters G_n and F_k being replaced by G_n^2 and F_k^2, respectively. In the following, we determine the number of Alices and Bobs sharing the entanglement, and we calculate the critical $G_{n,c}^{\mathrm{TS}}$ numerically, if any, under the conditions $G_l = G_{l,c}^{\mathrm{TS}}$ ($l = 1, \ldots, n-1$). Specifically, $G_{n,c}^{\mathrm{TS}}$ can be obtained using the same method as that for obtaining $G_{n,c}^{\mathrm{OS}}$. Nonetheless, from

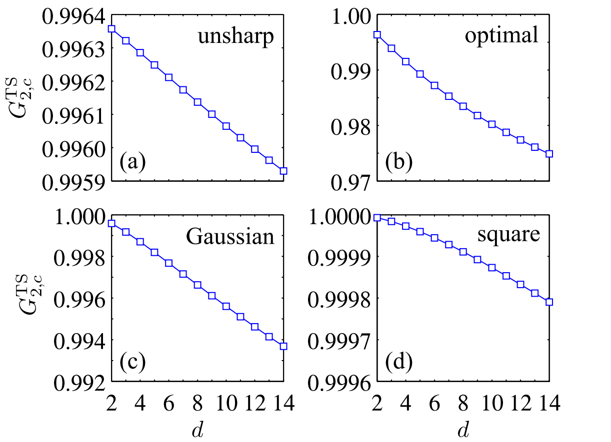

FIG. 4. The d dependence of the critical precision $G_{2,c}^{\rm TS}$ (obtained with $G_1 = G_{1,c}^{\rm TS}$) for the scenario TS2 with different pointers.

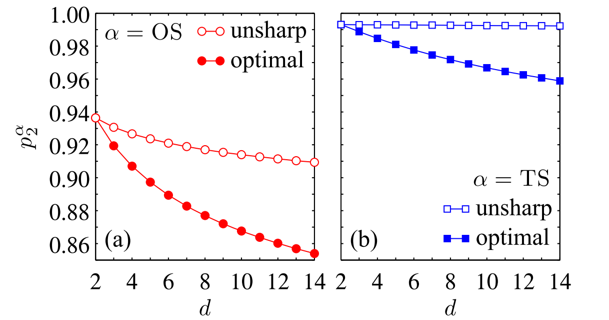

FIG. 5. The d dependence of p_2^α (α = OS or TS) for the scenarios OS2 and TS2 with unsharp and optimal pointers.

the results for scenario TS1, we know that the entanglement can always be sequentially shared by two Alices and Bobs.

(B2.1) For the unsharp pointer, $G_{2,c}^{\rm TS}$ decreases very slowly with the increase of d, see Fig. 4(a). When $d \to \infty$, we can obtain $(G_{2,c}^{\rm TS})^2 \to 2(5+2\sqrt{2})/17$. However, when the previous two Alices and Bobs can sequentially share the entanglement, we always have $\mathcal{U}_{A_3B_3}^{\rm TS} > \log_2 d$. Thus, at most, two Alices and Bobs can sequentially sharing the entanglement in this case.

(B2.2) For the optimal pointer, as shown in Fig. 4(b), $G_{2,c}^{\rm TS}$ still decreases with the increase of d, and when $d \to \infty$, we have $G_{2,c}^{\rm TS} \to \sqrt{6}/3$. Moreover, when the previous two Alices and Bobs can share the entanglement, $\mathcal{U}_{A_3B_3}^{\rm TS} > \log_2 d$. Thus Alice$_3$ and Bob$_3$ cannot share the entanglement even with the optimal pointer. This result also indicates that, for the weak measurement with arbitrary pointer state, the entanglement can be shared by, at most, two Alices and Bobs.

(B2.3) For the Gaussian pointer, the d dependence of $G_{2,c}^{\rm TS}$ is shown in Fig. 4(c). When $d \geqslant 10$, it becomes smaller than that for the unsharp pointer, indicating that for the two-qudit system with large local dimension, the Gaussian pointer performs better than the unsharp pointer on entanglement sharing.

(B2.4) For the square pointer, as illustrated in Fig. 4(d), $G_{2,c}^{\rm TS}$ is obviously larger than that for the other pointers. It decreases slowly with the increasing d, and when $d \to \infty$, it approaches the same asymptotic value as that with an unsharp pointer due to the same reason for the one-sided scenario.

C. Case of nonmaximally entangled states

Up to now, we demonstrated sequential sharing of entanglement for the initial maximally entangled state, then it is natural to ask whether the sequential sharing of entanglement is also possible for those nonmaximally entangled states. To this end, we consider the following isotropic state:

$$\rho_I = p|\Psi^+\rangle\langle\Psi^+| + \frac{1-p}{d^2}\mathbb{1}, \tag{30}$$

where $0 \leqslant p \leqslant 1$ and ρ_I is separable for $p \leqslant 1/(d+1)$. Note that ρ_I in Eq. (30) is equivalent to that given in Ref. [66].

For this state, assuming that all other conditions of the scenarios remain unchanged, then after some algebra we can obtain the expressions of $U_{A_nB}^\alpha$, $U_{A_nB_n}^\alpha$, $\mathcal{U}_{A_nB}^\alpha$, and $\mathcal{U}_{A_nB_n}^\alpha$ (α = OS or TS), which are similar to those listed in the above two subsections, and the only difference lies in that the parameters G_n for the one-sided case and G_n^2 for the two-sided case are multiplied by a factor p. Then one can show that for $p < p_1 = G_{1,c}^{\rm OS}$, the entanglement in ρ_I cannot be shared by any observer, irrespective of the measurement scenario.

For the scenario OS1, from Eq. (21) (note that G_n should be replaced by pG_n) we can obtain that if two Alices can share the entanglement in ρ_I with Bob, then the following inequalities must hold:

$$G_1 \geqslant \frac{G_{1,c}^{\rm OS}}{p}, \quad F_1 \geqslant \frac{2G_{1,c}^{\rm OS}}{p} - 1. \tag{31}$$

Then for the unsharp pointer, they hold simultaneously when $p \geqslant p_2^{\rm OS}$, where $p_2^{\rm OS} = (3d - \sqrt{2d} - 4)G_{1,c}^{\rm OS}/2d_2$. Similarly, we have $p_2^{\rm OS} = 5G_{1,c}^{\rm OS}/4$ for the optimal pointer and $p_2^{\rm OS} = 3G_{1,c}^{\rm OS}/2$ for the square pointer. For the Gaussian pointer, $p_2^{\rm OS}$ can be obtained numerically. They all decrease with the increasing d (for the conciseness of this paper, we do not plot them here). For the scenario TS1, following the same line of reasoning as above, one can obtain $p_2^{\rm TS} = 3G_{1,c}^{\rm OS}/2$ for the optimal pointer, while for the other three pointers, even for $p = 1$, the entanglement can be shared by two Alices and Bobs only for very large d, so we do not derive the corresponding $p_2^{\rm TS}$ here.

For the scenarios OS2 and TS2, the critical p_2^α larger than which the entanglement can be shared by two Alices and Bob can be obtained numerically. We illustrate the method for OS2 (it is the same for TS2). First, we assume $G_1 = G_{1,c}^{\rm OS}/p$ which ensures that Alice$_1$ and Bob share the entanglement. For given G_1, the associated F_1 can be obtained accordingly for different pointers. Then by substituting G_1, F_1, and $G_2 = 1$ into Eq. (26), $\mathcal{U}_{A_2B}^{\rm OS}$ will be transformed to a function of p, and by solving numerically the equation $\mathcal{U}_{A_2B}^{\rm OS} = \log_2 d$, we can obtain the critical p_2^α. For the unsharp and optimal pointers, the d dependence of p_2^α can be found from Fig. 5 (the d dependence of p_2^α for the square and Gaussian pointers are similar and we do not plot them here). They also decrease with the increase of d. For infinite large d, we have $p_2^{\rm OS} \to 2/3$ and $p_2^{\rm TS} \to 2(5+2\sqrt{2})/17$ for the unsharp pointer, while $p_2^{\rm OS} \to 4 - 2\sqrt{3}$ and $p_2^{\rm TS} \to 2/3$ for the optimal pointer.

The above results exemplify that even the state is not maximally entangled, it is also possible to sequentially sharing the entanglement in it. From Ref. [66] we can note that in the region of $p > p_2^{\alpha}$, the entanglement of formation (a measure of entanglement [67]) increases monotonically with the increase of p, thus the condition $p > p_2^{\alpha}$ for sequential sharing of the nonmaximal entanglement implies that the entanglement in ρ_I must be stronger than a critical value.

D. Other cases

Having clarified the number of observers sharing the entanglement for $\mathbb{X}$ and $\mathbb{Z}$ constructed via the MUBs given in Eq. (11), we now turn to the case of primes d, for which one can also construct $\mathbb{X}$ and $\mathbb{Z}$ via other MUBs. For this case, when considering the one-sided scenario, a direct calculation shows that the uncertainties and the corresponding critical measurement precision are the same as those discussed above. When considering the two-sided scenario, however, they remain the same as that discussed above only for $d = 2$. For primes $d \geqslant 3$, $U_{A_nB_n}^{\mathrm{TS}}$ and $\mathcal{U}_{A_nB_n}^{\mathrm{TS}}$ obtained with $\mathbb{X}$ and $\mathbb{Z}$ constructed by other MUBs are always larger than that obtained with $\mathbb{X}$ and $\mathbb{Z}$ constructed via Eq. (11). As a matter of fact, for this case even Alice$_1$ and Bob$_1$ cannot witness the entanglement.

Another issue that remains is whether the conclusion also holds when different observers choose different measurement settings for the primes $d \geqslant 2$, e.g., Alice$_1$ (or Alice$_1$ and Bob$_1$) chooses $\mathbb{X}$ and $\mathbb{Z}$ of Eq. (11) and the other Alices (Alices and Bobs) choose that constructed by other MUBs. For this case, a further calculation shows that for both the one- and two-sided scenarios, at most, one Alice can share the entanglement with Bob. Thus the measurement settings $\mathbb{X}$ and $\mathbb{Z}$ constructed via Eq. (11) perform better than the other cases.

Finally, it is also relevant to ask whether or not the number of observers sharing the entanglement will be changed if they measure the received qudits with equal precision, i.e., $G_n = G$ ($\forall n$). For this case, we performed calculations based on the formulas in the above subsections and it is found that only for the one-sided weak measurements with the optimal and Gaussian pointers can there exists a valid region of $G \in [G_L, G_U]$ in which the entanglement can be shared by two Alices and Bob. For the scenario OS1 with an optimal pointer, this region can be obtained by substituting $G_{1,2} = G$ and $F_{1,2} = (1 - G^2)^{1/2}$ into Eq. (21), then the two bounds G_L and G_U, if any, can be obtained by solving numerically the equation $U_{A_2B}^{\mathrm{OS}} = \log_2 d$. They exist for $d \geqslant 62$ and as illustrated in Fig. 6, the bound G_L (G_U) decreases (increases) with the increase of d. When $d \to \infty$, G_L approaches the asymptotic value of about 0.5437 and G_U approaches the asymptotic value 1. These asymptotic values are solutions of $G^4 - 2G + 1 = 0$, which was obtained from Eq. (21) for the optimal pointer. For the scenario OS2 with an optimal pointer, the region $G \in [G_L, G_U]$ can be obtained similarly. It exists for $d \geqslant 11$, where G_L and G_U approach the same asymptotic values as above; in particular, G_U approaches rapidly the asymptotic value 1 (e.g., it is of about 0.9931 for $d = 11$ and 0.9999 for $d = 15$). Similarly, for the scenario OS1 (OS2) with the Gaussian pointer, the region $G \in [G_L, G_U]$ exists for $d \gtrsim 6950$ ($d \gtrsim 640$), and now the asymptotic value of G_L is about 0.5861, while the asymptotic

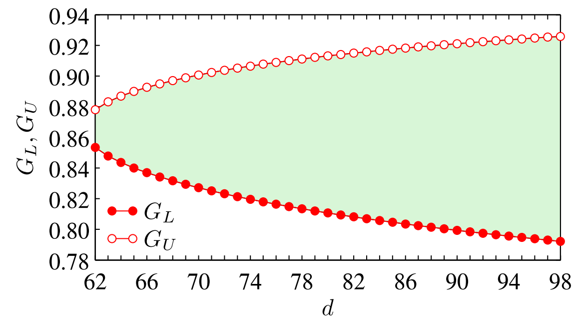

FIG. 6. The bounds G_L and G_U versus d for the scenario OS1 with an optimal pointer. When G locates in the green shaded region, the maximum entanglement can be shared by two Alices and Bob.

value of G_U is still 1. This shows that the sequential sharing of entanglement is possible even using weak measurements with near-highest precision (i.e., very strong but not ideal strong measurement). A similar phenomenon for nonlocality sharing was reported previously [15,19].

V. CONCLUSION

We explored entanglement sharing from the perspective of entanglement-assisted EUR, and this enabled us to consider a general two-qudit state which shows more fruitful characteristics compared with the two-qubit case. We considered two different scenarios, that is, the scenario in which multiple Alices and a single Bob share the entangled pair and the scenario in which multiple Alices and Bobs share the entangled pair. For both the scenarios, we considered the weak measurements with different pointers. The results showed that for the unsharp and square pointers, at most two Alices could sequentially share the entanglement with one Bob (one-sided scenario) or two Bobs (two-sided scenario). For the optimal and Gaussian pointers, however, the maximum number of Alices sequentially sharing the entanglement with one Bob increased discontinuously with the increasing dimension d of each qudit for the one-sided scenario, while the maximum number of Alices and Bobs remains two for the two-sided scenario. We also obtained the critical measurement precision higher than that which the entanglement could be shared, and it was found that it always decreased with the increasing d and approached a finite asymptotic value when d approached infinity.

By further considering the isotropic states, we exemplified that it is also possible to achieve the sequential sharing of nonmaximal entanglement, provided that it is strong enough. However, for a general nonmaximally entangled state, further study is still needed to clarify the interplay between the amount of entanglement and the number of observers sharing the entanglement. Furthermore, as the unsharp measurement is optimal (in the sense of achieving the highest precision for a given disturbance) only for $d = 2$, it is also of fundamental significance to explore what forms the optimal pointers take for general weak measurements on high-dimensional systems.

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[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[2] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[3] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. **69**, 2881 (1992).
[4] C. H. Bennett, D. P. DiVincenzo, P. W. Shor, J. A. Smolin, B. M. Terhal, and W. K. Wootters, Phys. Rev. Lett. **87**, 077902 (2001).
[5] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[6] M. Genovese, Phys. Rep. **413**, 319 (2005).
[7] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).
[8] D. Cavalcanti and P. Skrzypczyk, Rep. Prog. Phys. **80**, 024001 (2017).
[9] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, Rev. Mod. Phys. **92**, 015001 (2020).
[10] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, Rev. Mod. Phys. **84**, 1655 (2012).
[11] A. Streltsov, G. Adesso, and M. B. Plenio, Rev. Mod. Phys. **89**, 041003 (2017).
[12] M. L. Hu, X. Hu, J. C. Wang, Y. Peng, Y. R. Zhang, and H. Fan, Phys. Rep. **762-764**, 1 (2018).
[13] R. Silva, N. Gisin, Y. Guryanova, and S. Popescu, Phys. Rev. Lett. **114**, 250401 (2015).
[14] S. Mal, A. S. Majumdar, and D. Home, Mathematics **4**, 48 (2016).
[15] C. Ren, T. Feng, D. Yao, H. Shi, J. Chen, and X. Zhou, Phys. Rev. A **100**, 052121 (2019).
[16] D. Das, A. Ghosal, S. Sasmal, S. Mal, and A. S. Majumdar, Phys. Rev. A **99**, 022305 (2019).
[17] M. J. Hu, Z. Y. Zhou, X. M. Hu, C. F. Li, G. C. Guo, and Y. S. Zhang, npj Quantum Inf. **4**, 63 (2018).
[18] M. Schiavon, L. Calderaro, M. Pittaluga, G. Vallone, and P. Villoresi, Quantum Sci. Technol. **2**, 015010 (2017).
[19] T. Feng, C. Ren, Y. Tian, M. Luo, H. Shi, J. Chen, and X. Zhou, Phys. Rev. A **102**, 032220 (2020).
[20] P. J. Brown and R. Colbeck, Phys. Rev. Lett. **125**, 090401 (2020).
[21] T. Zhang and S. M. Fei, Phys. Rev. A **103**, 032216 (2021).
[22] S. Saha, D. Das, S. Sasmal, D. Sarkar, K. Mukherjee, A. Roy, and S. S. Bhattacharya, Quantum Inf. Process. **18**, 42 (2019).
[23] S. Sasmal, D. Das, S. Mal, and A. S. Majumdar, Phys. Rev. A **98**, 012305 (2018).
[24] A. Shenoy H., S. Designolle, F. Hirsch, R. Silva, N. Gisin, and N. Brunner, Phys. Rev. A **99**, 022317 (2019).
[25] D. Yao and C. Ren, Phys. Rev. A **103**, 052207 (2021).
[26] X. H. Han, Y. Xiao, H. C. Qu, R. H. He, X. Fan, T. Qian, and Y. J. Gu, Quantum Inf. Process. **20**, 278 (2021).
[27] Y.-H. Choi, S. Hong, T. Pramanik, H.-T Lim, Y.-S. Kim, H Jung, S.-W. Han, S. Moon, and Y.-M. Cho, Optica **7**, 675 (2020).
[28] A. Bera, S. Mal, A. Sen(De), and U. Sen, Phys. Rev. A **98**, 062304 (2018).
[29] S. Datta and A. S. Majumdar, Phys. Rev. A **98**, 042311 (2018).
[30] M. L. Hu, J. R. Wang, and H. Fan, Sci. China Phys. Mech. Astron. **65**, 260312 (2022).
[31] S. Cheng, L. Liu, T. J. Baker, and M. J. W. Hall, Phys. Rev. A **104**, L060201 (2021).
[32] S. Cheng, L. Liu, T. J. Baker, and M. J. W. Hall, Phys. Rev. A **105**, 022411 (2022).
[33] J. Zhu, M. J. Hu, C. F. Li, G. C. Guo, and Y. S. Zhang, Phys. Rev. A **105**, 032211 (2022).
[34] A. K. Das, D. Das, S. Mal, D. Home, and A. S. Majumdar, Quantum Inf. Process. **21**, 381 (2022).
[35] D. Das, A. Ghosal, A. G. Maity, S. Kanjilal, and A. Roy, Phys. Rev. A **104**, L060602 (2021).
[36] C. Ren, X. Liu, W. Hou, T. Feng, and X. Zhou, Phys. Rev. A **105**, 052221 (2022).
[37] W. Hou, X. Liu, and C. Ren, Phys. Rev. A **105**, 042436 (2022).
[38] S. S. Mahato and A. K. Pan, Phys. Rev. A **106**, 042218 (2022).
[39] J. H. Wang, Y. J. Wang, L. J. Wang, and Q. Chen, Phys. Rev. A **106**, 052412 (2022).
[40] T. Zhang, N. Jing, and S. M. Fei, Front. Phys. **18**, 31302 (2023).
[41] P. J. Coles, M. Berta, M. Tomamichel, and S. Wehner, Rev. Mod. Phys. **89**, 015002 (2017).
[42] H. P. Robertson, Phys. Rev. **34**, 163 (1929).
[43] D. Wang, F. Ming, M. L. Hu, and L. Ye, Ann. Phys. (Leipzig) **531**, 1900124 (2019).
[44] M. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner, Nat. Phys. **6**, 659 (2010).
[45] C. F. Li, J. S. Xu, X. Y. Xu, K. Li, and G. C. Guo, Nat. Phys. **7**, 752 (2011).
[46] R. Prevedel, D. R. Hamel, R. Colbeck, K. Fisher, and K. J. Resch, Nat. Phys. **7**, 757 (2011).
[47] M. L. Hu and H. Fan, Phys. Rev. A **86**, 032338 (2012).
[48] A. K. Pati, M. M. Wilde, A. R. Usha Devi, A. K. Rajagopal, and Sudha, Phys. Rev. A **86**, 042105 (2012).
[49] M. L. Hu and H. Fan, Phys. Rev. A **87**, 022314 (2013).
[50] M. L. Hu and H. Fan, Phys. Rev. A **88**, 014105 (2013).
[51] U. Singh, A. K. Pati, and M. N. Bera, Mathematics **4**, 47 (2016).
[52] X. Yuan, G. Bai, T. Peng, and X. Ma, Phys. Rev. A **96**, 032313 (2017).
[53] P. J. Coles, L. Yu, V. Gheorghiu, and R. B. Griffiths, Phys. Rev. A **83**, 062338 (2011).
[54] R. L. Frank and E. H. Lieb, Commun. Math. Phys. **323**, 487 (2013).
[55] O. Gühne, P. Hyllus, D. Bruß, A. Ekert, M. Lewenstein, C. Macchiavello, and A. Sanpera, Phys. Rev. A **66**, 062305 (2002).
[56] A. O. Pittenger and M. H. Rubin, Phys. Rev. A **67**, 012327 (2003).
[57] N. J. Cerf and C. Adami, Phys. Rev. Lett. **79**, 5194 (1997).
[58] I. Devetak and A. Winter, Proc. R. Soc. A. **461**, 207 (2005).
[59] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, NJ, 2018).

[60] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
[61] I. M. Duck, P. M. Stevenson, and E. C. G. Sudarshan, Phys. Rev. D **40**, 2112 (1989).
[62] W. K. Wooters, Found. Phys. **16**, 391 (1986).
[63] W. K. Wooters and B. D. Fields, Ann. Phys. (NY) **191**, 363 (1989).
[64] P. Busch, P. Lahti, and P. Mittelstaedt, *The Quantum Theory of Measurement* (Springer, Berlin, 1996).
[65] P. Busch, Phys. Rev. D **33**, 2253 (1986).
[66] B. M. Terhal and Karl Gerd H. Vollbrecht, Phys. Rev. Lett. **85**, 2625 (2000).
[67] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).