Physical realization of realignment criteria using the structural physical approximation

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Entanglement detection is an important problem in quantum information theory because quantum entanglement is a key resource in quantum information processing. Realignment criterion is a powerful tool for the detection of entangled states in bipartite and multipartite quantum systems. It works well not only for negative-partial-transpose entangled states (NPTESs) but also for positive-partial-transpose entangled states (PPTESs). Since the matrix corresponding to the realignment map is indefinite, the experimental implementation of the map is an obscure task. In this work, first, we approximate the realignment map to a positive map using the method of structural physical approximation, and then we show that the structural physical approximation of the realignment map (SPA-R) is completely positive. Positivity of the constructed map is characterized using moments which can be physically measured. Next, we develop a separability criterion based on our SPA-R map in the form of an inequality and show that the developed criterion not only detect NPTESs but also PPTESs. Further, we show that for a special class of states called Schmidt-symmetric states, the SPA-R separability criteria reduce to the original form of the realignment criteria. We provide some examples to support the results obtained. Moreover, we analyze the error that may occur because of approximating the realignment map.

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I. INTRODUCTION

Entanglement [1] is a key ingredient in quantum physics and the future of quantum technologies. It has advantages in various quantum information processing tasks such as quantum communication [2–4], quantum computation [5], remote state preparation [6], and quantum simulation [7], and thus, detection of entanglement is an important problem in quantum information theory. Detection of entanglement is also important because even if an experiment is carried out to generate an entangled state in a bipartite or multipartite quantum system, the generated state may not be entangled due to the presence of noise in the environment, and it is quite difficult to check whether the generated state is entangled or not. Despite much effort, a complete solution for the separability problem is still not known. Positive maps are strong detectors of entanglement. However, not every positive map can be regarded as physical; for example, in the case of describing a quantum channel or the reduced dynamics of an open system, a stronger positivity condition is required [8]. Completely positive maps play an important role in quantum information theory since a positive map is physical whenever it is completely positive. Completely positive maps were introduced by Stinespring in the study of dilation problems for operators [9]. Let $B(\mathcal{H}_A)$ and $B(\mathcal{H}_B)$ denote the set of bounded operators on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. If the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ has dimension k, we identify $B(\mathcal{H})$ with $M_k(\mathbb{C})$, the space of $k \times k$ matrices in $\mathbb{C}$. A linear map $\Phi : B(\mathcal{H}_A) \longrightarrow B(\mathcal{H}_B)$ is positive if $\Phi(\rho)$ is positive for each positive $\rho \in B(\mathcal{H}_A)$. The map Φ is completely positive if, for each positive integer k, the map $I_k \otimes \Phi : M_k(B(\mathcal{H}_A)) \longrightarrow M_k(B(\mathcal{H}_B))$ is positive. In quantum information theory, completely positive maps are important because they are used to characterize quantum operations [5]. Choi described the operator sum representation of completely positive maps [10]. In [11], necessary and sufficient conditions for the existence of completely positive maps are given. A physical way by which positive maps can be approximated by completely positive maps is called structural physical approximation (SPA) [12–17]. The idea is to mix a positive map Φ with a maximally mixed state, making the mixture $\tilde{\Phi}$ completely positive [12]. The resulting map can then be physically realized in a laboratory, and its action characterizes entanglement of the states detected by Φ. In addition, the resulting map keeps the structure of the output of the nonphysical map Φ since the direction of the generalized Bloch vector of the output state remains the same as the output state of the original nonphysical map; only the length of the vector is rescaled by some factor [18]. The SPA to the map Φ in $(d \otimes d)$-dimensional space is given by

$$\tilde{\Phi}(\rho) = \frac{p^*}{d^2} I_{d^2} + (1 - p^*)\Phi(\rho), \tag{1}$$

where I_{d^2} denotes the identity matrix of order d^2 and p^* is the minimum value of the probability p for which the approximated map $\tilde{\Phi}$ is completely positive [19].

Although various methods exist in the literature for the detection of entangled states, the first solution to this problem is connected to the theory of positive maps. It was proposed by Peres in the form of partial-transposition (PT) criteria [20]. Later, Horodecki proved that these criteria are necessary and sufficient for $(2 \otimes 2)$- and $(2 \otimes 3)$-dimensional quantum

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systems [21]. Although these criteria are some of the most important and widely used criteria, they suffer from serious drawbacks. One of the major drawback is that they are based on the negative eigenvalues of the partially transposed matrix and thus used to detect only negative-partial-transpose entangled states (NPTESs). Another drawback is that the partial transposition map is positive but not a completely positive map and hence may not be implemented in an experiment. In order to make it experimentally implementable, a partial-transposition map was approximated to a completely positive map using the method of SPA [12]. A lot of work was done on the structural physical approximation of partial transposition (SPA-PT) [12,19,22–24]. The SPA-PT has been used to detect and quantify entanglement [12,25], but until now, it could be used to detect and quantify only NPTESs.

As the positive partial transpose (PPT) criterion fails to detect bound entanglement in higher dimensions, certain other criteria have been proposed in the literature which can detect some positive-partial-transpose entangled states (PPTESs). These include the computable cross norm or realignment criterion (CCNR) [26,27], range criterion [28], and covariance-matrix criterion [29]. Moreover, it has been shown that the PPT criterion and the CCNR criterion are equivalent under permutations of the density matrix's indices [30]. The generalization of the CCNR criterion was investigated in [31]. The symmetric function of Schmidt coefficients was used to improve the CCNR criterion in [32]. Separability criteria based on the realignment of density matrices and reduced density matrices were proposed in [33]. In [34,35], witness operators using the realignment map were constructed which efficiently detect and quantify PPT entangled states. In [36], the rank of the realigned matrix was used to obtain necessary and sufficient product criteria for quantum states. Recently, methods for detecting bipartite entanglement based on estimating moments of the realignment matrix were proposed [37,38]. Realignment criteria are powerful criteria in the sense that they may be used to detect NPTESs as well as PPTESs. PPTESs are also known as bound entangled states, which are weak entangled states that cannot be distilled by performing local operations and classical communications (LOCC). Although they are some of the best for the detection of PPTESs, the problem with these criteria is that they may not be used to detect entanglement practically because the realignment map corresponds to a nonpositive map and it is known that nonpositive maps are not experimentally implementable. Also, it is known that completely positive maps may be realized in an experiment [39]. The defect that the realignment map may not be realized in an experiment may be overcome by approximating the nonpositive realignment map to a completely positive map. Our work is significant because although there has been considerable progress in entanglement detection using the SPA of a partial-transposition map, the idea of SPA of the realignment operation has still not been explored.

In this work we approximate the nonpositive realignment map to a completely positive map. To achieve this goal, we first approximate the nonpositive realignment map with a positive map, and then we show that the obtained positive map is also completely positive. We estimate the eigenvalues of the realignment matrix using moments which may be used physically in an experiment [40–43]. Further, we formulate separability criteria that we call SPA-R criteria, using our approximated map that detects not only negative partial transpose (NPT) entangled states but also PPT entangled states. Next, we show that the SPA-R criteria reduce to the original formulation of realignment criteria for a class of states called Schmidt-symmetric states. Moreover, we discuss the accuracy of our approximated realignment (SPA-R) map by calculating the error of the approximation in the trace norm. We also introduce an error inequality which holds for all separable states.

This paper is organized as follows: In Sec. II, we revisit the realignment criteria and review some preliminary results that we will use in later sections. In Sec. III, we approximate the nonpositive realignment map to a positive map, and further, we show that the approximated positive map is completely positive. In Sec. IV, we develop our separability criteria, called SPA-R criteria, based on the approximated realignment map. Furthermore, we show that the SPA-R criteria and the original form of the realignment criteria become the same for Schmidt-symmetric states. In Sec. V, we investigate the error generated due to the approximation procedure. In Sec. VI, we illustrate some examples to support the results obtained in this work. In Sec. VII, we discuss the efficiency of SPA-R criteria. Finally, we conclude in Sec. VIII.

II. PRELIMINARIES

In this section, we give the realignment criteria and some results which are discussed in the literature. We will use these results in the subsequent section to obtain the modified form of the realignment criteria that may be realizable in an experiment.

A. Realignment criteria

First, let us recall the definition of the realignment operation. For any $m \times m$ block matrix X, with each block X_{ij} of size $n \times n$, $i, j = 1, \ldots, m$, the realigned matrix $R(X)$ is defined by

$$R(X) = [\text{vec}(X_{11}), \ldots, \text{vec}(X_{m1}), \ldots, \text{vec}(X_{1m}), \ldots, \text{vec}(X_{mm})]^t, \quad (2)$$

where, for any $n \times n$ matrix X_{ij} with entries x_{ij}, $\text{vec}(X_{ij})$ is defined as

$$\text{vec}(X_{ij}) = [x_{11}, \ldots, x_{n1}, x_{12} \ldots, x_{n2}, \ldots, x_{1n}, \ldots, x_{nn}]^t. \quad (3)$$

Let us consider a bipartite quantum system described by a density operator ρ in an $(H_A^{d_1} \otimes H_B^{d_2})$-dimensional quantum system. The density operator ρ may be expressed as

$$\rho = \sum_{i,j,k,l} p_{ij,kl} |ij\rangle\langle kl|, \quad (4)$$

where d_1 and d_2 are the dimensions of the Hilbert spaces H_A and H_B, respectively. After applying the realignment operation on ρ, the realigned matrix $R(\rho)$ may be expressed as

$$R(\rho) = \sum_{i,j,k,l} p_{ij,kl} |ik\rangle\langle jl|. \quad (5)$$

Then the realignment criteria may be stated as follows: If ρ represents a separable state, then $\|R(\rho)\|_1 \leqslant 1$. Here $\| \cdot \|_1$

denotes the trace norm, and it may be defined as $\|T\|_1 = \mathrm{Tr}(\sqrt{TT^\dagger})$ [26].

B. A few well-known results

In this section, we mention a few important results that may be required in the following section. To proceed, we employ a useful theorem by Weyl [44] that connects the eigenvalues of the sum of the Hermitian matrices to those of the individual matrices. We use this theorem to prove the positivity of our approximated map. For convenience, Weyl's theorem can be stated as follows.

Result 1. Weyl's inequality [44]. Let $A, B \in M_n$ be two Hermitian matrices, and let $\{\lambda_i[A]\}_{i=1}^n$, $\{\lambda_i[B]\}_{i=1}^n$, and $\{\lambda_i[A+B]\}_{i=1}^n$ be the eigenvalues of A, B, and $A+B$, respectively, arranged in ascending order, i.e., $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. Then,

$$\lambda_k[A+B] \leqslant \lambda_{k+j}[A] + \lambda_{n-j}[B], \quad j = 0, \ldots, n-k, \tag{6}$$

$$\lambda_{k-j+1}[A] + \lambda_j[B] \leqslant \lambda_k[A+B], \quad j = 1, \ldots, k. \tag{7}$$

It may not be an easy task to directly compute the eigenvalues of a matrix; thus, bounds for eigenvalues are of great importance. Bounds for eigenvalues using were studied in [45]. Further, the bound of the eigenvalues expressed in terms of moments may be useful for the experimentalist to estimate eigenvalues in the laboratory. We now state the result [45] given below that determines a lower bound for the minimum eigenvalue of a matrix in terms of first- and second-order moments of the matrix. We will use Result 2 in the subsequent section to prove the positivity of our approximated map.

Result 2 [45]. Let $A \in M_n(\mathbb{C})$ be any matrix with real eigenvalues and $\lambda_{\min}^{\mathrm{lb}}[A]$ denote the lower bound of the minimum eigenvalue of A. Then,

$$\lambda_{\min}^{\mathrm{lb}}[A] \leqslant \lambda_{\min}[A] \tag{8}$$

where the lower bound is given by

$$\lambda_{\min}^{\mathrm{lb}}[A] = \frac{\mathrm{Tr}[A]}{n} - \sqrt{(n-1)\left[\frac{\mathrm{Tr}[A^2]}{n} - \left(\frac{\mathrm{Tr}[A]}{n}\right)^2\right]}. \tag{9}$$

The useful conditions for the existence of completely positive maps were studied in [11]. We have exploited the conditions to prove the complete positivity of the introduced approximated map. The conditions are expressed as Result 3.

Result 3 [11]. Consider a map $\Phi : M_n(\mathbb{C}) \longrightarrow M_m(\mathbb{C})$. Let $A \in M_n$ and $B \in M_m$ be Hermitian matrices such that $\Phi(A) = B$. Then the map Φ is completely positive iff there exist non-negative real numbers γ_1 and γ_2 such that the following conditions hold:

$$\lambda_{\min}[B] \geqslant \gamma_1 \lambda_{\min}[A], \tag{10}$$

$$\lambda_{\max}[B] \leqslant \gamma_2 \lambda_{\max}[A]. \tag{11}$$

III. STRUCTURAL PHYSICAL APPROXIMATION OF THE REALIGNMENT MAP: POSITIVITY AND COMPLETE POSITIVITY

In this section, we employ the method of structural physical approximation to approximate the realignment map. To proceed toward our aim, let us first recall the depolarizing map, which may be defined in the following way: A map $\Phi_d : M_n \longrightarrow M_n$ is said to be depolarizing if

$$\Phi(A) = \frac{\mathrm{Tr}[A]}{n} I_n. \tag{12}$$

In the method of structural physical approximation, we mix an appropriate proportion of the realignment map with a depolarizing map in such a way that the resulting map will be positive. This can happen because the lowest negative eigenvalues generated by the realignment map can be offset by the eigenvalues of the maximally mixed state generated by the depolarizing map.

Consider any quantum state ρ in a $(d \otimes d)$-dimensional system $\mathcal{D} \subset \mathcal{H}_A \otimes \mathcal{H}_B$ such that $\mathcal{D}$ contains the states ρ whose realignment matrix $R(\rho)$ has real eigenvalues and positive trace. The structural physical approximation of the realignment map may be defined as $\widetilde{R} : M_{d^2}(\mathbb{C}) \longrightarrow M_{d^2}(\mathbb{C})$ such that

$$\widetilde{R}(\rho) = \frac{p}{d^2} I_{d\otimes d} + \frac{(1-p)}{\mathrm{Tr}[R(\rho)]} R(\rho), \quad 0 \leqslant p \leqslant 1. \tag{13}$$

A. Positivity of the structural physical approximation of the realignment map

It is known that $R(\rho)$ forms an indefinite matrix; its eigenvalues may be negative or positive. Let us first consider the case when all the eigenvalues of $R(\rho)$ are non-negative. By the definition of $\widetilde{R}$ given in (13), $\widetilde{R}(\rho)$ is positive for all $p \in [0, 1]$, and hence, $\widetilde{R}$ defines a positive map. On the other hand, if $R(\rho)$ has negative eigenvalues, then $\widetilde{R}(\rho)$ may be positive under some conditions. But since the realignment operation is not physically realizable, it is not feasible to compute the eigenvalues of $R(\rho)$. To overcome this challenge, we find the range of p in terms of $\lambda_{\min}^{\mathrm{lb}}[R(\rho)]$ defined in (9), which can be expressed in terms of $\mathrm{Tr}[R(\rho)]$ and $\mathrm{Tr}\{[R(\rho)]^2\}$. The first and second moments of $R(\rho)$ may be measured experimentally [42]. Now the problem is how to determine the sign of the real eigenvalues of $R(\rho)$ experimentally without directly computing its eigenvalues. The method we develop here to tackle this problem is described below in detail.

1. *Method for determining the sign of real eigenvalues of* $R(\rho)$

Let $\rho \in \mathcal{D}$ be a $(d \otimes d)$-dimensional state such that $R(\rho)$ has real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{d^2}$. The characteristic polynomial of $R(\rho)$ is given as

$$f(x) = \prod_{i=1}^{d^2} (x - \lambda_i) = \sum_{k=0}^{d^2} (-1)^k a_k x^{d^2-k}, \tag{14}$$

where $a_0 = 1$ and $\{a_k\}_{k=1}^{d^2}$ are the functions of the eigenvalues of $R(\rho)$.

Let us now consider the polynomial $f(-x)$, which effectively replaces the positive eigenvalues of $R(\rho)$ by negative ones and vice versa. For a polynomial with real roots, Descartes's rule of sign states that the number of positive roots is given by the number of sign changes between consecutive elements in the ordered list of its nonzero coefficients [46]. The matrix $R(\rho)$ is positive semidefinite iff the number of sign changes in the ordered list of nonzero coefficients of $f(x)$ is equal to the degree of the polynomial $f(x)$. These nonzero coefficients can be determined in terms of moments of the matrix $R(\rho)$. The coefficients a_i are related to the moments of $R(\rho)$ by the recursive formula [47]

$$a_k = \frac{1}{k}\sum_{i=0}^{k}(-1)^{i-1}a_{k-i}m_i(R(\rho)), \tag{15}$$

where $m_i(R(\rho)) = \text{Tr}\{[R(\rho)]^i\}$ denotes the ith-order moment of the matrix $R(\rho)$. For convenience, we write $m_i(R(\rho))$ as m_i. The ith-order moment can be explicitly expressed as

$$m_i = (-1)^{i-1}ia_i + \sum_{k=1}^{i-1}(-1)^{i-1+k}a_{i-k}m_k. \tag{16}$$

Using (15), we get

$$a_1 = m_1, \tag{17}$$

$$a_2 = \tfrac{1}{2}\left(m_1^2 - m_2\right), \tag{18}$$

$$a_3 = \tfrac{1}{6}\left(m_1^3 - 3m_1m_2 + 2m_3\right), \tag{19}$$

and so on.

Therefore, the matrix $R(\rho)$ is positive semidefinite iff $a_i \geqslant 0$ for all $i = 1, \dots, d^2$.

2. Positivity of $\widetilde{R}(\rho)$

In this section, we derive the condition for which the approximated map $\tilde{R}(\rho)$ will be positive when (i) $R(\rho)$ is positive and (ii) $R(\rho)$ is indefinite. The obtained conditions are stated in the following theorem.

Theorem 1. Let ρ be a $(d \otimes d)$-dimensional state such that its realignment matrix $R(\rho)$ has real eigenvalues. The structural physical approximation of the realignment map $\widetilde{R}(\rho)$ is a positive operator for $p \in [l, 1]$, where l is given by

$$l = \begin{cases} 0 & \text{for } \lambda_{\min}[R(\rho)] \geqslant 0, \\ \frac{d^2k}{\text{Tr}[R(\rho)]+d^2k} \leqslant p \leqslant 1 & \text{for } \lambda_{\min}[R(\rho)] < 0, \end{cases} \tag{20}$$

where $k = \max[0, -\lambda_{\min}^{\text{lb}}[R(\rho)]]$ and $\lambda_{\min}^{\text{lb}}[R(\rho)]$ denotes the lower bound of the minimum eigenvalue of $R(\rho)$ defined in (9).

Proof. Recalling the definition (13) of the SPA of the realignment map, the minimum eigenvalue of $\widetilde{R}(\rho)$ is given by

$$\lambda_{\min}[\widetilde{R}(\rho)] = \lambda_{\min}\left[\frac{p}{d^2}I_{d\otimes d} + \frac{(1-p)}{\text{Tr}[R(\rho)]}R(\rho)\right], \tag{21}$$

where $\lambda_{\min}(\cdot)$ denote the minimum eigenvalue of $[\cdot]$. Using Weyl's inequality given in (7) on the right-hand side of (21), it reduces to

$$\begin{aligned}\lambda_{\min}[\widetilde{R}(\rho)] &\geqslant \lambda_{\min}\left[\frac{p}{d^2}I_{d\otimes d}\right] + \lambda_{\min}\left[\frac{(1-p)}{\text{Tr}[R(\rho)]}R(\rho)\right] \\ &= \frac{p}{d^2} + \frac{(1-p)}{\text{Tr}[R(\rho)]}\lambda_{\min}[R(\rho)].\end{aligned} \tag{22}$$

Now our task is to find the range of p for which $\widetilde{R}$ defines a positive map. Based on the sign of $\lambda_{\min}[R(\rho)]$, we consider the following two cases.

Case I. When $\lambda_{\min}[R(\rho)] \geqslant 0$, the right-hand side of the inequality in (22) is positive for every $0 \leqslant p \leqslant 1$, and hence, $\widetilde{R}(\rho)$ represents a positive map for all $p \in [0, 1]$.

Case II. If $\lambda_{\min}[R(\rho)] < 0$, then (22) may be rewritten as

$$\lambda_{\min}[\widetilde{R}(\rho)] \geqslant \frac{p}{d^2} + \frac{(1-p)}{\text{Tr}[R(\rho)]}\lambda_{\min}^{\text{lb}}[R(\rho)], \tag{23}$$

where $\lambda_{\min}^{\text{lb}}[R(\rho)]$ is given in (9) and may be reexpressed in terms of moments as

$$\lambda_{\min}^{\text{lb}}[R(\rho)] = \frac{m_1}{d^2} - \sqrt{(d^2-1)\left[\frac{m_2}{d^2} - \left(\frac{m_1}{d^2}\right)^2\right]}, \tag{24}$$

where $m_1 = \text{Tr}[R(\rho)]$ and $m_2 = \text{Tr}\{[R(\rho)]^2\}$.

Taking $\lambda_{\min}^{\text{lb}}[R(\rho)] = -k,\ \ k(>0) \in \mathbb{R}$, (23) reduces to

$$\lambda_{\min}[\widetilde{R}(\rho)] \geqslant \frac{p}{d^2} - k\frac{(1-p)}{\text{Tr}[R(\rho)]}. \tag{25}$$

Now, if we impose the condition on the parameter p as $p \geqslant \frac{d^2k}{\text{Tr}[R(\rho)]+d^2k} = l$, then $\lambda_{\min}[\widetilde{R}(\rho)] \geqslant 0$. Thus, combining the two above-discussed cases, we can say that the approximated map $\widetilde{R}(\rho)$ represents a positive map when (20) holds. Hence, the theorem is proved.

B. Complete positivity of the structural physical approximation of the realignment map

In order to show that the approximated map $\widetilde{R}(\rho)$ defined in (13) may be realized in an experiment, it is not enough to show that $\widetilde{R}(\rho)$ is positive; we also need to show that it is completely positive.

When $l \leqslant p \leqslant 1$, there exist non-negative real numbers γ_1 and γ_2 such that the following conditions hold:

$$\lambda_{\min}[\widetilde{R}(\rho)] \geqslant \gamma_1\lambda_{\min}[\rho], \tag{26}$$

$$\lambda_{\max}[\widetilde{R}(\rho)] \leqslant \gamma_2\lambda_{\max}[\rho]. \tag{27}$$

Hence, using Result 3, $\widetilde{R}(\rho)$ is a completely positive operator for $p \in [l, 1]$.

IV. DETECTION USING THE EXPERIMENTAL IMPLEMENTABLE FORM OF THE REALIGNMENT CRITERIA

In this section, we derive a separability condition for the detection of NPTESs and PPTESs that may be implemented in the laboratory. The separability condition obtained depends on the structural physical approximation of the realignment criterion, and thus, the condition may be termed the SPA-R criterion. We will then further identify a class of states known as Schmidt-symmetric states for which the SPA-R criterion is equivalent to the original form of the realignment criterion [30,31] and the weak form of the realignment criterion [48].

A. SPA-R criterion

We are now in a position to derive the laboratory-friendly (for clarification, see Appendix B) separability criterion that may detect the NPTESs and PPTESs. The proposed entanglement detection criterion is based on the structural physical approximation of the realignment criterion, and it is stated in the following theorem.

Theorem 2. If any quantum system described by a density operator ρ_{sep} in a $d \otimes d$ system is separable, then

$$\|\widetilde{R}(\rho_{sep})\|_1 \leqslant \frac{p\{\text{Tr}[R(\rho_{sep})]-1\}+1}{\text{Tr}[R(\rho_{sep})]} = \widetilde{R}(\rho_{sep})_{\text{UB}}. \quad (28)$$

Proof. Let us consider a two-qudit bipartite separable state described by the density matrix ρ_{sep}; then the approximated realignment map (13) may be recalled as

$$\widetilde{R}(\rho_{sep}) = \frac{p}{d^2} I_{d\otimes d} + \frac{1-p}{\text{Tr}[R(\rho_{sep})]} R(\rho_{sep}). \quad (29)$$

Taking the trace norm on both sides of (29) and using a triangular inequality on norm, it reduces to

$$\begin{aligned}\|\widetilde{R}(\rho_{sep})\|_1 &\leqslant \left\| \frac{p}{d^2} I_{d\otimes d} \right\|_1 + \left\| \frac{1-p}{\text{Tr}[R(\rho_{sep})]} R(\rho_{sep}) \right\|_1 \\ &= p + \frac{1-p}{\text{Tr}[R(\rho_{sep})]} \|R(\rho_{sep})\|_1. \quad (30)\end{aligned}$$

Since ρ_{sep} denotes a separable state, using the realignment criteria, we have $\|R(\rho_{sep})\|_1 \leqslant 1$ [26,27]. Therefore, (30) further reduces to

$$\begin{aligned}\|\widetilde{R}(\rho_{sep})\|_1 &\leqslant p + \frac{1-p}{\text{Tr}[R(\rho_{sep})]} \\ &= \frac{p\{\text{Tr}[R(\rho_{sep})]-1\}+1}{\text{Tr}[R(\rho_{sep})]}. \quad (31)\end{aligned}$$

Hence, Theorem 2 is proved.

Corollary 1. If, for any two-qudit bipartite state ρ, the inequality

$$\|\widetilde{R}(\rho)\|_1 > \frac{p\{\text{Tr}[R(\rho)]-1\}+1}{\text{Tr}[R(\rho)]} = \widetilde{R}(\rho)_{\text{UB}} \quad (32)$$

holds, then the state ρ is an entangled state.

We should note the important fact that $\widetilde{R}(\rho_{sep})_{\text{UB}}$ given in (28) and (32) depends on $\text{Tr}[R(\rho)]$, which can be considered the first moment of $R(\rho)$, and it may be measured in experiments [42] (see Appendix B).

Corollary 2. If, for any separable state $\rho_{sep}^{(1)}$, $\text{Tr}[R(\rho_{sep}^{(1)})] = 1$ holds, then (28) reduces to

$$\left\|\widetilde{R}\left(\rho_{sep}^{(1)}\right)\right\|_1 \leqslant 1. \quad (33)$$

B. Schmidt-symmetric states

Let us consider a class of states known as Schmidt-symmetric states, which may be defined as [48]

$$\rho_{sc} = \sum_i \lambda_i A_i \otimes A_i^*, \quad (34)$$

where A_i represent the orthonormal bases of the operator space and λ_i denote non-negative real numbers known as Schmidt coefficients.

We are considering this particular class of states because we will show in this section that the separability criteria using SPA-R map become equivalent to the original form of the realignment criteria for such a class of states. Hertz *et al.* [48] studied the Schmidt-symmetric states and proved that a bipartite state ρ_{sc} is Schmidt symmetric if and only if

$$\|R(\rho_{sc})\|_1 = \text{Tr}[R(\rho_{sc})]. \quad (35)$$

For any Schmidt-symmetric state described by the density operator ρ_{sc}, the realignment matrix $R(\rho_{sc})$ defines a positive semidefinite matrix. Hence, using Theorem 1, $\widetilde{R}(\rho_{sc})$ is positive $\forall\ p \in [0, 1]$. Also, using (26) and (27), $\widetilde{R}(\rho_{sc})$ can be shown to be a completely positive. To achieve the motivation of this section, let us start with the following lemma.

Lemma 1. For any Schmidt-symmetric state ρ_{sc},

$$\|\widetilde{R}(\rho_{sc})\|_1 = 1. \quad (36)$$

Proof. Let us recall (13), which may provide the structural physical approximation of the realignment of the Schmidt-symmetric state. The SPA-R of ρ_{sc} is denoted by $\widetilde{R}(\rho_{sc})$, and it is given by

$$\widetilde{R}(\rho_{sc}) = \frac{p}{d^2} I_{d\otimes d} + \frac{(1-p)}{\text{Tr}[R(\rho_{sc})]} R(\rho_{sc}). \quad (37)$$

Taking the trace norm on both sides and using the triangle inequality, we have

$$\|\widetilde{R}(\rho_{sc})\|_1 \leqslant p + \frac{(1-p)}{\text{Tr}[R(\rho_{sc})]} \|R(\rho_{sc})\|_1. \quad (38)$$

Using (35), the inequality (38) reduces to

$$\|\widetilde{R}(\rho_{sc})\|_1 \leqslant 1. \quad (39)$$

Again, using (13), the trace of the approximated map $\widetilde{R}(\rho_{sc})$ is given by

$$\text{Tr}[\widetilde{R}(\rho_{sc})] = \text{Tr}\left[\frac{p}{d^2} I_{d\otimes d} + \frac{(1-p)}{\text{Tr}[R(\rho_{sc})]} R(\rho_{sc})\right] = 1. \quad (40)$$

Moreover, it is known that the trace norm of an operator is greater than or equal to its trace. Therefore, applying this result to $\widetilde{R}(\rho_{sc})$, we get

$$\text{Tr}[\widetilde{R}(\rho_{sc})] \leqslant \|\widetilde{R}(\rho_{sc})\|_1. \quad (41)$$

Using (40), inequality (41) reduces to

$$\|\widetilde{R}(\rho_{sc})\|_1 \geqslant 1. \quad (42)$$

Both (39) and (B7) hold only when

$$\|\widetilde{R}(\rho_{sc})\|_1 = 1 \quad (43)$$

holds. Thus, Lemma 1 is proved.

We are now in a position to show that SPA-R criteria may reduce to the original form of the realignment criteria for Schmidt-symmetric states. It is expressed in the following theorem.

Theorem 3. For Schmidt-symmetric states, the SPA-R separability criterion reduces to the original form of the realignment criterion.

Proof. Let ρ_{sc}^{sep} be any separable Schmidt-symmetric state. The SPA-R separability criterion for ρ_{sc}^{sep} is given by

$$\left\|\widetilde{R}\left(\rho_{sc}^{sep}\right)\right\|_1 \leqslant \widetilde{R}\left(\rho_{sc}^{sep}\right)_{\text{UB}}. \quad (44)$$

Using (36), inequality (44) reduces to

$$\begin{aligned}\widetilde{R}(\rho_{sc}^{sep})_{UB} &= \frac{p\{\mathrm{Tr}[R(\rho_{sc}^{sep})]-1\}+1}{\mathrm{Tr}[R(\rho_{sc}^{sep})]} \geqslant 1\\ &\Rightarrow p\{\mathrm{Tr}[R(\rho_{sc}^{sep})]-1\}+1 \geqslant \mathrm{Tr}[R(\rho_{sc}^{sep})]\\ &\Rightarrow \mathrm{Tr}[R(\rho_{sc}^{sep})](p-1) \geqslant (p-1)\\ &\Rightarrow \mathrm{Tr}[R(\rho_{sc}^{sep})] \leqslant 1\\ &\Rightarrow \|R(\rho_{sc}^{sep})\|_1 \leqslant 1. \end{aligned} \tag{45}$$

The last step follows from (35). Hence, Theorem 3 is proved.

V. ERROR IN THE APPROXIMATED MAP

In this section, we study and analyze the error generated when $R(\rho)$ is approximated by its SPA. In the approximated map, we add an appropriate proportion of the maximally mixed state such that the approximated map has no negative eigenvalue. The error between the approximated map $\widetilde{R}(\rho)$ and the realignment map $R(\rho)$ may be calculated as

$$\begin{aligned}\|\widetilde{R}(\rho)-R(\rho)\|_1 &= \left\|\frac{p}{d^2}I_{d\otimes d}+\frac{(1-p)}{\mathrm{Tr}[R(\rho)]}R(\rho)-R(\rho)\right\|_1\\ &= \left\|\frac{p}{d^2}I_{d\otimes d}+\left[\frac{(1-p)}{\mathrm{Tr}[R(\rho)]}-1\right]R(\rho)\right\|_1.\end{aligned} \tag{46}$$

Using a triangular inequality for the trace norm, (46) reduces to

$$\|\widetilde{R}(\rho)-R(\rho)\|_1 \leqslant p+\frac{1-p-\mathrm{Tr}[R(\rho)]}{\mathrm{Tr}[R(\rho)]}\|R(\rho)\|_1. \tag{47}$$

Inequality (47) may be termed the error inequality. The error inequality holds for any two-qudit bipartite state.

Proposition 1. The equality relation

$$\|\widetilde{R}(\rho_{sep})-R(\rho_{sep})\|_1 = \frac{(1-p)\{1-\mathrm{Tr}[R(\rho_{sep})]\}}{\mathrm{Tr}[R(\rho_{sep})]} \tag{48}$$

holds for separable states described by the density operator ρ_{sep} such that $\|R(\rho_{sep})\|_1 = 1$.

Proof. The equality in (47) holds if and only if

$$\frac{p}{d^2}I_{d\otimes d} = \left[\frac{(1-p)}{\mathrm{Tr}[R(\rho)]}-1\right]R(\rho); \tag{49}$$

that is, the equality in (47) holds when the realigned matrix takes the form

$$R(\rho) = \frac{p\mathrm{Tr}[R(\rho)]}{1-p-\mathrm{Tr}[R(\rho)]}\frac{I}{d^2}, \quad 0 \leqslant p \leqslant 1. \tag{50}$$

Taking the trace norm, (50) reduces to

$$\|R(\rho)\|_1 = \frac{p\mathrm{Tr}[R(\rho)]}{1-p-\mathrm{Tr}[R(\rho)]}, \quad 0 \leqslant p \leqslant 1. \tag{51}$$

For separable states ρ_{sep}, (51) reduces to

$$1 = \frac{p\mathrm{Tr}[R(\rho)]}{1-p-\mathrm{Tr}[R(\rho)]}, \quad 0 \leqslant p \leqslant 1. \tag{52}$$

Simplifying (52), the values of p and $1-p$ may be expressed as

$$p = \frac{1-\mathrm{Tr}[R(\rho_{sep})]}{1+\mathrm{Tr}[R(\rho_{sep})]}, \quad 1-p = \frac{2\mathrm{Tr}[R(\rho_{sep})]}{1+\mathrm{Tr}[R(\rho_{sep})]}. \tag{53}$$

Substituting values of p and $1-p$ in (50), the realigned matrix for separable states $R(\rho_{sep})$ takes the form

$$R(\rho_{sep}) = \frac{1}{d^2}I. \tag{54}$$

Therefore, (54) holds only for separable states. This means that a separable state ρ_{sep} exists such that $\|\rho_{sep}\|_1 = 1$, for which the equality condition in the error inequality (47) holds.

Result 4. If any quantum system described by a density operator ρ in a $d\otimes d$ system is separable, then the error inequality is given by

$$\|\widetilde{R}(\rho)-R(\rho)\|_1 \leqslant \frac{(1-p)\{1-\mathrm{Tr}[R(\rho)]\}}{\mathrm{Tr}[R(\rho)]}. \tag{55}$$

Proof. Let us consider a separable state ρ_{sep}. Thus, we have $\|R(\rho_{sep})\|_1 \leqslant 1$. Therefore, the error inequality (47) reduces to

$$\begin{aligned}\|\widetilde{R}(\rho_{sep})-R(\rho_{sep})\|_1 &\leqslant p+\frac{1-p-\mathrm{Tr}[R(\rho_{sep})]}{\mathrm{Tr}[R(\rho_{sep})]}\\ &= \frac{(1-p)\{1-\mathrm{Tr}[R(\rho_{sep})]\}}{\mathrm{Tr}[R(\rho_{sep})]}.\end{aligned} \tag{56}$$

Hence, Result 4 is proved.

Corollary 3. If inequality (55) is violated by any bipartite $(d\otimes d)$-dimensional quantum state, then the state under investigation is entangled.

VI. ILLUSTRATIONS

Example 1. Consider the family of two-qubit states $\rho(r,s,t)$ discussed in [49]. For $r=\frac{1}{4}$ and $s=\frac{1}{2}$, the family is represented by

$$\rho_t = \frac{1}{2}\begin{pmatrix}\frac{5}{4} & 0 & 0 & t\\ 0 & 0 & 0 & 0\\ 0 & 0 & \frac{1}{4} & 0\\ t & 0 & 0 & \frac{1}{2}\end{pmatrix}. \tag{57}$$

ρ_t may be defined as a valid quantum state when $|t| \leqslant \frac{\sqrt{\frac{5}{2}}}{2} \approx 0.790569$. By the PPT criterion, ρ_t is entangled when $t \neq 0$. The realignment criteria detect the entangled states for $|t| > 0.116117$.

Using the prescription given in (13), we construct the SPA-R map $\widetilde{R}: M_4(\mathbb{C}) \longrightarrow M_4(\mathbb{C})$ as

$$\widetilde{R}(\rho_t) = \frac{p}{4}I_4+\frac{(1-p)}{\mathrm{Tr}[R(\rho_t)]}R(\rho_t), \tag{58}$$

where $0 \leqslant p \leqslant 1$.

Using Descartes's rule of sign, we find that $R(\rho_t)$ is positive semidefinite for $t \geqslant 0$ (detailed calculations are given in Appendix A).

Applying Theorem 1, it can be shown that the approximated map $\widetilde{R}(\rho_t)$ is positive as well as completely positive for $l \leqslant p \leqslant 1$, where

$$l = \begin{cases} p_1(t) & \text{if } -0.790569 \leqslant t < 0, \\ 0 & \text{if } 0 \leqslant t \leqslant 0.790569, \end{cases} \tag{59}$$

where

$$p_1(t) = \frac{2(13 - 24t + 8t^2) - \sqrt{3(67 - 112t + 64t^2)}}{(-5 + 4t)^2}. \tag{60}$$

Thus, the SPA-R map $\widetilde{R}(\rho_t)$, which is a completely positive map, may be suitable for detecting the entanglement in the family of states described by the density operator ρ_t. Now we apply our separability criterion discussed in Theorem 2 which involves the comparison of $\|\widetilde{R}(\rho_t)\|_1$ and the upper bound $\widetilde{R}(\rho_t)_{\mathrm{UB}}$ defined in (28). After a few steps of the simple calculation, we obtain

$$\|\widetilde{R}(\rho_t)\|_1 > \widetilde{R}(\rho_t)_{\mathrm{UB}} \tag{61}$$

for

$t \in (-0.790569, -0.665506]$ for $p_1(t) \leqslant p < p_2(t)$,
$t \in (0.116117, 0.125]$ for $0 \leqslant p < p_3(t)$,
$t \in (0.125, 0.790569]$ for $0 \leqslant p \leqslant 1$,

where

$$p_2(t) = \frac{(-91 - 48t - 64t^2) - \sqrt{u(t)}}{2(-7 + 48t)^2}, \tag{62}$$

$$p_3(t) = \frac{(14 - 128t + 64t^2)}{(7 - 80t + 128t^2)}. \tag{63}$$

The function $u(t)$ is given by $u = 8673 + 9632t - 8832t^2 - 6144t^3 + 4096t^4$. Thus, the inequality (28) is violated when $t > 0.116117$ and $t < -0.665506$, which implies that the state ρ_t is entangled for $t \in [-0.790569, -0.665506) \cup (0.116117, 0.790569]$.

A comparison of $\|\widetilde{R}(\rho_t)\|_1$ and $\widetilde{R}(\rho_t)_{\mathrm{UB}}$ for the two-qubit state ρ_t is studied in Fig. 1 for different ranges of t given in (61). From Fig. 1 it can be observed that inequality (61) holds for $t > 0.116117$, which implies that the entanglement of ρ_t is detected in this region.

Example 2. Consider the two-qutrit state defined in [50], which is described by the density operator

$$\rho_a = \frac{1}{5 + 2a^2} \sum_{i=1}^{3} |\psi_i\rangle\langle\psi_i|, \quad \frac{1}{\sqrt{2}} \leqslant a \leqslant 1, \tag{64}$$

where $|\psi_i\rangle = |0i\rangle - a|i0\rangle$ for $i = \{1, 2\}$ and $|\psi_3\rangle = \sum_{i=0}^{2} |ii\rangle$.

The state described by the density operator ρ_a is a NPTES [50]. Using the prescription given in (13), we construct the SPA-R map $\widetilde{R} : M_9(\mathbb{C}) \longrightarrow M_9(\mathbb{C})$ as

$$\widetilde{R}(\rho_a) = \frac{p}{9} I_9 + \frac{(1 - p)}{\mathrm{Tr}[R(\rho_a)]} R(\rho_a), \quad 0 \leqslant p \leqslant 1. \tag{65}$$

Using Descartes's rule of sign, we find that $R(\rho_a)$ is not a positive semidefinite operator. (A detailed calculation given in Appendix A). Using Theorem 1, the approximated map $\widetilde{R}(\rho_a)$ is positive as well as completely positive for $l_1 \leqslant p \leqslant 1$, where

$$l_1 = \frac{-1 + 15\sqrt{2}w + 6\sqrt{2}a^2 w}{3\sqrt{2}(5 + 2a^2)w},$$

$$w = \sqrt{\frac{1}{56 + 9a^2(5 + a^2)}}. \tag{66}$$

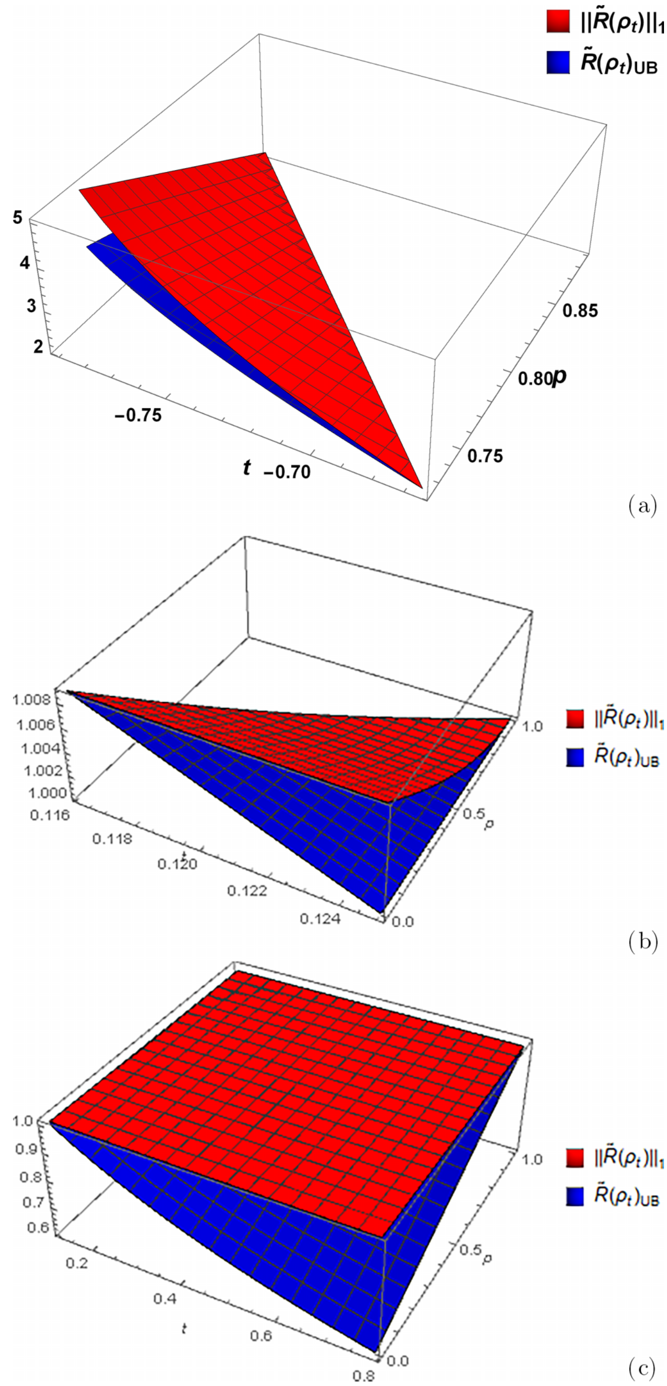

FIG. 1. A comparison between $\|\widetilde{R}(\rho_t)\|_1$ and $\widetilde{R}(\rho_t)_{\mathrm{UB}}$ for the two-qutrit state $\rho_{t>0}$ is displayed. In (a), one can observe that the inequality (28) obtained in Theorem 2 is violated when $-0.790569 \leqslant t < -0.665506$ for $p \in [p_1(t), p_2(t)]$, whereas in (b) the inequality is violated when $0.116117 < t \leqslant 0.125$ and p lies in the interval $[0, p_3(t))$. (c) shows the violation of inequality (28) when $t > 0.125$ and $0 \leqslant p \leqslant 1$.

Thus, the SPA-R map $\widetilde{R}(\rho_a)$ is suitable for detecting the entanglement in the state ρ_a experimentally. Now we apply our separability criterion discussed in Theorem 2, which involves a comparison of $\|\widetilde{R}(\rho_a)\|_1$ and the upper bound $\widetilde{R}(\rho_a)_{\mathrm{UB}}$

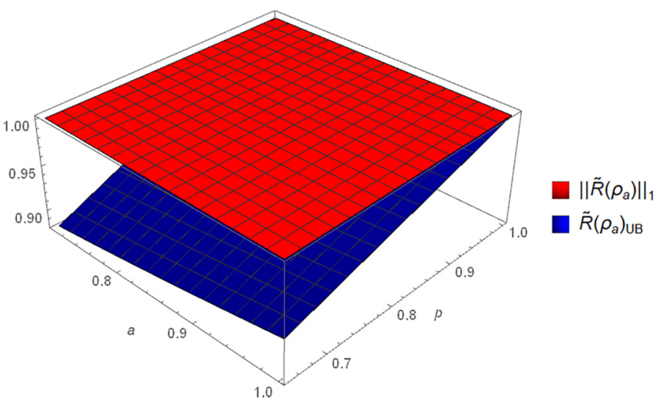

FIG. 2. A comparison between $\|\widetilde{R}(\rho_a)\|_1$ and $\widetilde{R}(\rho_a)_{\mathrm{UB}}$ for the two-qutrit state ρ_a is displayed. It is observed that the inequality (28) is violated for ρ_a in the whole range of a and for $p \in [l_1, 1]$

defined in (28). For $\frac{1}{\sqrt{2}} \leqslant a \leqslant 1$, we find that

$$\|\widetilde{R}(\rho_a)\|_1 > \widetilde{R}(\rho_a)_{\mathrm{UB}}. \tag{67}$$

The comparison of $\|\widetilde{R}(\rho_a)\|_1$ and $\widetilde{R}(\rho_a)_{\mathrm{UB}}$ for the two-qutrit state ρ_a is studied in Fig. 2. From Fig. 2, it is evident that inequality (28) obtained in Theorem 2 is violated. Thus, the state described by the density operator ρ_a is an entangled state.

Example 3. Let us consider a two-qutrit isotropic state described by the density operator ρ_β [51]:

$$\rho_\beta = \beta|\phi_+\rangle\langle\phi_+| + \frac{1-\beta}{9}I_9, \quad -\frac{1}{8} \leqslant \beta \leqslant 1, \tag{68}$$

where I_9 denotes an identity matrix of order 9 and the state $|\phi_+\rangle$ represents a Bell state in a two-qutrit system and may be expressed as

$$|\phi_+\rangle = \frac{1}{\sqrt{3}}(|11\rangle + |22\rangle + |33\rangle). \tag{69}$$

Using realignment criteria, the state ρ_β is an entangled state for $\frac{1}{3} < \beta \leqslant 1$. Using Descartes's rule of sign, we find that the realignment matrix $R(\rho_\beta)$ is positive semidefinite. A comparison between $\|\widetilde{R}(\rho_\beta)\|_1$ and $\widetilde{R}(\rho_\beta)_{\mathrm{UB}}$ is studied in Fig. 3. From Fig. 3, it can be observed that the inequality (28) is violated for $\frac{1}{3} < \beta \leqslant 1$ and $p \in [0, 1]$. Thus, using Theorem 2, the state described by the density operator ρ_β is an entangled state.

Example 4. An α state for $0 \leqslant \alpha \leqslant 1$ described by the density operator ρ_α may be defined as

$$\rho_\alpha = \frac{1}{8\alpha+1}\begin{pmatrix} \alpha & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & \alpha \\ 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & \alpha & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+\alpha}{2} & 0 & \frac{\sqrt{1-\alpha^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 \\ \alpha & 0 & 0 & 0 & \alpha & 0 & \frac{\sqrt{1-\alpha^2}}{2} & 0 & \frac{1+\alpha}{2} \end{pmatrix}. \tag{70}$$

It has been shown that this state is a PPTES for $0 < \alpha < 1$ [52]. Using Descartes's rule of sign, we find that the realignment matrix $R(\rho_\alpha)$ is positive semidefinite (see Appendix A for detailed calculations). Further, using Result 2, it can easily be shown that the SPA-R map $\widetilde{R}(\rho_\alpha)$ is completely positive for any $p \in [0, 1]$. It has been observed that inequality (28) is violated for different ranges of p for some values of α, which is shown in Table I. Thus, we have shown that the criterion given by Theorem 2 is violated by ρ_α, and thus, our criterion detects the bound entangled state given by (70).

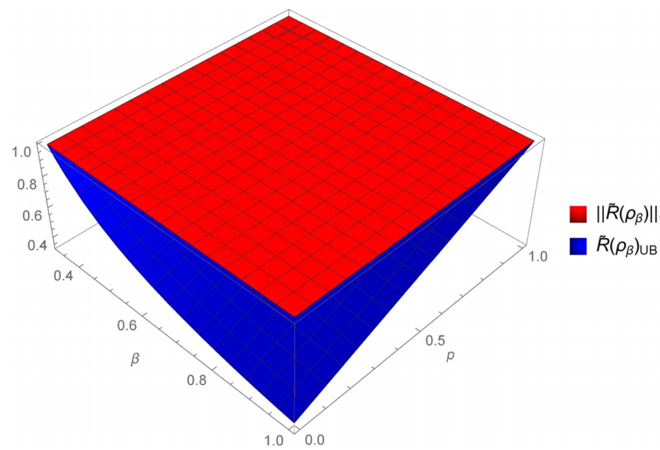

FIG. 3. A comparison between $\|\widetilde{R}(\rho_\beta)\|_1$ and $\widetilde{R}(\rho_\beta)_{\mathrm{UB}}$ for the two-qutrit state ρ_β is displayed. It is observed that the inequality (28) is violated for all values of $\beta \in (1/3, 1]$ and for any $p \in [0, 1]$.

VII. EFFICIENCY OF THE SPA-R CRITERION

In this section, we show how the SPA-R criterion is efficient in comparison to other entanglement detection criteria. In particular, we consider the following entanglement detection criteria for comparing the efficiency of the SPA-R criterion: (a) a separability criterion based on the realigned moment [37] and (b) a partially realigned moment criterion [38].

A. Comparing SPA-R and moment-based criterion (a)

To compare the SPA-R criterion with the moment-based criterion (a), we use Examples 1 and 4.

(i) Let us recall Example 1, in which the family of states is described by the density operator ρ_t. Interestingly, for this family of states, when $t > 0$, our SPA-R criteria detect

TABLE I. The range of the probability p for which inequality (28) is violated for different values of the state parameter α.

α	Range of p	Theorem 2
0.1	$0 \leqslant p \leqslant 0.019383$	Violated
0.2	$0 \leqslant p \leqslant 0.022143$	Violated
0.3	$0 \leqslant p \leqslant 0.021903$	Violated
0.4	$0 \leqslant p \leqslant 0.020444$	Violated
0.5	$0 \leqslant p \leqslant 0.018284$	Violated
0.6	$0 \leqslant p \leqslant 0.015611$	Violated
0.7	$0 \leqslant p \leqslant 0.012488$	Violated
0.8	$0 \leqslant p \leqslant 0.008904$	Violated
0.9	$0 \leqslant p \leqslant 0.004791$	Violated

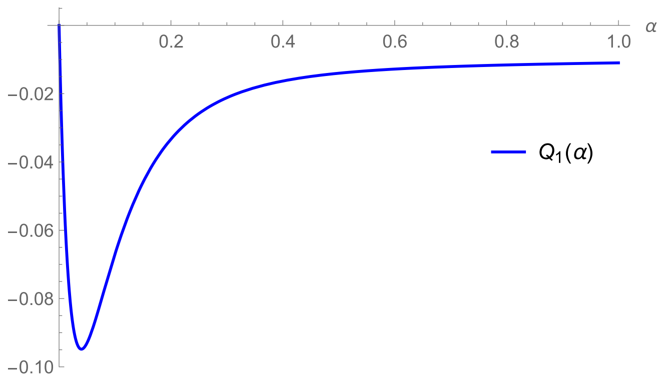

FIG. 4. The blue curve represents Q_1 for the state ρ_α, and the x axis depicts the state parameter α.

entanglement in the region $t \in (0.116117, 0.790569]$. But the realignment-moment-based criteria given in [37] detect the entangled state in the range $t \in (0.370992, 0.790569]$. Clearly, SPA-R criteria detect the NPTES ρ_t for $t > 0$ in a better range than the moment-based criterion (a).

(ii) Let us consider the bound entangled state (BES) studied in Example 4, which is described by the density operator ρ_α, $0 < \alpha < 1$. The realignment moment for a bipartite state ρ_α may be defined as [37]

$$r_k(R(\rho_\alpha)) = \mathrm{Tr}\{R(\rho_\alpha)[R(\rho_\alpha)]^\dagger\}^{k/2}, \quad k = 1, 2, 3, \ldots, n, \tag{71}$$

where n denotes the order of the matrix $R(\rho_\alpha)$.

The separability criterion based on realignment moments r_2 and r_3 may be stated as follows [37]: If a quantum state ρ_α is separable, then

$$Q_1 = [r_2(R(\rho_\alpha))]^2 - r_3(R(\rho_\alpha)) \leqslant 0. \tag{72}$$

$Q_1 > 0$ certifies that the given state is entangled.

Figure 4 shows that inequality (72) is not violated for the BES ρ_α in the whole range $0 < \alpha < 1$. Hence, the BES ρ_α is undetected by this realignment-moment-based criteria.

B. Comparing SPA-R and moment-based criterion (b)

Let us again recall Examples 1 and 4 to compare the SPA-R criterion with moment-based criterion (b).

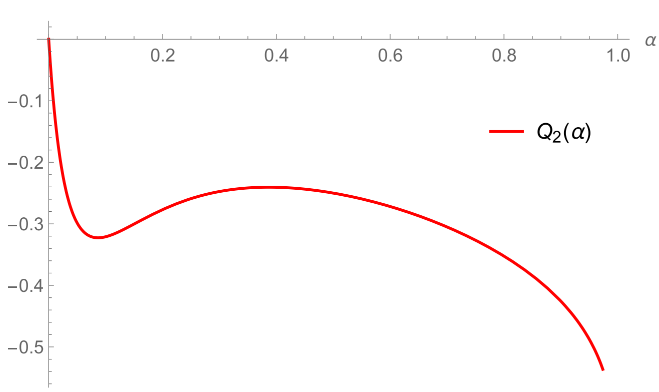

FIG. 5. The red curve represents Q_2 for the state ρ_α, and the x axis depicts the state parameter α.

(i) In Example 1, the family of states is described by the density operator ρ_t. By the R-moment criterion given in [38], ρ_t is detected when $t \in (0.214312, 0.790569] \subset (0.116117, 0.790569]$. Therefore, the SPA-R criteria detect more entangled states than the R-moment criterion.

(ii) Let us now consider the BES studied in Example 4. Applying the R-moment criterion [38] to the BES described by the density operator ρ_α, $0 < \alpha < 1$, we get

$$Q_2 \equiv 56 D_8^{1/8} + T_1 - 1 \leqslant 0 \ \ \forall\, \alpha \in (0, 1), \tag{73}$$

where $D_8 = \prod_{i=1}^{8} \sigma_i^2(\rho_\alpha)$ and $T_1 = \mathrm{Tr}[R(\rho_\alpha)]$. Here $\sigma_i(\rho_\alpha)$ represents the ith singular value of ρ_α. Since the above inequality is not violated for any α, the BES ρ_α is undetected by R-moment-based criteria. This is shown in Fig. 5.

VIII. CONCLUSION

To summarize, we have developed a separability criterion by approximating the realignment operation via structural physical approximation. Since the partial-transposition operation is limited to detect only NPTESs, we have studied here the realignment operation, which may detect both NPTESs and PPTESs. But since the realignment map is not a positive map and thus does not represent a completely positive map, it is difficult to implement it in a laboratory. Therefore, in order to make a realignment map completely positive, first, we approximated it to a positive map using the method of SPA, and then we showed that this approximated map is also completely positive. We showed that the positivity of the SPA-R map can be verified in an experiment because the lower bound of the fraction p can be expressed in terms of the first and second moments of the realignment matrix. Interestingly, we showed that the separability criterion derived in this work using the approximated (SPA-R) map detects NPT and PPT bipartite entangled states. Some examples were cited to support the result obtained in this work. Although there are other PPT criteria that may detect NPTESs and PPTESs, our result is interesting in the sense that it may be realized in an experiment. Our obtained results may be realized in an experiment, but to achieve this aim, we pay a price in terms of the short-range detection. This fact can be observed in Example 1, in which the range of the state parameter for the detection of the entangled state is smaller than the range obtained by the realignment operation (without approximation). We also analyzed the error that occurred during the structural physical approximation of the realignment map, and it was described by an inequality known as an error inequality. Last, we obtained an inequality which is satisfied by all bipartite $(d \otimes d)$-dimensional separable states, and the violation of this inequality guarantees that the state being probed is entangled. Interestingly, the SPA-R criteria coincide with the original realignment criteria for Schmidt-symmetric states.

ACKNOWLEDGMENTS

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APPENDIX A

1. Example 1

Consider the two-qubit state ρ_t defined in (57). The characteristic polynomial of the matrix $R(\rho_t)$ can be expressed as

$$f_1(x) = x^4 - a_1(t)x^3 + a_2(t)x^2 - a_3(t)x + a_4(t). \quad \text{(A1)}$$

Using (15), we get

$$a_1(t) = m_1 = t + \tfrac{7}{8}, \quad \text{(A2)}$$

$$a_2(t) = \tfrac{1}{2}\left(m_1^2 - m_2\right) = \tfrac{1}{32}(8t^2 + 28t + 5), \quad \text{(A3)}$$

$$a_3(t) = \tfrac{1}{6}\left(m_1^3 - 3m_1m_2 + 2m_3\right) = \tfrac{1}{32}(7t^2 + 5t), \quad \text{(A4)}$$

$$a_4(t) = \tfrac{1}{24}\left(m_1^4 - 6m_1^2m_2 + 8m_1m_3 + 3m_2^2 - 6m_4\right) = \tfrac{5}{128}t^2, \quad \text{(A5)}$$

where $m_k = \mathrm{Tr}\{[R(\rho_t)]^k\}$.

$R(\rho_t)$ is positive semidefinite iff $a_i(t) \geqslant 0$ for all $i = 1$ to 4. After simple calculation, we get

$$\begin{aligned} a_1(t) &> 0 \ \text{ for } \ t \in [-0.790569, 0.790569],\\ a_2(t) &\geqslant 0 \ \text{ for } \ t \in [-0.188751, 0.790569],\\ a_3(t) &\geqslant 0 \ \text{ for } \ t \in [-0.790569, -0.714286]\\ &\quad \cup [0, 0.790569],\\ a_4(t) &\geqslant 0 \ \text{ for } \ t \in [-0.790569, 0.790569]. \end{aligned}$$

From the above calculations, we observe the following.

Case 1. If $t \geqslant 0$, then all the coefficients of the characteristic polynomial $f_1(-x)$ are positive; that is, there is no sign change in the ordered list of coefficients of $f_1(-x)$. Thus, $R(\rho_t)$ has no negative eigenvalue for $t \geqslant 0$.

Case 2. If $t < 0$, then (i) $a_2(t) < 0$ for $t \in [-0.790569, -0.188751]$, and (ii) $a_3(t) < 0$ for $t \in [-0.714286, 0]$, Hence, for every t, at least one coefficient of $f_1(x)$ is negative. Hence, $R(\rho_t)$ has at least one negative eigenvalue; that is, $R(\rho_t)$ is not positive semidefinite (PSD) for $t < 0$.

From the above analysis, it is trivial that $\widetilde{R}(\rho_t)$ defines a positive map when $t \geqslant 0$. Now by Theorem 1, for $t < 0$, $\widetilde{R}(\rho_t) > 0$, when the lower bound l of the proportion p is given as

$$l = \frac{4k}{\mathrm{Tr}[R(\rho_t)] + 4k} = p_1(t), \quad \text{(A6)}$$

where $\mathrm{Tr}[R(\rho_t)] = a_1(t)$ is given in (A2), $p_1(t)$ is defined in (60), and k is given by

$$\begin{aligned} k &= -\lambda_{\min}^{\mathrm{lb}}[\rho_t]\\ &= \tfrac{-1}{32}\left[7 + 8t - \sqrt{3(67 - 112t + 64t^2)}\right]. \end{aligned} \quad \text{(A7)}$$

2. Example 2

Consider the two-qutrit state defined in (64).

Let $f_2(x)$ be a characteristic polynomial of ρ_a given as

$$\begin{aligned} f_2(x) = {}& x^9 - a_1(a)x^8 + a_2(a)x^7 - a_3(a)x^6 + a_4(a)x^5\\ & - a_5(a)x^4 + a_6(a)x^3 - a_7(a)x^2 + a_8(a)x - a_9(a), \end{aligned}$$

where the coefficients $a_i(a)$ calculated in terms of moments using (15) are given as

$$\begin{aligned} a_1(a) &= \frac{9}{5 + 2a^2},\\ a_2(a) &= -\frac{4(-9 + a^2)}{(5 + 2a^2)^2},\\ a_3(a) &= -\frac{28(-3 + a^2)}{(5 + 2a^2)^3},\\ a_4(a) &= \frac{126 - 84a^2 + 5a^4}{(5 + 2a^2)^4},\\ a_5(a) &= \frac{126 - 140a^2 + 25a^4}{(5 + 2a^2)^5},\\ a_6(a) &= -\frac{2(-42 + 70a^2 - 25a^4 + a^6)}{(5 + 2a^2)^6},\\ a_7(a) &= -\frac{2(-18 + 42a^2 - 25a^4 + 3a^6)}{(5 + 2a^2)^7},\\ a_8(a) &= -\frac{-9 + 28a^2 - 25a^4 + 6a^6}{(5 + 2a^2)^8}, \end{aligned} \quad \text{(A8)}$$

and

$$a_9(a) = -\frac{(-1 + a^2)^2(-1 + 2a^2)}{(5 + 2a^2)^9}. \quad \text{(A9)}$$

From the coefficients of $f_2(x)$, it can be observed that at least one coefficient of $f_2(x)$ is negative. This means $R(\rho_a)$ has at least one negative eigenvalue; that is, $R(\rho_a)$ is not PSD.

Using Theorem 1, the approximated map $\widetilde{R}(\rho_a)$ is positive as well as completely positive when the lower bound l of the proportion p is given as

$$l = \frac{9k}{\mathrm{Tr}[R(\rho_a)] + 9k}, \quad \text{(A10)}$$

where $\mathrm{Tr}[R(\rho_a)]$ is the trace of $R(\rho_a)$ and

$$\begin{aligned} k &= -\lambda_{\min}^{\mathrm{lb}}[\rho_a]\\ &= -\frac{1}{5 + 2a^2} + 3\sqrt{2}\sqrt{\frac{1}{56 + 45a^2 + 9a^4}}. \end{aligned} \quad \text{(A11)}$$

Substituting the values of k and $\mathrm{Tr}[R(\rho_a)]$, the lower bound l_1 may be expressed as

$$l_1 = \frac{-1 + 15\sqrt{2}w + 6\sqrt{2}a^2w}{3\sqrt{2}(5 + 2a^2)w}, \quad \text{(A12)}$$

where $w = \sqrt{\frac{1}{56 + 9a^2(5 + a^2)}}$.

3. Example 3

Let us consider a two-qutrit isotropic state described by the density operator ρ_β in (68).

Let $f_3(x)$ be a characteristic polynomial of ρ_β given as

$$f_3(x) = x^9 - a_1(\beta)x^8 + a_2(\beta)x^7 - a_3(\beta)x^6 + a_4(\beta)x^5 - a_5(\beta)x^4 + a_6(\beta)x^3 - a_7(\beta)x^2 + a_8(\beta)x - a_9(\beta),$$

where the coefficients $a_i(\beta)$, in terms of moments, may be expressed as

$$a_1(\beta) = \frac{1}{3}(1+8\beta), \quad a_2(\beta) = \frac{4}{9}f(2+7\beta),$$
$$a_3(\beta) = \frac{28}{27}f^2(1+2\beta), \quad a_4(\beta) = \frac{14}{81}f^3(4+5\beta),$$
$$a_5(\beta) = \frac{14}{243}f^4(5+4\beta), \quad a_6(\beta) = \frac{28}{729}f^5(2+\beta),$$
$$a_7(\beta) = \frac{4\beta^6(7+2\beta)}{2187}, \quad a_8(\beta) = \frac{\beta^7(8+\beta)}{6561}, \tag{A13}$$

and $a_9(\beta) = \frac{\beta^8}{19623}$. Since all the coefficients $a_i(\beta)$, $i = 1$–9, of $f_3(x)$ are positive, the realignment matrix $R(\rho_\beta)$ is positive semidefinite. Thus, $\widetilde{R}(\rho_\beta)$ is completely positive for $0 \leqslant p \leqslant 1$.

4. Example 4

Consider the α state defined in (70). Let $f_2(x)$ be a characteristic polynomial of ρ_α given as

$$f_4(x) = x^9 - a_1(\alpha)x^8 + a_2(\alpha)x^7 - a_3(\alpha)x^6 + a_4(\alpha)x^5 - a_5(\alpha)x^4 + a_6(\alpha)x^3 - a_7(\alpha)x^2 + a_8(\alpha)x - a_9(\alpha),$$

where the coefficients $a_i(\alpha)$ calculated in terms of moments using (15) are given as

$$a_1(\alpha) = \frac{1+17\alpha}{2(1+8\alpha)}, \quad a_2(\alpha) = \frac{\alpha(7+59\alpha)}{2(1+8\alpha)^2},$$
$$a_3(\alpha) = \frac{a^2(21+109\alpha)}{2(1+8\alpha)^3}, \quad a_4(\alpha) = \frac{5\alpha^3(7+23\alpha)}{2(1+8\alpha)^4},$$
$$a_5(\alpha) = \frac{\alpha^4(35+67\alpha)}{2(1+8\alpha)^5}, \quad a_6(\alpha) = \frac{\alpha^5(21+17\alpha)}{2(1+8\alpha)^6},$$
$$a_7(\alpha) = \frac{\alpha^6(7-\alpha)}{2(1+8\alpha)^7}, \quad a_8(\alpha) = \frac{\alpha^7(1-\alpha)}{2(1+8\alpha)^8}, \tag{A14}$$

and $a_9(\alpha) = 0$. Now, since $a_i(\alpha) \geqslant 0$ for $i = 1$–9, $R(\rho_\alpha)$ is PSD. Hence, by Theorem 1, $\widetilde{R}(\rho_\alpha)$ defines a positive map for $0 \leqslant p \leqslant 1$ and for all $\alpha \in (0, 1)$.

APPENDIX B: ESTIMATION OF THE FIRST MOMENT OF $R(\rho)$

Let ρ_{AB} be a $(d \otimes d)$-dimensional state. Reference [42] showed that the measurement of moments of a partially transposed matrix is technically possible using m copies of the state ρ_{AB} and SWAP operations. In this process, the matrix power is written as an expectation value of a permutation operator. We can apply the same method adopted in Refs. [12,53], but on the single copy of the realigned matrix, as

$$m_1 = \text{Tr}[R(\rho_{AB})P], \tag{B1}$$

where P is the normalized permutation operator. Now, since $R(\rho_{AB})$ is not physically realizable, we need to express the first moment m_1 of $R(\rho_{AB})$ in terms of a physically realizable operator. From the definition (13) of the SPA of the realigned matrix, we can write

$$R(\rho_{AB}) \propto \widetilde{R}(\rho_{AB}) - \frac{p}{d^2}I_{d\otimes d}. \tag{B2}$$

Therefore, the first moment of $R(\rho_{AB})$ may be expressed as

$$\begin{aligned} m_1 &\simeq \text{Tr}\left\{\left[\widetilde{R}(\rho_{AB}) - \frac{p}{d^2}I_{d\otimes d}\right]P\right\} \\ &= \text{Tr}[\widetilde{R}(\rho_{AB})P] - \frac{p}{d^2}\text{Tr}[P] \\ &= \text{Tr}[\widetilde{R}(\rho_{AB})P] - \frac{p}{d^2} \\ &\leqslant \text{Tr}[\widetilde{R}(\rho_{AB})P] - \frac{k}{m_1 + d^2k}. \end{aligned} \tag{B3}$$

In the last line, we have used $p \geqslant \frac{d^2k}{m_1+d^2k}$ and $k = \max[0, -\lambda_{\text{min}}^{\text{lb}}[R(\rho_{AB})]]$, which is defined in Theorem 1. The equality is obtained when all the eigenvalues of $R(\rho_{AB})$ are positive.

Inequality (B3) may be reexpressed as

$$m_1 + \frac{k}{m_1 + d^2k} \leqslant \text{Tr}[\widetilde{R}(\rho_{AB})P] := s. \tag{B4}$$

Since $\widetilde{R}(\rho_{AB})$ is a positive semidefinite operator with unit trace, $s = \text{Tr}[\widetilde{R}(\rho_{AB})P]$ can be measured using controlled SWAP operations [42].

Inequality (B4) can be reexpressed as

$$m_1^2 + m_1(d^2k - s) + k(1 - d^2s) \leqslant 0. \tag{B5}$$

Solving the above quadratic equation for m_1, we have

$$\frac{-(d^2k-s) - \sqrt{(d^2k-s)^2 - 4k(1-d^2s)}}{2} \leqslant m_1 \leqslant \frac{-(d^2k-s) + \sqrt{(d^2k-s)^2 - 4k(1-d^2s)}}{2}. \tag{B6}$$

For m_1 to be real, we have

$$(d^2k - s)^2 - 4k(1 - d^2s) \geqslant 0. \tag{B7}$$

Also, let us assume that $1 - d^2s \geqslant 0$. Inequality (B7) may be further simplified to

$$\begin{aligned} &(d^2k - s)^2 - 4k(1 - d^2s) \geqslant 0 \\ &\Rightarrow d^4k^2 + 2k(d^2s - 2) + s^2 \geqslant 0. \end{aligned} \tag{B8}$$

Inequality (B8) holds when either $k \geqslant \frac{2-d^2s+2\sqrt{1-d^2s}}{d^4}$ or $k \leqslant \frac{2-d^2s-2\sqrt{1-d^2s}}{d^4}$.

Case 1. If $2 - d^2s + 2\sqrt{1-d^2s} \leqslant d^4k \leqslant d^4$, then

$$f_l(s) \leqslant m_1 \leqslant f_u(s). \tag{B9}$$

Case 2. If $0 \leqslant d^4k \leqslant 2 - d^2s - 2\sqrt{1-d^2s}$, then

$$g_l(s) \leqslant m_1 \leqslant g_u(s). \tag{B10}$$

The functions $f_l(s)$, $f_u(s)$, $g_l(s)$, an $g_u(s)$ are given as follows:

$$f_l(s) = \frac{1}{2}(-d^2+s) - \frac{1}{2d^2}\sqrt{d^8+2d^6s+4d^2s+d^4s^2-8(1+\sqrt{x})}, \tag{B11}$$

$$f_u(s) = \frac{-1}{d^2}(x+\sqrt{x}) + \frac{1}{2d^2}(\sqrt{d^8+2d^6s+4d^2s+d^4s^2-8(1+\sqrt{x})}), \tag{B12}$$

$$g_l(s) = \frac{1}{d^2}(-x+\sqrt{x}-\sqrt{1+x-2\sqrt{x}}), \tag{B13}$$

$$g_u(s) = \frac{s}{2} + \frac{1}{d^2}\sqrt{1+x-2\sqrt{x}}, \tag{B14}$$

where $x = 1 - d^2 s$.

Hence, the first moment of $R(\rho_{AB})$ may be estimated using (B9)–(B14). Since the functions f_l, f_u, g_l, and g_u are expressed in terms of $s = \text{Tr}[\widetilde{R}(\rho_{AB})P]$, the first moment of $R(\rho_{AB})$ can be estimated experimentally.

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).
[2] C. H. Bennett and V. Brassard, in *Proceedings of IEEE International Conference on Computers, Systems and Signal Processing, Bangalore, India* (IEEE, Piscataway, NJ, 1984), p. 175.
[3] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).
[4] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).
[5] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th anniversary ed. (Cambridge University Press, New York, 2011).
[6] A. K. Pati, Phys. Rev. A **63**, 014302 (2000).
[7] S. Lloyd, Science **273**, 1073 (1996).
[8] K. Kraus, *States, Effects, and Operators: Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).
[9] W. F. Stinespring, Proc. Amer. Math. Soc. **6**, 211 (1955).
[10] M. D. Choi, Linear Algebra Appl. **10**, 285 (1975).
[11] C.-K. Li and Y.-T. Poon, Linear Multilinear Algebra **59**, 1159 (2011).
[12] P. Horodecki and A. Ekert, Phys. Rev. Lett. **89**, 127902 (2002).
[13] J. Fiurášek, Phys. Rev. A **66**, 052315 (2002).
[14] R. Augusiak, J. Bae, L. Czekaj, and M. Lewenstein, J. Phys. A **44**, 185308 (2011).
[15] K. C. Ha and S. H. Kye, Commun. Math. Phys. **328**, 131 (2014).
[16] R. Augusiak, J. Bae, J. Tura, and M. Lewenstein, J. Phys. A **47**, 065301 (2014).
[17] J. Bae, Rep. Prog. Phys. **80**, 104001 (2017).
[18] P. Horodecki, Phys. Rev. Lett. **90**, 167901 (2003).
[19] S. Adhikari, Phys. Rev. A **97**, 042344 (2018).
[20] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[21] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[22] H.-T. Lim, Y.-S. Kim, Y.-S. Ra, J. Bae, and Y.-H. Kim, Phys. Rev. Lett. **107**, 160401 (2011).
[23] H.-T. Lim, Y.-S. Kim, Y.-S. Ra, J. Bae, and Y.-H. Kim, Phys. Rev. A **86**, 042334 (2012).
[24] A. Kumari and S. Adhikari, Phys. Rev. A **100**, 052323 (2019).
[25] A. Kumari and S. Adhikari, Ann. Phys. (NY) **446**, 169113 (2022).
[26] K. Chen and L. A. Wu, Quantum Inf. Comput. **3**, 193 (2003).
[27] O. Rudolph, Quantum Inf. Process. **4**, 219 (2005).
[28] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Phys. Rev. Lett. **82**, 5385 (1999).
[29] O. Gühne, P. Hyllus, O. Gittsovich, and J. Eisert, Phys. Rev. Lett. **99**, 130504 (2007).
[30] O. Rudolph, Lett. Math. Phys. **70**, 57 (2004).
[31] K. Chen and L. A. Wu, Phys. Lett. A **306**, 14 (2002); S. Albeverio, K. Chen, and S. M. Fei, Phys. Rev. A **68**, 062313 (2003).
[32] C. Lupo, P. Aniello, and A. Scardicchio, J. Phys. A **41**, 415301 (2008); C. K. Li, Y. T. Poon, and N. S. Sze, *ibid.* **44**, 315304 (2011).
[33] S. Q. Shen, M. Y. Wang, M. Li, and S. M. Fei, Phys. Rev. A **92**, 042332 (2015).
[34] S. Aggarwal and S. Adhikari, Quantum Inf. Process. **20**, 83 (2021).
[35] S. Aggarwal and S. Adhikari, Ann. Phys. (NY) **444**, 169043 (2022).
[36] X. Qi, T. Gao, and F. Yan, Int. J. Theor. Phys. **56**, 3642 (2017).
[37] T. Zhang, N. Jing, and S.-M. Fei, Quantum Inf. Process. **21**, 276 (2022).
[38] S. Aggarwal, S. Adhikari, and A. S. Majumdar, arXiv:2302.04797.
[39] J. K. Korbicz, M. L. Almeida, J. Bae, M. Lewenstein, and A. Acin, Phys. Rev. A **78**, 062105 (2008).
[40] T. A. Brun, Quantum Inf. Comput. **4**, 401 (2004).
[41] T. Tanaka, Y. Ota, M. Kanazawa, G. Kimura, H. Nakazato, and F. Nori, Phys. Rev. A **89**, 012117 (2014).
[42] J. Gray, L. Banchi, A. Bayat, and S. Bose, Phys. Rev. Lett. **121**, 150503 (2018).
[43] X.-D. Yu, S. Imai, and O. Gühne, Phys. Rev. Lett. **127**, 060504 (2021).
[44] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 1999); A. Knutson and T. Tao, Notices Amer. Math. Soc. **48**, 175 (2001).
[45] H. Wolkowicz and G. Styan, Linear Algebra Appl. **29**, 471 (1980).
[46] X. Wang, The American Mathematical Monthly **111**, 525 (2004).

[47] D. Zeilberger, Discrete Math. **49**, 319 (1984).

[48] A. Hertz, M. Arnhem, A. Asadian, and N. J. Cerf, Phys. Rev. A **104**, 022427 (2021).

[49] O. Rudolph, Phys. Rev. A **67**, 032312 (2003).

[50] S. Rana, Phys. Rev. A **87**, 054301 (2013).

[51] R. A. Bertlmann and P. Krammer, J. Phys. A **41**, 235303 (2008).

[52] P. Horodecki, Phys. Lett. A **232**, 333 (1997).

[53] A. K. Ekert, C. M. Alves, D. K. L. Oi, M. Horodecki, P. Horodecki, and L. C. Kwek, Phys. Rev. Lett. **88**, 217901 (2002).